Automorphisms of finite soluble groups. Preliminary version

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AUTOMORPHISMS OF FINITE SOLUBLE GROUPS. PRELIMINARY VERSION OF 23 MARCH 1994

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Preface to Brian Hartley's preprint of 1994 "*Automorphisms of finite soluble groups. Preliminary version*"

This preprint was distributed by Brian Hartley among his colleagues in 1994, not long before his untimely death. He worked a lot on almost fixed-point-free automorphisms of finite groups, both soluble and unsoluble, and obtained fundamental results in this area. He also posed many interesting problems in his talks at conferences, in research and survey papers, and in "Kourovka Notebook". Some of these problems still remain open. In this note Hartley obtained a partial result, which is however still of interest to the specialists in the field. We think that it is most appropriate to make this note more widely available through publication as a MIMS preprint, which would allow scholars to give proper references to Hartley's theorems and stimulate further research.

We are grateful to Mrs Mary Hartley and Ms Catherine Barber-Brown for giving their kind permissions to publish a facsimile of Brian Hartley's original text as a MIMS preprint.

Please note updated references on the additional page attached to a facsilile.

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Preliminary version

Brian Hartley

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1 Section 1.

Let's try to begin with the following. Let $h(G)$ denote the Fitting height of a finite soluble group G.

Theorem 1.1 *Let G be a finite soluble group admitting an automorphism α of order n. Let $C = C_G(\alpha)$. Suppose that every abelian subgroup of C has rank at most r, and that there is at most one prime p that divides both $|G|$ and n. Then $h(G)$ is n, r-bounded.*

Proof.
Case 1. $\alpha = 1$. The hypothesis is that every abelian subgroup of G has rank at most r. Then the Fitting subgroup F of F has r-bounded rank s, say. Let Φ be the Frattini subgroup of G. Then $V = F/\Phi$ is a direct product of elementary abelian groups of bounded rank, and so as $C_G(V) = F$, we have that G/F is a subdirect product of soluble linear groups of s-bounded degree. By Mal'cev's Theorem,it follows that G/F is soluble of bounded derived length.
Case 2. $(n, |G|) = 1$. By Case 1, C has r-bounded Fitting height t, say. By Turull's Theorem (or the earlier versions due to Thompson or Kurzweil), the result follows.
Case 3. General case. Let $n = p^s m$, where $(p, m) = (m, |G|) = 1$. Let $\alpha = \alpha_p \alpha_m$, where α_p is a p-element and α_m is a p'-element. Let $D = C_G(\alpha_m)$. By Turull's Theorem as above, it suffices to show that $h(D)$ is n, r-bounded. Considering the action of α on D, we see that we may assume that α is a p-element and $D = G$. Let $Q = O_{p'}(G)$. By Case 2, $h(Q)$ is r, n-bounded. We have $C_{G/Q}(\alpha) = CQ/Q$, and with a little argument we see that the abelian subgroups of CQ/Q have r-bounded rank. So we may assume that $Q = 1$.

Let $P = O_p(G)$. Every abelian subgroup of $C_P(\alpha)$ has rank at most r. It follows that the rank of P itself is r, n-bounded, and arguing as in Case 1 with P/Φ gives the result.

Remark. It follows from the argument of Case 1 that the abelian section rank is bounded in terms of the abelian subgroup rank, as is well known.

Invariance of centralizers.

Lemma 1.2 *Let G be a finite soluble group acted on by an element α of order n. Let $C = C_G(\alpha)$. Assume that every abelian section of C has rank at most r. Let N be a normal α-invariant subgroup of G of Fitting height at most h, and let $D/N = C_{G/N}(\alpha)$. Then there exists a number s depending only on n, r and h, such that every abelian section of D/N has rank at most s.*

Proof. By induction on h, we may assume that $h = 1$. Let π be the set of prime divisors of n and $Q = O_{\pi'}(N)$. Let $B/Q = C_{G/Q}(\alpha)$. Then $B = CQ$, so it follows that we may assume that $Q = 1$. By a further induction we may assume that N is a p-group, where $p|n$.

Next we may assume that $D = G$, so that α operates trivially on G/N. Now let $\alpha = \beta\gamma$, where β is a p-element and γ a p'-element. Put $E = C_G(\gamma)$. Then $D = EN$. It is enough to prove that E has abelian section rank bounded in terms of the given parameters. Hence we may assume that $E = G$, in other words, that α is of p-power order.

Let $R = O_{p'}(G)$ and $P = O_p(G) \geq N$. Then $[R, \alpha] \leq R \cap N = 1$, and so $R \leq C$. Hence R has abelian section rank at most r. Also $C/R = C_{G/R}(\alpha)$. So we may assume that $R = 1$. Let $V = P/\Phi$, where Φ is the Frattini subgroup of G. Then $C_G(V) = P$. Now P admits the automorphism α and every abelian subgroup of $C_P(\alpha)$ has rank at most r. It follows that the rank of P is bounded in terms of n and r, and hence V is a vector space whose dimension m is n, r-bounded, over a field of order p, a prime in π. Therefore G/P is a group of order bounded in terms of n and r, and hence involves only a set of primes bounded in terms of the given parameters. Since we have seen that P has bounded rank, this completes the proof.

2 Section 2.

Lemma 2.1 *Let p and q be primes, let Q be a finite q-group, and let x be an element of Q of order q^n. Let $y = x^{q^{n-1}}$. Let V be a faithful F_pQ-module and let $\dim C_V(x) = m$. Let A be any abelian subgroup of Q normalized by x. Then*
(i) $|[A, y]|$ is p, q, n, m-bounded.
(ii) The normal closure $< y^Q >$ has p, q, n, m-bounded class.

