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The Path Formulation of Bifurcation Theory

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Abstract

We show how the path formulation of bifurcation theory can be made to work, and that it is (essentially) equivalent to the usual parametrized contact equivalence of Golubitsky and Schaeffer.

Introduction

In their original paper on imperfect bifurcation theory [GS79], Golubitsky and Schaeffer consider the so-called path formulation of bifurcation theory. However they had to abandon this approach as the calculations were mostly intractable, and they replaced it by their now standard distinguished parameter formulation. In this paper I describe how the path approach can be made to work thanks to recent advances in Singularity Theory, and I will show that it is (almost) equivalent to the distinguished parameter formulation. The new technology available, allowing the path formulation to work is two-fold: firstly, computations are facilitated by the development of computer algebra packages, and secondly the path-formulation itself is clarified by the introduction by J. Damon of $\mathcal{K}_V$-equivalence [D87].

As far as the computations go, Golubitsky and Schaeffer found the distinguished parameter approach more tractable thanks to a lemma ensuring that a bifurcation problem is finitely determined with respect to distinguished parameter contact equivalence if and only if it is finitely determined with respect to a restricted form of equivalence which is easier to compute. However, this lemma fails to hold as soon as there is more than 1 parameter, and in that case the computations of the full distinguished parameter equivalence are considerably harder than those of the path formulation. See for example [P].

There is one drawback at present to a coherent path formulation, and that is the distinction between the smooth (C^∞) and analytic theories. The problem arises as the modules of smooth vector fields tangent to certain varieties (discriminants) are not necessarily finitely generated. However, in the analytic category, all such modules are finitely generated. One can argue that this is not a problem, since one is dealing with finitely determined bifurcation problems, so that after a change of coordinates, they are analytic, and even polynomial. It seems that this shortcoming may be able to be overcome, but the details are still to be worked out.

Part of the object of this paper is to give a general description of Singularity Theory for the non-specialist; this is done in Section 1. This point of view which groups all the different equivalence relations together and puts "bifurcation equivalence" in a wider perspective, is not evident in Golubitsky and Schaeffer [GS85]. The remainder of the paper is organized as follows. Section 2 introduces Damon's notion of $\mathcal{K}_V$-equivalence. In Section 3, we give the main theorem (paragraph (3.3)), and in Section 4 we describe how one calculates generators of Derlog(Δ), necessary for calculating $\mathcal{K}_\Delta$-tangent spaces. We conclude with some remarks and questions for symmetric bifurcations.

1 Singularities, Bifurcations and Paths

In this section we give a brief overview of the salient points of singularity theory necessary for understanding the results of this paper. We will be considering families of maps (or map-germs) from $\mathbf{R}^n$ to $\mathbf{R}^p$, and occasionally families of families of maps. The parameter spaces for the families will be denoted Λ, V or U according to the interpretation. Λ will be for the parameter space of a given bifurcation problem, U will always denote the parameter space (or base space) of a versal deformation, and V will be the base space for an arbitrary deformation. An excellent reference for the main results of singularity theory is C.T.C. Wall's survey paper [W], although much progress and consolidation has been made since then, and in particular Damon's introduction of Geometric Subgroups of $\mathcal{A}$ and $\mathcal{K}$ [D84], as the general class for singularity theoretic equivalences.

It should be borne in mind that we are really considering germs of maps and germs of deformations, so that all spaces such as $\mathbf{R}^n$ and $\mathbf{R}^p$ should really be considered as (small) neighbourhoods of the origin in $\mathbf{R}^n$ and $\mathbf{R}^p$ respectively. Similarly, although we say $\Lambda = \mathbf{R}^k$, we really mean that Λ is a neighbourhood of the origin in $\mathbf{R}^k$. We will not usually refer explicitly to germs, though there are occasional lapses — either through inconsistency or to remind the reader!

It should also be borne in mind that although we only make explicit reference in this section to real (C^∞) maps, one could equally well consider real analytic or complex analytic maps (or germs!). However, in Sections 2 and 3, there are certain results that only hold in the analytic categories.

(1.1) Bifurcation problems
For the purposes of this paper, a bifurcation problem is an equation of the form

$$g(x, \lambda) = 0,$$

where $g : \mathbf{R}^n \times \Lambda \to \mathbf{R}^p$ is a map-germ defined at $(0, 0) \in \mathbf{R}^n \times \Lambda$, and $\Lambda = \mathbf{R}^k$. We view Λ as parameter space, and this distinction between Λ and $\mathbf{R}^n$ is reflected in the notion of equivalence used in bifurcation theory. Thus, a bifurcation problem is a system of p equations in n unknowns, with k parameters. In applications, it is common that $n = p$; however it makes no difference to the theory. We often refer to the map g as the bifurcation problem, with the equation $g = 0$ understood.

(1.2) Organizing centre
The organizing centre of (1.1) is obtained by putting $\lambda = 0$:

$$g_0(x) = g(x, 0),$$

so that $g_0 : \mathbf{R}^n \to \mathbf{R}^n$.

(1.3) Equivalence of bifurcation problems
Following Golubitsky and Schaeffer [GS85], two bifurcation problems

$$f, g : \mathbf{R}^n \times \Lambda \to \mathbf{R}^n$$

are said to be equivalent if there is a diffeomorphism $(x, \lambda) \mapsto (H(x, \lambda), h(\lambda))$ and an invertible $p \times p$ matrix $S(x, \lambda)$ depending on x and λ such that

$$f(H(x, \lambda),\, h(\lambda)) = S(x, \lambda) g(x, \lambda).$$

We say f and g are *bifurcation equivalent*, or $\mathcal{B}$-equivalent. In the special case that h is the identity, the equivalence is called *restricted bifurcation equivalence.*

Putting $\lambda = 0$ we arrive at a natural equivalence of organizing centres,

$$f_0(H(x)) = S(x) g_0(x),$$

where now S is a $p \times p$ matrix depending only on x, and H is a change of coordinates on $\mathbf{R}^n$. This equivalence is called *contact equivalence*, or $\mathcal{K}$-equivalence; it was introduced into singularity theory by J. Mather in the late 1960's. (It is also sometimes known as V-equivalence [AGV].) Thus, bifurcation equivalence is a parametrized version of contact equivalence — see the next paragraph.

