MANCHESTER
1824

The University
of Manchester

Precise description of the different far fields encountered in the problem of diffraction of acoustic waves by a quarter-plane

Assier, Raphael and Peake, Nigel

2012

MIMS EPrint: **2014.61**

Manchester Institute for Mathematical Sciences
School of Mathematics

The University of Manchester

Reports available from: http://eprints.maths.manchester.ac.uk/
And by contacting: The MIMS Secretary
School of Mathematics
The University of Manchester
Manchester, M13 9PL, UK

ISSN 1749-9097

IMA Journal of Applied Mathematics (2012) **77**, 605–625
doi:10.1093/imamat/hxs042
Advance Access publication on July 15, 2012

The winner of the 1st IMA Lighthill-Thwaites Prize

Precise description of the different far fields encountered in the problem of diffraction of acoustic waves by a quarter-plane

R. C. Assier* and N. Peake
DAMTP, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK
*Corresponding author: rca32@cam.ac.uk n.peake@damtp.cam.ac.uk

[Received on 18 August 2011; revised on 2 February 2012; accepted on 24 May 2012]

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

This paper provides a review of important results concerning the Geometrical Theory of Diffraction and Geometrical Optics. It also reviews the properties of the existing solution for the problem of diffraction of a time harmonic plane wave by a half-plane. New mathematical expressions are derived for the wave fields involved in the problem of diffraction of a time harmonic plane wave by a quarter-plane, including the secondary radiated waves. This leads to a precise representation of the diffraction coefficient describing the diffraction occurring at the corner of the quarter-plane. Our results for the secondary radiated waves are an important step towards finding a formula giving the corner diffraction coefficient everywhere.

Keywords: geometrical theory of diffraction; acoustic diffraction; quarter-plane.

1. Introduction

The diffraction of acoustic waves by an ideal quarter-plane is a complex problem, which has so far proved insoluble via classical techniques. This problem is an important canonical model for the Geometrical Theory of Diffraction (GTD) due to Keller (1962). Following the GTD, further work has been done by Kraus & Levine (1961), Satterwhite (1974) and Hansen (1991), by considering the quarter-plane as a degenerated elliptic cone and expressing the field as a spherical-wave multipole series. However, these series are poorly convergent when the source and the observer are located far from the vertex of the cone. A more extensive review has been undertaken by Blume (1996). In the same paper, Blume proposes a method to accelerate the convergence of these series and obtains some numerical results. Radlow (1961) claimed to have found an analytic solution of the acoustic quarter-plane problem using the Wiener–Hopf method. However, this solution has long been suspected to be wrong (see Meister, 1987), and indeed it has recently been disproved by Albani (2007). A different way to attack the quarter-plane problem has been introduced by Smyshlyaev (1990), followed by some work by Babich *et al.* (1995), but in this case the solution is still difficult to evaluate numerically. Despite this difficulty, Babich *et al.* (2000) describe a numerical method based on the Abel–Poisson-type summation method and a boundary integral equation that gives the diffraction coefficient for smooth convex cones in the non-singular directions. However, Shanin (2005a,b), following Smyshlyaev's work, presents a new analytical/numerical method, which partially solves the acoustic quarter-plane problem in the Dirichlet case. The main advantage of this method compared with the one mentioned previously is that in this case the formulae giving the diffraction coefficient are 'naturally convergent' in the sense that they do not require a special treatment to regularize or accelerate the convergence. This method has been extensively described, adapted to the Neumann case and implemented by Assier & Peake (2012). The main

© The authors 2012. Published by Oxford University Press on behalf of the Institute of Mathematics and its Applications. All rights reserved.

achievement of this method is to allow one to have access to the diffraction coefficient describing the diffraction by the corner of the quarter-plane.

The present paper has two main motivations. The first motivation is the inconsistency of the representation of the diffraction coefficient of the corner. Indeed, for example, two very important papers on the subject by Shanin (2005b) and Skelton *et al.* (2010) give a different representation of this coefficient. The second motivation is the conjecture formulated at the end of Assier & Peake (2012), where the possibility of an ultimate modified Smyshlyaev formula was expressed. The existence of such a formula is closely related to the other diffracted fields (not only the diffraction from the corner but also the scattering and re-scattering by the two edges) involved in the quarter-plane problem. This is why the present paper aspires to describe precisely those other fields and to precisely define the corner diffraction coefficient. After a brief reminder (mainly inspired by Borovikov & Kinber (1994)) of the laws of Geometrical Optics (GO) and the Geometrical Theory of Diffraction (GTD) in Section 2, a presentation of the main results concerning the problem of diffraction by a half-plane is outlined in Section 3. The total far field emanating from the diffraction of a time-harmonic plane acoustic wave by a quarter-plane is then studied in detail in Section 4. Namely, an approach mixing the exact theory of the half-plane and the postulates of both GO and the GTD is applied to the quarter-plane. Throughout this paper, the cases of Dirichlet (soft surface) and Neumann (hard surface) boundary conditions shall, when possible, be treated simultaneously. Hence, following the notation of Assier & Peake (2012), we introduce the indexes ^{d,n} such that ^{d} refers to the Dirichlet case and ^{n} refers to the Neumann case. This approach is similar in spirit to the approach taken by Budaev & Bogy (2005), in the first part of their paper, when dealing with the plane sector. Similar ideas have also been used by Shanin (2011) applied to the diffraction of waves propagating on the surface of the unit sphere with a cut. In Section 4, the far field will be divided into separate wave fields emanating from the GTD and an exact mathematical expression will be found for each of them. The wave fields considered are the incident and reflected waves, the primary waves diffracted by the edges and the secondary waves diffracted by the edges. Exact formulae are given for each of these wave fields, including for the secondary diffracted waves, which to the authors' knowledge is a result that has not been published previously. Finally, the wave diffracted by the corner of the quarter-plane is described by introducing the diffraction coefficient $f^{d,n}$. The evaluation of this diffraction coefficient was the main topic of Assier & Peake (2012).

2. Theory background

2.1 *Laws of geometrical optics*

Let us consider an infinite homogeneous medium with sound speed c_0. A time harmonic GO wave field u (with time frequency Ω) propagating in this medium with wave number k_0, such that $k_0 = \Omega/c_0$, should satisfy the Helmholtz equation

$$\Delta u + k_0^2 u = 0. \tag{2.1}$$

It can be written as a slowly varying amplitude A multiplied by a rapidly oscillating function and is hence given by

$$u = A\,\mathrm{e}^{\mathrm{i}k_0 s}, \tag{2.2}$$

where A and s depend on the spatial coordinates used to describe the space and s is called the eikonal. The GO law of energy conservation allows one to have more information about the slow variation of the amplitude A. It states that the energy flux in an elementary ray tube with ray-formed walls confining

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

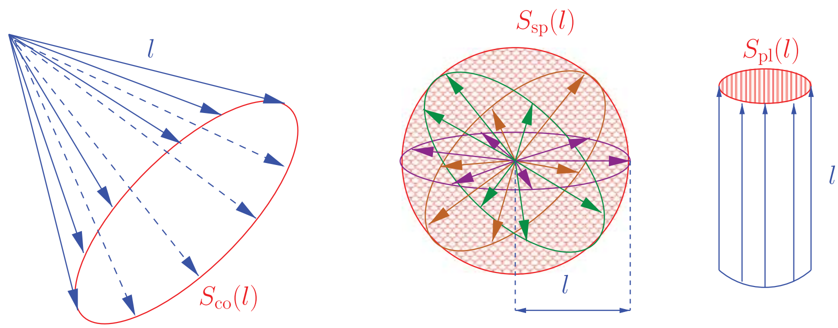

FIG. 1. Illustration of different ray tubes and cross-sections.

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

an elementary cross-section should be conserved. Let us apply the law of energy conservation to three 'simple' examples. Let us consider a plane wave, a spherical wave and a conical wave, in a 3D space, for which the shape of the ray tubes are described in Fig. 1.

The cylindrical wave is propagating along rays on the surface of a cone and the cross-section $S_{\rm co}$ is a circle. Hence, its slowly varying amplitude $A_{\rm co}$ is given by $A_{\rm co} = A^0_{\rm co}/\sqrt{l}$, where l is the distance from the source along a ray. In the case of the spherical wave, the wave is propagating throughout 3D space, and the cross-section $S_{\rm sp}$ is the surface of the sphere with radius l. Hence, the slowly varying amplitude $A_{\rm sp}$ is given by $A_{\rm sp} = A^0_{\rm sp}/l$. In the case of the plane wave, the cross-section $S_{\rm pl}$ is independent of l, which leads its slowly varying amplitude to be constant (independent of l). Let us now consider a perfectly reflecting plane surface within the infinite medium. The surface can be either soft, i.e. subject to the Dirichlet boundary condition or hard, i.e. subject to the Neumann boundary condition. The rays obey the law of reflection, that is, if an incident ray $u_i = A_i\,{\rm e}^{{\rm i}k_0 s_i}$ encounters the surface, then a reflected ray $u^{d,n}_r = A^{d,n}_r\,{\rm e}^{{\rm i}k_0 s_r}$ is generated. The angle θ_i between the incident ray and the normal of the surface at the point of incidence is equal to the angle θ_r between the reflected ray and the same normal. Moreover, the constant amplitudes A_i and A_r are related by $A^{d,n}_r = -\mathfrak{d}^{d,n} A_i$, where $\mathfrak{d}^d = 1$ and $\mathfrak{d}^n = -1$. As efficient as the laws of geometrical optics are in describing the propagation of a wave, they however fail to provide an explanation for the phenomenon of diffraction. The following subsection describes some of the laws of the GTD that gives a ray approach to this phenomenon.

2.2 *Laws of geometrical theory of diffraction*

The GTD, first introduced by Keller (1962), proposes a set of postulates that explain diffraction using ray theory. Borovikov & Kinber (1994) classify the postulates of the GTD into two groups, one group concerning the direction of the diffracted rays and one group concerning their amplitudes. In what follows, we shall quote and comment on the postulates (taken from Borovikov & Kinber, 1994) that are relevant to our problem. The first postulate concerning the directions is given as follows.

