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2013

MIMS EPrint: **2013.44**

ISSN 1749-9097

Submitted exclusively to the London Mathematical Society
doi:10.1112/0000/000000

The Maximal Subgroups of $E_7(2)$

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Abstract

Here we determine up to conjugacy all the maximal subgroups of the finite exceptional group of Lie-type $E_7(2)$.

1. *Introduction*

Interest in maximal subgroups of finite simple groups can be traced back to the dawn of group theory, in the guise of primitive permutation groups. The letter Galois wrote to Chevallier [**20**] on May 29, 1832, the evening before the fatal duel, which includes statements on the minimal (non-trivial) permutation degrees of $L_2(p)$, p a prime, being one such example. Later Dickson [**18**] in 1901 determined all the subgroups of $L_2(q)$ (q a power of a prime), while those for $L_3(q)$ were similarly classified by Mitchell [**37**] in 1911 (for q odd) and Hartley [**23**] in 1925 (for q even). As the work on the finite simple groups, which would ultimately lead to their classification, gathered pace so too did investigations into the maximal subgroups of the simple groups. For many of the sporadic groups machine calculations figure prominently – at the moment all maximal subgroups of the sporadic groups, apart from those of the Monster, are known. Not surprisingly, there is still much to be learnt about the maximal subgroups of the simple groups of Lie-type. This area has been the subject of intense investigation by numerous authors over the last sixty years. The approaches used have been wide and diverse, ranging from exploiting certain multilinear forms associated with the groups ([**3**], [**4**], [**16**]), through the theory of linear algebraic groups ([**31**], [**34**]), to a mixture of both theoretical and computational work ([**25**], [**38**]). Among the many milestones we mention Aschbacher [**5**] which introduced the Aschbacher classes for the classical groups and the recent work by Liebeck, Saxl and Seitz amongst others on the maximal subgroups of exceptional algebraic and finite groups of Lie-type. This latter body of work forms the starting point for our work here, so we summarize this as Theorem 2.1 in our next section. Our interest here is in the exceptional group of Lie-type $E_7(2)$ – for the state of play with the other exceptional groups see Chapter 4 of [**40**] and references therein. The main result of this paper is as follows (notation will be discussed later in this section).

Theorem 1.1. *Let H be a maximal subgroup of $E_7(2)$. Then H has one of the following shapes:*

2000 *Mathematics Subject Classification* 20E28, 20E32.

Partial support for Bates (EPSRC Studentship) and Ballantyne (EPSRC Mathematics Platform Engagement grant EP/I01912X/1) is gratefully acknowledged.

$2^{1+32} : \Omega_{12}^{+}(2)$	$[2^{47}] : (\mathrm{Sym}(3) \times L_6(2))$
$[2^{53}] : (\mathrm{Sym}(3) \times L_3(2) \times L_4(2))$	$[2^{50}] : (L_3(2) \times L_5(2))$
$[2^{42}] : (\mathrm{Sym}(3) \times \Omega_{10}^{+}(2))$	$2^{27} : E_6(2)$
$[2^{42}] : L_7(2)$	$\mathrm{Sym}(3) \times \Omega_{12}^{+}(2)$
$E_6(2) : 2$	$3^{\cdot 2}E_6(2) : \mathrm{Sym}(3)$
$U_8(2) : 2$	$L_8(2) : 2$
$3^{\cdot}(U_3(2) \times U_6(2)) : \mathrm{Sym}(3)$	$(L_3(2) \times L_6(2)) : 2$
$(L_2(8) \times {}^3D_4(2)) : 3$	$((\mathrm{Sym}(3))^3 \times \Omega_8^{+}(2)) : \mathrm{Sym}(3)$
$L_2(128) : 7$	$U_3(3) : 2 \times Sp_6(2)$
$3^7 : (2 \times Sp_6(2))$	

Furthermore, maximal subgroups of each shape above exist in $E_7(2)$, and these maximal subgroups are determined uniquely up to conjugacy.

We shall use $\mathrm{Sym}(n)$, respectively $\mathrm{Alt}(n)$, to denote the symmetric group, respectively alternating group, of degree n. Also $\Omega_n^{\pm}(q)$ will stand for the simple projective orthogonal group of either plus type or minus type, $Sp_n(q)$ for the symplectic groups and $\mathrm{Dih}(m)$ the dihedral group of order m. Apart from these exceptions, our notation for group structures follows the Atlas [**15**]. Moreover the Atlas will be a significant source for information on many of the groups we encounter. For general group theoretic notation see either [**2**] or [**22**], and we use the symbol $\sim$ to indicate that two groups have the same shape.

We now give an indication of the strategy and methods used to prove Theorem 1.1, as well as the layout of the paper. We shall see that, as a consequence of Theorem 2.1, to prove Theorem 1.1 there is a finite list (see List 1) of groups for which we must determine whether or not they are maximal in $E_7(2)$. As is turns out only one group, up to conjugacy in $E_7(2)$, arising from this list is in fact maximal in $E_7(2)$. Section 3 is devoted to assembling an extensive catalogue of information concerning the involutions and semisimple elements of $E_7(2)$. This data is used in a variety of ways, such as for example in Lemma 4.2, to demonstrate that $L_2(64)$, $Sp_4(8)$, $L_4(8)$ and $Sp_6(8)$ cannot be subgroups of $E_7(2)$. While Lemma 3.6, used in Lemma 4.21 to eliminate H being a maximal subgroup with $\mathrm{Soc}(H) \cong L_4(4)$, gives us an $\Omega_8^{-}(2)$ subgroup of $E_7(2)$ within which we can work. The 56-dimensional irreducible $\mathbb{F}_2$-module for $E_7(2)$ is used at almost every turn in our analysis, either for computational attacks or in considering the possible restrictions for subgroups of $E_7(2)$ upon this module. Use of Brauer characters will be seen to be a potent weapon for either eliminating certain possible subgroups of $E_7(2)$ (for example see Lemmas 4.3 and 5.2) or in pinpointing those cases needing further investigation. Thus Appendix A contains a large number of (partial) Brauer character tables (with each irreducible Brauer character labelled by χ_i for some i) while Sections 4 and 5 are where we consider, respectively, those possible maximal subgroups H where $\mathrm{Soc}(H) \in \mathrm{Lie}(2)$, the Lie-type groups of characteristic 2, and $\mathrm{Soc}(H) \notin \mathrm{Lie}(2)$.

For those cases when H is a possible maximal subgroup of $E_7(2)$ with $\mathrm{Soc}(H)$ being isomorphic to a small simple group such as $L_3(2)$ or $L_2(8)$, their elimination is a protracted campaign involving a mainly computational approach. For the former case this is played out in Lemmas 4.13 to 4.19. The latter case is dealt with in Lemma 4.20 and also makes heavy use of facts about normalisers of cyclic subgroups of order 7 given in Section 4.2. As a further comment, we remark that the method in [**8**] for finding (computationally) generators for centralisers of strongly real elements is used extensively.

Where extensive computation has been employed, we have provided downloadable data files which document these computations. Names of specific files associated to a particular result are given at the beginning of the relevant proof.

Finally, we would like to thank the referee, whose comments and suggestions have greatly improved this paper.

2. *Preliminary Results*

The first theorem of this section, already mentioned in the previous section, summarizes the work of a variety of authors in [**26**], [**27**], [**28**], [**29**], [**30**], [**32**], [**33**], [**34**] and [**39**]. For Theorem 2.1, G denotes an adjoint simple algebraic exceptional group of Lie-type over $\overline{\mathbb{F}_q}$ and σ is a standard Frobenius homomorphism of G.

THEOREM 2.1. *Let H be a maximal subgroup of the finite exceptional group G_σ over $\mathbb{F}_q$, $q = p^a$ where p is a prime. Then one of the following holds:*

(i) *$H = M_\sigma$ where M is a maximal closed σ-stable subgroup of positive dimension in G; the possibilities are as follows;*
  (a) *Both M and H are parabolic subgroups;*
  (b) *M is a reductive group of maximal rank. The possibilities for M are determined in [**28**].*
  (c) *$G = E_7$, $p > 2$ and $H = (2^2 \times \Omega_8^+(q).2^2).\mathrm{Sym}(3)$ or ${}^3D_4(q).3$;*
  (d) *$G = E_8$, $p > 5$ and $H = PGL_2(q) \times \mathrm{Sym}(5)$;*
  (e) *M is as in Table 1 of [**31**], and $H = M_\sigma$ as in Table 3 of [**31**].*

(ii) *H is of the same type as G;*
(iii) *H is an exotic local subgroup (see [**34**]);*
(iv) *G is of type E_8, $p > 5$ and $H \sim (\mathrm{Alt}(5) \times \mathrm{Alt}(6)).2^2$;*
(v) *$F^*(H) = H_0$ is simple, and not in $\mathrm{Lie}(p)$: the possibilities for H_0 are given up to isomorphism by [**32**];*
(vi) *$F^*(H) = H(q_0)$ is simple and in $\mathrm{Lie}(p)$; moreover $rk(H(q_0)) \le \frac{1}{2}rk(G)$, and one of the following holds:*
  (a) *$q_0 \le 9$;*
  (b) *$H(q_0) \cong A_2(16)$ or ${}^2A_2(16)$;*
  (c) *$q_0 \le (2, p-1)u(G)$ and $H(q_0) \cong A_1(q_0)$, ${}^2B_2(q_0)$ or ${}^2G_2(q_0)$, where the values of $u(G)$ for each type of exceptional group are as follows:*

G	G_2	F_4	E_6	E_7	E_8
$u(G)$	12	68	124	388	1312

In cases $(i)-(iv)$, H is determined up to G_σ-conjugacy.

For the remainder of this paper, G will denote $E_7(2)$ and V its (minimal) 56-dimensional $\mathbb{F}_2G$-module. Thus, from [**15**],

$$|G| = 2^{63}.3^{11}.5^2.7^3.11.13.17.19.31.43.73.127.$$

Let H be a maximal subgroup of G. In view of Theorem 2.1 and calling on the finite simple group classification, to prove Theorem 1.1 we must consider the following possibilities.

LIST 1.

(i) $F^*(H)$ is isomorphic to one of the following groups.

$L_4(2)$	$L_4(4)$	$L_4(8)$
$U_4(2)$	$U_4(4)$	$U_4(8)$
$Sp_6(2)$	$Sp_6(4)$	$Sp_6(8)$
$L_3(2)$	$L_3(4)$	$L_3(8)$
$U_3(4)$	$U_3(8)$	$Sp_4(4)$
$Sp_4(8)$	$G_2(4)$	$G_2(8)$
$Sz(8)$	$L_2(8)$	

These groups arise from Theorem 2.1(vi)(a).

(ii) $F^*(H)$ is isomorphic to $L_3(16)$ or $U_3(16)$ (these possibilities arise from Theorem 2.1(vi)(b)).

(iii) $F^*(H)$ is isomorphic to $L_2(16)$, $L_2(32)$, $L_2(64)$, $L_2(128)$, $L_2(256)$, $Sz(32)$ or $Sz(128)$ (these possibilities arise from Theorem 2.1(vi)(c)).

(iv) $F^*(H)$ is isomorphic to one of the following groups.
  - (a) $\mathrm{Alt}(n)$ for $n = 5, 6, 7, 8, 9, 10, 11, 12$ and 13.
  - (b) $L_2(q)$ for $q = 7, 11, 13, 17, 19, 25, 27, 29$ and 37.
  - (c) $L_3(3)$, $L_4(3)$, $U_3(3)$, $U_4(3)$, $\Omega_7(3)$ and $G_2(3)$.
  - (d) M_{11}, M_{12}, J_2.

  These possibilities arise from Theorem 2.1(v).

When analysing potential subgroups H of G, we shall often show the non-maximality of H by demonstrating that H must fix a non-zero vector in V. The next four results are used in this regard.

PROPOSITION 2.2. *The vector stabilisers of G on $V \setminus \{0\}$ have the following shapes.*

(i) $2^{27} : E_6(2)$
(ii) $E_6(2) : 2$
(iii) $3 \cdot {}^2E_6(2) : 2$
(iv) $2^{1+32} : Sp_{10}(2)$
(v) $2^{26} : F_4(2)$.

Proof. The point-stabilisers of $E_7(q)$ (for arbitrary q) in its action on its minimal 56-dimensional module are determined in [**27**]. In order to obtain the structures given in the statement of the proposition for the special case where $q = 2$ we can use information from [**6**]. □

LEMMA 2.3. *Let p be a prime and let P and Q be non-trivial finite p-groups. Suppose $\alpha : P \to \mathrm{Aut}(Q)$ is a group homomorphism. Then $C_Q(P) \neq 1$.*

Proof. See 5.15 in [**2**], for example. □

LEMMA 2.4. *Suppose that S is a non-abelian simple subgroup of G such that $\mathrm{Out}(S)$ is a 2-group. Suppose further that S fixes a non-zero vector in V. Then one of the following must hold.*

(i) $C_G(S) \neq 1$.
(ii) *$N_G(S)$ is contained in one of the vector stabilisers described in Lemma 2.2.*

Proof. Assume that $C_G(S) = 1$. Then since the automizer $N_G(S)/C_G(S)$ is a subgroup of $\mathrm{Aut}(S)$ we know that $N_G(S)$ embeds in $\mathrm{Aut}(S)$. Further, since S is simple we have $Z(S) = 1$, and so $\mathrm{Inn}(S) \cong S/Z(S) \cong S$. Thus

$$\mathrm{Out}(S) = \mathrm{Aut}(S)/\mathrm{Inn}(S) \cong \mathrm{Aut}(S)/S$$

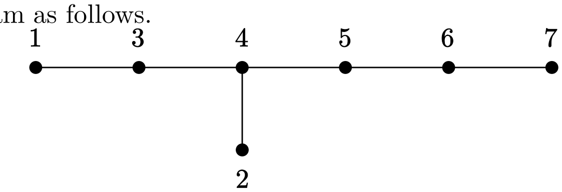
and so $N_G(S)/S$ is a 2-group. Since S fixes a non-zero vector in V and $N_G(S)/S$ acts on this subspace we know that $N_G(S)$ fixes a non-zero vector in V by Lemma 2.3. Hence $N_G(S)$ must lie in a vector stabiliser as claimed. □

The following result allows us to use knowledge of the dimensions of certain cohomology groups to deduce information regarding the action of a potential subgroup $H \leq G$ on V.

PROPOSITION 2.5. *Suppose that S is a finite group, with k a field of characteristic $p > 0$. Let W be a finite dimensional kS-module, with V the projective indecomposable module corresponding to W. Then the number of trivial modules in an S-composition series of* $\mathrm{Soc}^2(V)/\mathrm{Soc}(V)$ *is equal to* $\dim(H^1(S, W))$.

Proof. See [**9**]. □

We conclude this section with a brief description of the construction of $E_7(2)$ which we use for the majority of our computations. Not surprisingly the root system of type E_7 will feature in this construction and at various points in our arguments. So we first set up the labelling of the E_7 Dynkin diagram as follows.



We shall employ the ordering of the positive roots as given in Table 1. This ordering is first by height and then by lexicographic order with respect to the labelling of the fundamental roots. Notationally α_i will denote the i-th root in this ordering system. Also we set $\alpha_0 := \alpha_{63}$, the highest root in the root system. For $64 \leq i \leq 126$ we then set α_i to be the negative of α_{i-63}.

Following the construction of Chevalley groups over arbitrary fields as described in [**13**], we first produce generators for $E_8(2)$ acting on its 248-dimensional adjoint module V_{248} (with respect to a Chevalley basis). These generators correspond to roots from the root system of type E_8. We now use the fact that $E_7(2)$ is a subgroup of $E_8(2)$ (see [**40**], for example). Taking the canonical subgroup $E_7(2)$ which corresponds to the first seven nodes of the E_8 Dynkin diagram, we use the implementation of Parker's Meataxe in MAGMA [**11**] to decompose V_{248} as an $E_7(2)$-module. The minimal 56-dimensional module for $E_7(2)$ appears as an $E_7(2)$-submodule of V_{248}, while the irreducible 132-dimensional module, which we denote V_{132}, appears as a compositon factor of V_{248}. It is now straightforward to map our canonical root element generators to those for the 56 and 132-dimensional representations of $E_7(2)$.