(1.4) Deformations and their equivalence
One of the important applications of singularity theory is to the study of how maps deform. One is able to deal in the same way with many types of equivalence (contact, bifurcation, right, left-right, equivariant, . . .).

Let $f : X \to Y$ be a map (e.g. $X = \mathbf{R}^n$ for contact equivalence, $X = \mathbf{R}^n \times \Lambda$ for bifurcation equivalence). A deformation of f is a map

$$F : X \times U \to Y$$

satisfying $F(\cdot, 0) = f$. The deformed (or perturbed) map $x \mapsto F(x, u)$ is denoted F_u. If F is a deformation of f, then any map $\gamma : V \to U$ defines another deformation of f, denoted $\gamma^* F$, by

$$\gamma^* F(x, v) := F(x, \gamma(v)).$$

The deformation $\gamma^* F$ is said to be *induced from F by γ.* This idea is central to what follows. (Note that since we are really talking about germs, we automatically have $\gamma(0) = 0$.)

Suppose now that $\mathcal{G}$ is one of the equivalence relations of singularity theory (contact, bifurcation, . . .). Then $\mathcal{G}$ defines an equivalence of deformations, sometimes denoted $\mathcal{G}_{\mathrm{un}}$, as follows. Let $F_1 : X \times U_1 \to Y$ and $F_2 : X \times U_2 \to Y$ be two deformations of f. Then F_1 and F_2 are said to be $\mathcal{G}_{\mathrm{un}}$-equivalent if there is a diffeomorphism $h : U_1 \to U_2$ such that $F_{1,u}$ is $\mathcal{G}$-equivalent to $F_{2,h(u)}$ for all $u \in U_1$. Moreover, the equivalences must depend smoothly on the parameter.

For contact equivalence ($\mathcal{G} = \mathcal{K}$), equivalence of deformations is precisely bifurcation equivalence. Thus $\mathcal{K}_{\mathrm{un}} = \mathcal{B}$.

For bifurcation equivalence, equivalence of deformations is a little more complex. A deformation of a bifurcation problem $g(x, \lambda)$ is a map

$$\tilde{g} : \mathbf{R}^n \times \Lambda \times V \to \mathbf{R}^p$$

such that $g(x, \lambda) = \tilde{g}(x, \lambda, 0)$. Two deformations $\tilde{g}_1$ and $\tilde{g}_2$ of a bifurcation problem g are deformation bifurcation equivalent ($\mathcal{B}_{\mathrm{un}}$-equivalent) if there are changes of coordinates $(x, \lambda, v) \mapsto (H(x, \lambda, v), h_1(\lambda, v), h_2(v))$ and a matrix $S(x, \lambda, v)$ such that

$$\tilde{g}_1(x, \lambda, v) = S(x, \lambda, v)\, \tilde{g}_2(H(x, \lambda, v), h_1(\lambda, v), h_2(v)).$$

(1.5) Versal deformations

One of the basic notions of singularity theory is that of a versal deformation; it applies to all the usual equivalences. A versal deformation is a deformation which contains (up to the equivalence in question) any deformation of the singularity. For contact equivalence, this reads as follows.

Let $g_0 : \mathbf{R}^n \to \mathbf{R}^p$ be given, and let $G : \mathbf{R}^n \times U \to \mathbf{R}^p$ be a deformation of g_0 (so that $g_0 = G(\cdot, 0)$). One says that G is a *versal deformation* of g_0 if for any deformation $g : \mathbf{R}^n \times V \to \mathbf{R}^p$ of g_0 there is a map $\gamma : V \to U$ such that $g(x, v)$ is parametrized contact equivalent to $G(x, \gamma(v))$.

A deformation

$$G : \mathbf{R}^n \times \Lambda \times U \to \mathbf{R}^p$$

of a bifurcation problem $g : \mathbf{R}^n \times \Lambda \to \mathbf{R}^p$ is said to be a *versal deformation* of g if for every deformation $\tilde{g} : \mathbf{R}^n \times \Lambda \times V \to \mathbf{R}^p$ of g there is a map $\gamma : V \to U$ such that $\tilde{g}(x, \lambda, v)$ and $G(x, \lambda, \gamma(v))$ are parametrized bifurcation equivalent.

There is a simple algebraic criterion for deciding whether a given deformation is versal, in terms of the tangent or normal spaces — see paragraph (1.9).

(1.6) Example

Consider the organizing centre $g_0(x) = x^3$ (here $n = p = 1$). There are several well-known bifurcation problems with this organizing centre. For example,

Pitchfork: $g(x, \lambda) = x^3 - \lambda x$;

Hysteresis: $g(x, \lambda) = x^3 - \lambda$.

A versal deformation of g_0 is given by

$$G(x, u_1, u_2) = x^3 + u_1 x + u_2,$$

where $U = \mathbf{R}^2$. The two bifurcation problems are induced by the maps $\gamma : \mathbf{R} \to \mathbf{R}^2$ given by,

Pitchfork: $\gamma(\lambda) = (-\lambda, 0)$;

Hysteresis: $\gamma(\lambda) = (0, -\lambda)$.

Versal deformations of the two bifurcation problems are given by

Pitchfork: $\tilde{G}(x, \lambda, u_1, u_2) = x^3 - \lambda x + u_1 + u_2 \lambda$;

Hysteresis: $\tilde{G}(x, \lambda, u) = x^3 - \lambda + ux$.

Versal deformations are often called universal unfoldings [GS85]. The word versal is used in singularity theory rather than universal, since the prefix 'uni' refers to uniqueness, and versal deformations are not unique. The difference between a deformation and an unfolding is mainly notational, and need not concern us here.

(1.7) Tangent spaces
Associated to any map (germ), and any equivalence relation in singularity theory, is the 'tangent space' of the map in question. It is essentially the tangent space to the equivalence class containing the map. To calculate it, one uses the given class of diffeomorphisms, and differentiates to obtain a tangent space. It is a subset of all infinitesimal deformations of the given map.