> POSTULATE 1 Diffracted rays are induced only by those incident rays which are incident on inhomogeneous areas of the body, such as tips, edges and lines of curvature discontinuity, or touch the body (tangentially). In other words, diffracted rays are induced only by those rays of the incident field which form light-shadow boundaries.

The diffracted rays emanating from incident rays tangential to a smooth body are called creeping waves and are of extreme importance when considering canonical problems involving smooth surfaces, such as the problem of diffraction by a cone. In this paper, we shall be mainly interested in the quarter-plane geometry, which does not involve smooth-curved surfaces. Therefore, the concept of creeping waves shall not be developed further. Instead, the primary focus will be edges and tips diffraction. The second postulate concerning the propagation directions of the diffracted ray is quoted below

> POSTULATE 2 Each ray of the primary field satisfying the above conditions produces an infinite set of diffracted rays. When an incident ray hits a corner, it gives rise to diffracted rays travelling away in all directions, thus generating an outgoing spherical wave. When a ray impinges an edge, the fanning out diffracted rays form a cone at each point of the edge. The angle [...] of spread of the cone at each point of an edge is equal to the angle between the tangent to the edge and the incident ray.

The latter postulate is important and involves two of the ray propagation patterns discussed in the previous subsection, namely the conical and spherical waves. From the geometrical optics, as seen in the previous subsection, we know the behaviour of the rays belonging to such wave fields up to a scalar factor. This is the topic of the next postulate.[1]

> POSTULATE 3 The amplitude of a diffracted ray is proportional to the amplitude of the inducing prime ray at the point of incidence. Recalling that the diffracted fields obey the [...] Geometrical Optics laws, it follows that they may be written in the form
>
> $$u_d = A_i^0 D(\omega_0, \omega) A_d \,\mathrm{e}^{\mathrm{i}k_0 s},$$
>
> where ω_0 and ω are the unit vectors of the incident and diffracted rays, A_i^0 is the amplitude of the incident wave at a point of the edge or apex from which the diffracted ray emanates. $D(\omega, \omega_0)$ is called the diffraction coefficient. A_d is defined as the varying part of the amplitude that depends on the geometry of the diffracted field, as seen in the previous subsection, i.e. $1/l$ for a spherical wave and $1/\sqrt{l}$ for a conical wave.

The latter postulate introduces the important notion of diffraction coefficient. Its definition is refined by the final postulate.

> POSTULATE 4 The coefficient of diffraction is contingent on the local geometry of the body in the vicinity of the incident ray in the case of corner and edges.

This final postulate, also known as the locality principle, will be used extensively in this paper. The main advantage of the GTD is that it allows one to describe geometrically the far field structure of a wave field emanating from the scattering of an incident wave by an obstacle. There are three main limitations to the GTD. The first one, obviously, is that the description of the field is only valid when the far field approximation is possible, that is when the product $k_0 l$ is large. Hence, this theory cannot give any

[1] Here the mathematical notations have been modified to fit better with rest of the paper. Also A_d has been defined in a slightly different way to Borovikov & Kinber (1994).

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

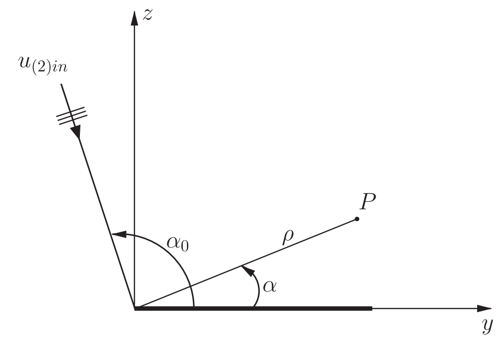

FIG. 2. Description of the polar coordinates used to solve the half-plane problem.

information about what happens in a small neighbourhood of an edge or a corner. Another limitation is that the GTD approximation is not valid in the vicinity of a light-shadow boundary. Such a region in which the GTD approximation stops to be valid is called a penumbral zone. Finally, the last limitation is that the GTD does not give any information concerning the value of the diffraction coefficient.

But before considering the quarter-plane problem, let us focus on one of the first (and few) canonical problems for which an exact solution is known, the problem of diffraction by a half-plane. This problem should emphasize both the strengths and the limitations of the GTD.

3. An important canonical problem: the half-plane

3.1 *The 2D case*

3.1.1 *The exact solution.* Let us consider a plane wave incidence on a Dirichlet or Neumann half-plane in the cylindrical coordinate system (ρ, α) described in Fig. 2. The total field $u_{(2)}^{d,n}(\rho, \alpha)$ should satisfy the 2D cylindrical Helmholtz equation

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u_{(2)}^{d,n}}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u_{(2)}^{d,n}}{\partial \alpha^2} + k_0^2 u_{(2)}^{d,n} = 0 \tag{3.1}$$

throughout the fluid.

Let the incident plane wave $u_{(2)\,\mathrm{in}}$ have an amplitude equal to one. Hence, according to the laws of GO, the incident plane wave at a point $P = (\rho\cos(\alpha), \rho\sin(\alpha))$ is given by

$$u_{(2)\,\mathrm{in}}(\rho, \alpha) = \exp\{-\mathrm{i}k_0\rho\cos(\alpha - \alpha_0)\}. \tag{3.2}$$

According to the GO law of reflection, this incident plane wave will only be present in the illuminated zone and will be zero in the shadow zone. The shadow zone $\mathfrak{S}(\alpha_0)$ is defined as follows:

$$\mathfrak{S}(\alpha_0) = \begin{cases} \{\alpha, \alpha - \alpha_0 > \pi\} & \text{if } \alpha_0 < \pi, \\ \{\alpha, \alpha_0 - \alpha > \pi\} & \text{if } \alpha_0 > \pi. \end{cases} \tag{3.3}$$

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

Hence the contribution of the incident wave to the total field will be

$$u_{(2)\,\mathrm{in}}(\rho,\alpha) = \mathrm{e}^{-\mathrm{i}k_0\rho\cos(\alpha-\alpha_0)}\mathbb{H}[\pi - |\alpha-\alpha_0|], \tag{3.4}$$

where $\mathbb{H}$ is the Heaviside function. Similarly, according to the same law of reflection, there should be a reflected wave $u^{d,n}_{(2)\,\mathrm{re}}$ with wave vector $\overrightarrow{k_{\mathrm{re}}}$, where $\overrightarrow{k_{\mathrm{re}}} = k_0(-\cos(\alpha_0), \sin(\alpha_0))$, and an amplitude $-\mathfrak{d}^{d,n}$ such that $\mathfrak{d}^d = 1$ and $\mathfrak{d}^n = -1$. Again, according to the laws of GO, this reflected wave should only be present in the reflection zone $\mathfrak{R}(\alpha_0)$, defined by

$$\mathfrak{R}(\alpha_0) = \begin{cases} \{\alpha, \alpha+\alpha_0 < \pi\} & \text{if } \alpha_0 < \pi, \\ \{\alpha, \alpha_0+\alpha > 3\pi\} & \text{if } \alpha_0 > \pi. \end{cases} \tag{3.5}$$

Hence the contribution of the reflected wave to the total field should be

$$u^{d,n}_{(2)\,\mathrm{re}}(\rho,\alpha) = -\mathfrak{d}^{d,n}\,\mathrm{e}^{-\mathrm{i}k_0\rho\cos(\alpha+\alpha_0)}\mathbb{H}[\Pi(\alpha_0)],$$

where $\Pi(\alpha_0) = \pi - (\alpha+\alpha_0)$ if $\alpha_0 < \pi$ and $\Pi(\alpha_0) = \alpha+\alpha_0-3\pi$ if $\alpha_0 > \pi$. Now, according to the Postulates 1 and 2 of the GTD, the total field (in the far field approximation) should look like

$$u_{(2)} = u_{(2)\,\mathrm{in}} + u^{d,n}_{(2)\,\mathrm{re}} + u^{d,n}_{(2)D}, \tag{3.6}$$

where $u^{d,n}_{(2)D}$ is the wave diffracted by the edge of the half-plane. Moreover, in 2Ds, the edge of the half-plane is equivalent to a corner. Hence, according to Postulate 2, the rays of the diffracted wave should propagate in all directions of the 2D plane. Moreover, according to Postulate 3, we know that the diffracted wave should take the form

$$u^{d,n}_{(2)D}(\rho,\alpha) = \frac{\mathrm{e}^{\mathrm{i}k_0\rho}}{\sqrt{k_0\rho}}D^{d,n}(\alpha,\alpha_0). \tag{3.7}$$

Independently of the laws of the GTD, it is possible to find the exact solution of the problem. It can be solved using various techniques, such as the Wiener–Hopf technique or the Sommerfeld integral. This problem is considered one of the classic problems of diffraction by canonical geometries and its solution can be found in many books such as Jones (1989), Felsen & Marcuvitz (1973), Bowman *et al.* (1987) or Borovikov & Kinber (1994). Surprisingly, each book seems to have a different way to write the solution. Here we will choose the notation used by Borovikov & Kinber (1994), as we consider this is the notation best reflecting the structure of the solution:

$$\begin{aligned} u^{d,n}_{(2)}(\rho,\alpha) = {} & \mathrm{e}^{-\mathrm{i}k_0\rho\cos(\alpha-\alpha_0)}F\left[\sqrt{2k_0\rho}\cos\left(\frac{\alpha-\alpha_0}{2}\right)\right] \\ & - \mathfrak{d}^{d,n}\,\mathrm{e}^{-\mathrm{i}k_0\rho\cos(\alpha+\alpha_0)}F\left[\sqrt{2k_0\rho}\cos\left(\frac{\alpha+\alpha_0}{2}\right)\right], \end{aligned} \tag{3.8}$$

where

$$F(\xi) = \frac{\mathrm{e}^{-\mathrm{i}(\pi/4)}}{\sqrt{\pi}}\int_{-\infty}^{\xi}\mathrm{e}^{\mathrm{i}s^2}\,\mathrm{d}s \tag{3.9}$$

is a Fresnel-type integral. This solution is valid everywhere, including the penumbral zone, which is the zone in the neighbourhood of the GO boundaries, as presented in Fig. 3.