3. *Involutions and Semisimple Elements*

3.1. *Involutions in G*

The following result is found in [**6**], and gives information on the G-conjugacy of involutions.

TABLE 1. *Ordering of the root system of type E_7*

α_1	(1 0 0 0 0 0 0)
α_2	(0 1 0 0 0 0 0)
α_3	(0 0 1 0 0 0 0)
α_4	(0 0 0 1 0 0 0)
α_5	(0 0 0 0 1 0 0)
α_6	(0 0 0 0 0 1 0)
α_7	(0 0 0 0 0 0 1)
α_8	(1 0 1 0 0 0 0)
α_9	(0 1 0 1 0 0 0)
α_{10}	(0 0 1 1 0 0 0)
α_{11}	(0 0 0 1 1 0 0)
α_{12}	(0 0 0 0 1 1 0)
α_{13}	(0 0 0 0 0 1 1)
α_{14}	(1 0 1 1 0 0 0)
α_{15}	(0 1 1 1 0 0 0)
α_{16}	(0 1 0 1 1 0 0)
α_{17}	(0 0 1 1 1 0 0)
α_{18}	(0 0 0 1 1 1 0)
α_{19}	(0 0 0 0 1 1 1)
α_{20}	(1 1 1 1 0 0 0)
α_{21}	(1 0 1 1 1 0 0)
α_{22}	(0 1 1 1 1 0 0)
α_{23}	(0 1 0 1 1 1 0)
α_{24}	(0 0 1 1 1 1 0)
α_{25}	(0 0 0 1 1 1 1)
α_{26}	(1 1 1 1 1 0 0)
α_{27}	(1 0 1 1 1 1 0)
α_{28}	(0 1 1 2 1 0 0)
α_{29}	(0 1 1 1 1 1 0)
α_{30}	(0 1 0 1 1 1 1)
α_{31}	(0 0 1 1 1 1 1)
α_{32}	(1 1 1 2 1 0 0)
α_{33}	(1 1 1 1 1 1 0)
α_{34}	(1 0 1 1 1 1 1)
α_{35}	(0 1 1 2 1 1 0)
α_{36}	(0 1 1 1 1 1 1)
α_{37}	(1 1 2 2 1 0 0)
α_{38}	(1 1 1 2 1 1 0)
α_{39}	(1 1 1 1 1 1 1)
α_{40}	(0 1 1 2 2 1 0)
α_{41}	(0 1 1 2 1 1 1)
α_{42}	(1 1 2 2 1 1 0)
α_{43}	(1 1 1 2 2 1 0)
α_{44}	(1 1 1 2 1 1 1)
α_{45}	(0 1 1 2 2 1 1)
α_{46}	(1 1 2 2 2 1 0)
α_{47}	(1 1 2 2 1 1 1)
α_{48}	(1 1 1 2 2 1 1)
α_{49}	(0 1 1 2 2 2 1)
α_{50}	(1 1 2 3 2 1 0)
α_{51}	(1 1 2 2 2 1 1)
α_{52}	(1 1 1 2 2 2 1)
α_{53}	(1 2 2 3 2 1 0)
α_{54}	(1 1 2 3 2 1 1)
α_{55}	(1 1 2 2 2 2 1)
α_{56}	(1 2 2 3 2 1 1)
α_{57}	(1 1 2 3 2 2 1)
α_{58}	(1 2 2 3 2 2 1)
α_{59}	(1 1 2 3 3 2 1)
α_{60}	(1 2 2 3 3 2 1)
α_{61}	(1 2 2 4 3 2 1)
α_{62}	(1 2 3 4 3 2 1)
$\alpha_{63} = \alpha_0$	(2 2 3 4 3 2 1)

Theorem 3.1. *Each involution in $G = E_7(2)$ is conjugate to one of the following.*
(i) $t_1 = x_{\alpha_0}(1)$
(ii) $t_2 = x_{\alpha_{59}}(1)x_{\alpha_{58}}(1)$
(iii) $t_3 = x_{\alpha_{53}}(1)x_{\alpha_{55}}(1)x_{\alpha_{54}}(1)$
(iv) $t_4 = x_{\alpha_{48}}(1)x_{\alpha_{47}}(1)x_{\alpha_{49}}(1)$
(v) $t_5 = x_{\alpha_{53}}(1)x_{\alpha_{49}}(1)x_{\alpha_{47}}(1)x_{\alpha_{48}}(1)$

Proof. See [**6**], 16.1, using the fact that $\alpha_0 = r_{27}$, $\alpha_{59} = r_{21}$, $\alpha_{58} = r_{23}$, $\alpha_{53} = r_{48}$, $\alpha_{55} = r_{18}$, $\alpha_{54} = r_{19}$, $\alpha_{48} = r_{13}$, $\alpha_{47} = r_{14}$ and $\alpha_{49} = r_{15}$ (r_j as defined in [**6**]). □

Let P_i denote the maximal parabolic subgroup of G obtained by the removal of the i^{th} node from the Dynkin diagram. Also we use Q_i to denote the maximal normal unipotent subgroup of P_i, and L_i its Levi complement.

Theorem 3.2. *The maximal parabolic subgroups containing the centralisers of the involutions of G are as follows;*
(i) $C_G(t_1) \leq P_1$
(ii) $C_G(t_2) \leq P_6$
(iii) $C_G(t_3) \leq P_3$
(iv) $C_G(t_4) \leq P_7$
(v) $C_G(t_5) \leq P_2, P_7$

Proof. See [**6**], 16.20. □

Theorem 3.3. *For t an involution of G, let U denote the maximal normal unipotent subgroup of $C_G(t)$. The possible structures of $C_G(t)$ are as follows.*
(i) $C_G(t_1) = Q_1L_1 = P_1$. *Moreover, Q_1 is an extraspecial 2-group and so $Q_1/Z(Q_1)$ has the structure of an orthogonal space, upon which $L_1 \cong \Omega^+_{12}(2)$ acts irreducibly.*
(ii) $C_G(t_2) = UL$, *where $U = Q_6$ and $L \cong Sp_8(2) \times \mathrm{Sym}(3)$.*
(iii) $C_G(t_3) = UL$, *where $U = C_{Q_3}(t_3) \leq Q_3$ and $L \cong \mathrm{Sym}(3) \times Sp_6(2) \leq L_3$.*
(iv) $C_G(t_4) = UL$, *where $U = Q_7$ and $F_4(2) \cong L \leq L_7$.*
(v) $C_G(t_5) = UL$, *where $U \leq Q_2Q_7$ and $Sp_6(2) \cong L \leq L_2 \cap L_7 \cong L_6(2)$.*

Proof. See [**6**], 16.20. □

In accordance with the Atlas [**15**] convention of largest centralisers having precedence, we label the t_1, t_2, t_3, t_4 and t_5 G-conjugacy classes by, respectively, $2A$, $2B$, $2D$, $2C$ and $2E$.

Lemma 3.4. *Let t be an involution in the classes $2A$, $2B$, $2C$, $2D$ or $2E$ of G. Then the rank of $1 + t$ on the module V is 12, 20, 24, 28 or 28, respectively.*

Proof. Firstly let us take the subgroup

$$L = \langle x_{\pm\alpha_0}(1), x_{\pm\alpha_1}(1), x_{\pm\alpha_3}(1), x_{\pm\alpha_4}(1), x_{\pm\alpha_5}(1), x_{\pm\alpha_6}(1), x_{\pm\alpha_7}(1)\rangle \cong L_8(2),$$

as the structure of involutions in the linear groups is well-known. Now using the Meataxe we know that as an L-module we have $V \cong \Lambda^2(V_8) \oplus \Lambda^2(V_8)^*$, where V_8 denotes the natural 8-dimensional module for $L_8(2)$ in characteristic 2 and $\Lambda^2(V_8)$ denotes its exterior square. Let

t be an involution of $L_8(2)$ with one Jordan block of size 2 and six Jordan blocks of size 1 on V_8. Now t is centralised by an $L_6(2)$ in L and so must be a $2A$-element since no other involution is centralised by an element of order 31. Now such an involution has six Jordan blocks when represented on both $\Lambda^2(V_8)$ and $\Lambda^2(V_8)^*$ and hence the rank of $1+t$ is 12. Let s be an involution of $L_8(2)$ with two Jordan blocks of size 2 and four Jordan blocks of size 1 on V_8. The element s is centralised by an $\mathrm{Alt}(8) \times \mathrm{Sym}(3)$ in L and so using [**15**] we see that it must be a $2B$-element. This involution has ten Jordan blocks when represented on both $\Lambda^2(V_8)$ and $\Lambda^2(V_8)^*$, and hence the rank of $1+s$ is 20.

Now we take a non-diagonal involution r in $N_G(7B) \sim (7:2 \times {}^3D_4(2)):3$ (see Section 4.2) which inverts the $7B$ element (that is an involution in the dihedral group $7:2$). Since r centralises an element of order 13 it must be a $2C$-element, just by consideration of the centraliser orders. We find that $\dim(C_V(r)) = 28$ and the rank of $1+r$ is 28. Now from [**25**] we know there is an involution v in $E_6(2)$ whose centraliser contains $\mathrm{Sym}(3) \times L_3(2)$ and for which $1+v$ has rank 12 on the minimal $E_6(2)$-module V_{27}. Now as an $E_6(2)$-module we have $V = 1 \oplus 1 \oplus V_{27} \oplus V_{27}^*$, and so on V we see that the rank of $1+v$ is 24. This differs from the ranks so far accounted for and so the corresponding involution v in G is either in $2D$ or $2E$. However, $Sp_6(2)$ contains no subgroups $\mathrm{Sym}(3) \times L_3(2)$ and so v must be in the class $2D$. Finally we need to find the rank of a $2E$-involution. For this we turn to the 56-dimensional matrix representation of G over $\mathbb{F}_2$. We take a $2E$-involution u as described in [**6**] and find $\dim(C_V(t_4)) = 28$. Thus $1+u$ has rank 28 and we are done. □

3.2. *Semisimple Classes*

In preparation for the proof of Theorem 1.1, we require information on the structure of centralisers of semisimple elements of $E_7(2)$. Frank Lübeck has produced a parametrization of the conjugacy classes of $E_7(2)$, which is stored at [**35**]. It is well-known that there are 128 conjugacy classes of semisimple elements in $E_7(2)$, and [**35**] gives the orders of the centralisers of such elements. As we require more detail regarding the structure of these centralisers, we produce this here. The results are recorded in Table 2. We also record, in Tables 3 and 4 respectively, the Brauer character values and fixed-space dimensions of certain semisimple elements of G for both the 56- and 132-dimensional modules for G, as these values will be referenced in later proofs.

Table 2: Semisimple classes of $E_7(2)$

x	$C_G(x)$	$\lvert C_G(x)\rvert$
$3A$	$3^{\cdot 2}E_6(2).3$	$2^{36}.3^{11}.5^2.7^2.11.13.17.19$
$3B$	$3 \times \Omega_{12}^+(2)$	$2^{30}.3^9.5^2.7^2.11.17.31$
$3C$	$3 \times U_7(2)$	$2^{21}.3^9.5.7.11.43$
$3D$	$3 \times \Omega_{10}^-(2) \times \mathrm{Sym}(3)$	$2^{21}.3^8.5^2.7.11.17$
$3E$	$3^{\cdot}(U_3(2) \times U_6(2)).3$	$2^{18}.3^{10}.5.7.11$
$5A$	$5 \times \Omega_8^-(2) \times \mathrm{Sym}(3)$	$2^{13}.3^5.5^2.7.17$
$7A$	$7 \times L_6(2)$	$2^{15}.3^4.5.7^3.31$
$7B$	$7 \times {}^3D_4(2)$	$2^{12}.3^4.7^3.13$
$7C$	$7 \times L_3(2) \times L_2(8)$	$2^6.3^3.7^3$
$9A$	$9 \times {}^3D_4(2)$	$2^{12}.3^6.7^2.13$
$9B$	$9 \times U_5(2)$	$2^{10}.3^7.5.11$
$9C$	$9 \times U_3(8)$	$2^9.3^6.7.19$
$9D$	$9 \times U_4(2) \times \mathrm{Sym}(3)$	$2^7.3^7.5$
$9E$	$9 \times L_2(8) \times \mathrm{Sym}(3)$	$2^4.3^5.7$
$9F$	$[3^3].2.3.[2^2].[3^3].2$	$2^4.3^7$

Table 2: Semisimple classes of $E_7(2)$

x	$C_G(x)$	$\|C_G(x)\|$
$11A$	$11 \times 3^{1+2} : 2.\mathrm{Alt}(4)$	$2^3.3^4.11$
$13A$	$13 \times L_2(8)$	$2^3.3^2.7.13$
$15A$	$15 \times \Omega_8^-(2)$	$2^{12}.3^5.5^2.7.17$
$15B$	$15 \times \mathrm{Alt}(8) \times \mathrm{Sym}(3)$	$2^7.3^4.5^2.7$
$15C$	$15 \times \mathrm{Alt}(8) \times 3$	$2^6.3^4.5^2.7$
$15D$	$15 \times \mathrm{Alt}(5) \times (\mathrm{Sym}(3))^2$	$2^4.3^4.5^2$
$15E$	$5 \times 3_+^{1+2} : 2.\mathrm{Alt}(4) \times \mathrm{Sym}(3)$	$2^4.3^5.5$
$15F$	$15 \times \mathrm{Alt}(5) \times \mathrm{Sym}(3) \times 3$	$2^3.3^4.5^2$
$15G$	$5 \times 3_+^{1+2} : 2.\mathrm{Alt}(4) \times 3$	$2^3.3^5.5$
$17AB$	$17 \times \mathrm{Alt}(5) \times \mathrm{Sym}(3)$	$2^3.3^2.5.17$
$19A$	19×9	$3^2.19$
$21A$	$21.L_3(4).3$	$2^6.3^4.5.7^2$
$21B$	$21 \times \mathrm{Alt}(8)$	$2^6.3^3.5.7^2$
$21CD$	$21 \times L_2(8)$	$2^3.3^3.7^2$
$21E$	$63 \times L_3(2)$	$2^3.3^3.7^2$
$21F$	$7 \times 3_+^{1+2}.2.\mathrm{Alt}(4)$	$2^3.3^4.7$
$21G$	$21 \times \mathrm{Alt}(5) \times \mathrm{Sym}(3)$	$2^3.3^3.5.7$
$21H$	$7 \times 3 \times 9$	$3^3.7$
$31ABC$	$31 \times L_3(2)$	$2^3.3.7.31$
$33AB$	$11 \times 3^{1+2} : 2.\mathrm{Alt}(4)$	$2^3.3^4.11$
$33CDEFGH$	$11 \times 3^{1+2} : 2$	$2.3^3.11$
$33I$	$11 \times 3^{1+2}$	$3^3.11$
$35A$	$35 \times 3 \times \mathrm{Sym}(3)$	$2.3^2.5.7$
$39A$	117	$3^2.13$
$43ABC$	43×3	3.43
$45AB$	$5 \times 9 \times \mathrm{Sym}(3)$	$2.3^3.5$
$51AB$	$51 \times \mathrm{Alt}(5)$	$2^2.3^2.5.17$
$51CD$	$51 \times \mathrm{Sym}(3)$	$2.3^2.17$
$51EF$	51×3	$3^2.17$
$57AB$	19×9	$3^2.19$
$63ABC$	$63 \times L_3(2)$	$2^3.3^3.7^2$
$63D$	63×7	$3^2.7^2$
$63EFG$	$63 \times \mathrm{Sym}(3)$	$2.3^3.7$
$63HIJ$	63×3	$3^3.7$
$73ABCD$	73	73
$85ABCD$	$85 \times \mathrm{Sym}(3)$	$2.3.5.17$
$91ABC$	91	91
$93ABC$	93	93
$99AB$	99	$3^2.11$
$105AB$	$35 \times 3 \times \mathrm{Sym}(3)$	$2.3^2.5.7$
$105CDE$	$35 \times 3 \times 3$	$3^2.5.7$
$117ABC$	117	$3^2.13$
$127ABCDEFGHI$	127	127
$129ABCDEF$	129	3.43
$171ABCDEF$	171	$3^2.19$
$217ABCDEF$	217	7.31
$255ABCD$	255	$3.5.17$

LEMMA 3.5. *There are five classes of elements of order 3 in G, with centralisers as in Table 2.*

Proof. See [**30**], for example. □

LEMMA 3.6. *There is one G-conjugacy class of elements of order 5 in G. Any element in this class has centraliser $5 \times \Omega_8^-(2) \times \mathrm{Sym}_3$.*

Proof. The subgroup $E_6(2)$ contains a full Sylow 5-subgroup of G (see [**15**]). This is an elementary abelian group of order 5^2. Further, since $E_6(2)$ contains a unique class of elements of order 5 we must have one class in G. The structure of the centraliser follows from [**19**] and [**25**]. □

It is clear that to find the G-conjugacy classes of elements g of order pq in G (where $(p, q) = 1$) we need to determine the $C_G(g^p)$-classes of elements of order p.