For the proof we need the following.

Lemma 2.2 *Let q be a prime, and let $H = A < x >$ be the product of a finite normal abelian q-group A and a cyclic subgroup $< x >$ of order q^n. Let $y = x^{q^{n-1}}$. Let k be a field of characteristic different from q, and let V be a finite dimensional kH-module such that $V = [V, [A, y]]$.. Then $\dim C_V(x) = \frac{1}{q^n} \dim V$.*

Proof. We may assume that V is irreducible and that k is algebraically closed. Then we apply Clifford's Theorem to see that V is induced from a one-dimensional kA-module.

Proof of Lemma 2.1. (i) We may assume that $Q = A < x >$ and that $V = [V, [A, y]]$. By Lemma 2.2, we find that $\dim V$ is bounded in terms of the relevant parameters, and hence so is $|Q|$.
(ii) Let $R =< y^Q >$ have class t, let $s = [\frac{t}{2}] + 1$, and let $A = \gamma_s(R)$. Here, $\{\gamma_i(X)\}$ denotes the lower central series of a group X. Then A is abelian and normal in Q. By (i), $|[A, y]|$ is bounded. Let $|[A, y]| = q^u$ and define $A_0 = A$, and $A_{i+1} = [A_i, R, R]$. Suppose that $[A_i, R] \neq 1$ for some i. Then y does not centralize A_i/A_{i+1}. For if $[A_i, y] \leq A_{i+1}$, then as A_i and A_{i+1} are normal subgroups of Q, we find that $[A_i, R] \leq A_{i+1}$, so $[A_i, R] = [A_i, R, R]$ and $[A_i, R] = 1$. Since $[y, A_i] \neq A_{i+1}$, we may choose an element $a_i \in A_i$ such that $b_i = [a_i, y] \notin A_{i+1}$. Thus,

$$b_i \in ([A, y] \cap A_i) \setminus ([A, y] \cap A_{i+1}.).$$

It follows that $[A_u, R] = 1$. This is the claim.

We apply this to q-soluble linear groups. If X is a finite group, let $\lambda_{q,0}(X) = 1$, $\lambda_{q,1}(X) = O_{q',q}(X)$, and $\lambda_{q,i+1}(X)/\lambda_{q,i}(X) = O_{q',q}(X/\lambda_{q,i}(X))$.

Proposition 2.3 *Let p and q be primes and let G be a finite q- soluble linear group acting faithfully on a vector space V over F_p. Let x be an element of order q^n in G and let $y = x^{q^{n-1}}$. Let $\dim C_V(x) = m$. Then there is a number s, bounded in terms of p, q, n and m, such that $y \in \lambda_{q,s}(G)$.*

Proof. Let Q be a Sylow q-subgroup of G containing X. Then by Lemma 2.1, the normal closure R of y in Q has bounded class. Now if A is any abelian normal subgroup of Q, then it is known that $A \leq \lambda_{q,1}(G)$ if $q \geq 5$ [1, Hall and Higman], while if $q = 3$ then $A \leq \lambda_{q,2}(G)$ and if $q = 2$ then $A \leq \lambda_{q,3}(G)$. [2]. From these facts and a simple induction the claim follows.

Now the aim is to prove the following.

Theorem 2.4 *Let G be a finite soluble group containing an element u of order $p^m q^n$. where p and q are distinct primes. Let $C = C_G(u)$ and suppose that C has abelian section rank at most r. Then $h(G)$ is p^m, q^n, r-bounded.*

Proof. We will prove this by induction on n. If $n = 0$ then it follows from Theorem 1.1. So suppose that $n > 0$. Let $Q = O_{p'}(G)$. By Theorem 1.1, Q has bounded Fitting height, and by Lemma 1.2, the hypotheses are inherited by G/Q (with a different value of r.) Hence we may assume that $Q = 1$. Let $P = O_p(G)$. In a similar way we may assume that $\Phi(G) = 1$, so that we can view $\overline{G} = G/P$ as a linear group acting on $V = P$. Let x be the q-part of u, let z be the p-part, and let $y = x^{q^{n-1}}$. Let $\dim C_V(x) = d$. Considering the action of z on $C_V(x)$, we see that $\dim C_V(u) \geq d/p^m$. This tells us that d is bounded.

Note .This would not work if z were an arbitrary q'-element.
By Proposition 2.3, we have $y \in \lambda_{q,c}(G)$, where c is bounded in terms of the available parameters. Now we show the following.
(*) *The Fitting height of $\lambda_{q,e}(G)$ is bounded in terms of e and the other parameters.*
For suppose that we have this for some value of e. By Lemma 1.2, we have that if $W = G/\lambda_{q,e}(G)$, then $C_W(u)$ has bounded abelian section rank, where the bound is in terms of e and the other parameters. Considering in particular the action of u on $O_{q'}(W)$, we may apply Theorem 1.1 to conclude that $O_{q'}(W)$ has Fitting height bounded in terms of the various parameters. This gives the inductive step.
Since c is bounded in terms of p , q , m , n , r, we may apply (*) to conclude firstly that $\lambda_{c,q}(G)$ has Fitting height bounded in terms of these parameters, and secondly that in $C_{G/\lambda_{q,e}}(G)$, the centralizer of u has abelian section rank bounded in terms of these parameters. Now the order of the image of u in this group divides $p^m q^{n-1}$, and so by induction on n, this group has bounded Fitting height. This concludes the proof of the theorem.

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