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

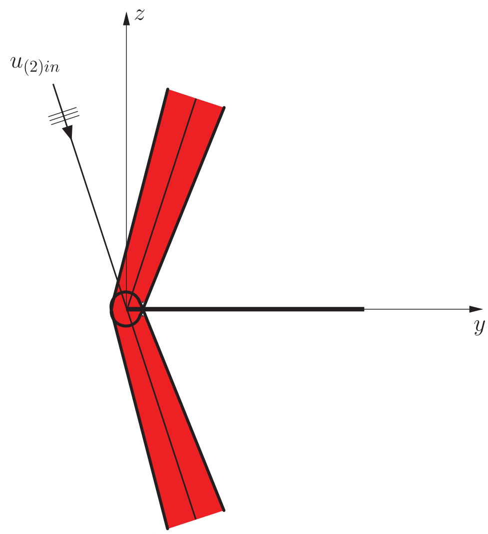

FIG. 3. Illustration of the penumbral zones.

3.1.2 *The far field behaviour.* The useful fact about this function F is that its integral representation leads to an easy way to evaluate its behaviour as ξ tends to infinity away from the penumbral zone. Hence, considering the case $\alpha_0 < \pi$, as $k_0\rho$ tends to infinity, away from the penumbral zones, we obtain

$$F\left[\sqrt{2k_0\rho}\cos\left(\frac{\alpha\pm\alpha_0}{2}\right)\right] \approx \mathbb{H}[\pi-(\alpha\pm\alpha_0)] - \frac{\exp(\mathrm{i}k_0\rho[1+\cos(\alpha\pm\alpha_0)])\,\mathrm{e}^{\mathrm{i}(\pi/4)}}{2\sqrt{2\pi k_0\rho}}\sec\left(\frac{\alpha\pm\alpha_0}{2}\right). \tag{3.10}$$

Putting the two asymptotic expansions (3.10) back into the solution (3.8), we obtain

$$u_{(2)}^{d,n}(k_0,\rho,\alpha) \approx \mathrm{e}^{-\mathrm{i}k_0\rho\cos(\alpha-\alpha_0)}\mathbb{H}[\pi-(\alpha-\alpha_0)] - \mathfrak{d}^{d,n}\,\mathrm{e}^{-\mathrm{i}k_0\rho\cos(\alpha+\alpha_0)}\mathbb{H}[\pi-(\alpha+\alpha_0)] - \frac{\exp(\mathrm{i}\{k_0\rho+\pi/4\})}{2\sqrt{2\pi k_0\rho}}\left\{\sec\left(\frac{\alpha-\alpha_0}{2}\right) - \mathfrak{d}^{d,n}\sec\left(\frac{\alpha+\alpha_0}{2}\right)\right\}. \tag{3.11}$$

This last expansion (3.11) can be compared with the GTD result (3.6) and we can then obtain an expression for the scattered cylindrical edge wave

$$u_{(2)\,\mathrm{D}}^{d,n} \approx -\frac{\mathrm{e}^{\mathrm{i}}(\pi/4)}{2\sqrt{2\pi}}T^{d,n}(\alpha,\alpha_0)\frac{\mathrm{e}^{\mathrm{i}k_0\rho}}{\sqrt{k_0\rho}}, \tag{3.12}$$

where the edge diffraction coefficient $T^{d,n}$ is given by

$$T^{d,n}(\alpha,\alpha_0) = \sec\left(\frac{\alpha-\alpha_0}{2}\right) - \mathfrak{d}^{d,n}\sec\left(\frac{\alpha+\alpha_0}{2}\right). \tag{3.13}$$

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

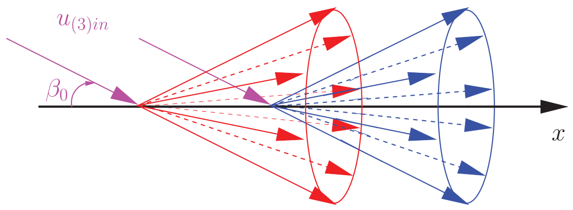

FIG. 4. Illustration of the conical diffracted field.

3.1.3 *The near-field behaviour.* Let us now try to evaluate the behaviour of the total field near the edge. In order to do that, let us expand the function F around zero as follows:

$$F(\xi) \approx \frac{1}{2} + \frac{e^{-i(\pi/4)}}{\sqrt{\pi}}\xi + O(\xi^3). \tag{3.14}$$

Substituting (3.14) into (3.8) and expanding the exponential into its Taylor series leads to the following behaviour for the total field:

$$u^d_{(2)}(\rho,\alpha) \approx 2\,e^{-i(\pi/4)}\sqrt{\frac{2k_0}{\pi}}\sin\left(\frac{\alpha_0}{2}\right)\sin\left(\frac{\alpha}{2}\right)\rho^{1/2} - ik_0\sin(\alpha_0)\sin(\alpha)\rho + O(\rho^{3/2}). \tag{3.15}$$

$$u^n_{(2)}(\rho,\alpha) \approx 1 + 2\,e^{-i(\pi/4)}\sqrt{\frac{2k_0}{\pi}}\cos\left(\frac{\alpha_0}{2}\right)\cos\left(\frac{\alpha}{2}\right)\rho^{1/2} - ik_0\cos(\alpha_0)\cos(\alpha)\rho + O(\rho^{3/2}). \tag{3.16}$$

This expansion, up to the term in $\rho^{1/2}$, can be found, for example, in Jones (1989).

3.2 *The 3D case*

The extension to the 3D case is surprisingly easy. In this subsection, we shall remind the reader of the reasoning used by Jones (1989). Let us consider a wave with an incidence angle β_0 with the edge (x-axis) as shown in Fig. 4. The 3D Helmholtz equation can be written as follows in the (x,ρ,α) cylindrical coordinates

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u^{d,n}_{(3)}}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u^{d,n}_{(3)}}{\partial\alpha^2} + \frac{\partial^2 u^{d,n}_{(3)}}{\partial x^2} + k_0^2 u^{d,n}_{(3)} = 0. \tag{3.17}$$

Then, in the Cartesian coordinates, the incident plane wave has a wave vector $\overrightarrow{k_0}$ given by $\overrightarrow{k_0} = -k_0(\cos(\beta_0), \cos(\alpha_0)\sin(\beta_0), \sin(\alpha_0)\sin(\beta_0))$ and so the incident wave at a point $\vec{r}$ described by $\vec{r} = (r\cos(\beta), r\cos(\alpha)\sin(\beta), r\sin(\alpha)\sin(\beta))$ has the form

$$u_{(3)\,\mathrm{in}} = \exp(-ik_0(r\cos(\beta_0)\cos(\beta) + r\sin(\beta)\cos(\alpha)\cos(\alpha_0)\sin(\beta_0) + r\sin(\alpha)\sin(\beta)\sin(\alpha_0)\sin(\beta_0))).$$

But note that $r\sin(\beta) = \rho$ and $r\cos(\beta) = x$ in the 3D cylindrical coordinates. Hence we can simplify $u_{(3)\,\mathrm{in}}$ by writing

$$u_{(3)\,\mathrm{in}} = \exp(-ik_0\cos(\beta_0)x)\exp(-ik_0\sin(\beta_0)\rho\cos(\alpha-\alpha_0)). \tag{3.18}$$

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

At this stage, it is interesting to note two things. Firstly, there seems to be a natural separation of variables between x and (ρ,α) and secondly the second exponential looks exactly like the incident wave (3.2) in the 2D case when k_0 has been replaced by $k_0\sin(\beta_0)$. Therefore, let us assume that we can separate the solution as follows:

$$u_{(3)}^{d,n} = \exp(-\mathrm{i}k_0\cos(\beta_0)x)v^{d,n}(k_0,\rho,\alpha). \tag{3.19}$$

Putting the expression (3.19) back into (3.17), we obtain

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial v^{d,n}}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 v^{d,n}}{\partial\alpha^2} + (k_0\sin(\beta_0))^2 v^{d,n} = 0,$$

which is exactly equation (3.1) when k_0 is replaced by $k_0\sin(\beta_0)$. Hence we know the solution of this problem from the 2D case because the boundary conditions do not depend on x. It is given by $v^{d,n}(k_0,\rho,\alpha) = u_{(2)}^{d,n}(k_0\sin(\beta_0),\rho,\alpha)$ and hence, the global solution for the 3D case is

$$u_{(3)}^{d,n}(k_0,\rho,\alpha) = \mathrm{e}^{-\mathrm{i}k_0\cos(\beta_0)x}u_{(2)}^{d,n}(k_0\sin(\beta_0),\rho,\alpha). \tag{3.20}$$

Again, this is an exact solution valid everywhere, including within the penumbral zones. Once again, it is possible to have an asymptotic expansion of this expression away from the penumbral zones that will correspond to the GTD and gives

$$u_{(3)}^{d,n} = u_{(3)\,\mathrm{in}} + u_{(3)\,\mathrm{re}}^{d,n} + u_{(3)d}^{d,n}.$$

Hence, at a point $P(x,\rho,\alpha)$, by taking the leading order term (3.12), the diffracted field emanating from the edge diffraction takes the form

$$u_{(3)d}^{d,n}(x,\rho,\alpha) \approx -\exp(-\mathrm{i}k_0\cos(\beta_0)x)\frac{\exp(\mathrm{i}\{k_0\sin(\beta_0)\rho+\pi/4\})}{2\sqrt{2\pi k_0\sin(\beta_0)\rho}}T^{d,n}(\alpha,\alpha_0). \tag{3.21}$$

Let us attempt to interpret this wave geometrically, assuming that $\beta_0 < \pi/2$. For any point $P(x,\rho,\alpha)$, there is a unique pair (X,R) such that X is the intersection between a straight line making an angle β_0 with the x-axis and passing through P, and R is the distance between X and P along this line. In other word, P is on the semi-cone $\mathfrak{C}(\beta_0,X)$ with apex X and semi-angle β_0, at a distance R from the apex. The relationship between (ρ,x) and (R,X) is given by the equations

$$X - x = R\cos(\beta_0) \quad \text{and} \quad \rho = R\sin(\beta_0).$$

Using these relations, it is easy to show that

$$\exp\{-\mathrm{i}k_0\cos(\beta_0)x\}\exp\{\mathrm{i}k_0\sin(\beta_0)\rho\} = \mathrm{e}^{\mathrm{i}k_0R}\exp\{-\mathrm{i}k_0\cos(\beta_0)X\}.$$

Hence we can rewrite $u_{(3)d}^{d,n}$ at the same point P in terms of X and R. This leads to

$$u_{(3)d}^{d,n}(X,R,\alpha) \approx -\frac{\mathrm{e}^{\mathrm{i}k_0R}}{\sqrt{k_0R}}\left\{\frac{\mathrm{e}^{-\mathrm{i}k_0\cos(\beta_0)X}\,\mathrm{e}^{\mathrm{i}(\pi/4)}}{2\sin(\beta_0)\sqrt{2\pi}}T^{d,n}(\alpha,\alpha_0)\right\}. \tag{3.22}$$

Note that all the points P with the same (X,R) have the same phase. Hence the wave fronts are circles belonging to the cone and we can then see the wave 'hitting' P as a wave propagating along the cone

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

$\mathfrak{C}(\beta_0, X)$. Hence the total diffracted field can be thought of as a set of parallel cones emanating from the edge, with a semi-angle equal to β_0. When $\beta_0 \to \pi/2$, the cone becomes a plane perpendicular to the edge and including the point $x = X$ on the edge, which is consistent with the 2D solution. Moreover, the conical form of the diffracted wave is then in complete agreement with the Postulate 2 of the GTD.