TABLE 3. *Brauer character values of certain semisimple classes of G*

X	$\chi_{56}(g)$	$\chi_{132}(g)$
$3A$	-25	51
$3B$	20	33
$3C$	-7	6
$3D$	2	6
$3E$	2	-3
$5A$	6	7
$7A$	14	20
$7B$	-7	13
$7C$	0	-1
$9A$	-1	6
$9B$	-10	18
$9C$	-1	-3
$9D$	8	9
$9E$	2	0
$9F$	-1	0
$11A$	1	0

TABLE 4. *Dimensions of $C_V(g)$ for certain semisimple elements*

X	$C_{V_{56}}(g)$	$C_{V_{132}}(g)$
$3A$	2	78
$3B$	32	66
$3C$	14	48
$3D$	20	48
$3E$	20	42
$5A$	16	32
$7A$	20	36
$7B$	2	30
$7C$	8	18
$9A$	0	30
$9B$	0	26
$9C$	0	24
$9D$	12	20
$9E$	8	14
$9F$	6	14
$11A$	6	12

Lemma 3.7. *There are seven classes of elements of order* 15 *in* G*, with centralisers as in Table 2. Furthermore the fifth-powers of elements from these classes lie in the following classes of* G.

x	$15A$	$15B$	$15C$	$15D$	$15E$	$15F$	$15G$
x^5	$3B$	$3D$	$3A$	$3B$	$3E$	$3D$	$3C$

Proof. Any element of order 15 in G must cube to an element in $5A$, since this is the unique class of elements of order 5 in G. By Lemma 3.6 we have $C_G(5A) \cong 5 \times \Omega_8^-(2) \times \mathrm{Sym}(3)$. Now from [**15**] we know that $\Omega_8^-(2)$ has three classes of elements of order 3, and we can deduce the structure of their centralisers. There is a class $3A$ with centraliser $3 \times \mathrm{Alt}(8)$, a class $3B$ with centraliser $3 \times \mathrm{Alt}(5) \times \mathrm{Sym}(3)$ and a class $3C$ with soluble centraliser $3_+^{1+2} : 2.\mathrm{Alt}(4)$. Also, there is clearly a unique class of elements of order 3 in Sym(3) and such elements are self-centralising. Hence we find there are seven classes of elements of order 15 in G, with centralisers as in the statement of the lemma. The G-conjugacy classes in which the fifth powers of elements x lie were determined computationally. □

Lemma 3.8. *There are two classes of elements of order* 45 *in* G*, with centralisers as in Table 2. In both cases the fifteenth power of such an element lies in the class* $3E$ *of* G.

Proof. As in Lemma 3.7, any element of order 45 in G must power down to a $5A$-element. Again by Lemma 3.6, $C_G(5A) \cong 5 \times \Omega_8^-(2) \times \mathrm{Sym}(3)$ and from [**15**] we see that $\Omega_8^-(2)$ has a unique class of elements of order 9, and these are self-centralising. Hence there are two classes of elements of order 45 in G, namely $45A$ with centraliser of order $2.3^3.5$ and $45B$ with centraliser of order $3^3.5$. Furthermore, $9A$-elements in $\Omega_8^-(2)$ cube to $3C$-elements in $\Omega_8^-(2)$ with centraliser of order $2^3.3^4$. Hence, from Lemma 3.7 we deduce that the elements in $45A$ and $45B$ both have fifteenth power in $3E$ of G. □

Lemma 3.9. *There is a unique class of elements of order* 35 *in* G*, with centraliser as in Table 2. These elements power to* $7A$*-elements in* G.

Proof. From [**15**] we see that $\Omega_8^-(2)$ has a unique class of elements of order 7 with cyclic centraliser of order 21. This yields a unique class $35A$ in G with centraliser $35 \times 3 \times \mathrm{Sym}(3)$. Hence, any element of order 35 in G lies in this class, and upon finding such an element $g \in G$ and calculating $\dim C_V(g)$, we find that the fifth power of g lies in $7A$. □

Lemma 3.10. *There are five classes of elements of order* 105 *in* G*, with centralisers as in Table 2.*

Proof. By Lemma 3.9 any element of order 105 in G cubes to an element in $35A$, and $C_G(35A) \cong 35 \times 3 \times \mathrm{Sym}(3)$. Hence we obtain five classes of elements of order 105 - two classes with centraliser $35 \times 3 \times \mathrm{Sym}(3)$ corresponding to elements $(35A, 3A, 1A)$ in $C_G(35A)$, two classes with centraliser $35 \times 3 \times 3$ corresponding to elements $(35A, 3A, 3A)$ in $C_G(35A)$, and one class with centraliser $35 \times 3 \times 3$ corresponding to elements $(35A, 1A, 3A)$ in $C_G(35A)$. □

LEMMA 3.11. *There are four classes of elements of order* 85 *in* G*, with centralisers as in Table 2.*

Proof. From [**15**] we see that $\Omega_8^-(2)$ has four classes of elements of order 17, all of which are self-centralising. Using notation from [**15**], these form a master class and subsequent slave classes, $17A$, $17B \star 2$, $17C \star 3$ and $17D \star 6$, which immediately yield G-conjugacy classes $85A$, $85B \star 2$, $85C \star 3$ and $85D \star 6$, all of which have centraliser of the form $85 \times \mathrm{Sym}(3)$. □

LEMMA 3.12. *There are four classes of elements of order* 255 *in* G*, with centralisers as in Table 2.*

Proof. Any element of order 255 must cube to an 85-element. Using Lemma 3.11 we see that each class of elements of order 85 in G yield a self-centralising class of elements of order 255, and so we have $255A$, $255B \star 2$, $255C \star 3$ and $255D \star 6$. □

LEMMA 3.13. *There are three classes of elements of order* 7 *in* G*, with centralisers as in Table 2.*

Proof. Since $|G|_7 = 7^3$ and $|E_6(2)|_7 = 7^3$, the subgroup $E_6(2)$ contains a full Sylow 7-subgroup of G. From [**25**] we know that $S \in \mathrm{Syl}_7(E_6(2))$ is elementary abelian of order 7^3. There are four classes of elements of order 7 in $E_6(2)$, namely $7AB$ with centraliser $7 \times {}^3D_4(2)$, $7C$ with centraliser $7 \times L_3(2) \times L_3(2)$ and $7D$ with centraliser $7^2 \times L_3(2)$. On the minimal module V_{27} for $E_6(2)$ we have $\dim C_{V_{27}}(7AB) = 0$, $\dim C_{V_{27}}(7C) = 9$ and $\dim C_{V_{27}}(7D) = 3$. Now as an $E_6(2)$-module we know that $V = 1 \oplus 1 \oplus V_{27} \oplus V_{27}^*$. Hence these elements in G have fixed spaces of dimensions 2, 2, 20 and 8 on V for elements in $7A$, $7B$, $7C$ and $7D$, respectively. Hence the only possible fusion is between $7A$ and $7B$ classes in $E_6(2)$. Indeed these classes fuse under the graph automorphism of $E_6(2)$, and since $E_6(2):2 \leq G$ we have three classes $7A$, $7B$ and $7C$ in G. The structure of their centralisers follows from [**19**]. □

LEMMA 3.14. *There are eight classes of elements of order* 21 *in* G*, with centralisers as in Table 2.*

Proof. Consider first those 21-elements whose cube is a $7A$-element. Now

$$C_G(7A) \cong 7 \times L_6(2)$$

and $L_6(2)$ has three classes of elements of order 3. There is a class $3A$ with centraliser $3.L_3(4).3$, a class $3B$ with centraliser $3 \times \mathrm{Alt}(8)$, and a class $3C$ with centraliser $3 \times \mathrm{Alt}(5) \times \mathrm{Sym}(3)$. Each of these yields a corresponding class of elements of order 21 in G. Now $7B$-elements in G have centraliser $7 \times {}^3D_4(2)$, and from [**15**] we see that ${}^3D_4(2)$ has two classes of elements of order 3. These are a class $3A$ with centraliser $3 \times L_2(8)$ and a class $3B$ with centraliser $3_+^{1+2} \cdot 2.\mathrm{Alt}(4)$. Finally, there are 21-elements in G whose cube is in the class $7C$. Recall that $C_G(7C) \cong 7 \times L_3(2) \times L_2(8)$. Now $L_3(2)$ contains a unique class of elements of order 3, and these are self-centralising. Futher $L_2(8)$ also contains a unique class of elements of order 3, whose centraliser is a cyclic subgroup of order 9. Hence there are three G-conjugacy classes of elements of order 21 whose cube is in $3C$, a class with centraliser $21 \times L_2(8)$, a class with centraliser $63 \times L_3(2)$ and a class with centraliser $7 \times 3 \times 9$. □

Lemma 3.15. *There are ten classes of elements of order* 63 *in* G*, with centralisers as in Table 2.*

Proof. Any element of order 63 has ninth power in either $7A$, $7B$ or $7C$ in G. We begin by looking at the $7A$ case. Now $C_G(7A) \cong 7 \times L_6(2)$, and there is a unique class of elements of order 9 in $L_6(2)$. These have centraliser of order $3^2.7$ in $L_6(2)$, and so we obtain a class of elements of order 63 with centraliser of shape 63×7. In the $7B$ case we have $C_G(7B) \cong 7 \times {}^3D_4(2)$. Now the group ${}^3D_4(2)$ has three classes $9A$, $9B \star 2$ and $9C \star 4$, all of which have centraliser of $63 \times \mathrm{Sym}(3)$ in ${}^3D_4(2)$. Thus we obtain three classes of elements of order 63, each with centraliser of size $2.3^3.7$. Finally, we look at the case where the element of order 63 power to a $7C$-element in G. We know that $C_G(7C) \cong 7 \times L_3(2) \times L_2(8)$. Now $L_3(2)$ contains no elements of order 9 but does have a unique class of elements of order 3, which are self-centralising. The group $L_2(8)$ has three self-centralising classes of elements of order 9, namely $9A$, $9B \star 2$ and $9C \star 4$. Hence we obtain three classes of elements of order 63 with centraliser $63 \times L_3(2)$ and three class of elements of order 63 with centraliser 63×3. □

Lemma 3.16. *There are six classes of elements of order* 217 *in* G*. Each class is self-centralising.*

Proof. Examining the orders of the centralisers of elements of order 7 in G we see that only $7A$-elements are centralised by elements of order 31. Furthermore, $L_6(2)$ has six classes of elements of order 31 (a master class and its slaves), all of which are self-centralising. Thus we see immediately that there are six classes of elements of order 217 in G, and these are all self-centralising. □

Lemma 3.17. *There is a unique class of elements of order* 13 *in* G*, and such elements have centraliser* $13 \times L_2(8)$.

Proof. Note that $E_6(2)$ contains a full Sylow 13-subgroup of G. From [**25**] we know there is a single conjugacy class of elements of order 13 in $E_6(2)$, and hence a single class $13A$ of elements of order 13 in G. Moreover, we can deduce from [**19**] that these have centraliser $13 \times L_2(8)$, as claimed. □

Lemma 3.18. *There is a unique class of elements of order* 39 *in* G*, and such elements have centraliser* 117*.*

Proof. Any element of order 39 in G must cube to a $13A$-element. Using Lemma 3.17 and the fact that $L_2(8)$ contains a unique conjugacy class of elements of order 3 yields a single class $39A$ in G, with cyclic centraliser of order 117. □

Lemma 3.19. *There are three classes of elements of order* 91 *in* G*. These are all self-centralising.*

Proof. The seventh power of any element of order 91 in G must lie in the class $13A$. Now Lemma 3.17 and knowledge that $L_2(8)$ contains three self-centralising classes of elements of order 9 yields the result. □

Lemma 3.20. *There are three classes of elements of order* 117 *in* G.

Proof. Any element of order 117 in G powers down to a 13A-element. Again, using Lemma 3.17 and the fact that $L_2(8)$ contains three classes of elements of order 9 allows us to deduce that G contains three classes of elements of order 117 in G, all of which are self-centralising. □

Lemma 3.21. *There are two classes of elements of order* 17 *in* G. *These both have centraliser* $17 \times \mathrm{Alt}(5) \times \mathrm{Sym}(3)$.

Proof. Using [**15**] we see that the subgroup $E_6(2)$ contains a full Sylow 17-subgroup of G, and contains two classes of elements of order 17 - in fact there is a unique class of cyclic subgroups of order 17 in $E_6(2)$. Hence there are at most two classes of elements of order 17 in G and the only possibility is that these two classes fuse in G. However, computation of Brauer character values on V yields 17-elements g_1 and g_2 with $\chi(g_1) \neq \chi(g_2)$, where χ is the character afforded by the G-module V. Indeed, we find elements g_1 and g_2 of order 17 in G with Brauer character values

$$\begin{aligned}\chi(g_1) = &-4\xi - 4\xi^2 - 6\xi^3 - 4\xi^4 - 6\xi^5 - 6\xi^6 - 6\xi^7 - 4\xi^8 - 4\xi^9 \\ &-6\xi^{10} - 6\xi^{11} - 6\xi^{12} - 4\xi^{13} - 6\xi^{14} - 4\xi^{15} - 4\xi^{16}\end{aligned}$$

and

$$\begin{aligned}\chi(g_2) = &-6\xi - 6\xi^2 - 4\xi^3 - 6\xi^4 - 4\xi^5 - 4\xi^6 - 4\xi^7 - 6\xi^8 - 6\xi^9 \\ &-4\xi^{10} - 4\xi^{11} - 4\xi^{12} - 6\xi^{13} - 4\xi^{14} - 6\xi^{15} - 6\xi^{16},\end{aligned}$$

where ξ is a primitive seventeenth root of unity. Hence the potential fusion does not occur. Finally, we see from [**19**] that any element of order 17 in G has centraliser $17 \times \mathrm{Alt}(5) \times \mathrm{Sym}(3)$. □

Lemma 3.22. *There are six classes of elements of order* 51 *in* G.

Proof. We know that Alt(5) contains a single conjugacy class of elements of order 3, as does Sym(3). Furthermore, in both these groups the 3-elements are self-centralising. Thus there are three classes of elements of order 51 which cube to 17A-elements. These have centralisers $51 \times \mathrm{Alt}(5)$, $51 \times \mathrm{Sym}(3)$ and 51×3. Likewise, we get three classes for elements of order 51 which cube to 17B-elements, accounting for the six classes claimed in the statement of the lemma. □

Lemma 3.23. *There is a unique class of elements of order* 11 *in* G, *and such elements have centraliser* $11 \times 3^{1+2} : 2.\mathrm{Alt}(4)$.

Proof. Using the Dynkin diagram of type E_7, we see that G contains a subgroup $Sp_{10}(2)$. Such a subgroup contains a full Sylow 11-subgroup of G, and contains a unique conjugacy class of elements of order 11. Thus there is a unique class 11A in G. From Lemma 3.5 we have $C_G(3A) \sim 3 {}^{\cdot 2}E_6(2).3$. Further, this lemma implies that 3A-elements are conjugate in G to their inverses, and so we have $N_G(3A) \sim 3 {}^{\cdot 2}E_6(2).\mathrm{Sym}(3)$. From the character table of ${}^2E_6(2).\mathrm{Sym}(3)$ (which is stored in [**21**] for example) we deduce that $N_G(3A)$ contains a unique class of elements of order 11. Now using Lemma 3.5 again we see that $N_G(\langle z \rangle) \sim$

$3^{\cdot}(U_3(2) \times U_6(2)).3$ for any element $z \in 3A$. Moreover, using the character table library in [**21**] we see that for any 11-element x in $N_G(\langle z \rangle)$ we have $C_{N_G(\langle z \rangle)}(x) \sim 11 \times 3^{1+2} : 2.\mathrm{Alt}(4)$. Hence $C_{N_G(\langle z \rangle)}(x)$ has order $2^3.3^4.11$, and we know from [**19**] that this is the order of the full centraliser in G. □

Lemma 3.24. *There are nine classes of elements of order* 33 *and two classes of elements of order* 99 *in* G.