Notation: We denote by $\mathcal{E}_n$ the ring of C^∞ functions on $\mathbf{R}^n$, by $\mathcal{E}_\Lambda$ the functions on Λ, and $\mathcal{E}_{n,\lambda}$ consists of the functions on $\mathbf{R}^n \times \Lambda$. Similarly, Θ_n denotes the $\mathcal{E}_n$-module of vector fields on $\mathbf{R}^n$. The (maximal) ideal of functions vanishing at $0 \in \mathbf{R}^p$ is denoted $\boldsymbol{m}_p$, and consequently $\boldsymbol{m}_p\Theta_p$ is the $\mathcal{E}_p$-module of vector fields on $\mathbf{R}^p$ that vanish at the origin. Finally, we denote by $\Theta_{n,\lambda}$ the $\mathcal{E}_{n,\lambda}$-module of vector fields on $\mathbf{R}^n$ parametrized by $\lambda \in \Lambda$.

Let f be a map (organizing centre, bifurcation problem, or whatever), and $\mathcal{G}$ an equivalence relation (contact, bifurcation, or whatever). The space of infinitesimal deformations of f is denoted Θ_f consists of vector fields along f, that is, vector fields on $\mathbf{R}^a$ with values in $\mathbf{R}^b$ (more brutally, if $f \in C^\infty(\mathbf{R}^a, \mathbf{R}^b)$ then $\Theta_f = C^\infty(\mathbf{R}^a, \mathbf{R}^b)$). The $\mathcal{G}$-tangent space of f is a subspace of Θ_f, denoted $T\mathcal{G}_e \cdot f$ (the $_e$ is for 'extended'[1]). Note that Θ_f is a module over the ring of smooth functions on $\mathbf{R}^a$.

For $g_0 : \mathbf{R}^n \to \mathbf{R}^p$, and $\mathcal{K}$-equivalence, one finds that

$$T\mathcal{K}_e \cdot g_0 = tg_0(\Theta_n) + g_0^*(\boldsymbol{m}_p\Theta_p).$$

The term $tg_0(\Theta_n)$ is the image of vector fields under the tangent mapping tg_0 of g_0; the term $g_0^*(\boldsymbol{m}_p\Theta_p)$ is the $\mathcal{E}_n$-module generated by the pull-backs of vector fields on $\mathbf{R}^p$, that is by the set of vector fields of the form $v \circ g_0$, with $v \in \boldsymbol{m}_p\Theta_p$. Such composites are vector fields along g_0.

For a bifurcation problem $g : \mathbf{R}^n \times \Lambda \to \mathbf{R}^p$, the tangent space for bifurcation equivalence is given by

$$T\mathcal{B}_e \cdot g = t_1 g(\Theta_{n,\lambda}) + g^*(\Theta_{p,\lambda}) + t_2 g(\Theta_\Lambda).$$

Here $t_1 g$ and $t_2 g$ mean differentiating with respect to the first ($\mathbf{R}^n$) and second (Λ) variables, respectively. Note that each of the first two terms is an $\mathcal{E}_{n,\lambda}$-module, while the third term is merely an $\mathcal{E}_\Lambda$-module. The whole is therefore only an $\mathcal{E}_\Lambda$-module. Golubtsky and Schaeffer [GS85] denote this tangent space by $T(g)$. Their restricted tangent space $RT(g)$ is given by the first two terms only (the third is omitted by forbidding changes in the parameter) and is therefore an $\mathcal{E}_{n,\lambda}$-module. In [MM], $RT(g)$ is denoted $T\mathcal{K}_{\text{rel}} \cdot g$.

[1] The 'unextended' tangent space $T\mathcal{G} \cdot f$ is defined in the same way, but using only the vector fields that vanish at 0; it is used in conditions for finite determinacy.

(1.8) Normal spaces
Given the (extended) tangent space $T\mathcal{G}_e{\cdot}f \subset \Theta_f$ one defines the normal space as the quotient:

$$N\mathcal{G}{\cdot}f = \frac{\Theta_f}{T\mathcal{G}_e{\cdot}f}.$$

This of course holds for $\mathcal{G} = \mathcal{K}, \mathcal{B}$ etc. The *codimension* of f with respect to $\mathcal{G}$-equivalence is defined to be,

$$\operatorname{cod}_{\mathcal{G}} f = \dim_{\mathbf{R}} N\mathcal{G}{\cdot}f.$$

In the case that $\dim \Lambda = 1$, one has that $T(g)$ has finite codimension if and only if $RT(g)$ does [GS85, p. 127]. This fact allows Golubitsky and Schaeffer to make their theory computable: being an $\mathcal{E}_{n,\lambda}$-module makes $RT(g)$ much easier to compute than $T(g) = T\mathcal{B}_e{\cdot}g$.

(1.9) Versality theorem
One of the basic theorems of singularity theory gives a simple criterion for determining whether a given deformation is versal (which works for all equivalence relations $\mathcal{G}$ such as contact, bifurcation, . . .all "geometric subgroups" of $\mathcal{A}$ and $\mathcal{K}$ [D84]).

Let f be a map (germ) and $\mathcal{G}$ one of the singularity theory equivalences appropriate to f. Let $F = f + u_1\phi_1 + \cdots u_r\phi_r$ be a deformation of f, with $\phi_1, \ldots, \phi_r \in \Theta_f$. Then F is a versal deformation if and only if $\{\phi_1, \ldots, \phi_r\}$ spans $N\mathcal{G}{\cdot}f$ as a real vector space. In other words, F is versal if and only if

$$T\mathcal{G}_e{\cdot}f + \mathbf{R}\{\phi_1, \ldots, \phi_r\} = \Theta_f.$$

The codimension of a singularity is thus the number of parameters needed for a versal deformation. The space U is called the *base space* of the versal deformation.

(1.10) Discriminant
Let $G : \mathbf{R}^n \times U \to \mathbf{R}^p$ be a versal deformation of the map $g_0 : \mathbf{R}^n \to \mathbf{R}^p$ (with respect to $\mathcal{K}$-equivalence). For each $u \in U$, let $G_u : \mathbf{R}^n \to \mathbf{R}^p$ be the map given by

$$G_u(x) = G(x, u).$$

The following conditions on $u \in U$ are equivalent:

(i) *there is an $x \in \mathbf{R}^n$ such that $G(x, u) = 0$ and G_u is singular at x, and*

(ii) *u is a singular value of the projection $\pi_G : G^{-1}(0) \to U$ given by $\pi_G(x, u) = u$.*

This fact is easy to prove. The set of all such u is called the *discriminant* of the versal deformation G, denoted $\Delta = \Delta_G$. It is the basic geometric object for the remainder of this paper. It is a hypersurface in U, i.e. given by one equation $h(u) = 0$ with $h : U \to \mathbf{R}$.