REMARK 3.1 If $\beta_0 > \pi/2$, the semi-angle of the cone is actually $\pi - \beta_0$. However, as one can easily verify, this would not affect the relationship between (ρ, x) and (R, X).

The far field structure of the wave field resulting from the incidence of a time-harmonic plane wave on a half-plane is then completely understood. In the next section, we shall use these results together with the laws of the GTD to formulate an accurate description of the far field structure of the wave field resulting from the incidence of a time-harmonic plane wave on a quarter-plane.

4. Description of the far field for the quarter-plane problem

4.1 *Coordinate systems and notation*

Let us consider an incident time-harmonic plane wave scattered by a quarter-plane defined by $(x > 0, y > 0, z = 0)$. The geometry of the problem is described in Fig. 5. Let us denote the two edges of the quarter-plane as Λ_1 and Λ_2. According to the GTD, in the far field approximation away from the penumbral zones, the total field should have the form

$$u_{\rm tot}^{d,n} = u_{\rm in} + u_{\rm re}^{d,n} + u_{\rm co\,1}^{d,n} + u_{\rm co\,2}^{d,n} + u_{\rm co\,21}^{d,n} + u_{\rm co\,12}^{d,n} + u_{\rm sp}^{d,n}, \tag{4.1}$$

where $u_{\rm in}$ and $u_{\rm re}^{d,n}$ are the incident and reflected fields, respectively, and $u_{\rm co\,1}^{d,n}$ and $u_{\rm co\,2}^{d,n}$ are the primary conical waves emanating from the edges Λ_1 and Λ_2. The wave fields $u_{\rm co\,21}^{d,n}$ and $u_{\rm co\,12}^{d,n}$ represent the secondary radiated conical waves. Namely, as will be explained in more detail in Section 4.4, and illustrated in Fig. 5, it is possible for a ray of the primary conical wave $u_{\rm co\,2}^{d,n}$ coming from Λ_2 to hit the opposite edge Λ_1 and to be diffracted by this edge. The wave field thus created is called $u_{\rm co\,21}^{d,n}$. Finally, in (4.1), $u_{\rm sp}^{d,n}$ represents the spherical wave emanating from the diffraction of the incident field by the corner of the quarter-plane. One of the reasons why (4.1) is exact is that, for the quarter-plane, the number of times a ray can be diffracted is limited to two. This is not the case, for example, for a plane sector with an internal angle smaller than $\pi/2$.

The aim of this section is to describe each of the different wave fields involved in (4.1) as accurately as possible in the spirit of what has been attempted for the plane sector by Budaev & Bogy (2005), who only described the primary diffracted waves. In order to do so, we need to equip the space with efficient sets of coordinates as described in Fig. 6. The aim is to describe the position of a point P belonging to the space. First of all, consider the classic Cartesian coordinates (x, y, z), with the corner of the quarter-plane being the origin, Λ_1 being along the positive x-axis and Λ_2 being along the positive y-axis. Consider also the classic spherical coordinates (r, θ, φ), r being the distance to the corner, the inclination angle θ being measured from the z-axis and the azimuthal angle φ being measured from the x-axis. In addition, let us consider a system of coordinates (ω, r), where ω is defined as the unit vector $\overrightarrow{\rm OP}/r$ and can also be considered a point on the unit sphere. The point/vector ω can be described either by its spherical coordinates (θ, φ) or its projection coordinates (ξ, η) as can be seen in Fig. 6. As hinted by Fig. 5, the edges are also equipped with their own systems of cylindrical coordinates. A point can be either described with respect to Λ_1 by the coordinates (x, ρ_1, α_1) or with respect to Λ_2 by the coordinates (y, ρ_2, α_2). The angle α_1 is measured from the upper surface of the quarter-plane, while the angle α_2 is

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

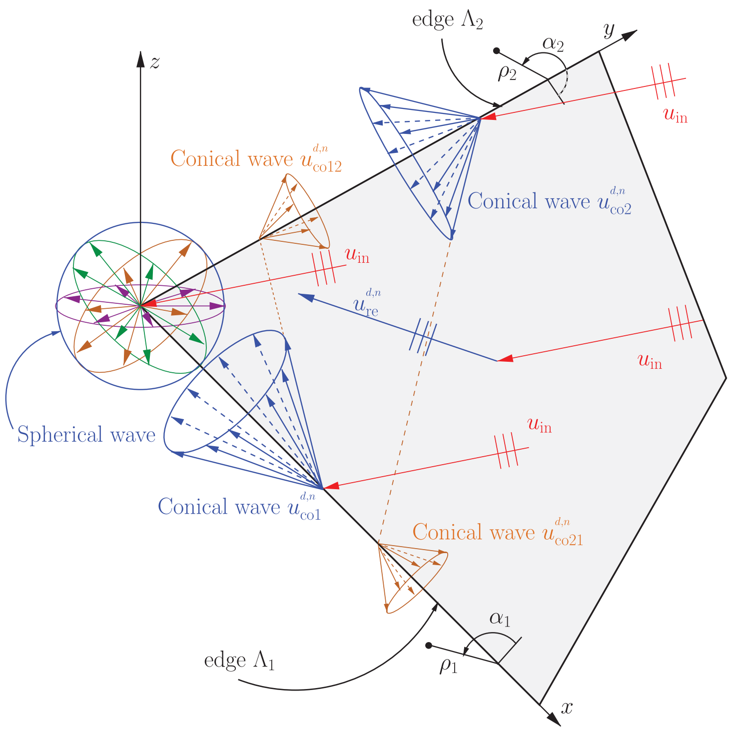

FIG. 5. Overall structure of the total far field of the quarter-plane problem, taken from Assier & Peake (2012).

measured from the lower surface of the quarter-plane. Finally, let β_1 and β_2 denote the angles between the line (OP) and the edge Λ_1 and Λ_2, respectively.

Obviously, these coordinates are related to each other. In order for our argument to be slightly smoother in the following subsections, we state in Table 1 a few of the relationships between the different systems of coordinates used.

4.2 *The incident and reflected wave field*

Let us define the incident harmonic plane wave by a point ω_0 and a wave vector $\overrightarrow{k_0} = -k_0(\xi_0, \eta_0, \cos(\theta_0))$ in the Cartesian coordinates described in Fig. 6. Note that without the loss of generality, we can restrict the study to incident waves coming from the upper half space, that is, we restrict the problem to $\cos(\theta_0) > 0$. At a point P defined by the spherical coordinates $(r, \theta, \varphi) = (\omega, r)$ or similarly by the vector $\vec{r} = r(\xi, \eta, \cos(\theta))$, we can write

$$\begin{aligned} u_{\text{in}}(r, \omega) &= \mathrm{e}^{\mathrm{i}\overrightarrow{k_0}.\vec{r}} \\ &= \mathrm{e}^{-\mathrm{i}k_0 r[\xi\xi_0 + \eta\eta_0 + \cos(\theta)\cos(\theta_0)]}. \end{aligned} \tag{4.2}$$

Using Table 1, it is possible to write the incident wave using the spherical coordinates only as

$$u_{\text{in}}(r, \theta, \varphi) = \mathrm{e}^{-\mathrm{i}k_0 r[\cos(\theta)\cos(\theta_0) + \sin(\theta)\sin(\theta_0)\cos(\varphi - \varphi_0)]}. \tag{4.3}$$

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

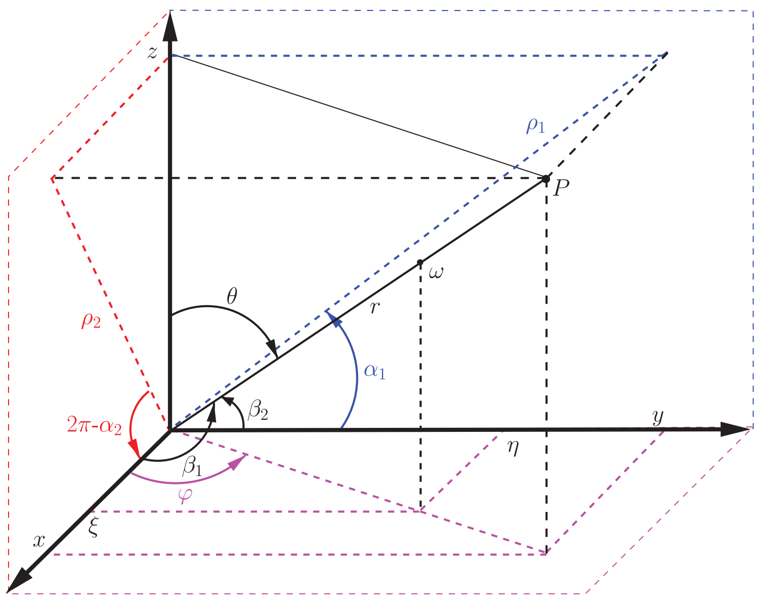

FIG. 6. Description of the coordinates used to describe the quarter-plane problem.