Proof. From Lemma 3.23 we know that any element of order 33 in G must cube to an $11A$-element. Recall that $C_G(11A) \sim 11 \times 3^{1+2} : 2.\mathrm{Alt}(4)$. Now let g be an $11A$-element in G and set $K = C_G(g)$. Let C denote the direct factor $3^{1+2} : 2.\mathrm{Alt}(4)$ of K, with $\overline{C}$ the factor group $C/Z(C)$. Now $\overline{C} \sim 3^2 : 2.\mathrm{Alt}(4)$, and $\overline{C}$ has five classes of elements of order 3. One of these lies in the core of $\overline{C}$ and has centraliser of order 3^3. This pulls back to C to yield a single class of elements of order 33 in G with centraliser of order $3^3.11$. There are two classes which have centraliser of size 2.3^2 in $\overline{C}$. Each one of these classes pulls back to give three classes of elements of order 33 in G with centraliser of size $2.3^3.11$ (that is, there are six such classes in total). Finally, we need to consider pulling back the identity element of $\overline{C}$. This yields two classes of elements of order 33 in G, both with centraliser $11 \times 3^{1+2} : 2.\mathrm{Alt}(4)$. This accounts for nine classes of elements of order 33 in G. We now look at the elements of order 99 in G. Now there are two classes of elements of order 3 in $\overline{C}$ with centraliser of size 3^2. These both pull back to give two classes of elements of order 9 and hence two G-classes of elements of order 99 in G with centraliser of size $3^2.11$. □

Lemma 3.25. *There are three classes of elements of order* 43 *in* G. *These have centraliser* 43×3.

Proof. We know from Lemma 3.5 that for $3C$-elements we have

$$N_G(3C) \sim (3 \times U_7(2)) : 2.$$

Now $U_7(2)$ contains a full Sylow 43-subgroup of G, and calculations in $U_7(2) : 2$ reveal that for any element g of order 43 the normaliser $N_{U_7(2):2}(\langle g \rangle)$ is a Frobenius group $43 : 14$. Hence we know that there are at most three classes of elements of order 43 in G. That these classes do not fuse in G is proved by the fact that we can compute three different Brauer character values for elements of order 43 in G on V, in a similar manner to the proof of Lemma 3.21. Finally, elements in each of these classes are clearly centralised by a $3C$-element and that this is the full centraliser is apparent from [**19**]. □

Lemma 3.26. *There are six classes of elements of order* 129 *in* G *and all are self-centralising.*

Proof. This follows immediately from Lemma 3.25. □

Lemma 3.27. *There are nine classes of elements of order* 127 *in* G *and all are self-centralising.*

Proof. We know from [**28**] that G contains (maximal) subgroups of the form $L_2(128) : 7$. Now $L_2(128) : 7$ contains elements of order 127, and if g is such an elements then $\langle g \rangle$ must be a Sylow 127-subgroup of G. Thus any element of order 127 in G lies in a subgroup $L_2(128) : 7$.

Now any 127-element $g \in L_2(128):7$ has normaliser $127:14$ and hence contains 9 classes of elements of order 127. These do not fuse in G by Brauer character value considerations and account for all of the non-trivial semisimple classes of G whose centralisers are divisible by 127. That they are self-centralising again follows from [**19**]. □

LEMMA 3.28. *There is a unique class of elements of order* 19 *in* G*, and these have centraliser* 19×9.

Proof. From [**15**] we see that 19 divides $|{}^2E_6(2)|$, and hence

$$N_G(3A) \sim 3^{\cdot 2}E_6(2).\mathrm{Sym}(3)$$

contains a full Sylow 19-subgroup of G. The character table of the group ${}^2E_6(2).\mathrm{Sym}(3)$ is in the GAP[**21**] character table library and we find that $N_G(3A)$ contains a unique class of elements of order 19. Hence we have a single class $19A$ of elements of order 19 in G. Now an element of order 19 in $N_G(3A)$ has centraliser 9×19 and using [**19**] we see that this is the order of the full centraliser in G. □

LEMMA 3.29. *There are two classes of elements of order* 57 *in* G*, and these both have centraliser of order* $3^2.19$.

Proof. Any element of order 57 cubes to a $19A$-element and $C_G(19A) \cong 19 \times 9$ from Lemma 3.28. Now a cyclic subgroup of order 9 contains two elements of order 3 which are non-conjugate since the group is abelian. Hence we obtain two classes of order 57 in G with centraliser as claimed. □

LEMMA 3.30. *There are six classes of elements of order* 171 *in* G *and these are all self-centralising.*

Proof. We argue as in the proof of Lemma 3.29 but taking the six elements of order 9 in the cyclic subgroup. □

LEMMA 3.31. *There are three classes of elements of order* 31 *in* G*. These all have centraliser* $31 \times L_3(2)$.

Proof. Note that $Sp_{10}(2)$ contains a full Sylow 31-subgroup of G. From the character table of $Sp_{10}(2)$ we know that it contains three classes $31ABC$ (master and slave classes). The possibility that these three classes fuse in $E_7(2)$ is negated by Brauer character computations, and the centraliser is found by referring to [**19**]. □

LEMMA 3.32. *There are three classes of elements of order* 93 *in* G*. These are all self-centralising.*

Proof. Any element of order 93 cubes into one of the three classes of elements of order 31 in G. Suppose this class is $31A$. Then $C_G(31A) \cong 31 \times L_3(2)$, and since $L_3(2)$ has a unique class of elements of order 3 which are self-centralising, this yields a G-conjugacy class $93A$ which is self-centralising. Similarly, we get a class each for the cases $31B$ and $31C$ and so a total of three classes as claimed. □

LEMMA 3.33. *There are four classes of elements of order* 73 *in* G*, all of which are self-centralising.*

Proof. Since 73 divides $|E_6(2)|$ we have that $E_6(2)$ contains a full Sylow 73-subgroup of G. From the character table of $E_6(2)$ we know it contains eight classes of elements of order 73 which are all self-centralising. Since for any $E_6(2)$-subgroup of G the automorphism group $E_6(2):2$ is also contained in G, these fuse to four classes in $E_7(2)$. Any further fusion is ruled out by the computation of Brauer characters. That these elements are self-centralising is seen by referring to [**19**]. □

LEMMA 3.34. *There are six classes of elements of order* 9 *in* G*, with centralisers as in Table 2. Moreover, elements from classes* $9A$ *and* $9C$ *cube into the class* $3A$*, while elements from the remaining classes of elements of order* 9 *cube into the class* $3E$*.*

Proof. From [**35**] there are exactly 128 G-conjugacy classes of semisimple elements, and thus far we have accounted for 122 such classes. Therefore there are at most six classes of elements of order 9 in G. We attack this problem computationally, finding six representatives of elements of order 9 which are non-conjugate in G. We distinguish between elements in differing classes by comparing the dimensions of their fixed spaces on V or their Brauer character values. In each case we generate the centraliser by using the method of [**8**]. □

4. *Almost Simple Subgroups in* Lie(2)

4.1. *Non-existence of certain subgroups*

LEMMA 4.1. *Suppose that* H *is isomorphic to* $L_2(256)$, $U_3(16)$, $Sz(32)$, $Sz(128)$, $U_4(4)$ *or* $Sp_6(4)$*. Then* H *is not a subgroup of* G*.*

Proof. In each case we apply Lagrange's Theorem to deduce that H cannot be contained in G. □

LEMMA 4.2. *Suppose that* H *is isomorphic to* $L_2(64)$, $Sp_4(8)$, $L_4(8)$, $U_4(8)$ *or* $Sp_6(8)$*. Then* H *is not a subgroup of* G*.*

Proof. First suppose that $H \cong L_2(64)$. Then H contains an element of order 13, and such an element has cyclic centraliser of order 65 in H. However, by Lemma 3.17 there is a unique class of elements of order 13 in G and $C_G(13A) \cong 13 \times L_2(8)$, which does not contain any elements of order 5. Thus H cannot be contained in G. Now observe that $L_2(64) \cong Sp_2(64) \leq Sp_4(8)$, whence $Sp_4(8)$ cannot be a subgroup of G. However, $Sp_4(8)$ is a subgroup of $L_4(8)$ and $Sp_6(8)$, and is also contained in $U_4(8)$ (see [**40**], for example), so these latter three groups also cannot be contained in G. □

LEMMA 4.3. *Suppose that H is isomorphic to $L_3(4)$, $U_3(4)$, $Sz(8)$, $G_2(4)$ or $L_3(16)$. Then H is not a subgroup of G.*

Proof. In each case we find that there is no possible restriction of Brauer characters of H to V, thus showing that H cannot be a subgroup of G. For the majority of cases this can be easily demonstrated using the Brauer character table of H, the relevant pieces of which are contained in Tables A.13, A.18, A.30 and A.32, and the values of the Brauer characters of G given in Table 3. However if $H \cong L_3(16)$, then we note that H contains a maximal subgroup which is isomorphic to $U_3(4)$. It is now easily seen from the Brauer character table of $U_3(4)$ that such a subgroup cannot be contained in G, and so $H \not\leq G$. □

4.2. *Normalisers of cyclic subgroups of order 7*

(Electronic files folder `/Cyclic7Normalisers`*)*
In proving a number of results which follow we make use of normalisers of certain cyclic subgroups of order 7. Here we describe how to construct computationally the normalisers of subgroups $X \leq G$, where $X = \langle x \rangle$ and x is an element from class $7B$ or $7C$. First we have the case $x \in 7C$. We first construct the subgroup

$$\begin{aligned} H &= \langle x_{\alpha_{\pm 0}}(1), x_{\alpha_{\pm 1}}(1) \rangle \times \langle x_{\alpha_{\pm 6}}(1), x_{\alpha_{\pm 7}}(1) \rangle \\ &\cong L_3(2) \times L_3(2) \end{aligned}$$

before finding $x \in H$ such that x has order 7 and $C_V(x)$ has dimension 8 (such an x projects non-trivially onto both factors of H). Thus $x \in 7C$, and x is inverted by an involution t which lies in the Weyl group of G and swaps the root α_0 with α_1 and the root α_6 with α_7. Set $X = \langle x \rangle$, and note that $[N_G(X) : C_G(X)] \leq 6$. To construct $N_G(X)$ computationally, we first construct the normaliser of X in a variety of subgroups generated by fundamental root generators, along with $x_{\pm\alpha_0}$. Adding t to our generating set, this yields a subgroup $K \leq N_G(X)$ where

$$K \sim ((7 \times L_3(2)) : 2 \times 7) : 3.$$

At this stage, to complete the construction of the full normaliser we might look to apply the method of [**8**] with the elements x and t. However, experimental evidence suggests that the method of [**8**] is unsuccessful when applied with these elements (or at least very inefficient). We therefore take an element y of order 7 in K such that $C_V(y)$ has dimension 20. Note that y may be chosen to also be inverted by t, so we may apply [**8**] with this pair of elements and construct

$$C_G(y) \cong 7 \times L_6(2).$$

We now check that

$$\langle K, N_{C_G(y)}(X) \rangle \sim ((7 \times L_3(2)) : 2 \times L_2(8)) : 3,$$

which must be all of $N_G(X)$ by order considerations.

For $x \in 7B$ the method is very similar. In fact, we find that a direct application of [**8**] (using x and the inverting involution t) yields the full normaliser

$$N_G(X) \sim (7 : 2 \times {}^3D_4(2)) : 3.$$

LEMMA 4.4. *Let $X = \langle x \rangle$ be a cyclic group of prime order, with W a finite dimensional KX-module, where K is some field. Suppose R, S are isomorphic irreducible X-submodules of W with dimension n, and that X acts transitively on the set of nonzero vectors of R, respectively S. Set $u := r_0 + s_0$, where $u \neq 0$, $r_0 \in R$, $s_0 \in S$, and set $U = \langle u, ux, \ldots, ux^{n-1} \rangle \subseteq W$. Then U is an X-module, and is isomorphic to R (and S).*

Proof. When either $r_0 = 0$ or $s_0 = 0$ the result is clear, so assume that this is not the case. If $R = S$, then since X acts transitively on the non-zero vectors of R we have $r_0 x^i = u$ for some i, and multiplication by x^i is an X-module isomorphism between R and U, as required. Suppose then that $R \neq S$. If $\dim(U) < n$ then in particular $ux^{n-1} = \sum_{i=0}^{n-2} k_i ux^i$ for some $k_i \in K$. Since both R and S are irreducible we have $R \cap S = \{0\}$, and so we must have $r_0 x^{n-1} = \sum_{i=0}^{n-2} k_i r_0 x^i$, a contradiction since R is irreducible with dimension n. Thus $\dim(U) = n$. Let $\theta : R \to S$ be an X-module isomorphism, which exists by assumption. Since X acts transitively on the nonzero vectors of S, we must have $r_0 \theta x^i = s_0$ for a suitable choice of i, and since $\theta x^i : R \to S$ is also an X-module isomorphism, without loss of generality we may assume that $r_0 \theta = s_0$.

We now show that U is X-invariant. Clearly it suffices to show that ux^n is a linear combination of $\{u, ux, \ldots, ux^{n-1}\}$. Using the fact that θ is an X-module isomorphism, we have

$$ux^n = (r_0 + s_0)x^n = r_0 x^n + s_0 x^n = r_0 x^n + r_0 \theta x^n = r_0 x^n + r_0 x^n \theta.$$

Now, since $\dim(R) = n$ we have

$$\begin{aligned} r_0 x^n + r_0 x^n \theta &= \sum_{i=0}^{n-1} a_i r_0 x^i + \left(\sum_{i=0}^{n-1} a_i r_0 x^i \right) \theta \\ &= \sum_{i=0}^{n-1} a_i r_0 x^i + \left(\sum_{i=0}^{n-1} a_i r_0 \theta x^i \right) \\ &= \sum_{i=0}^{n-1} a_i r_0 x^i + \left(\sum_{i=0}^{n-1} a_i s_0 x^i \right) \\ &= \sum_{i=0}^{n-1} a_i (r_0 + s_0) x^i \\ &= \sum_{i=0}^{n-1} a_i u x^i. \end{aligned}$$

Thus $\dim(U) = n$, as claimed. We now define a map $\phi : R \to U$ by setting $(r_0 x^i)\phi := ux^i$, for $0 \leq i \leq n-1$, and extending linearly. It is straightforward to check that ϕ is an X-module homomorphism, and the result follows. □

Before stating our next results we note that if X is a cyclic group of order 7 then there are exactly three isomorphism classes of irreducible X-modules over $\mathbb{F}_2$, namely one consisting of the trivial module, and two isomorphism classes of 3-dimensional modules.

LEMMA 4.5. *Suppose $X = \langle x \rangle$, where $x \in 7C$. As an X-module we have*

$$V = V_1 \oplus V_2 \oplus C_V(X)$$

where $V_1 = \bigoplus_{i=1}^{8} U_i$ and $V_2 = \bigoplus_{j=1}^{8} \tilde{U}_j$. The subspaces U_i lie in one isomorphism class of irreducible 3-dimensional X-modules, and the subspaces $\tilde{U}_j$ lie in the other. Denote by $\mathcal{U}_k$ the set of irreducible 3-dimensional X-submodules of V which are contained in V_k, for $k = 1, 2$. Then $\mathrm{Stab}_{N_G(X)}(V_1) = \mathrm{Stab}_{N_G(X)}(V_2)$, and $\mathcal{U}_k$ breaks up into 35 orbits under the action of this subgroup, with lengths as follows.