For example, in the case $n = p$ (central to bifurcation theory), for each $u \in U$ the set $G_u^{-1}(0)$ is finite (otherwise g_0 would not be of finite codimension, and so would not have a versal deformation). The number of elements in $G_u^{-1}(0)$ is locally constant on an open dense set in U, whose complement is precisely the discriminant. Thus, the discriminant consists of those points u for which G_u has multiple roots. Over $\mathbf{C}$, the number of elements in $G_u^{-1}(0)$ is constant, for $u \notin \Delta$, not merely locally constant.

It is a central observation for this paper that:
Suppose $g = \gamma^*G$, then bifurcation points of g correspond under γ to points of image$(\gamma) \cap \Delta_G$.

2 $\mathcal{K}_\Delta$-equivalence

A new equivalence relation on maps was introduced a few years ago by J. Damon [D87], called $\mathcal{K}_V$-equivalence (or $\mathcal{K}_\Delta$-equivalence). It is a generalization of Mather's contact equivalence (see (1.3) above), which has been finding many applications. For an application to caustics see [M] and to bifurcations of periodic points see [BF].

Consider maps $\gamma : \Lambda \to U$ and a subset (subvariety) $\Delta \subset U$. The geometrical notion captured by $\mathcal{K}_\Delta$ equivalence is the *contact of* γ *with* Δ, which is clearly important since bifurcations correspond to points of intersection of γ with the discriminant Δ.

(2.1) Definition
Two maps $\gamma_1, \gamma_2 : \Lambda \to U$ are said to be $\mathcal{K}_\Delta$-equivalent if there exist diffeomorphisms h of Λ, and H of $\Lambda \times U$ satisfying

- $H(\lambda, u) = (h(\lambda), \bar{H}(\lambda, u))$, for some $\bar{H} : \Lambda \times U \to U$,
- $u \in \Delta \Rightarrow \bar{H}(\lambda, u) \in \Delta$, and
- $\gamma_1(h(\lambda)) = \bar{H}(\lambda, \gamma_2(\lambda))$, for all $\lambda \in \Lambda$.

In other words, H maps the graph of γ_2 to the graph of γ_1, whilst preserving Δ. In the case that $\Delta = \{0\}$, then $\mathcal{K}_\Delta$-equivalence reduces to $\mathcal{K}$-equivalence. It is clear that if γ_1 and γ_2 are $\mathcal{K}_\Delta$ equivalent, then $\gamma_1^{-1}(\Delta)$ and $\gamma_2^{-1}(\Delta)$ are diffeomorphic; however in general the converse is not true (it is true if Δ is smooth).

(2.2) Derlog(Δ)
For most varieties $\Delta \subset U$, it is not easy to characterize the set of diffeomorphisms preserving Δ. However, the infinitesimal version is often not so hard. For a vector field $\xi \in \Theta_U$ to integrate to a 1-parameter family of diffeomorphisms preserving Δ, it is necessary and sufficient that ξ be tangent to Δ. Note that if Δ is singular, then tangent to Δ means tangent to each stratum of some natural stratification of Δ.

The $\mathcal{E}_U$-module of vector fields tangent to Δ has the unfortunate name Derlog(Δ), for reasons that go well beyond this paper [S].

A few words about the structure of Derlog(Δ) are in order. Firstly, Δ is a hypersurface, given by the equation $h(u) = 0$, so that

$$\mathrm{Derlog}(\Delta) = \{\theta \in \Theta_U \mid \theta(h) \in \langle h \rangle\},$$

where $\langle h \rangle$ is the ideal generated by h. This is because if θ is tangent to Δ and as h is constant on Δ, then $\theta(h) = 0$ on Δ. In other words, $\theta \in \mathrm{Derlog}(\Delta)$ if and only if there exists $f \in \mathcal{E}_U$ for which $\theta(h) = fh$.

We can define a submodule $\mathrm{Derlog}(h) \subset \mathrm{Derlog}(\Delta)$, by

$$\mathrm{Derlog}(h) = \{\theta \in \Theta_U \mid \theta(h) = 0\}.$$

It consists of those vector fields that are tangent to all level sets of h, and not just to the zero level set Δ. Clearly, Derlog(h) depends on the choice of function used to define Δ, whereas Derlog(Δ) does not.

Suppose now that h is weighted homogeneous, so that there are integers $w_1, \ldots, w_\ell$ (where $\ell = \dim U$), such that

$$h(t^{w_1}u_1, \ldots, t^{w_\ell}u_\ell) = t^d h(u_1, \ldots, u_\ell),$$

for some d — the *degree* of h with respect to the given weights. Then Euler's formula states

$$\sum_{j=1}^{\ell} w_j u_j \frac{\partial}{\partial u_j} h = d.h.$$

The vector field

$$\boldsymbol{e} = \sum_{j=1}^{\ell} w_j u_j \frac{\partial}{\partial u_j},$$

is called the Euler vector field for these weights, and we have $\boldsymbol{e}(h) = d.h$.

Suppose now that $\theta \in \mathrm{Derlog}(\Delta)$, with $\theta(h) = fh$. Then the vector field $\bar{\theta} = \theta - d^{-1}f\boldsymbol{e}$ satisfies $\bar{\theta} \in \mathrm{Derlog}(h)$, as is easy to see. Consequently, there is a natural projection $\mathrm{Derlog}(\Delta) \longrightarrow \mathrm{Derlog}(h)$, $\theta \to \bar{\theta}$ whose kernel is precisely $\mathcal{E}_U.\boldsymbol{e}$. Thus, in the weighted homogeneous case,

$$\mathrm{Derlog}(\Delta) = \mathcal{E}_U.\boldsymbol{e} \oplus \mathrm{Derlog}(h),$$

a direct sum of $\mathcal{E}_U$-modules. It is generally easier to calculate $\mathrm{Derlog}(h)$ than $\mathrm{Derlog}(\Delta)$.

(2.3) Liftable vector fields

There is an important geometric characterization of elements of $\mathrm{Derlog}(\Delta)$ when Δ is the discriminant of a map $\pi_G : G^{-1}(0) \to U$, namely they are the *liftable* vector fields. However, this only holds in the analytic categories (real and complex), and not in general for C^∞ maps and vector fields.