TABLE 1 *Some relationships between the different systems of coordinates.*

$y = \rho_1 \cos(\alpha_1) = r\eta$	$\rho_1 \sin(\alpha_1) = r \sin(\theta)$	$\xi = \cos(\beta_1)$
$x = \rho_2 \cos(\alpha_2) = r\xi$	$\rho_2 \sin(\alpha_2) = -r \sin(\theta)$	$\eta = \cos(\beta_2)$
$\xi = \sin(\theta)\cos(\varphi)$	$\eta = \sin(\theta)\sin(\varphi)$	$\rho_{1,2} = \sin(\beta_{1,2})$

However, this incident wave should only be present in the illuminated zone and thus should be excluded form the shadow zone. In order to describe the shadow zone more efficiently, using the results obtained for the half-plane in Section 3, let us try to express the incident wave in terms of the cylindrical coordinates (x, ρ_1, α_1) and (y, ρ_2, α_2) attached to the axes Λ_1 and Λ_2. It is also interesting to consider the angles β_1^0 and β_2^0 between the incidence direction and the axes Λ_1 and Λ_2. Table 1 leads to two different ways of writing (4.2) in terms of the cylindrical coordinates,

$$u_{\text{in}}(x, \rho_1, \alpha_1; \beta_1^0, \alpha_1^0) = \exp[-\mathrm{i}k_0 \cos(\beta_1^0)x] \exp[-\mathrm{i}k_0 \sin(\beta_1^0)\rho_1 \cos(\alpha_1 - \alpha_1^0)] \tag{4.4}$$

$$u_{\text{in}}(y, \rho_2, \alpha_2; \beta_2^0, \alpha_2^0) = \exp[-\mathrm{i}k_0 \cos(\beta_2^0)y] \exp[-\mathrm{i}k_0 \sin(\beta_2^0)\rho_2 \cos(\alpha_2 - \alpha_2^0)]. \tag{4.5}$$

At this stage, it is very important to note the similarity between the incident field in (4.4) and (4.5) and the way with which we have written the incident wave in the 3D case for the half-plane in (3.18). Here one must remember that we consider only the case $\cos(\theta_0) > 0$, which implies that $\alpha_1^0 < \pi$ and $\alpha_2^0 > \pi$. Hence, by similarity with the half-plane case, it is now very easy to describe the shadow zone. Indeed, according to (3.3), we are in the shadow zone if $(\alpha_1 - \alpha_1^0) > \pi$ and $(\alpha_2^0 - \alpha_2) > \pi$ simultaneously,

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

which leads us to the exact contribution of the incident field to the total field at a point P,

$$u_{\text{in}}(P) = \mathrm{e}^{-\mathrm{i}k_0[\xi_0 x + \eta_0 y + \cos(\theta_0)z]}\mathbb{H}(\pi - (\alpha_1 - \alpha_1^0))\mathbb{H}(\pi - (\alpha_2^0 - \alpha_2^0)). \tag{4.6}$$

Following similar reasoning, but using (3.5) this time, it is also possible to define precisely the contribution of the reflected wave field to the total field at a point P,

$$u_{\text{re}}^{d,n}(P) = -\eth^{d,n}\,\mathrm{e}^{-\mathrm{i}k_0[\xi_0 x + \eta_0 y - \cos(\theta_0)z]}\mathbb{H}(\pi - (\alpha_1 + \alpha_1^0))\mathbb{H}((\alpha_2 + \alpha_2^0) - 3\pi). \tag{4.7}$$

4.3 *The primary radiated conical waves*

Let us try to apply the results of the half-plane theory to determine the form of the primary radiated conical waves $u_{\text{co}\,1}^{d,n}$. First of all, let us note the similarity of the incident waves described by (3.18) and (4.4) and then apply the locality principle (Postulate 4) of the GTD. We can then use the result stated for the half-plane in the 3D case. Hence we can write the value of $u_{\text{co}\,1}^{d,n}$ at a point $P(x, \rho_1, \alpha_1)$ as being

$$u_{\text{co}\,1}^{d,n}(x, \rho_1, \alpha_1; \beta_1^0, \alpha_1^0) = \frac{-\exp\{-\mathrm{i}k_0\cos(\beta_1^0)x\}\exp\{\mathrm{i}k_0\sin(\beta_1^0)\rho_1\}\,\mathrm{e}^{\mathrm{i}(\pi/4)}T^{d,n}(\alpha_1, \alpha_1^0)}{2\sqrt{2\pi k_0\sin(\beta_1^0)\rho_1}}\mathbb{H}(X), \tag{4.8}$$

where X is the projection of P onto the x-axis along the line passing through P and making an angle β_1^0 with the x-axis (this is true if $\beta_1^0 < \pi/2$, otherwise the angle to consider is $\pi - \beta_1^0$). The $\mathbb{H}(X)$ factor has been added in order to take into account that the edge Λ_1 is only semi-infinite. Indeed, if $X < 0$, then no diffracted cone emanating from X will be generated simply because for $X < 0$, there is no edge to interact with the incident wave. Similarly, we can formulate an expression for the diffracted rays coming from a point Y of the edge Λ_2 as

$$u_{\text{co}\,2}^{d,n}(y, \rho_2, \alpha_2; \beta_2^0, \alpha_2^0) = \frac{-\exp\{-\mathrm{i}k_0\cos(\beta_2^0)y\}\exp\{\mathrm{i}k_0\sin(\beta_2^0)\rho_2\}\,\mathrm{e}^{i(\pi/4)}T^{d,n}(\alpha_2, \alpha_2^0)}{2\sqrt{2\pi k_0\sin(\beta_2^0)\rho_2}}\mathbb{H}(Y). \tag{4.9}$$

As mentioned in Section 4.1, a most interesting point is that these diffracted rays can travel on the surface of the quarter-plane, hit the opposite edge and be diffracted again, leading to the creation of the secondary radiated waves $u_{\text{co}\,12}^{d,n}$ and $u_{\text{co}\,21}^{d,n}$. These fields will be considered in the following subsection.

4.4 *The secondary radiated conical waves*

The advantage of the quarter-plane compared with plane sectors with a smaller internal angle is that the number of times a ray can be diffracted is limited to two. Moreover, it is important to note that these secondary radiated waves do not necessary appear for all incidence directions. Indeed, a simple geometric approach shows that $u_{\text{co}\,12}^{d,n}$ only appears if $\xi_0 > 0$. Similarly, $u_{\text{co}\,21}^{d,n}$ only appears if $\eta_0 > 0$. Table 2 summarizes the occurrence (✓) or absence (O) of the secondary radiated conical waves in terms of the azimuthal angle φ_0, which characterizes the direction of the incident wave.

Despite the geometric similarities and the fact that both problems have been treated simultaneously so far, when it comes to secondary radiated waves, the Neumann and the Dirichlet cases are very different and will now be treated separately.

4.4.1 *The Neumann case.* Now, let us try to describe $u_{\text{co}\,21}^{n}$ mathematically. First of all, we have seen that for it to occur, we need to have $\eta_0 > 0$. According to Table 1, it implies that $0 < \beta_2^0 < \pi/2$. In this

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

TABLE 2 *Occurrence of the secondary radiated waves* $u^{d,n}_{\text{co}\,12}$ *and* $u^{d,n}_{\text{co}\,21}$.

φ_0 domain	$[0,\pi/2]$	$[\pi/2,\pi]$	$[\pi,3\pi/2]$	$[3\pi/2,2\pi]$
$u^{d,n}_{\text{co}\,21}$	✓	✓	O	O
$u^{d,n}_{\text{co}\,12}$	✓	O	O	✓

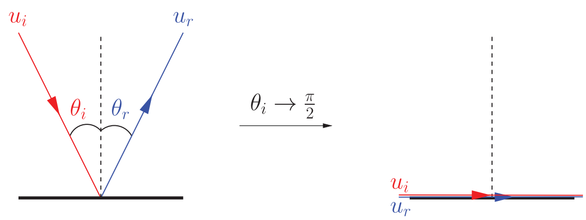

FIG. 7. Interpretation of a surface wave as a limit case of reflection.

case, the diffracted rays of $u^n_{\text{co}\,2}$ propagating on the surface of the quarter-plane will intersect the edge Λ_1. From now on, despite the misnomer, we shall use the term 'ray' to describe both the field on the ray and the ray itself. There are two types of rays satisfying these conditions, the rays propagating on the upper surface of the quarter plane and those propagating on the lower surface. At this stage, one has to realize that a surface-propagating wave can be interpreted as a limit case of an incident and reflected wave system when the angle of incidence (θ_i on Fig. 7) tends to $\pi/2$. For this reason, as illustrated in Fig. 7, a surface-propagating wave is actually the sum of two components: an 'incident' surface wave and a 'reflected' surface wave, both components having the same direction and the same amplitude.

In our case, we are only interested in the 'incident' component of the surface wave. Indeed, one can interpret the edge diffraction of a surface wave as the limit of the diffraction of an incident wave with same amplitude, when the angle between this incident wave and the surface tends to zero. Moreover, if one considers a point $P(x,\rho,2\pi)$ on the upper surface ray, the total value of the surface wave at P is given by inputting $\alpha_2 = 2\pi$ into (4.9). Hence, in order to represent the 'incident' part of the surface wave, one needs to divide the amplitude by a factor 2. Let us consider the two 'incident' rays R^{n+} and R^{n-} emanating from the edge Λ_2 at Y as shown in Fig. 8. The point P can also be expressed in terms of the coordinates (R_2, Y) (see Fig. 8). Hence, using (4.9), the same reasoning used to obtain (3.22), setting $\alpha_2 = 2\pi$ for R^{n+} and $\alpha_2 = 0$ for R^{n-} and dividing by 2, we obtain

$$R^{n+}(R_2, Y; \beta^0_2, \alpha^0_2) \approx -\frac{\mathrm{e}^{\mathrm{i}k_0 R_2}}{\sqrt{k_0 R_2}} \left\{ \frac{\mathrm{e}^{-\mathrm{i}k_0\cos(\beta^0_2)Y}\,\mathrm{e}^{\mathrm{i}(\pi/4)}}{4\sin(\beta^0_2)\sqrt{2\pi}} T^n(2\pi,\alpha^0_2) \right\}, \tag{4.10}$$

$$R^{n-}(R_2, Y; \beta^0_2, \alpha^0_2) \approx -\frac{\mathrm{e}^{\mathrm{i}k_0 R_2}}{\sqrt{k_0 R_2}} \left\{ \frac{\mathrm{e}^{-\mathrm{i}k_0\cos(\beta^0_2)Y}\,\mathrm{e}^{\mathrm{i}(\pi/4)}}{4\sin(\beta^0_2)\sqrt{2\pi}} T^n(0,\alpha^0_2) \right\}. \tag{4.11}$$

Let us attempt to express these rays in terms of the coordinates (R_1, X, β^1_1) associated with the edge Λ_1 and described in Fig. 8.