Orbit length	*No. of orbits*
254016	6
84672	4
63504	4
36288	2
31752	2
21168	4
12096	3
10584	1
3528	2
2646	1
1512	1
441	1
378	1
216	1
63	1
9	1

Moreover, representatives for these orbits are included in the accompanying electronic file `/RepsNx7C`.

Proof. It is straightforward to check that $\mathrm{Stab}_{N_G(X)}(V_1) = \mathrm{Stab}_{N_G(X)}(V_2)$, and this subgroup has index 2 in $N_G(X)$, with $V_1^t = V_2$, where t is any involution which inverts x. Since $|X| = 7$ and any 3-dimensional subspace of V contains exactly 7 nonzero vectors, X must act transitively on the nonzero vectors of any 3-dimensional X-submodule of V. Therefore we may apply Lemma 4.4 to see that if $0 \neq v \in V_1$, then $U = \langle v, vx, vx^2 \rangle \in \mathcal{U}_1$. Moreover, since U is irreducible it is the unique submodule in $\mathcal{U}_1$ which contains v. Since there are $2^{24} - 1$ nonzero vectors in V_1, we deduce that there are $(2^{24} - 1)/7 = 2396745$ irreducible 3-dimensional X-submodules of V_1 (and similarly of V_2). Write $S = \mathrm{Stab}_{N_G(X)}(V_1)$. By making suitable choices of v and constructing the orbits of the corresponding U under the action of S (using Magma's `Orbit` command), we find a complete set of S-orbit representatives for $\mathcal{U}_1$, with lengths as given in the statement of the lemma. Note that since also $S = \mathrm{Stab}_{N_G(X)}(V_2)$, and V_2 is $N_G(X)$-conjugate to V_1, the S-orbits of $\mathcal{U}_1$ will be in one-to-one correspondence with those of $\mathcal{U}_2$. □

Lemma 4.6. *Suppose $X = \langle x \rangle$, where $x \in 7B$. As an X-module we have*

$$V = V_1 \oplus V_2 \oplus C_V(X)$$

where $V_1 = \bigoplus_{i=1}^{9} U_i$ and $V_2 = \bigoplus_{j=1}^{9} \tilde{U}_j$. The subspaces U_i lie in one isomorphism class of irreducible 3-dimensional X-modules, and the subspaces $\tilde{U}_j$ lie in the other. Denote by $\mathcal{U}_k$ the set of irreducible 3-dimensional X-submodules of V which are contained in V_k, for $k = 1, 2$. Then $\mathrm{Stab}_{N_G(X)}(V_1) = \mathrm{Stab}_{N_G(X)}(V_2)$, and $\mathcal{U}_k$ breaks up into 19 orbits under the action of this subgroup, with lengths as follows.

Orbit length	*No. of orbits*
3302208	2
2935296	2
1257984	1
1100736	2
978432	2
825552	1
179712	1
117936	1
52416	2
17472	2
17199	1
2457	1
1	1

Moreover, representatives for these orbits are included in the accompanying electronic file `RepsNx7B`*.*

Proof. We follow the same process as in the proof of Lemma 4.5, noting that in this case V_1 and V_2 contain $(2^{27}-1)/7 = 19173961$ irreducible 3-dimensional X-submodules of V. □

4.3. *Subgroups which fix a vector or hyperplane*

Since V is a self-dual G-module, a subgroup H of G fixes a non-zero vector of V if and only if it fixes a hyperplane of V. Unless otherwise stated, the dimensions (as $\mathbb{F}_2$-spaces) of the cohomology groups referred to in this section were calculated using Magma's `CohomologicalDimension` command. We use Proposition 2.5 throughout this section and without comment.

Lemma 4.7. *Suppose that $H \leq G$ with $\mathrm{Soc}(H) \cong Sp_4(4)$. Then H fixes a non-zero vector of V.*

Proof. First suppose that $H \cong Sp_4(4)$. Consulting Table A.27, we see that there are only two possible $\mathbb{F}_2$-character restrictions of H to V, namely

$$8\chi_1 + 2\chi_2 + 2\chi_5$$

and

$$8\chi_1 + 2\chi_3 + 2\chi_4.$$

We have

$$\dim(H^1(Sp_4(4), \chi_2)) = 2,$$

$$\dim(H^1(Sp_4(4), \chi_3)) = 2,$$

$$\dim(H^1(Sp_4(4), \chi_4)) = 0$$

and

$$\dim(H^1(Sp_4(4), \chi_5)) = 0$$

(the first two equalities are found in [**14**]). Thus we see that in both cases H must fix a non-zero vector of V. Now suppose that $H \sim Sp_4(2).2$. Then by the above and Lemma 2.4 we have that

H again fixes a non-zero vector of V. Finally we note that there is no possible $\mathbb{F}_2$-character restriction of $\mathrm{Aut}(Sp_4(4)) \sim Sp_4(4).4$ to V, and so this group cannot embed in G. □

LEMMA 4.8. *Suppose that $H \leq G$ with $\mathrm{Soc}(H) \cong U_3(8)$. Then H fixes a non-zero vector of V.*

Proof. Using Table A.20, the only possible character restriction is $2\chi_1 + \chi_3$, and it is immediate in this case that if $H \cong U_3(8)$ then H must fix a vector or hyperplane. Now let $A = \mathrm{Aut}(H)$, and note that

$$A \sim U_3(8).(\mathrm{Sym}(3) \times 3).$$

In view of Lemma 2.4, to complete the proof we must show that any subgroups of A which contain H as an odd-index subgroup fix a non-zero vector of V. Up to A-conjugacy there are four such subgroups of A, which we label $U_3(8) : 3_1$, $U_3(8) : 3_2$, $U_3(8).3_3$ and $U_3(8).3^2$, following [**41**]. Portions of the $\mathbb{F}_2$-character tables of these groups are given in Tables A.21, A.22, A.23 and A.24, respectively. For $U_3(8) : 3_1$ we see that the only possible character restrictions are sums of two trivial characters and one other character, and so any $U_3(8) : 3_1$ subgroup must fix a non-zero vector of V, while for $U_3(8) : 3_2$, $U_3(8).3_3$ and $U_3(8).3^2$ we find there are no possible character restrictions, and so these groups cannot be contained in G. □

LEMMA 4.9. *Suppose that $H \leq G$ with $\mathrm{Soc}(H) \cong L_3(8)$. Then H fixes a non-zero vector of V.*

Proof. Suppose first that $H \cong L_3(8)$. Using Table A.14, the only possible character restrictions of H to V consist of two trivial characters and two characters of degree 27. Duality considerations mean the pair of 27-dimensional characters must be (χ_5, χ_6), (χ_7, χ_8) or (χ_9, χ_{10}). Now for $x \in H$ an element of order 7 we have $(\chi_5 + \chi_6)(x) = 5$, $(\chi_7 + \chi_8)(x) = -9$ and $(\chi_9 + \chi_{10})(x) = 12$, leading us to deduce that the only possible restriction is $2\chi_1 + \chi_7 + \chi_8$. Since $\dim(H^1(L_3(8), \chi_7)) = 0$ and $\dim(H^1(L_3(8), \chi_8)) = 0$, we deduce that H must fix a non-zero vector of V. A similar method, using Table A.15, proves the result for $H \sim L_3(8) : 3$. □

LEMMA 4.10. *Suppose $H \leq G$ with $\mathrm{Soc}(H) \cong L_2(16)$. Then H fixes a non-zero vector of V.*

Proof. First assume that $H \cong L_2(16)$. The information in Table A.4 shows that there are only two possible $\mathbb{F}_2$-character restrictions of H to V. These are

$$8\chi_1 + 2\chi_2 + \chi_6$$

and

$$8\chi_1 + 2\chi_3 + 2\chi_4.$$

The character χ_6 corresponds to the Steinberg module of $L_2(16)$ in characteristic 2, so is projective. Moreover we have $\dim(H^1(L_2(16), \chi_2)) = 4$. Thus for the first possible restriction given above we see that H must fix a non-zero vector of V. For the second possible restriction we have

$$\dim(H^1(L_2(16), \chi_3)) = \dim(H^1(L_2(16), \chi_4)) = 0,$$

and so in this case H must also fix a non-zero vector of V.

Since $\mathrm{Aut}(L_2(16)) \sim L_2(16).4$, we may use Lemma 2.4 to show that any group H with $\mathrm{Soc}(H) \cong L_2(16)$ must also fix a non-zero vector of V. □

LEMMA 4.11. *Suppose $H \leq G$ with $H \cong L_2(32)$. Then H fixes a non-zero vector of V.*

Proof. *(Electronic files folder* `/L2(32)`*)* First suppose that $H \cong L_2(32)$. We first use Table A.9 so deduce that there is only one possible character restriction of H to V, namely $4\chi_1 + 2\chi_2 + \chi_5$. We now consider $\mathrm{Soc}(V_H)$. The character χ_5 corresponds to the Steinberg module and hence is projective. Consequently, $\mathrm{Soc}(V_H)$ must contain an irreducible H-module corresponding to χ_5, along with at least one further irreducible H-module corresponding to either χ_1 or χ_2. If such a module corresponds to χ_1 then H must fix a non-zero vector, and we are done. We therefore suppose that $\mathrm{Soc}(V_H)$ contains a module W corresponding to χ_2.

Our aim is to consider possible candidate modules for W, and investigate the possibility that $H \leq \mathrm{Stab}_G(W)$. We first take the subgroup

$$L = \langle x_{\pm\alpha_4}(1), x_{\pm\alpha_5}(1), x_{\pm\alpha_6}(1), x_{\pm\alpha_7}(1) \rangle \cong L_5(2)$$

and find a subgroup $X = \langle x \rangle$ of L with order 31. Note that this lies in the unique G-conjugacy class of subgroups of order 31. We next generate the group

$$K = (31 : 5 \times L_3(2)) : 2$$

which contains all the involutions of G which invert x in a single K-conjugacy class. Now taking t to be such an involution we have that H must contain some G-conjugate of $\langle X, t \rangle$, so without loss of generality we assume $\langle X, t \rangle \leq H$.

As an X-module, V decomposes as

$$V_X = C_V(X) \oplus V_1 \oplus V_1^t \oplus V_2 \oplus V_2^t \oplus V_3 \oplus V_3^t.$$

Here V_1 and V_2 are 5-dimensional irreducible X-modules, while $V_3 = \bigoplus_{i=1}^{3} U_i$ with the U_i also 5-dimensional irreducible X-modules. Moreover, the isomorphism classes of the irreducible modules contained in V_1, V_1^t, V_2, V_2^t, V_3 and V_3^t are all pairwise disjoint. By considering the possible irreducible modules for $L_2(32)$ over $GF(2)$, we see that the irreducible H-module W must decompose as an X-module as

$$W = U \oplus U^t,$$

where U is an irreducible X-module of dimension 5. Thus, using the decomposition of V_X above, we may construct a set of candidate subspaces for W. We find that, up to K-conjugacy, there are only five possible candidates for W. We denote this set of five subspaces by $\mathcal{W}$.

For each subspace $W \in \mathcal{W}$ we now wish to consider $\mathrm{Stab}_G(W)$. Due to the size of G, it is not possible to simply construct these stabilisers using standard MAGMA commands. We therefore proceed as follows. We observe that the centraliser of an involution in $L_2(32)$ is an elementary abelian group of order 32. Hence, if $H \leq \mathrm{Stab}_G(W)$, we would expect $\mathrm{Stab}_G(W) \cap C_G(t)$ to contain such a subgroup. We therefore construct $C_G(t)$ using Bray's method [**12**], before finding a set $\mathcal{S}$ of Sylow 2-subgroups of $C_G(t)$ such that all involutions of $C_G(t)$ are contained in the union of the subgroups from $\mathcal{S}$. Since the subgroups from $\mathcal{S}$ are 2-groups they are unipotent subgroups of G, and for each $S \in \mathcal{S}$ we may calculate $\mathrm{Stab}_S(W)$ using the MAGMA command `UnipotentStabiliser`. We then construct the subgroups Q of $\mathrm{Stab}_G(W)$ which are generated by the groups $\mathrm{Stab}_S(W)$, as S runs through $\mathcal{S}$.

The results are as follows. For all but one subspace $W \in \mathcal{W}$ we find that no elementary abelian subgroup of order 32 lies in Q, and hence H cannot stabilise W; while for the remaining candidate we find that both Q and X fix a non-zero vector in V. Since an $L_2(32)$ is generated by an element of order 31 and the centraliser of any of its involutions, we deduce that H must also fix a non-zero vector in V, which completes the proof.

□

LEMMA 4.12. *Suppose $H \leq G$ with $H \cong L_2(32) : 5$. Then H fixes a non-zero vector of V.*

Proof. Let $H' \leq H$ be isomorphic to $L_2(32)$, and let $K \leq H$ be a subgroup of order 5. By Lemma 4.11 there exists a non-zero vector $v \in V$ which is fixed by H'. Let $W = \langle v^K \rangle$. Then W is a module for H, and consulting Table A.10 we deduce that W is either irreducible corresponding to the character χ_2, or is the direct sum of four trivial modules. Now by considering Brauer character values on elements of order 5 we see that the latter case must hold, and so H must fix a non-zero of vector of V.

□

Our next set of results are concerned with the case of potential subgroups of G with socle isomorphic to $L_3(2)$. It is possible to show non-maximality of such subgroups by using similar arguments to those in the proof of Lemma 4.20. However, we require more detailed results for use in later proofs.

LEMMA 4.13. *There are two classes of Frobenius group $7C : 3$ in G. One of these classes has elements of order 3 in class $3D$, while the other has elements of order 3 in class $3E$.*

Proof. For $z \in 7C$, we generate $N = N_G(\langle z \rangle) \sim ((7 \times L_3(2)) : 2 \times L_2(8)) : 3$ using the method of Section 4.2. We may now check that within N there are two classes of elements of order 3 which act non-trivially on $\langle z \rangle$, which lie in the classes $3D$ and $3E$ of G, respectively.
□

LEMMA 4.14. *There are two classes of Frobenius group $7B : 3$ in G. One of these classes has elements of order 3 in class $3D$, while the other has elements of order 3 in class $3E$.*

Proof. Let $z \in 7B$ in G. As in Section 4.2 we generate

$$N_G(\langle z \rangle) \sim (7 : 2 \times {}^3D_4(2)) : 3.$$

From the character table of ${}^3D_4(2) : 3$ in [**15**] we see that there are four classes of outer elements of order 3, with any such element x lying in a distinct ${}^3D_4(2) : 3$-conjugacy class from x^2. Since x will clearly lie in the same Frobenius group as x^2, we deduce that there are at most two G-conjugacy classes of Frobenius groups $7B : 3$. Now computationally we determine that there are 3-elements in such groups which lie in the G-conjugacy classes $3D$ and $3E$, and the result follows. □

LEMMA 4.15. *Suppose $H \leq G$ with $H \cong L_3(2)$, where elements of orders 3 and 7 in H fuse to the classes $3D$ and $7C$ of G, respectively. Then H fixes a non-zero vector in V.*

Proof. *(Electronic files folder* `/L3(2)`*)* Let $\langle z, x \rangle \cong 7C : 3D$, where $z \in 7C$ and $x \in 3D$. By Lemma 4.13, we see that up to G-conjugacy, any $L_3(2)$ subgroup of G of the form stated in the lemma must contain $\langle z, x\rangle$. We first generate the normaliser

$$N_G(\langle x\rangle) \sim (3 \times \Omega_{10}^-(2)) : 2 \times \mathrm{Sym}(3)$$

by constructing the normaliser $N_S(\langle x\rangle)$ in various small well-known subgroups $S \le G$ (primarily those we can easily generate from the root elements), and using the method of [**8**] to complete generation of the full normaliser. We now need to consider involutions in $N_G(\langle x\rangle)$ which invert x. From [**15**] we see that the subgroup $\Omega_{10}^-(2) : 2$ has three classes of outer involutions. This yields six such classes in $N = N_G(\langle x\rangle)$ as given below, when taking into account the unique class of involutions in the subgroup Sym(3).