In general, let $f : X \to U$ be a map. A vector field $\eta \in \Theta_U$ is said to be *liftable over* f (or via f) if there is a vector field $\xi \in \Theta_X$ such that $df_x(\xi_x) = \eta_{f(x)}$. It is not hard to show that any liftable vector field must be tangent to the discriminant $\Delta(f)$ of f (integrating ξ and η give diffeomorphisms r of X and ℓ of U such that $f \circ r = \ell \circ f$, so that ℓ must preserve $\Delta(f)$).

For certain maps the converse is also true. In particular, Looijenga proved [L] that if $G : \mathbf{R}^n \times U \to \mathbf{R}^p$ is a versal deformation, and $\pi_G : G^{-1}(0) \to U$ the associated projection, then a vector field $\eta \in \Theta_U$ is liftable over π_G if and only if $\eta \in \mathrm{Derlog}(\Delta_G)$ (recall that Δ_G is the discriminant of π_G, see (1.10)).

More recently the general relationship between liftable vector fields and vector fields tangent to a discriminant has been clarified by Bruce, du Plessis and Wilson [BdPW].

(2.4) Example

Let $U = \mathbf{R}^2$ and Δ be defined by the equation $h(u_1, u_2) = 4u_1^3 + 27u_2^2 = 0$ (this is the equation for the discriminant of the versal deformation of $g_0(x) = x^3$ given in (1.6)). Then $\mathrm{Derlog}(\Delta)$ is generated over $\mathcal{E}_U$ by the two vector fields

$$\boldsymbol{e} = \begin{pmatrix} 2u_1 \\ 3u_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 9u_2 \\ -2u_1^2 \end{pmatrix}.$$

It is easy to show that any vector field annihilating h is a multiple of the second generator. Here $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha\frac{\partial}{\partial u_1} + \beta\frac{\partial}{\partial u_2}$.

Note that as discussed in the previous paragraph, these two vector fields are indeed liftable. The Euler field $\boldsymbol{e}$ lifts to $\begin{pmatrix} x \\ 2u_1 \\ 3u_2 \end{pmatrix}$, while the other generator lifts to $\begin{pmatrix} 3x^2 + 2u_1 \\ 9u_2 \\ -2u_1^2 \end{pmatrix}$. These vector fields on $\mathbf{R} \times U$ are both tangent to $G^{-1}(0)$, as is easily checked.

For futher examples, see Section 4 below.

(2.5) $\mathcal{K}_\Delta$ tangent and normal spaces
Let $\gamma : \Lambda \to U$, and $\Delta \subset U$ a subvariety. Then the extended $\mathcal{K}_\Delta$-tangent space to γ is

$$T\mathcal{K}_{\Delta,e}\cdot\gamma = t\gamma(\Theta_\Lambda) + \gamma^* \operatorname{Derlog}(\Delta).$$

Notice how this is very similar to the ordinary $\mathcal{K}$-tangent space, except that $\boldsymbol{m}_p\Theta_p$ has been replaced by $\operatorname{Derlog}(\Delta)$. This is because instead of diffeomorphisms preserving the origin in $\mathbf{R}^p$, here we are considering diffeomorphisms preserving Δ.

The $\mathcal{K}_\Delta$-normal space is of course defined by

$$N\mathcal{K}_\Delta\cdot\gamma = \Theta_\gamma / T\mathcal{K}_{\Delta,e}\cdot\gamma.$$

(2.6) Example
For the paths defining the pitchfork and hysteresis bifurcations (1.6), and the discriminant $\Delta \subset U$, we can compute the $\mathcal{K}_\Delta$ tangent and normal spaces. Generators of the module $\operatorname{Derlog}(\Delta)$ are given in (2.4).

For the pitchfork $\gamma(\lambda) = (-\lambda, 0)$, so that

$$t\gamma(\Theta_\Lambda) = (\mathcal{E}_\Lambda, 0),$$

while

$$\gamma^*(\operatorname{Derlog}(\Delta)) = (\lambda\mathcal{E}_\Lambda, \lambda^2\mathcal{E}_\Lambda).$$

A similar calculation for the hysteresis bifurcation gives

$$t\gamma(\Theta_\Lambda) = (0, \mathcal{E}_\Lambda),$$

$$\gamma^*(\operatorname{Derlog}(\Delta)) = (\lambda\mathcal{E}_\Lambda, \lambda\mathcal{E}_\Lambda).$$

Pitchfork: $T\mathcal{K}_{\Delta,e}\cdot\gamma = (\mathcal{E}_\Lambda, \lambda^2\mathcal{E}_\Lambda)$,

Hysteresis: $T\mathcal{K}_{\Delta,e}\cdot\gamma = (\lambda\mathcal{E}_\Lambda, \mathcal{E}_\Lambda)$.

The normal spaces are thus given by

Pitchfork: $N\mathcal{K}_\Delta\cdot\gamma \simeq \mathbf{R}\{(0,1),\, (0,\lambda)\}$,

Hysteresis: $N\mathcal{K}_\Delta\cdot\gamma \simeq \mathbf{R}\{(1,0)\}$.

The $\mathcal{K}_\Delta$-codimension of the first path is thus 2, while that of the second is only 1.

(2.7) $\mathcal{K}_\Delta$-versal deformations
As with other equivalence relations, a deformation $\Gamma : \Lambda \times V \to U$ of γ is said to be $\mathcal{K}_\Delta$-versal if any deformation of γ is equivalent to one induced from Γ. Also as with other equivalence relations (of the singularity theory type) one has the following result:

The deformation Γ of γ given by

$$\Gamma(\lambda, v_1, \ldots, v_r) = \gamma(\lambda) + \sum_j v_j \phi_j(\gamma)$$

is versal if and only if the ϕ_j span $N\mathcal{K}_\Delta \cdot \gamma$.

In the example above, versal deformations of γ are given by

Pitchfork: $\Gamma(\lambda, u_1, u_2) = (-\lambda, u_1 + u_2\lambda)$,

Hysteresis: $\Gamma(\lambda, u) = (u, -\lambda)$.

These expressions should be compared to the versal deformations of the two bifurcation problems given in (1.6)

(2.8) Finite determinacy
Another property of maps considered in singularity theory is finite determinacy. For $\mathcal{K}_\Delta$-equivalence, this reads as follows.