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

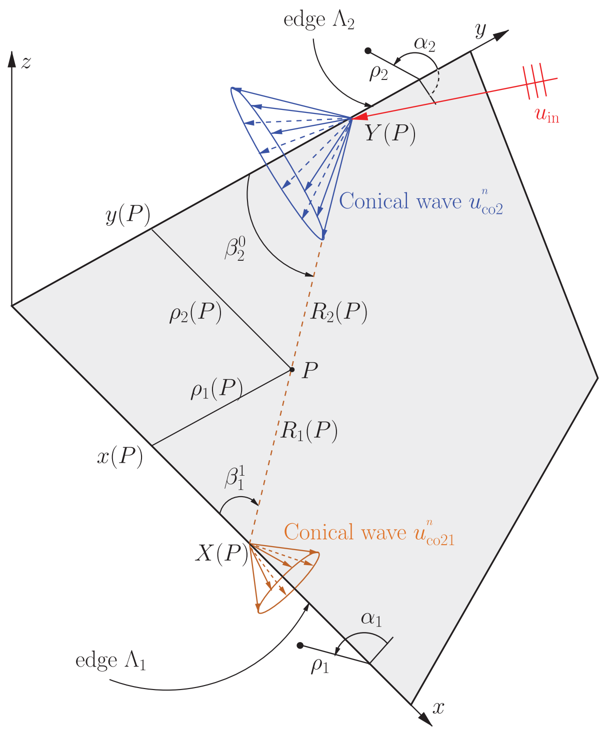

FIG. 8. Detail of the generation of $u^n_{\text{co}\,21}$.

With simple geometrical considerations, it is possible to derive the following relations between the two sets of coordinates:

$$\beta^1_1 = \frac{\pi}{2} + \beta^0_2, \quad Y = -\tan(\beta^1_1)X, \quad R_2 = \frac{-X}{\cos(\beta^1_1)} - R_1. \tag{4.12}$$

Using these relations, one can easily show that

$$\mathrm{e}^{\mathrm{i}k_0R_2}\exp\{-\mathrm{i}k_0\cos(\beta^0_2)Y\} = \mathrm{e}^{-\mathrm{i}k_0R_1}\exp\{-\mathrm{i}k_0\cos(\beta^1_1)X\}.$$

Hence, substituting (4.12) into (4.10), we obtain

$$R^{n+}(R_1, X; \beta^1_1, \alpha^0_2) = \frac{\mathrm{e}^{-\mathrm{i}k_0R_1}}{\sqrt{k_0(-X/\cos(\beta^1_1) - R_1)}}\left\{\frac{\mathrm{e}^{-\mathrm{i}k_0\cos(\beta^1_1)X}\,\mathrm{e}^{\mathrm{i}(\pi/4)}}{4\cos(\beta^1_1)\sqrt{2\pi}}T^n(2\pi, \alpha^0_2)\right\}. \tag{4.13}$$

Now, let us apply the locality principle (Postulate 4) to interpret (4.13) as a ray of an incident wave on the edge $\varLambda_1$ and to reuse the results of the half-plane. Let us consider the rapidly oscillating part of (4.13).

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

For a point P on the ray R^{n+}, we can write $x(P) = X + \rho_1 \cot(\beta_1^1)$ and $\rho_1(P) = R_1 \sin(\beta_1^1)$, and obtain:

$$
\begin{aligned}
\mathrm{e}^{-\mathrm{i}k_0 R_1}\, \mathrm{e}^{-\mathrm{i}k_0 \cos(\beta_1^1) X} &= \mathrm{e}^{-\mathrm{i}k_0 \cos(\beta_1^1) x}\, \mathrm{e}^{-\mathrm{i}k_0 \sin(\beta_1^1) \rho_1} \\
&= u_{\mathrm{in}}(x, \rho_1, 0; \beta_1^1, 0),
\end{aligned} \tag{4.14}
$$

where the expression of u_{in} is given by (4.4). Moreover, when getting close to the edge Λ_1, it is legitimate to make the assumption that $X \gg R_1$. Hence, using (4.13) and (4.14), as $R_1 \ll X$, we can write

$$
R^{n+}(x, \rho_1, X; \beta_1^1, \alpha_2^0) \sim \frac{-\mathrm{e}^{\mathrm{i}(\pi/4)} T^n(2\pi, \alpha_2^0)}{4\sqrt{-2\pi k_0 \cos(\beta_1^1) X}} u_{\mathrm{in}}(x, \rho_1, 0; \beta_1^1, 0). \tag{4.15}
$$

So locally, for the edge Λ_1, it is like being hit by a plane wave $u_{\mathrm{in}}(x, \rho_1, \alpha; \beta_1^1, 0)$, with an amplitude $-\mathrm{e}^{\mathrm{i}(\pi/4)} T^n(2\pi, \alpha_2^0)/4\sqrt{-2\pi k_0 \cos(\beta_1^1) X}$. As seen before, using the GTD, a ray of an incident plane wave generates a diffracted cone. This means that the cone itself depends solely on the amplitude of this particular ray. By the locality principle, we can re-use the result of the half-plane to say that at a point P_2 (see Fig. 8) with coordinates $(x^{(2)}, \rho_1^{(2)}, \alpha_1^{(2)}) = (X - R_1^{(2)} \cos(\beta_1^1), R_1^{(2)} \sin(\beta_1^1), \alpha_1^{(2)})$, we obtain:

$$
\begin{aligned}
u^{n+}_{\mathrm{co}\,21}(P_2) &= \frac{-\mathrm{e}^{\mathrm{i}(\pi/4)} T^n(2\pi, \alpha_2^0)}{4\sqrt{-2\pi k_0 \cos(\beta_1^1) X}} \left[-\frac{\mathrm{e}^{\mathrm{i}k_0 R_1^{(2)}}}{\sqrt{k_0 R_1^{(2)}}} \left\{ \frac{\mathrm{e}^{-\mathrm{i}k_0 \cos(\beta_1^1) X}\, \mathrm{e}^{\mathrm{i}(\pi/4)} T^n(\alpha_1^{(2)}, 0)}{2 \sin(\beta_1^1) \sqrt{2\pi}} \mathbb{H}(X) \right\} \right] \\
&= \frac{i T^n(2\pi, \alpha_2^0) T^n(\alpha_1^{(2)}, 0)}{16\pi \sin(\beta_1^1)} \frac{\mathrm{e}^{-\mathrm{i}k_0 \cos(\beta_1^1) X}}{\sqrt{-k_0 \cos(\beta_1^1) X}} \frac{\mathrm{e}^{\mathrm{i}k_0 R_1^{(2)}}}{\sqrt{k_0 R_1^{(2)}}} \mathbb{H}(X).
\end{aligned} \tag{4.16}
$$

So, once more, this looks like a conical wave emanating from the point X of the edge Λ_1. The main difference between the primary and secondary radiated cones is that this time the slowly oscillating part of the wave has a dependency in X. Hence, it is possible to describe the secondary radiated wave emanating from Λ_1 as follows. For any point P_2 lying within the cone of semi-angle $\pi - \beta_1^1$ with apex at the origin and the edge Λ_1 as axis, there exists a unique set of coordinates $(X, R_1^{(2)}, \alpha_1^{(2)})$, such that $(x^{(2)}, \rho_1^{(2)}, \alpha_1^{(2)}) = (X - R_1^{(2)} \cos(\beta_1^1), R_1^{(2)} \sin(\beta_1^1), \alpha_1^{(2)})$. At this point, the contribution of the ray is given by (4.16). It is possible to express this in terms of $\rho_1^{(2)}$ and $x^{(2)}$, but no major simplification arises. If P_2 is outside this cone, then the wave is equal to zero. Following a similar reasoning leads to a corresponding expression for $u^{n-}_{\mathrm{co}\,21}$,

$$
u^{n-}_{\mathrm{co}\,21}(P_2) = \frac{i T^n(0, \alpha_2^0) T^n(\alpha_1^{(2)}, 2\pi)}{16\pi \sin(\beta_1^1)} \frac{\mathrm{e}^{-\mathrm{i}k_0 \cos(\beta_1^1) X}}{\sqrt{-k_0 \cos(\beta_1^1) X}} \frac{\mathrm{e}^{\mathrm{i}k_0 R_1^{(2)}}}{\sqrt{k_0 R_1^{(2)}}} \mathbb{H}(X).
$$

Finally, the overall contribution of $u^n_{\mathrm{co}\,21}$ at the point P_2 is given by $u^n_{\mathrm{co}\,21}(P_2) = u^{n+}_{\mathrm{co}\,21}(P_2) + u^{n-}_{\mathrm{co}\,21}(P_2)$, which leads to

$$
u^n_{\mathrm{co}\,21}(P_2) = \frac{i}{2\pi \sin(\beta_1^1)} \frac{\mathrm{e}^{-\mathrm{i}k_0 \cos(\beta_1^1) X}}{\sqrt{-k_0 \cos(\beta_1^1) X}} \frac{\mathrm{e}^{\mathrm{i}k_0 R_1^{(2)}}}{\sqrt{k_0 R_1^{(2)}}} D^n_{21}(\alpha_1^{(2)}, \alpha_2^0) \mathbb{H}_{21}, \tag{4.17}
$$

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

where

$$D^n_{21}(\alpha_1,\alpha_2^0)=\frac{1}{8}(T^n(2\pi,\alpha_2^0)T^n(\alpha_1,0)+T^n(0,\alpha_2^0)T^n(\alpha_1,2\pi))$$
$$=-\sec\left(\frac{\alpha_2^0}{2}\right)\sec\left(\frac{\alpha_1}{2}\right), \tag{4.18}$$

and

$$\mathbb{H}_{21}=\mathbb{H}(X)\mathbb{H}(\eta_0). \tag{4.19}$$

Applying the same reasoning and switching the edges would lead to a similar mathematical description of $u^n_{\mathrm{co}\,12}$.