$x \in X$	Class in G	$\lvert x^N\rvert$
$(2E, 1)$	$2B$	1584
$(2F, 1)$	$2D$	8482320
$(2G, 1)$	$2E$	101787840
$(2E, 2A)$	$2C$	4752
$(2F, 2A)$	$2E$	25446960
$(2G, 2A)$	$2E$	305363520

We wish to consider the subgroups $\langle z, x, t\rangle$, where t is an involution lying in one of the classes given above, and check whether they are isomorphic to $L_3(2)$. Note that we need only take one representative from each orbit of each class under $N_G(\langle z, x\rangle)$. The size of many of these classes prohibits simply running through them directly. Therefore we take a subgroup $O_P \sim \Omega_{10}^-(2) : 2$ on its standard generators c_P, d_P in its 495-degree permutation representation (see [**41**]). We then take $O_M \le N_G(\langle x\rangle)$ with $O_M \sim \Omega_{10}^-(2) : 2$ as a 56-dimensional matrix group, and construct an isomorphism $\phi : O_P \to O_M$ using standard MAGMA commnands. The motivation for doing this is because computation and storage is far easier in this permutation group than in the matrix group. However, the classes $2F$ and $2G$ of outer involutions in O_P are still too large to easily store. Now, in this 495-degree permutation representation we find that $2F$-elements fix 63 points and $2G$-elements fix 15 points. Hence all outer involutions from these classes fix a point in $\Omega = \{1, \ldots, 495\}$. From [**15**] we see that the stabiliser S of a point in this representation has the form $2^8 : \Omega_8^-(2) : 2$. This group has 12 classes of involutions, four of which correspond to outer involutions in $\Omega_{10}^-(2) : 2$. Let us label these four classes by $2A$, $2B$, $2C$ and $2D$, slightly defying the usual convention. Then we have the following table.

X	Class in O_P	$\lvert x^S\rvert$
$2A$	$2E$	272
$2B$	$2F$	17136
$2C$	$2F$	342720
$2D$	$2G$	1028160

For a given stabiliser these sets are now manageable. To run through, for example, the $2F$-involutions in $\mathrm{Stab}_{O_P}(1)$ we form the set $X_1 = 2B \cup 2C$ of size 359856. Now we pull each of these elements r back into the matrix group using ϕ^{-1} and check whether $\langle z, x, r^{\phi^{-1}}\rangle$ is isomorphic to $L_3(2)$. (Note that we do not need to consider the two other outer involutions $zr^{\phi^{-1}}$ and $z^2r^{\phi^{-1}}$ since they clearly generate the same group as $r^{\phi^{-1}}$). We further check whether $\langle z, x, r^{\phi^{-1}}v_i\rangle \cong L_3(2)$, for $i = 1, 2, 3$, where the v_i are the three involutions in the group $\mathrm{Sym}(3) \le N_G(\langle x\rangle)$. We then repeat this process for the stabilisers $\mathrm{Stab}_{O_P}(j)$ for $j \in \Sigma$, where Σ is a set of representatives of the orbits of $O_P \cap N_G(\langle z, x\rangle)^{\phi^{-1}}$ on Ω. This ensures we have exhausted the $2F$-conjugacy class. After completing the process for the classes $2E$ and $2G$, we find that in total there are eleven $N_G(\langle z, x\rangle)$-orbits of subgroups of G which contain $\langle z, x\rangle$ and are isomorphic to $L_3(2)$. Subgroups from three of these orbits fix (pointwise) 4-dimensional subspaces of V, while subgroups from the remaining eight orbits fix 2-dimensional subspaces of V. Hence the result holds. □

LEMMA 4.16. *There are no subgroups $L_3(2)$ whose elements of orders 3 and 7 fuse to $3E$ and $7C$ elements in G, respectively.*

Proof. Let $\langle z, x\rangle \cong 7C : 3E$, where $z \in 7C$ and $x \in 3E$. As previously, Lemma 4.13 implies that, up to G-conjugacy, any $L_3(2)$ subgroup of G of the form stated in the lemma must contain $\langle z, x\rangle$. We proceed as in the proof of Lemma 4.15, by first constructing the normaliser

$$N_G(\langle x\rangle) \sim 3^{\cdot}(U_3(2) \times U_6(2)) : \mathrm{Sym}(3).$$

There are two classes of outer involutions in $N_G(\langle x\rangle)$, and these have sizes 684288 and 43110144 and lie in the G-conjugacy classes $2C$ and $2E$, respectively. Now we find the subgroups $U_1 = 3^{\cdot}(U_3(2) \times 1) : \mathrm{Sym}(3)$ and $U_2 = 3^{\cdot}(1 \times U_6(2)) : \mathrm{Sym}(3)$ of $N_G(\langle x\rangle)$ and construct smaller permutation representations of the latter (the former group is small enough to deal with directly as a 56-dimensional matrix group). This is done by splitting the module V (using the Meataxe) to yield a semisimple module with three constituents of dimension 12 and one of dimension 20. Now the induced 12-dimensional matrix group $3^{\cdot}U_6(2) : \mathrm{Sym}(3)$ has three orbits on the vectors of this 12-dimensional module, having sizes 1, 2016 and 2079. The permutation group obtained from the action on the orbit of length 2016 yields a group $U \sim 3^{\cdot}U_6(2) : \mathrm{Sym}(3)$ within which we can compute and store the involutions. We find from the character table stored in GAP [**21**] that there are two classes of outer involutions in $3^{\cdot}U_6(2).\mathrm{Sym}(3)$, of sizes 57024 and 3592512. We now create an isomorphism $\phi : U \to U_2$. Thus in a similar way to the proof of Lemma 4.15 we may create the classes of involutions and pull them back into the 56-dimensional matrix group. Note that elements in the class of size 57024 do not fix any points of $\Gamma = \{1, \ldots, 2016\}$ whereas elements in the class of size 3592512 fix 32 points on Γ. Thus in the latter case we use the stabiliser method from Lemma 4.15 while in the form case we simply store all the involutions directly. Now, also using the subgroup U_1, we run through all the outer involutions in $N_G(\langle z\rangle)$ to deduce that there are no subgroups of G which contain $\langle z, x\rangle$ and are isomorphic to $L_3(2)$. □

LEMMA 4.17. *Suppose $H \leq G$ with $H \cong L_3(2)$, where elements of orders 3 and 7 in H fuse to the classes $3D$ and $7B$ of G, respectively. Then H fixes a non-zero vector in V.*

Proof. *(Electronic files folder* `/L3(2)`) Here we follow the same method as in the proof of Lemma 4.15. Taking $\langle z, x\rangle \cong 7B : 3D$, we find five $N_G(\langle z, x\rangle)$-orbits of subgroups which are isomorphic to $L_3(2)$ and contain $\langle z, x\rangle$, with each such subgroup fixing a non-zero vector of V. Using Lemma 4.14, we see the result holds. □

LEMMA 4.18. *There are no subgroups $L_3(2)$ whose elements of orders 3 and 7 fuse to $3E$ and $7B$ elements in G, respectively.*

Proof. This follows by using a similar technique to that employed in the proof of Lemma 4.16. □

LEMMA 4.19. *Suppose $H \leq G$ with $\mathrm{Soc}(H) \cong L_3(2)$. Then H fixes a non-zero vector of V.*

Proof. Let $H \cong L_3(2)$. Using Table A.11, there are only four possible character restrictions of H to V. These are listed below.

(i) $2\chi_1 + 9\chi_2 + 9\chi_3$, and the $7A$ and $7B$ elements in H fuse to $7B$ elements in G.
(ii) $4\chi_1 + 6\chi_2 + 6\chi_3 + 2\chi_4$, and the $7A$ and $7B$ elements in H fuse to $7C$ elements in G.
(iii) $8\chi_1 + 6\chi_4$, and the $7A$ and $7B$ elements in H fuse to $7A$ elements in G.
(iv) $20\chi_1 + 6\chi_2 + 6\chi_3$, and the $7A$ and $7B$ elements in H fuse to $7A$ elements in G.

Note that the character χ_4 is the Steinberg character of $L_3(2)$, so is projective. Moreover, the projective indecomposable modules corresponding to χ_2 and χ_3 are found in [**9**], and in particular we have $\dim(H^1(L_3(2), \chi_2)) = 1$ and $\dim(H^1(L_3(2), \chi_3)) = 1$. We deduce that any H which appears in cases (iii) or (iv) above must fix a vector or hyperplane of V. If we are in case (i), then we may apply Lemmas 4.17 and 4.18, while in case (ii) we apply Lemmas 4.15 and 4.16. Since $\mathrm{Aut}(L_3(2)) \sim L_3(2):2$, the result now follows by Lemma 2.4. □

4.4. *Other potential subgroups*

LEMMA 4.20. *Suppose that $H \leq G$ with $\mathrm{Soc}(H) \cong L_2(8)$. Then H is not a maximal subgroup of G.*

Proof. Write H_0 for the subgroup of H which is isomorphic to $L_2(8)$. Note that any element of order 3 in $L_2(8)$ is the cube of an element of order 9. By Lemma 3.34, elements of order 9 in G must cube into class $3A$ or $3E$. Using this information, along with Table A.1, we see that there are the following possible $\mathbb{F}_2$-character restrictions of H_0 to V.

(i) $2\chi_2 + \chi_3 + 3\chi_4$
(ii) $8\chi_2 + \chi_3$
(iii) $4\chi_3 + 2\chi_4$
(iv) $2\chi_1 + 9\chi_2$
(v) $6\chi_1 + 3\chi_2 + \chi_3 + 2\chi_4$
(vi) $8\chi_1 + 6\chi_3$
(vii) $8\chi_1 + 4\chi_2 + 2\chi_4$
(viii) $12\chi_1 + 2\chi_2 + 4\chi_3$

Moreover, we deduce from the resulting character values that in cases (vi) and (vii) elements of order 7 in H must lie in the class $7A$ of G, while in the other cases such elements must lie in either class $7B$ or $7C$.

Note that the character χ_3 of $L_2(8)$ (and $\mathrm{Aut}(L_2(8)) \sim L_2(8):3$) is projective, and so in case (vi) we immediately have that H must fix a non-zero vector of V. Furthermore we have $\dim(H^1(L_2(8):3, \chi_2)) = 1$ (and $\dim(H^1(L_2(8), \chi_2)) = 3$), and so in case (viii) H must also fix a vector or hyperplane. Thus we may proceed on the assumption that elements of order 7 in H_0 lie in either class $7B$ or $7C$ of G.

Suppose that H does not fix a non-zero vector of V. Let $x \in H_0$ be an element of order 7, and write $X = \langle x \rangle$. Let $W \subseteq V$ be a minimal H_0-submodule. Thus W must correspond to either χ_2, χ_3 or χ_4, and we deal with each of these possibilities in turn.

Case A, W corresponds to χ_2

By considering the action of $L_2(8)$ on its irreducible 6-dimensional $\mathbb{F}_2$-module, as an X-module we must have

$$W = U \oplus U^*,$$

where $\dim(U) = \dim(U^*) = 3$ and U^* is dual to U. Moreover, for any involution $t \in H$ which inverts x we must have $U^t = U^*$. We can produce a set of $N_G(X)$-orbit representatives of candidate subspaces W using our representatives from Lemmas 4.6 and 4.5. Denote by $\mathcal{W}$

the set of such orbit representatives. For each representative U for we must consider spaces of the form $\langle U, U^t \rangle$, where t is an involution which inverts x. Denote by w an involution from the subgroup $7:2$ of $N_G(X)$. Then any involution which inverts x must take the form wc, where c is an involution in $C_G(x) \cap C_G(w)$. It suffices to consider only representatives of $N_G(X)$-conjugacy classes of involutions which invert x. Indeed, suppose that $t' = t^g$, where $g \in N_G(X)$. Then

$$U \oplus U^{t'} = U \oplus U^{g^{-1}tg} = (U^{g^{-1}} \oplus U^{g^{-1}t})^g,$$

so $U \oplus U^{t'}$ lies in the same $N_G(X)$-orbit as $U^{g^{-1}} \oplus U^{g^{-1}t}$. If $x \in 7C$ then there are two such involution classes in $N_G(X)$, and so we find a complete set of $N_G(X)$-orbit representatives of $\mathcal{W}$ consisting of 70 elements. While if $x \in 7B$ there are three such involution classes in $N_G(X)$, and we have a complete set of $N_G(X)$-orbit representatives of $\mathcal{W}$ consisting of 57 elements.

We now show that for each representative W of the form $W \in \mathcal{W}$, $\mathrm{Stab}_G(W)$ is not isomorphic to $L_2(8)$ or $L_2(8):3$. If H is a maximal subgroup of G and $H \leq \mathrm{Stab}_G(W)$ for some W, it must be the case that $H = \mathrm{Stab}_G(W)$. For any given involution of $L_2(8)$ the centraliser of this involution is an elementary abelian group of order 8, while for $L_2(8):3$ the centraliser of an involution has structure $2^3:3$. Thus, if H were maximal in G, we would have $|\mathrm{Stab}_G(W) \cap C_G(t)| = 8$ or 24. Using Bray's method [**12**] we construct $C_G(t)$, and then find a set $\mathcal{S}$ of Sylow 2-subgroups of $C_G(t)$ such that all involutions of $C_G(t)$ are contained in $\mathcal{S}$. Of course, $\mathcal{S}$ consists of unipotent subgroups of G. For each $S \in \mathcal{S}$, we then calculate $\mathrm{Stab}_S(W)$ using the MAGMA command `UnipotentStabiliser`. In Tables 5 and 6 we list the cardinality of the subgroups Q of $\mathrm{Stab}_G(W)$ which are generated by the groups $\mathrm{Stab}_S(W)$, as S runs through $\mathcal{S}$.

TABLE 5. $x \in 7B$, $\dim(W) = 6$

Size of subgroup Q	Occurences
301989888	1
8640	2
6144	3
2048	1
96	1
32	2
16	2
12	3
8	2
4	17
2	23

TABLE 6. $x \in 7C$, $\dim(W) = 6$

Size of subgroup Q	Occurences
2886218022912	1
412316860416	1
49152	1
32768	2
16384	2
8192	2
6144	1
192	1
96	1
32	1
12	1
4	13
2	43

If $|Q| < 8$ then $H \neq \mathrm{Stab}_G(W)$, since Q must contain all the involutions of H. Moreover, if $|Q| > 24$ then $H \neq \mathrm{Stab}_G(W)$ also. A glance at Tables 5 and 6 shows this leaves only two cases, with $|Q| = 8$ (the cases where $|Q| = 12$ or 16 are eliminated by Lagrange's Theorem, since $8 \nmid 12$ and $16 \nmid 24$). However, further investigation of these cases shows that $Q \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, so Q does not have the required structure. We deduce that there are no candidate subspaces W, and so H is not a maximal subgroup of G in Case A.

Case B, W corresponds to χ_3

Here our method is similar to that for Case A. As an X-module, we have $W = W' \oplus C_W(X)$, where $W' \in \mathcal{W}$ as in Case A. Moreover, $N_G(X)$ must preserve this decomposition of W, and t must act non-trivially on $C_W(X)$. Since we already have $N_G(X)$-orbit representatives for $\mathcal{W}$, to obtain orbit representatives for potential 8-dimensional modules for H we take all subspaces of the form $\langle W, J\rangle$, where $J \subset C_V(x)$ has dimension 2 and is stabilised, but not fixed pointwise, by t. We then follow the same procedure as in Case A. In Tables 7 and 8 we list the sizes of the subgroups of $\mathrm{Stab}_G(W)$ we generate by considering unipotent stabilisers, as above. Again when $|Q| = 8$ we have that Q is not elementary abelian. Note that for the case $x \in 7C$ only one $N_G(X)$-conjugacy class of involutions acts non-trivially on $C_W(X)$.