A path $\gamma : \Lambda \to U$ is k-determined with respect to $\mathcal{K}_\Delta$-equivalence if

$$\boldsymbol{m}_\Lambda^{k+1} \Theta_\gamma \subset T\mathcal{K}_\Delta \cdot \gamma,$$

where $T\mathcal{K}_\Delta \cdot \gamma \subset T\mathcal{K}_{\Delta,e} \cdot \gamma$ is the tangent space given by

$$T\mathcal{K}_\Delta \cdot \gamma = t\gamma(\boldsymbol{m}_n \Theta_n) + \gamma^* \operatorname{Derlog}_0(\Delta).$$

Here $\operatorname{Derlog}_0(\Delta) = \operatorname{Derlog}(\Delta) \cap \boldsymbol{m}_U \Theta_U$ consists of those vector fields tangent to Δ that vanish at 0. Note that if G is a miniversal deformation (i.e. $\dim U$ is as small as possible) then $\operatorname{Derlog}_0(\Delta) = \operatorname{Derlog}(\Delta)$, since then $\{0\}$ is a stratum of Δ. The proof of this is similar to the standard proofs of finite determinacy for $\mathcal{R}$- and $\mathcal{K}$-equivalence using the homotopy method and Nakayama's lemma, see for example [AGV].

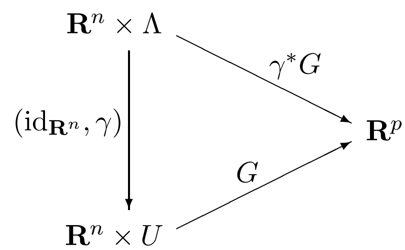
3 Paths and bifurcation problems

To recapitulate, let $g_0 : \mathbf{R}^n \to \mathbf{R}^p$ be a $\mathcal{K}$-finite map (germ), and $G : \mathbf{R}^n \times U \to \mathbf{R}^p$ be a versal deformation of g_0. Any path (map) $\gamma : \Lambda \to U$, induces a deformation (or bifurcation problem) $\gamma^* G$ of g_0 given by

$$(\gamma^* G)(x, \lambda) = G(x, \gamma(\lambda)).$$

Moreover, the bifurcation points of $\gamma^* G$ are the points $\lambda \in \Lambda$ for which $\gamma(\lambda) \in \Delta_G$. For this section, we assume that all maps are (real or complex) analytic.

Since G is versal, any deformation $g(x, \lambda)$ of g_0 is (bifurcation) equivalent to one of the form $\gamma^* G$ for some path γ, as explained in (1.5).



Thus, for a given organizing centre g_0, we have a map

$$\text{paths } [\Lambda \to U] \quad \longrightarrow \quad \text{bifurcation problems with organizing centre } g_0.$$

The precise form of this map depends, of course, on our choice of versal deformation.

(3.1) Deformations of paths and bifurcation problems
Suppose now that we deform the path γ. Let $\Gamma : \Lambda \times V \to U$ be such a deformation (V is the parameter space), so that $\Gamma(\lambda, 0) = \gamma(\lambda)$. This then induces a deformation $\Gamma^* G$ of the bifurcation problem $\gamma^* G$ by

$$\begin{array}{rcl} \Gamma^* G : \mathbf{R}^n \times \Lambda \times V & \longrightarrow & \mathbf{R}^p \\ (x, \lambda, v) & \mapsto & G(x, \Gamma(\lambda, v)). \end{array}$$

(3.2) The morphism Ψ_γ
This correspondence from deformations of γ to deformations of $\gamma^* G$ can be infinitesimalized, to obtain a map associating to any infinitesimal deformation of γ an infinitesimal deformation of the bifurcation problem $g = \gamma^* G$:

$$\Psi_\gamma : \Theta_\gamma \longrightarrow \Theta_g,$$

where $g = \gamma^* G$. If $G(x, u) = g_0(x) + \sum_{j=1}^{d} u_j \phi_j(x)$ then it is easy to see that

$$\Psi_\gamma(\xi_1(\lambda), \ldots, \xi_d(\lambda)) = \xi_1(\lambda)\phi_1(x) + \cdots + \xi_d(\lambda)\phi_d(x).$$

The map Ψ_γ is $\mathcal{E}_\Lambda$-linear; in other words, it is a morphism of $\mathcal{E}_\Lambda$-modules. Note that *a priori* Θ_g is an $\mathcal{E}_{n,\lambda}$-module, and can therefore be considered as an $\mathcal{E}_\Lambda$-module, although as such it is not finitely generated.

As has already been pointed out, the important geometry of a perturbation of γ is how it meets the discriminant Δ: bifurcation points of $\gamma^* G$ correspond to points of $\gamma^{-1}(\Delta)$. It is thus reasonable to consider $\mathcal{K}_\Delta$-equivalence of paths γ, as an alternative to bifurcation equivalence of bifurcation problems g. The following theorem shows that given γ and $g = \gamma^* G$, the notions of codimension of the two coincide, and moreover Γ is a versal deformation of γ if and only if $\Gamma^* G$ is a versal deformation of g.

(3.3) Isomorphism Theorem
Suppose $g_0 : \mathbf{R}^n \to \mathbf{R}^p$ is an analytic $\mathcal{K}$-finite map-germ (at 0), and that $G : \mathbf{R}^n \times U \to \mathbf{R}^p$ is a versal deformation of g_0. Let $\gamma : \Lambda \to U$ be an analytic map-germ, and let $g = \gamma^* G$ be the bifurcation problem induced by γ (with organizing centre g_0). The morphism $\Psi_\gamma : \Theta_\gamma \to \Theta_g$ of $\mathcal{E}_\Lambda$-modules (defined above) induces an isomorphism

$$\psi_\gamma : N\mathcal{K}_\Delta \cdot \gamma \xrightarrow{\simeq} N\mathcal{B} \cdot g.$$

(3.4) Discussion of proof
To begin with of course it is necessary to prove that the map ψ is well-defined; that is, that $\Psi_\gamma(T\mathcal{K}_{\Delta,e} \cdot \gamma) \subset T\mathcal{B}_e \cdot g$.