4.4.2 *The Dirichlet case.* The aim of this subsection is to describe $u^d_{\mathrm{co}\,21}$ mathematically. Concerning the secondary radiated waves, the Dirichlet case is more complicated than the Neumann case. Indeed, in the previous subsection about the Neumann case, we have seen that, close to the edge Λ_1, the ray along the upper surface emanating from a first diffraction by Λ_2 is given by (4.15). The main problem occurring in the Dirichlet case is that $T^d(2\pi,\alpha_2^0)=0$, and hence we cannot use a similar expression to (4.15) in this case.

An equivalent formulation. Let us rewrite (4.15) as

$$R^{n+}\underset{\alpha_2\to 2\pi}{\sim}\frac{-\mathrm{e}^{\mathrm{i}(\pi/4)}T^n(\alpha_2,\alpha_2^0)}{4\sqrt{-2\pi k_0\cos(\beta_1^1)X}}u_{\mathrm{in}}(x,\rho_1,\alpha_1).$$

We can write this because of (4.15) and the fact that

$$T^n(\alpha_2,\alpha_2^0)\underset{\alpha_2\to 2\pi}{\sim}T^n(2\pi,\alpha_2^0)\quad\text{and}\quad u_{\mathrm{in}}(x,\rho_1,\alpha_1;\beta_1^1,0)\underset{\alpha_2\to 2\pi}{\sim}u_{\mathrm{in}}(x,\rho_1,0;\beta_1^1,0).$$

Indeed, when α_2 tends to 2π, α_1 tends to 0, as shown in Fig. 9. The Dirichlet field incidence on Λ_1 can then a priori be written in a similar form:

$$R^{d+}\underset{\alpha_2\to 2\pi}{\sim}\frac{-\mathrm{e}^{\mathrm{i}(\pi/4)}T^d(\alpha_2,\alpha_2^0)}{4\sqrt{-2\pi k_0\cos(\beta_1^1)X}}u_{\mathrm{in}}(x,\rho_1,\alpha_1;\beta_1^1,0). \tag{4.20}$$

Now, let us approximate T^d by its Taylor expansion as $\alpha_2\to 2\pi$, to obtain

$$T^d(\alpha_2,\alpha_2^0)\underset{\alpha_2\to 2\pi}{\sim}(\alpha_2-2\pi)\partial_1 T^d(2\pi,\alpha_2^0), \tag{4.21}$$

where ∂_1 represents the partial derivative with respect to the first argument. At this stage, it is important to note that $\partial_1 T^d(2\pi,\alpha_2^0)\neq 0$. Hence, as $\alpha_2\to 2\pi$ and $\alpha_1\to 0$, we have an incident wave of the form

$$R^{d+}\underset{\alpha_2\to 2\pi}{\sim}\frac{-\mathrm{e}^{\mathrm{i}(\pi/4)}\partial_1 T^d(2\pi,\alpha_2^0)}{4\sqrt{-2\pi k_0\cos(\beta_1^1)X}}\{(\alpha_2-2\pi)u_{\mathrm{in}}(x,\rho_1,\alpha_1;\beta_1^1,0)\} \tag{4.22}$$

at the point P defined in Fig. 9. $P(x,\rho_1,\alpha_1)$ is chosen to be on the diffracted cone emanating from the point Y of the edge Λ_2, and in a small neighbourhood of the point X of the edge Λ_1 that would be hit by the tangential rays.

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

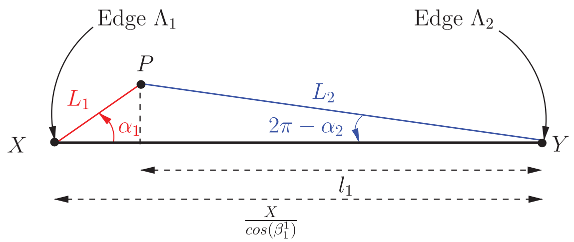

FIG. 9. Illustration of the geometry in the vicinity of the point $x = X$ of the edge Λ_1.

Now, let us operate a change of variable, by expressing everything in terms of α_1 (instead of α_2). Using the fact that $\alpha_2 - 2\pi$ is small, that we are in a small neighbourhood of X (i.e. we have $L_2 \sim l_1 \sim -X/\cos(\beta_1^1)$) and according to simple trigonometry relations related to Fig. 9, we have

$$\begin{aligned}\alpha_2 - 2\pi &\underset{\alpha_2 \to 2\pi}{\sim} \sin(\alpha_2 - 2\pi)\\ &\underset{\alpha_1 \to 0}{\sim} \frac{L_1 \sin(\alpha_1 - 0)}{l_1}\\ &\underset{\alpha_1 \to 0}{\sim} \frac{\rho_1/\sin(\beta_1^1)}{-X/\cos(\beta_1^1)} \sin(\alpha_1 - 0)\\ &\underset{\alpha_1 \to 0}{\sim} \frac{-1}{X \tan(\beta_1^1)} \rho_1 \sin(\alpha_1 - 0),\end{aligned}$$

and hence, (4.22) can be rewritten as follows, when $\alpha_1 \to 0$

$$R^{d+} \underset{\alpha_1 \to 0}{\sim} \frac{e^{i(\pi/4)} \partial_1 T^d(2\pi, \alpha_2^0)}{4X \tan(\beta_1^1)\sqrt{-2\pi k_0 \cos(\beta_1^1) X}} u_{\text{in}}^{☼}(x, \rho_1, \alpha_1; \beta_1^1, 0), \tag{4.23}$$

where

$$u_{\text{in}}^{☼}(x, \rho_1, \alpha_1; \beta_1^1, 0) = \rho_1 \sin(\alpha_1 - 0) u_{\text{in}}(x, \rho_1, \alpha_1; \beta_1^1, 0),$$

where the expression of u_{in} is given by (4.4).

A new two-dimensional half-plane diffraction problem.

Expression (4.23) leads us to consider a new 2D diffraction problem. Indeed, let us consider the problem of an incident wave $u_{(2)\,\text{in}}^{☼}$ on a half-plane, where $u_{(2)\,\text{in}}^{☼}$ is written as follows:

$$u_{(2)\,\text{in}}^{☼}(\rho, \alpha; \alpha_0) = \rho \sin(\alpha - \alpha^0) u_{(2)\,\text{in}}(\rho, \alpha; \alpha_0), \tag{4.24}$$

where $u_{(2)\,\text{in}}$ is defined by (3.2). The solution to this problem is a priori unknown. However, remembering that $u_{(2)\,\text{in}}$ also depends on α_0, following a similar argument as Shanin (2011), we can write

$$u_{(2)\,\text{in}}^{☼}(\rho, \alpha; \alpha_0) = \frac{i}{k_0} \frac{\partial u_{(2)\,\text{in}}}{\partial \alpha_0}(\rho, \alpha; \alpha_0). \tag{4.25}$$

It is now important to note that the half-plane diffraction problem does not depend on α_0 in the sense that neither the Helmholtz equation nor the boundary conditions depend on α_0. Therefore, the diffracted

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

wave $u^{☼ d}_{(2)D}$ of this new problem, using the notations of Section 3 (and equation (3.12) in particular), is given by

$$u^{☼ d}_{(2)D}(\rho,\alpha;\alpha_0) = \frac{i}{k_0}\frac{\partial u^d_{(2)D}}{\partial \alpha_0}(\rho,\alpha;\alpha_0),$$
$$= \frac{-i}{k_0}\frac{\mathrm{e}^{\mathrm{i}(\pi/4)}}{2\sqrt{2\pi}}\frac{\partial T^d}{\partial \alpha_0}(\alpha,\alpha_0)\frac{\mathrm{e}^{\mathrm{i}k_0\rho}}{\sqrt{k_0\rho}}.$$

Final expression for the Dirichlet case. We can now apply the reasoning of the previous paragraph to the case we are interested in by remarking that

$$u^{☼}_{\mathrm{in}}(x,\rho_1,\alpha_1;\beta_1^1,0) = \frac{i}{k_0\sin(\beta_1^1)}\frac{\partial u_{\mathrm{in}}}{\partial \alpha_1^0}(x,\rho_1,\alpha_1;\beta_1^1,0),$$

and by the locality principle (Postulate 4) we can re-use the result of the half-plane to show that at a point P_2 with coordinates $(x^{(2)},\rho_1^{(2)},\alpha_1^{(2)}) = (X - R_1^{(2)}\cos(\beta_1^1), R_1^{(2)}\sin(\beta_1^1),\alpha_1^{(2)})$, the diffraction of the ray R^{d+} by the edge Λ_1 leads to the contribution

$$u^{d+}_{\mathrm{co}\,21}(P_2) \approx \frac{i}{k_0\sin(\beta_1^1)}\frac{\mathrm{e}^{\mathrm{i}(\pi/4)}\partial_1 T^d(2\pi,\alpha_2^0)}{4X\tan(\beta_1^1)\sqrt{-2\pi k_0\cos(\beta_1^1)X}}\left\{-\mathrm{e}^{-\mathrm{i}k_0\cos(\beta_1^1)X}\frac{\mathrm{e}^{\mathrm{i}k_0R_1^{(2)}}}{\sqrt{k_0R_1^{(2)}}}\frac{\mathrm{e}^{\mathrm{i}(\pi/4)}\partial_2 T^d(\alpha_1^{(2)},0)}{2\sin(\beta_1^1)\sqrt{2\pi}}\right\},$$
$$\approx \frac{\mathrm{e}^{-\mathrm{i}k_0\cos(\beta_1^1)X}\partial_1 T^d(2\pi,\alpha_2^0)\partial_2 T^d(\alpha_1^{(2)},0)}{16\pi(k_0X)^{3/2}\sin^2(\beta_1^1)\tan(\beta_1^1)\sqrt{-\cos(\beta_1^1)}}\frac{\mathrm{e}^{\mathrm{i}k_0R_1^{(2)}}}{\sqrt{k_0R_1^{(2)}}}.$$