Case C, W corresponds to χ_4

Denote by V_H the 56-dimensional module for $E_7(2)$, considered as an H-module, and consider

TABLE 7. $x \in 7B$, $\dim(W) = 8$

Size of subgroup Q	Occurences
589824	1
24576	3
1152	2
256	1
32	2
16	1
12	4
8	2
4	18
2	23

TABLE 8. $x \in 7C$, $\dim(W) = 8$

Size of subgroup Q	Occurences
12884901888	1
4294967296	3
6291456	4
786432	16
262144	4
32768	24
24576	1
16384	72
256	24
128	28
96	1
32	1
16	64
12	120
4	719
2	3118

$S := \mathrm{Soc}(V_H)$. The irreducible summands of S must have dimensions 1, 6, 8 or 12, and since we have dealt with the first three cases already we may assume the summands of S all have dimension 12. From the possible character restrictions we now see that we must be in one of the cases (i), (iii), (v) or (vii). However, the 8-dimensional irreducible $\mathbb{F}_2$-module for $L_2(8)$ is the Steinberg module, so it is projective. Hence we may assume that V_H contains no 8-dimensional composition factors (since otherwise we would be in Case B). This implies we are in case (vii). In particular we deduce that S must contain at most two submodules, each corresponding to χ_4. Furthermore, if S were to contain only one summand, then V_H would be indecomposable, so would be a quotient of the projective cover of the 12-dimensional module for $L_2(8)$, which has dimension 48, a contradiction. Thus S is the direct sum of two irreducible modules corresponding to χ_4. However V is self-dual, and so the top composition factor of V_H must be isomorphic to S, implying that V_H contains at least four 12-dimensional composition factors. This contradicts the fact that we have the character restriction in case (vii). □

LEMMA 4.21. *Suppose that $H \leq G$ with $\mathrm{Soc}(H) \cong L_4(4)$. Then H is not a maximal subgroup of G.*

Proof. *(Electronic files folder* `/L4(4)`*)* First suppose that $H \cong L_4(4)$. Using Table A.17, we see that there are only two possible character restrictions of H to V. These are as follows.

(i) $2\chi_4 + \chi_8$
(ii) $2\chi_4 + \chi_5 + \chi_6$

As there are no trivial characters in either of these restrictions such a subgroup certainly does not fix a vector in V and so does not lie in a vector stabiliser. Thus we cannot deduce the non-maximality of any $L_4(4)$-subgroup from this analysis - we must carry out some computational work to achieve this. Firstly we need some results regarding the group $L_4(4)$. Note that this group has a permutation representation of degree 85 in which computations are straightforward. Now let $L \cong L_4(4)$ and suppose $z \in L$ is an element in the L-conjugacy class $5A$. Then

$$C_L(z) = \langle z \rangle \times K \cong 5 \times L_2(16)$$

where $K = C_L(z)' \cong L_2(16)$. From [**15**] we know that $L_2(16)$ only has a single class of cyclic subgroups of order 5. Let $x \in K$ be an arbitrary element of order 5 and consider $C_L(x) \cong 5 \times L_2(16)$. We find computationally that there exist involutions in $C_L(x)$ which, together with K, generate the whole group L.

Now G contains a unique class of elements of order 5 (see Table 2) and, by Lemma 3.6, these have centraliser of the form $5 \times \Omega_8^-(2) \times \mathrm{Sym}(3)$. From [**15**] we see that the only maximal subgroup of $\Omega_8^-(2)$ which can contain an $L_2(16)$ is the subgroup $L_2(16):2$. Hence there is a unique class of subgroups of the form $L_2(16)$ in $\Omega_8^-(2)$. We can thus fix an element $g \in G$ of order 5 and a subgroup $X \cong L_2(16)$ in $C_G(g)$ from which we can generate a representative H of any G-conjugacy class of subgroups isomorphic to $L_4(4)$ using the method described in the preceding paragraph.

Computationally we generate $C_G(g)$ as follows. We take

$$\langle x_{\pm\alpha_0}(1), x_{\pm\alpha_1}(1), x_{\pm\alpha_3}(1) \rangle \cong L_4(2)$$

and choose our $5A$-element g from this group. Then we know immediately from the Dynkin diagram of type E_7 that the groups

$$\langle x_{\pm\alpha_5}(1), x_{\pm\alpha_6}(1), x_{\pm\alpha_7}(1) \rangle \cong L_4(2)$$

and

$$\langle x_{\pm\alpha_2}(1) \rangle \cong \mathrm{Sym}(3)$$

lie in $C_G(g)$. We now use the method of [**8**] to complete the generation of the full centraliser. Next, within the group $O := C_G(g)'' \cong \Omega_8^-(2)$ we find a representative subgroup $L_2(16)$ (by using the standard generators for $L_2(16)$ given in [**41**]) and take this to be our fixed group X. We take $x \in X$ to be any element of order 5 and generate $C_G(x) \cong 5 \times \Omega_8^-(2) \times \mathrm{Sym}(3)$ using [**8**].

What now remains is to run through all involutions in $C_G(x)$ to determine for which involutions t we have $\langle X, t\rangle \cong L_4(4)$. Now $\Omega_8^-(2)$ contains three classes of involutions of sizes 1071, 4284 and 64260 (see [**15**]). Hence it is easy to see that $C_G(x)$ contains seven conjugacy classes of involutions of sizes 3, 1071, 3213, 4284, 12852, 64260 and 192780. Let us label these as $\mathcal{C}_i$, for $i \in \{1, \dots, 7\}$, respectively. We store these seven classes in Magma and run through them as detailed above. Note that to check whether a subgroup $\langle X, t\rangle$ is isomorphic to $L_4(4)$ we firstly carry out a check on the orders of some random elements. Secondly, we check that the module V has either three or four composition factors under $\langle X, t\rangle$ - if not, it cannot be isomorphic to $L_4(4)$ because of our character restriction analysis above. Finally, we check the order and simplicity of the group to confirm it is isomorphic to $L_4(4)$.

The results of this investigation are as follows. The classes $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_5$ and $\mathcal{C}_7$ do not yield any subgroup isomorphic to $L_4(4)$. However, in the class $\mathcal{C}_6$ there are 2160 involutions t for which $\langle X, t\rangle \cong L_4(4)$. Of these involutions 1080 generate a subgroup $L_4(4)$ which appears on the module V as in case (i) above, and the remaining 1080 generate a subgroup $L_4(4)$ which appears on the module V as in case (ii) above. For all these involutions t we find that $C_{C_G(x)}(\langle X, t\rangle) \cong \mathrm{Sym}(3)$, where the elements of order 3 lie in these Sym(3) subgroups lie in the class $3B$ of G. In particular none of these subgroups $L_4(4)$ is maximal in G.

We can now use this information to eliminate the possibility that any automorphism group of $L_4(4)$ is a maximal subgroup of G. Indeed, since $C_{C_G(x)}(\langle X, t\rangle) = C_G(\langle X, t\rangle)$, we deduce that any given subgroup H of G which is isomorphic to $L_4(4)$ must centralise a unique Sym(3) subgroup of G, say S, where the elements of order 3 in S lie in the class $3B$ of G. Now suppose that $g \in N_G(H)$. Then H must centralise S^g, whence $g \in N_G(S)$ by the uniqueness of S. Thus $N_G(H) \leq N_G(3B)$, and after consulting Table 4.2 we deduce that no automorphism group of $L_4(4)$ can be a maximal subgroup of G. □

Lemma 4.22. *Suppose that H is such that* $\mathrm{Soc}(H) \cong G_2(8)$. *Then H is not a subgroup of G.*

Proof. Again we wish to use character restriction to determine the possible embeddings of H in G. As the $\mathbb{F}_2$-character table of $G_2(8)$ is not currently available in the literature, we produce the portion given in Table A.33, along with the additional values listed below, as follows. We first construct $G_2(8)$ as a permutation group using standard Magma commands, and then construct its permutation module over $\mathbb{F}_2$. By decomposing this module with the Meataxe it is then possible to produce $\mathbb{F}_2$-representations of $G_2(8)$ acting on irreducible modules of dimensions 18, 42 and 108, from which it is straightforward to calculate the necessary Brauer character values. Consultation with [**36**] confirms that these are the only non-trivial irreducible modules for $G_2(8)$ of dimension at most 132 which are realisable over $\mathbb{F}_2$. For further details on the 2-modular character table of $G_2(8)$ we refer the reader to [**7**].

Now using Table A.33 we find that the only possible character restriction of H to V is $2\chi_1 + 3\chi_2$. In particular, this implies that $3A$-elements in H must fuse to $3A$-elements in G. We now consider possible $\mathbb{F}_2$-character restrictions of H to V_{132}. The Brauer character values on elements of order 3 for the character χ_4 of degree 108 are as follows.

	$1A$	$3A$	$3B$
χ_4	108	27	0

Using these values, we find that there is only one possible $\mathbb{F}_2$-character restriction of H to V_{132} whose $3A$-elements fuse to $3A$-elements in G, namely $6\chi_1 + 3\chi_3$. However, evaluating this on the $7A$-elements gives a value of 27, a contradiction. Thus $G_2(8)$ cannot be a subgroup of G. □

LEMMA 4.23. *Suppose that $H \leq G$ with $\mathrm{Soc}(H) \cong U_4(2)$. Then H is not a maximal subgroup of G.*

Proof. *(Electronic files folder* `/U4(2)`*)* Suppose that $H \cong U_4(2)$. Using Table A.25, along with our knowledge of the cubes of elements of order 9 in G from Lemma 3.34, we see the following possible character restrictions of H to V.

(i) $8\chi_2 + \chi_3$
(ii) $4\chi_1 + 4\chi_2 + 2\chi_4$
(iii) $12\chi_1 + \chi_2 + 4\chi_3$

The fusion from $U_4(2)$ to G in these three cases is given below.

$U_4(2)$		$3A$	$3B$	$3C$	$3D$	$5A$	$9A$	$9B$
	(i)	$3A$	$3A$	$3B$	$3D/3E$	$5A$	$9A/9C$	$9A/9C$
G	(ii)	$3E$	$3E$	$3B$	$3D/3E$	$5A$	$9E$	$9E$
	(iii)	$3E$	$3E$	$3D/3E$	$3B$	$5A$	$9D$	$9D$

We have $\dim(H^1(U_4(2), \chi_2)) = 0$, $\dim(H^1(U_4(2), \chi_3)) = 2$ and $\dim(H^1(U_4(2), \chi_4)) = 1$. In particular we deduce that if H is as in cases (ii) or (iii) then H must fix a non-zero vector of V. In these cases, if $\mathrm{Aut}(H) \sim U_4(2) : 2$ is a subgroup of G, then by Lemma 2.4 this subgroup will also fix a non-zero vector of V.

Therefore we may assume that H is as in case (i). Note that our knowledge of the possible fusion in this case is only partial and we cannot uniquely determine it for the $3D$, $3E$, $9A$ and $9B$-elements of $U_4(2)$ using only the analysis on the module V. To this end we now calculate the possible character restrictions of $U_4(2)$ to V_{132} in exactly the same way as above. We get four cases.

(i) $14\chi_1 + \chi_2 + 8\chi_4$
(ii) $18\chi_1 + 6\chi_2 + 8\chi_3 + \chi_4$
(iii) $8\chi_1 + 2\chi_2 + \chi_3 + \chi_5 + \chi_6$
(iv) $6\chi_1 + 9\chi_2 + 2\chi_3 + 4\chi_4$

The fusion from $U_4(2)$ to G in these cases is tabulated below.

$U_4(2)$		$3A$	$3B$	$3C$	$3D$	$5A$	$9A$	$9B$
	(i)	$3A$	$3A$	$3B$	$3D$	$5A$	$9A$	$9A$
G	(ii)	$3E$	$3E$	$3C/3D$	$3B$	$5A$	$9D$	$9D$
	(iii)	$3E$	$3E$	$3C/3D$	$3C/3D$	$5A$	$9D$	$9D$
	(iv)	$3E$	$3E$	$3B$	$3E$	$5A$	$9E/9F$	$9E/9F$

Comparison of the two sets of restrictions to the modules V and V_{132} immediately specifies the possible fusion uniquely in each case. Furthermore it reveals that case (iii) for V_{132} cannot exist.

The case which remains then has the following possible fusion of elements of $U_4(2)$ in G.

$U_4(2)$	$3A$	$3B$	$3C$	$3D$	$5A$	$9A$	$9B$
G	$3A$	$3A$	$3B$	$3D$	$5A$	$9A$	$9A$

This is represented as $8\chi_3 + \chi_2$ on V. A subgroup $U_4(2)$ can be generated as follows. We take an element z of order 9 and let x be its cube. Now in the centraliser in $U_4(2)$ of x we find an element y of order 3 which centralises x and is such that $\langle x, y\rangle \cong 3^2$. The main point here is then that the element y normalises the cyclic subgroup $\langle z\rangle$. Now there exists t which inverts y and yields $\langle z, y, t\rangle \cong U_4(2)$. Indeed we take $z \in 9A \cup 9B$ and so $x = z^3 \in 3A \cup 3B$. Now $|C_{U_4(2)}(x)| = 648$ and contains a $3D$-element r for which $|N_{U_4(2)}(\langle r\rangle)| = 108$. There are 15 involutions which invert r, composed of six in class $2A$ and nine in class $2B$. In the former case three of the involutions generate $U_4(2)$ whilst in the latter case they all do.

Hence our strategy is as follows. Take a $9A$-element z in G and let $x = z^3$. We use the method of [**8**] to generate $C_G(x) \sim 3 \cdot {}^2E_6(2).3$. Now we take a representative y from each class of elements of order 3, which fuse to $3D$-elements in G, in $3 \cdot {}^2E_6(2).3$ under the action of $C_G(z)$. We generate the normaliser $N_G(\langle y\rangle) \sim (3 \times \Omega_{10}^-(2)) : 2 \times \mathrm{Sym}(3)$ and run through the outer involutions in here to check for generation of $U_4(2)$.

We find from [**15**] that in $E = {}^2E_6(2).3$ there are eleven classes of elements of order 3. These have centraliser orders as follows.

X	$\lvert C_G(x)\rvert$
$3A$	$2^{15}.3^8.5.7.11$
$3B$	$2^{12}.3^8.5^2.7$
$3C$	$2^9.3^{10}$
$3D$	$2^{20}.3^7.5.7.11.17$
$3E$	$2^{20}.3^7.5.7.11.17$
$3F$	$2^{12}.3^6.7^2.13$
$3G$	$2^{12}.3^6.7^2.13$
$3H$	$2^{11}.3^7.5.11$
$3I$	$2^{11}.3^7.5.11$
$3J$	$2^9.3^6.7.19$
$3K$	$2^9.3^6.7.19$

Note that the structure of the centralisers can be deduced from [**15**] or from [**17**]. The classes $3A$, $3B$ and $3C$ lie in the derived group $E' \cong {}^2E_6(2)$ whilst the remaining classes lie outside E'. Now apart from the central classes (which do not interest us here) every element of order 3 in $3 \cdot {}^2E_6(2).3$ maps to an element of order 3 in the factor group ${}^2E_6(2).3$. Now we know that $|C_G(y)| = 2^{21}.3^8.5^2.7.11.17$ for $y \in 3D$ of G and this already eliminates the cases $3C$, $3F$, $3G$, $3J$ and $3K$ from above. Furthermore, from [**15**] we see that $C_E(3A) \cong 3 \times U_6(2)$ and so this cannot be a $3D$-element in G. However, determination of which of the remaining classes lie in the $3D$ class of G is more challenging. Thus, we reverse our point of view and look at the classes of $3A$-elements in $C_G(3D)$. Now from [**15**] we know that $\Omega_{10}^-(2)$ has classes $3A$, $3B$, $3C$, $3D$, $3E$ and $3F$, and hence $C_G(3D) \cong 3 \times \Omega_{10}^-(2) \times \mathrm{Sym}(3)$ has 27 classes of elements of order 3. Note that the classes in $\Omega_{10}^-(2)$ can all be distinguished by their centraliser sizes except for the classes $3B$ and $3C$ which fuse under the outer automorphism of $\Omega_{10}^-(2)$. We now take the group $C_G(3D)$ as a 56-dimensional matrix group over $\mathbb{F}_2$ and find the simple subgroup $\Omega_{10}^-(2)$ on its standard generators (from [**41**]). Now we take a copy $O \cong \Omega_{10}^-(2)$ on standard generators in its 495-degree permutation representation and construct an explicit isomorphism. Upon doing this we find that, for a given $x \in 3D$, there are three $C_G(x)$-conjugacy classes of elements of order 3 which centralise x. These correspond to the classes $(1, 3B, 3A)$, $(1, 3C, 3A)$ and $(3A, 3A, 1)$ in $C_G(x) \cong 3 \times \Omega_{10}^-(2) \times \mathrm{Sym}(3)$. We note further that since $3B$ and $3C$-elements in $\Omega_{10}^-(2)$ have centraliser of order $2^{10}.3^6.5.11$ and $3A$-elements in $\Omega_{10}^-(2)$ have centraliser of order $2^{12}.3^6.5^2.7$, comparison with the table of elements of order 3 in ${}^2E_6(2).3$ tells us that the $3D$-elements in $C_G(3A)$ must factor to $3H$, $3I$ and $3B$ elements. This now allows us to carry out the computational analysis. We find that every subgroup $U_4(2)$ in G which appear on V as in case (i) embeds in a subgroup $Sp_6(2)$ and hence is non-maximal. Moreover, since $\mathrm{Aut}(U_4(2)) \sim U_4(2):2$ also lies in $Sp_6(2)$, such subgroups must also be non-maximal. □

Lemma 4.24. *Suppose that $H \leq G$ with $\mathrm{Soc}(H) \cong Sp_6(2)$. Then H is not a maximal subgroup of G.*

Proof. Using Table A.29, there are the following possible character restrictions of H to V.
(i) $12\chi_1 + 2\chi_2 + 4\chi_3$
(ii) $4\chi_1 + 4\chi_2 + 2\chi_4$
(iii) $8\chi_2 + \chi_3$.
Now
$$\dim(H^1(Sp_6(2), \chi_2)) = 1, \dim(H^1(Sp_6(2), \chi_3)) = 0$$
and
$$\dim(H^1(Sp_6(2), \chi_4)) = 0.$$
Therefore in cases (i) and (ii) H must fix a non-zero vector of V. Now consider case (iii). For this character restriction we have that elements of order 7 in H must lie in the class $7B$ of G.