Firstly, it is clear from the definitions that $\Psi_\gamma(t\gamma(\Theta_\Lambda)) = t_2 g(\Theta_\Lambda)$. It therefore remains to show that $\Psi_\gamma(\gamma^* \operatorname{Derlog}(\Delta)) \subset T\mathcal{K}_{\mathrm{rel}} \cdot g$, where $T\mathcal{K}_{\mathrm{rel}} \cdot g = RT(g)$ is the sum of the first two terms in the expression for $T\mathcal{B}_e \cdot g$ given in (1.7). One can actually show more, see [MM, Lemma 3.2]:

$$\gamma^* \operatorname{Derlog}(\Delta) = \Psi_\gamma^{-1}(T\mathcal{K}_{\mathrm{rel}} \cdot g).$$

This relies heavily on the characterization of elements of $\operatorname{Derlog}(\Delta)$ as liftable vector fields over the map $\pi_G : G^{-1}(0) \to U$, as described in paragraph (2.3).

The map Ψ_γ thus descends to an injective map

$$\psi_\gamma : N\mathcal{K}_\Delta \cdot \gamma \longrightarrow N\mathcal{B} \cdot g.$$

The surjectivity of ψ follows from the preparation theorem. See [MM, Section 3].

(3.5) Example
Consider the organizing centre $g_0(x) = x^3$, its versal deformation $G(x, u_1, u_2) = x^3 + u_1 x + u_2$ and the pitchfork and hysteresis bifurcations (paragraphs (1.6), (2.6) and (2.7)).

Applying Ψ_γ to each of the $\mathcal{K}_\Delta$-versal deformations in (2.7) (i.e. substituting for Γ in G) we get:

Pitchfork: $(x, \lambda, u_1, u_2) \mapsto x^3 - \lambda x + u_1 + u_2 \lambda$,

Hysteresis: $(x, \lambda, u) \mapsto x^3 - \lambda + u$.

These agree with the versal deformations $\tilde{G}$ of the bifurcation problems given in (1.6).

(3.6) Equivalence of path and parametrized-contact formulations
We have been concentrating on the equivalence between the unfolding theories for g and for γ. However there is a more fundamental question that we have not addressed. Namely, whether $\mathcal{K}_\Delta$-equivalence of paths is *equivalent to* bifurcation equivalence of the induced bifurcation problems.

Suppose that γ_1 and γ_2 induces two bifurcation problems g_1 and g_2 from a versal deformation G, with all maps assumed to be analytic. One can show the following.

If γ_1 and γ_2 are $\mathcal{K}_\Delta$-equivalent, then g_1 and g_2 are bifurcation equivalent.

The proof of this fact is based on the fact that a diffeomorphism of U that preserves the discriminant Δ of π_G is liftable over π_G, which is a particular case of general results of du Plessis, Gaffney and Wilson. See for example [dPGW].

On the other hand, although it is probably true, I do not have a proof of the converse.

4 Calculations of $\mathrm{Derlog}(\Delta)$

There are some theoretical results giving more or less explicit generators for $\mathrm{Derlog}(\Delta)$, where Δ is the discriminant of a versal deformation, *without* calculating an equation for the discriminant. In the case $p = 1$, this is due to Bruce [B], and in the general case (with $n \geq p$) to Goryunov [G]. The basic result in these cases is Looijenga's theorem that $\mathrm{Derlog}(\Delta)$ is a *free* $\mathcal{E}_U$-module [L], and so has $\dim(U)$ generators. (It has at least that many, otherwise one could not obtain all vector fields away from Δ. If it had more, then there would be relations between the generators, and the module would not be free.)

In spite of the existence of theoretical results, calculations of $\mathrm{Derlog}(\Delta)$ are more easily done by brute force using computer algebra packages. The two most adapted to the sort of calculations necessary are *Macaulay* and *Singular*[2], though with some extra work it is possible to adapt other packages to do this type of computation.

The calculation proceeds as follows. First calculate the (an) equation $h(u) = 0$ for Δ. This is done by eliminating x from the equations $G(x,u) = \partial G(x,u)/\partial x_j = 0$. Using Grobner bases, this can be done very efficiently (finding the Grobner bases though can use a great deal of computer time).

To find elements of $\mathrm{Derlog}(\Delta)$, one uses the fact that a vector field

$$\theta(u) = \sum_j^d a_j(u) \frac{\partial}{\partial u_j}$$

is in $\mathrm{Derlog}(\Delta)$ if and only if

$$\sum_j^d a_j(u) \frac{\partial h}{\partial u_j} - fh = 0$$

for some function $f \in \mathcal{E}_U$. The $(d+1)$-tuple

$$(a_1, \ldots, a_d, -f) \in \mathcal{E}_U^{d+1}$$

defines a relation between the elements $\left(\frac{\partial h}{\partial u_1}, \ldots, \frac{\partial h}{\partial u_d}, h\right)$. Thus relations between the partial deriviatives of h and h itself correspond to elements of $\mathrm{Derlog}(\Delta)$; the correspondence being given by omitting the last term (here called f). It is also possible to take advantage of the decomposition

$$\mathrm{Derlog}(\Delta) = \mathcal{E}_U.\boldsymbol{e} \oplus \mathrm{Derlog}(h)$$

described in (2.2), by omitting the last term throughout.

Finding relations between elements of a ring (here $\mathcal{E}_U$) is also easy once Grobner bases have been calculated, and Macaulay and Singular are both purpose built for this type of task.

Some results of such calculations are listed in the table below. A column vector is identified with a vector field in alphabetical order, so that the first row is the coefficient of $\frac{\partial}{\partial a}$ the second of $\frac{\partial}{\partial b}$ and so on. The first four singularities are all corank 1, while the other two are of corank 2. Note that the first of the generators of $\mathrm{Derlog}(\Delta)$ is the Euler field, while the others are generators of $\mathrm{Derlog}(h)$ (for a suitable choice of h, namely for h quasihomogeneous).