A similar study leads to the following expression of $u^{d-}_{\mathrm{co}\,21}$:

$$u^{d-}_{\mathrm{co}\,21}(P_2) \approx \frac{\mathrm{e}^{-\mathrm{i}k_0\cos(\beta_1^1)X}\partial_1 T^d(0,\alpha_2^0)\partial_2 T^d(\alpha_1^{(2)},2\pi)}{16\pi(k_0X)^{3/2}\sin^2(\beta_1^1)\tan(\beta_1^1)\sqrt{-\cos(\beta_1^1)}}\frac{\mathrm{e}^{\mathrm{i}k_0R_1^{(2)}}}{\sqrt{k_0R_1^{(2)}}}.$$

Moreover, $u^d_{\mathrm{co}\,21}$ is given by $u^{d+}_{\mathrm{co}\,21} + u^{d-}_{\mathrm{co}\,21}$, which leads to the following description of $u^d_{\mathrm{co}\,21}$: for any point P_2 lying within the cone of semi-angle $\pi - \beta_1^1$ with apex at the origin and the edge Λ_1 as axis, there exists a unique set of coordinates $(X, R_1^{(2)},\alpha_1^{(2)})$, such that $(x,\rho_1^{(2)},\alpha_1^{(2)}) = (X - R_1^{(2)}\cos(\beta_1^1), R_1^{(2)}\sin(\beta_1^1),\alpha_1^{(2)})$, and at this point we have

$$u^d_{\mathrm{co}\,21}(P_2) \approx \frac{\mathrm{e}^{-\mathrm{i}k_0\cos(\beta_1^1)X}}{8\pi(k_0X)^{3/2}\sin^2(\beta_1^1)\tan(\beta_1^1)\sqrt{-\cos(\beta_1^1)}}\frac{\mathrm{e}^{\mathrm{i}k_0R_1^{(2)}}}{\sqrt{k_0R_1^{(2)}}}D^d_{21}(\alpha_1^{(2)},\alpha_2^0)\mathbb{H}_{21}, \tag{4.26}$$

where $\mathbb{H}_{21}$ is defined by (4.19) and

$$D^d_{21}(\alpha_1^{(2)},\alpha_2^0) = \frac{1}{2}(\partial_1 T^d(2\pi,\alpha_2^0)\partial_2 T^d(\alpha_1^{(2)},0) + \partial_1 T^d(0,\alpha_2^0)\partial_2 T^d(\alpha_1^{(2)},2\pi)) \tag{4.27}$$

$$= -\sec\left(\frac{\alpha_2^0}{2}\right)\tan\left(\frac{\alpha_2^0}{2}\right)\sec\left(\frac{\alpha_1^{(2)}}{2}\right)\tan\left(\frac{\alpha_1^{(2)}}{2}\right). \tag{4.28}$$

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

Finally note that the expressions (4.26) and (4.17) are valid in the far field in the sense that both k_0X and $k_0R_1^{(2)}$ should be large.

4.5 *The vertex spherical wave*

According to Postulate 2 of the GTD, when an incident ray hits a corner, it gives rise to diffracted rays travelling in all directions, thus generating an outgoing spherical wave. Therefore, in the far field, we can write down an asymptotic form for the spherical wave $u_{\mathrm{sp}}^{d,n}$ that is given by

$$u_{\mathrm{sp}}^{d,n}(\omega, r) \approx 2\pi \frac{\mathrm{e}^{\mathrm{i}k_0 r}}{k_0 r} f^{d,n}(\omega, \omega_0), \tag{4.29}$$

where $f^{d,n}$ is the diffraction coefficient, which depends on both the incident and the observer directions. As previously stated, this asymptotic expansion can only be valid away from the penumbral zones, and $f^{d,n}$ is singular when crossing the penumbral zones. We can, however, define a unique function $f^{d,n}$ in the regions where the expansion is valid, by describing $u_{\mathrm{sp}}^{d,n}$ as the difference between the total field and the other wave fields determined previously,

$$u_{\mathrm{sp}}^{d,n} = u_{\mathrm{tot}}^{d,n} - (u_{\mathrm{in}} + u_{\mathrm{re}}^{d,n} + u_{\mathrm{co}\,1}^{d,n} + u_{\mathrm{co}\,2}^{d,n} + u_{\mathrm{co}\,21}^{d,n} + u_{\mathrm{co}\,12}^{d,n}).$$

To date, no explicit mathematical expression for the diffraction coefficient $f^{d,n}$ exists.

5. Concluding remarks

In this paper, we have derived some explicit mathematical expressions for all the waves emanating from edge diffraction in the quarter-plane problem. This also led to a precise representation of the diffraction coefficient describing the spherical wave resulting from the diffraction by the corner of the quarter-plane. As explained by Shanin (2005b), in order to derive the three modified Smyshlyaev formulae, one needs to derive three embedding formulae first. The key to the derivation of these three embedding formulae is that the primary radiated waves can be annihilated by some simple differential operators with constant coefficients. In the conclusion of Assier & Peake (2012), we conjectured the existence of an ultimate modified Smyshlyaev formula giving the corner diffraction coefficient everywhere. In order to obtain such a formula, if one wishes to follow a similar approach as in Shanin (2005b), one needs to derive an ultimate embedding formula. This implies that a simple differential operator with constant coefficients that annihilate the secondary radiated waves $u_{\mathrm{co}\,21}^{d,n}$ and $u_{\mathrm{co}\,12}^{d,n}$ should be found. However, the complex structure of these secondary radiated waves obtained in the present paper makes this task more complicated than anticipated in Assier & Peake (2012) and success seems quite unlikely. The constructive approach used by Craster & Shanin (2005) to define the differential operators used in the case of the wedge embedding formulae also gives a clue to why operators killing the secondary radiated waves are unlikely to exist in our case. Hence, in order to obtain an ultimate modified Smyshlyaev formula, new techniques should probably be developed.

Acknowledgements

The authors are also very grateful to A.V. Shanin for very helpful discussions and comments regarding the subject of this paper.

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014

Funding

R.C.A. acknowledges the support of the EPSRC and the Cambridge European Trust.

References

Albani, M. (2007) On Radlow's quarter-plane diffraction solution. *Radio Sci.*, **42**, 1–10.

Assier, R. C. & Peake, N. (2012) On the diffraction of acoustic waves by a quarter-plane. *Wave Motion*, **49**, 64–82.

Babich, V. M., Dement'ev, D. B. & Samokish, B. A. (1995) On the diffraction of high-frequency waves by a cone of arbitrary shape. *Wave Motion*, **21**, 203–207.

Babich, V. M., Dement'ev, D. B., Samokish, B. A. & Smyshlyaev, V. P. (2000) On evaluation of the diffraction coefficients for arbitrary 'nonsingular' directions of a smooth convex cone. *SIAM J. Appl. Math.*, **60**, 536–573.

Blume, S. (1996) Spherical-multipole analysis of electromagnetic and acoustical scattering by a semi-infinite elliptic cone. *IEEE Antennas Propag. Mag.*, **38**, 33–44.

Borovikov, V. A. & Kinber, B. Y. (1994) *Geometrical Theory of Diffraction*. London: IEE.

Bowman, J. J., Senior, T. B. A. & Uslenghi, P. L. E. (1987) *Electromagnetic and Acoustic Scattering by Simple Shapes*. New York: Hemisphere Publishing Corporation.

Budaev, B. & Bogy, D. (2005) Diffraction of a plane wave by a sector with Dirichlet or Neumann boundary conditions. *IEEE Trans. Antennas Propag.*, **53**, 711–718.

Craster, R. V. & Shanin, A. V. (2005) Embedding formulae for diffraction by rational wedge and angular geometries. *Proc. R. Soc. A Math. Phys. Eng. Sci.*, **461**, 2227–2242.

Felsen, L. B. & Marcuvitz, N. (1973) *Radiation and Scattering of Waves*. Piscataway NJ/US: Prentice-Hall, Inc.

Hansen, T. B. (1991) Corner diffraction coefficients for the quarter plane. *IEEE Trans. Antennas Propag.*, **39**, 976–984.

Jones, D. S. (1989) *Acoustic and Electromagnetic Waves*. USA: Oxford University Press, pp. 548–562.

Keller, J. B. (1962) Geometrical theory of diffraction. *J. Opt. Soc. Am.*, **52**, 116–130.

Kraus, L. & Levine, L. M. (1961) Diffraction by an elliptic cone. *Commun. Pure Appl. Math.*, **14**, 49–68.

Meister, E. (1987) Some solved and unsolved canonical problems of diffraction theory. *Lect. Notes Math.*, **1285**, 320–336.

Radlow, J. (1961) Diffraction by a quarter-plane. *Arch. Ration. Mech. Anal.*, **8**, 139–158.

Satterwhite, R. (1974) Diffraction by a quarter plane, exact solution, and some numerical results. *IEEE Trans. Antennas Propag.*, **AP22**, 500–503.

Shanin, A. V. (2005a) Coordinate equations for a problem on a sphere with a cut associated with diffraction by an ideal quarter-plane. *Q. J. Mech. Appl. Math.*, **58**, 289–308.

Shanin, A. V. (2005b) Modified Smyshlyaev's formulae for the problem of diffraction of a plane wave by an ideal quarter-plane. *Wave Motion*, **41**, 79–93.

Shanin, A. V. (2011) Asymptotics of waves diffracted by a cone and diffraction series on a sphere. *Zapiski Nauch Sem POMI RAN (in Russian, to be translated in Journal of Mathematical Sciences)*, **393**, 234–258.

Skelton, E. A., Craster, R. V., Shanin, A. V. & Valyaev, V. Y. (2010) Embedding formulae for scattering by three-dimensional structures. *Wave Motion*, **47**, 299–317.

Smyshlyaev, V. P. (1990) Diffraction by conical surfaces at high-frequencies. *Wave Motion*, **12**, 329–339.

Downloaded from http://imamat.oxfordjournals.org/ by guest on December 7, 2014