[2] Both *Macaulay* and *Singular* are free and can be obtained by anonymous ftp; the first from zariski.harvard.edu, and the second from Kaiserslautern

Generators for Derlog(Δ) for organizing centres of low codimension: see text for explanations

Type	$G(x,u)$	Generators of Derlog(Δ)
A_1	x^2+a	$a.$
A_2	x^3+ax+b	$\begin{pmatrix}2a\\3b\end{pmatrix};\begin{pmatrix}9b\\-2a^2\end{pmatrix}.$
A_3	x^4+ax^2+bx+c	$\begin{pmatrix}2a\\3b\\4c\end{pmatrix};\begin{pmatrix}6b\\8c-2a^2\\-ab\end{pmatrix},\begin{pmatrix}48c-4a^2\\-12ab\\16ac-9b^2\end{pmatrix}.$
A_4	$x^5+ax^3+bx^2+cx+d$	$\begin{pmatrix}2a\\3b\\4c\\5d\end{pmatrix};\begin{pmatrix}15b\\20c-6a^2\\25d-4ab\\-2ac\end{pmatrix},\begin{pmatrix}40c-6a^2\\50d-17ab\\8ac-12b^2\\15ad-6bc\end{pmatrix},\begin{pmatrix}50d-2ab\\-4ac-3b^2\\30ad-10bc\\15bd-8c^2\end{pmatrix}.$
A_5	$x^6+ax^4+bx^3+cx^2+dx+e$	$\begin{pmatrix}2a\\3b\\4c\\5d\\6e\end{pmatrix};\begin{pmatrix}9b\\12c-4a^2\\15d-3ab\\18e-2ac\\-ad\end{pmatrix},\begin{pmatrix}40c-8a^2\\50d-22ab\\60e+4ac-15b^2\\10ad-10bc\\16ae-5bd\end{pmatrix},\begin{pmatrix}75d-6ab\\90e-10ac-9b^2\\45ad-27bc\\15bd+30ae-20c^2\\27be-10cd\end{pmatrix},\begin{pmatrix}180e-4ac\\-6bc-10ad\\120ae-8c^2-15bd\\90be-30cd\\48ce-25d^2\end{pmatrix}.$
$I_{2,2}$	$(x^2-y^2+ax+by+c,\ xy+d)$	$\begin{pmatrix}a\\b\\2c\\2d\end{pmatrix};\begin{pmatrix}3b\\-3a\\4d+2ab\\-c\end{pmatrix},\begin{pmatrix}16c-3a^2\\32d+ab\\2ac-24bd\\6ad\end{pmatrix},\begin{pmatrix}32d+ab\\-16c-3b^2\\2bc+24ad\\6bd\end{pmatrix}.$
$II_{2,2}$	$(x^2+ay+c,\ y^2+bx+d)$	$\begin{pmatrix}a\\b\\2c\\2d\end{pmatrix};\begin{pmatrix}3a\\-3b\\2c\\-2d\end{pmatrix},\begin{pmatrix}0\\8d\\3a^2b\\-8bc\end{pmatrix},\begin{pmatrix}8c\\0\\-8ad\\3ab^2\end{pmatrix}.$

5 Symmetric Bifurcations

Throughout this section we assume that H is a finite group acting linearly on $\mathbf{R}^n$ and $\mathbf{R}^p$ and that $g_0 : \mathbf{R}^n \to \mathbf{R}^p$ is H-equivariant. We also assume that g_0 is of finite $\mathcal{K}$-codimension as usual (which is the reason we assume H is finite).

One can then choose the versal deformation $G : \mathbf{R}^n \times U \to \mathbf{R}^p$ to be equivariant, with a suitable action on U. The discriminant $\Delta_G \subset U$ is then invariant under H.

If H acts on a set S, we let S^H denote $\mathrm{Fix}(H; S)$, the subset of S consisting of all points fixed by H.

The actions of H on $\mathbf{R}^n$, $\mathbf{R}^p$ and U induce actions on each of the spaces Θ_n, Θ_p, and on Θ_γ if $\gamma : \Lambda \to U$ is equivariant, as well as on the tangent spaces $T\mathcal{K}{\cdot}g_0$, $T\mathcal{B}_e{\cdot}g$, $T\mathcal{K}_{\Delta,e}{\cdot}\gamma$, and consequently on the normal spaces $N\mathcal{B}{\cdot}g$ and $N\mathcal{K}_\Delta{\cdot}\gamma$. The proof of Theorem 3.3 shows that provided γ is equivariant, then the isomorphism ψ_γ is also H-equivariant.

Consider the situation where the H-action on Λ is trivial, so that for γ to be equivariant it is necessary (and sufficient) that $\gamma(\Lambda) \subset U^H$. If one only considers perturbations of $g = \gamma^* G$ that are equivariant, then it is natural to consider the subspace

$$(T\mathcal{B}_e{\cdot}g)^H = T\mathcal{B}_e{\cdot}g \cap \Theta_g^H \subset T\mathcal{B}_e{\cdot}g,$$

which is isomorphic to $(T\mathcal{K}_{\Delta,e}{\cdot}\gamma)^H$.

However, it usually happens that equivariant organizing centres have high codimension, so that $\dim(U)$ is large, and the calculations of $\mathrm{Derlog}(\Delta)$ become impractical. It is therefore natural to ask whether it is possible to restrict to U^H before calculating the normal spaces. This comes down to the following:

Question For γ as above, are $(T\mathcal{K}_{\Delta,e}{\cdot}\gamma)^H$ and $T\mathcal{K}_{\Delta^H,e}{\cdot}\gamma$ isomorphic?

It is known (simple linear algebra) that $(T\mathcal{K}_{\Delta,e}{\cdot}\gamma)^H$ is that part of $T\mathcal{K}_{\Delta,e}{\cdot}\gamma$ obtained by using only equivariant vector fields: $(T\mathcal{K}_{\Delta,e}{\cdot}\gamma)^H = t\gamma(\Theta_\Lambda^H) + \gamma^* \,\mathrm{Derlog}(\Delta)^H$. Moreover, any element of $\mathrm{Derlog}(\Delta)^H$ (i.e. any equivariant vector field tangent to Δ) restricts to a vector field on U^H tangent to $\Delta \cap U^H = \Delta^H$. Thus there is a natural map given by restriction to U^H,

$$\mathrm{Derlog}(\Delta)^H \longrightarrow \mathrm{Derlog}(\Delta^H),$$

and the question would be answered if one knew that this map was surjective.

Similar questions arise if the action of H is not trivial, corresponding to forced symmetry breaking bifurcation problems. Further problems arise if g_0 does not have a finite dimensional versal deformation, but it does have a finite dimensional equivariant deformation.

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