From [**15**] we see that $Sp_6(2)$ contains a subgroup $L_2(8)$, and so this subgroup $L \leq H$ must stabilise either a 6-dimensional H-submodule of V corresponding to χ_2, or an 8-dimensional H-submodule of V corresponding to χ_3. These are also irreducible L-modules, and we have previously studied such situations in Lemma 4.20. Taking any involution t in this subgroup $L \leq H$, we have that $|C_H(t)| = 284$, and $C_H(t)$ contains an elementary abelian subgroup of order 2^6. We may now use the information in Tables 5 and 7 from Lemma 4.20. In all cases we see that the subgroup orders listed in these tables are either not divisible by 2^6, or do not divide 284. In the former case, since these subgroups contains all the involutions of $C_G(t) \cap \mathrm{Stab}_G(W)$, we deduce that H is not a subgroup of $\mathrm{Stab}_G(W)$. In the latter case we deduce that $H \neq \mathrm{Stab}_G(W)$ and so H is not a maximal subgroup of G. □

5. *Almost Simple Subgroups not in* Lie(2)

Lemma 5.1. *Suppose that H is isomorphic to $L_2(29)$ or $L_2(37)$. Then H is not a subgroup of G.*

Proof. This follows immediately by Lagrange's theorem. □

The dimensions of the cohomology groups referred to in this section have again been calculated using Magma's `CohomologicalDimension` command.

Lemma 5.2. *Suppose that H is isomorphic to $L_2(19)$, $L_2(27)$, $L_4(3)$, $U_4(3)$, $\Omega_7(3)$, $G_2(3)$, $\mathrm{Alt}(13)$ or J_2. Then H is not a subgroup of G.*

Proof. Using Tables A.6, A.16, A.31, A.42 and A.45, along with the fact (which may be easily verified in Magma) that $\Omega_7(3)$ has no non-trivial $\mathbb{F}_2$-characters of degree at most 56, we see that in all but one case there are no possible character restrictions of H to V. The outstanding case is where $H \cong L_2(27)$. Here, Table A.8 reveals that the only possible $\mathbb{F}_2$-character restriction of H to V is then $4\chi_1 + 2\chi_2$. We have $\dim(H^1(L_2(27), \chi_2)) = 2$, and hence any such subgroup must fix a non-zero vector of V. However, inspection of the vector stabilisers in Proposition 2.2 together with [**15**] reveals that this in not possible, so $H \not\leq G$. □

Lemma 5.3. *Suppose that $H \leq G$ with $\mathrm{Soc}(H) \cong \mathrm{Alt}(5)$. Then H fixes a non-zero vector of V.*

Proof. With Table A.34 to hand we see that the only character restrictions of Alt(5) to V are

$$16\chi_1 + 2\chi_2 + 8\chi_3$$

and

$$16\chi_1 + 8\chi_2 + 2\chi_3.$$

The Loewy structure of the projective indecomposable modules for Alt(5) in characteristic 2 can be found in [**9**]. Using this structure we see that, in particular, $\dim(H^1(\mathrm{Alt}(5), \chi_2)) = 2$ and the character χ_3 is projective. Thus H must fix a non-zero vector of V, and the result follows using Lemma 2.4. □

Lemma 5.4. *Suppose that $H \leq G$ with $H \cong \mathrm{Alt}(6)$. Then H fixes a non-zero vector of V.*

Proof. Using Table A.35, there are three possible $\mathbb{F}_2$-character restrictions of H to V, namely

(i) $8\chi_1 + 2\chi_2 + 2\chi_3 + 2\chi_4$
(ii) $16\chi_1 + 2\chi_2 + 8\chi_3$
(iii) $16\chi_1 + 8\chi_2 + 2\chi_3$.

We find that

$$\dim(H^1(\mathrm{Alt}(6), \chi_2)) = 1, \dim(H^1(\mathrm{Alt}(6), \chi_3)) = 1$$

and

$$\dim(H^1(\mathrm{Alt}(6), \chi_4)) = 0.$$

Therefore in each case we see that H must fix a non-zero vector of V. Since H has index 4 in Aut(Alt(6)) we apply Lemma 2.4 to deduce the result. □

Lemma 5.5. *Suppose $H \leq G$ is such that $\mathrm{Soc}(H) \cong \mathrm{Alt}(7)$. Then H is not a maximal subgroup of G.*

Proof. *(Electronic files folder* `/Alt(7)`*)* The information in Table A.36 allows the following four possible character restrictions:

(i) $\chi_2 + \chi_3 + 8\chi_4$
(ii) $4\chi_1 + 2\chi_4 + 2\chi_6$
(iii) $4\chi_1 + 4\chi_4 + 2\chi_5$
(iv) $12\chi_1 + 4\chi_2 + 4\chi_3 + 2\chi_4$.

We have that $\dim(H^1(\mathrm{Alt}(7), \chi_i)) = 0$ for $i = 2, 3, 4$, while $\dim(H^1(\mathrm{Alt}(7), \chi_5)) = \dim(H^1(\mathrm{Alt}(7), \chi_6)) = 1$. We therefore see that in cases (ii), (iii) and (iv) above such a subgroup Alt(7) must fix a vector or hyperplane.

Consider then case (i). The Brauer character values obtained in this case are as follows for the five non-trivial conjugacy classes of odd order in Alt(7).

$3A$	$3B$	$5A$	$7A$	$7B$
20	2	6	−7	−7

Hence we see that the fusion of the classes $3A$, $5A$, $7A$ and $7B$ in Alt(7) to G is $3B$, $5A$, $7B$ and $7B$, respectively, with the only ambiguity being whether or not $3B$ in Alt(7) fuses to $3D$ or $3E$ in G. To resolve this issue we look at the possible $\mathbb{F}_2$-representations on the module V_{132} with the fusion of conjugacy classes as given above. Now if $3B$-elements in Alt(7) fuse to $3E$ in G then we find the only possible solution is

$$14\chi_1 - 3\chi_2 - 3\chi_3 - 3\chi_4 + 10\chi_5 + \chi_6$$

which is a virtual character and hence such subgroups Alt(7) do not exist in G. However, if $3B$-elements in Alt(7) fuse to $3D$ in G then we find a possible solution is

$$14\chi_1 + \chi_4 + 8\chi_5$$

and so this is the only fusion pattern we need to check.

Note that both Alt(7) and Sym(7) may be generated by taking a Frobenius subgroup $\langle z, x\rangle \cong 7:3$ and an involution t which inverts x. We may therefore proceed in a similar manner to that of Lemma 4.17, the only difference being that of course we check whether $\langle z, x, t\rangle \cong \mathrm{Alt}(7)$ or Sym(7), rather than $L_3(2)$. We find that there is a single $N_G(\langle z, x\rangle)$-conjugacy class of involutions which generate an Alt(7) of the type in case (i), and this subgroup Alt(7) is in fact unique. These involutions are in the G-conjugacy class $2B$, and for such an $H \cong \mathrm{Alt}(7)$ we find $C_G(H) \sim U_3(3):2$. Thus H is not maximal in G. Furthermore, there are $2D$-involutions which generate a group isomorphic to Sym(7). Let $H \leq G$ be such a subgroup. Then we either have $C_G(H) \sim U_3(3):2$ or $C_G(H)$ is a soluble group of order 192. Again we deduce that H is not a maximal subgroup of G, which completes the proof. □

Corollary 5.6. *There is only a single G-conjugacy class of subgroups* Alt(7) *which are represented as in case (i) on the module V.*

Lemma 5.7. *Suppose that $H \leq G$ with* $\mathrm{Soc}(H) \cong \mathrm{Alt}(8)$. *Then H fixes a non-zero vector of V.*

Proof. *(Electronic files folder* `/Alt(8)`*)* Using Table A.37 we have the following potential character restrictions:

(i) $\chi_2 + \chi_3 + 8\chi_4$
(ii) $4\chi_1 + 2\chi_4 + 2\chi_7$
(iii) $4\chi_1 + 2\chi_4 + 2\chi_6$
(iv) $4\chi_1 + 2\chi_4 + \chi_6 + \chi_7$
(v) $4\chi_1 + 4\chi_4 + 2\chi_5$
(vi) $12\chi_1 + 4\chi_2 + 4\chi_3 + 2\chi_4$

We can immediately eliminate the cases (ii) and (iii) from being potential character restiction because the two modules corresponding to the 20-dimensional character are not self-dual. We have $\dim(H^1(\mathrm{Alt}(8), \chi_2)) = 0$, $\dim(H^1(\mathrm{Alt}(8), \chi_3)) = 0$, $\dim(H^1(\mathrm{Alt}(8), \chi_4)) = 1$, $\dim(H^1(\mathrm{Alt}(8), \chi_5)) = 1$, $\dim(H^1(\mathrm{Alt}(8), \chi_6)) = 1$ and $\dim(H^1(\mathrm{Alt}(8), \chi_7)) = 1$, and we deduce that in the cases (iv) and (vi) any subgroup Alt(8) must fix a vector or hyperplane. We are left to deal with the cases (i) and (v).

Let us look firstly at case (v). Denote by P_{χ_4} and P_{χ_5} the projective indecomposable modules associated to the two characters χ_4 and χ_5. The Loewy structure of these modules can be found in [**10**], and we see that the modules V_{χ_4} and V_{χ_5} associated to χ_4 and χ_5, respectively, can both be extended by exactly one trivial module. Also, the third socle layer of P_{χ_4} contains three modules isomorphic to V_{χ_4} and none isomorphic to V_{χ_5}, whilst the third socle layer of P_{χ_5} contains one module isomorphic to V_{χ_5} and none isomorphic to V_{χ_4}. We can deduce from this that any subgroup Alt(8) of G in case (v) must fix a non-zero vector of V.

Now we shall examine case (i). The Brauer character values obtained in this case for the non-trivial conjugacy classes of odd order in Alt(8) are given below.

$3A$	$3B$	$5A$	$7A$	$7B$	$15A$	$15B$
20	2	6	−7	−7	−15	−15

Now Alt(8) contains a unique conjugacy class of maximal subgroups Alt(7), and it is clear in this case that these subgroups are of type (i) in the analysis of Lemma 5.5. Thus we may use the results from the proof of Lemma 5.5 in the following way. If we take an element of order 5 from a representative subgroup Alt(7) and generate its centraliser in Alt(8), we find there are elements of order 3 which, together with the subgroup Alt(7), must generate Alt(8). We carry out this calculation in G and find that, even though such subgroups exist, they are not maximal subgroups in G. A more detailed explanation of the similar (but more involved) calculation for Sym(8) is given below.

Note that Sym(8) can be generated as follows. We take an alternating group $A \cong \mathrm{Alt}(7)$ and an element z of order 5 in A. Then there exist involutions t which commute with z and for which $\langle A, t\rangle \cong \mathrm{Sym}(8)$.

Hence we let A be a fixed Alt(7) and choose an element $z \in A$ of order 5. Since there exists $s \in A$ which inverts z we can use [**8**] to generate

$$C_G(z) = \langle z\rangle \times O \times S \cong 5 \times \Omega_8^-(2) \times \mathrm{Sym}(3).$$

Using [**15**] we see that $C_G(z)$ contains seven classes of involutions. We now want to run through all involutions t in $C_G(z)$ to determine which of these yields $\langle A, t\rangle \cong \mathrm{Sym}(8)$. Now $\Omega_8^-(2)$ has three classes of involutions $2A$, $2B$ and $2C$ and these classes have sizes 1071, 4284 and 64260, respectively, and of course Sym(3) has a single class of involutions of size 3. Hence there are seven classes of involutions g in $C_G(x)$ which are as follows.

g	$\lvert g^{C_G(x)}\rvert$	Class in G
$(1,1,2A)$	3	$2A$
$(1,2A,1)$	1071	$2A$
$(1,2B,1)$	4284	$2B$
$(1,2C,1)$	64260	$2D$
$(1,2A,2A)$	3213	$2B$
$(1,2B,2A)$	12852	$2C$
$(1,2C,2A)$	192780	$2E$

We now run through all the involutions in $C_G(x)$, and discover that the only classes which yields involutions which generate a subgroup Sym(8) are $(1,1,2A)$ and $(1,2C,2A)$. All three involutions in class $(1,1,2A)$ generate a subgroup Sym(8) and in each case the group is centralised by an element of order 3 in G, while the subgroups Sym(8) resulting from suitable involutions in the class $(1,2C,2A)$ have soluble centralisers of order 48 in G. □

LEMMA 5.8. *Suppose $H \leq G$ with* $\mathrm{Soc}(H) \cong \mathrm{Alt}(9)$. *Then H fixes a non-zero vector of V.*

Proof. Suppose that $H \cong \mathrm{Alt}(9)$. From Table A.38, there are only two possible character restrictions of H to V.

(i) $8\chi_1 + 2\chi_2 + 2\chi_3 + 2\chi_4$
(ii) $4\chi_1 + 2\chi_7$

We find that

$$\dim(H^1(\mathrm{Alt}(9), \chi_2)) = 0, \dim(H^1(\mathrm{Alt}(9), \chi_3)) = 0$$

and

$$\dim(H^1(\mathrm{Alt}(9), \chi_7)) = 2,$$

so we deduce that in both cases (i) and (ii) the subgroup H must fix a vector or hyperplane of V. To complete the proof for the case $H \cong \mathrm{Sym}(9)$ we apply Lemma 2.4. □

LEMMA 5.9. *Suppose that $H \leq G$ with $\mathrm{Soc}(H) \cong \mathrm{Alt}(10)$. Then H fixes a non-zero vector of V.*

Proof. Suppose that $H \cong \mathrm{Alt}(10)$. Table A.39 yields the following possible character restrictions.

(i) $8\chi_1 + 2\chi_2 + 2\chi_3$
(ii) $4\chi_1 + 2\chi_4$.

We have

$$\dim(H^1(\mathrm{Alt}(10), \chi_2)) = 1, \dim(H^1(\mathrm{Alt}(10), \chi_3)) = 0$$

and

$$\dim(H^1(\mathrm{Alt}(10), \chi_4)) = 1.$$

Therefore in both cases (i) and (ii) we see that H must fix a vector or hyperplane. Now if $H \cong \mathrm{Sym}(10)$ we apply Lemma 2.4. □

LEMMA 5.10. *Suppose that $H \leq G$ with $\mathrm{Soc}(H) \cong \mathrm{Alt}(11)$. Then H fixes a non-zero vector of V.*

Proof. Suppose that $H \cong \mathrm{Alt}(11)$. Using Table A.40 we see only one possible character restriction, namely

$$4\chi_1 + 2\chi_2 + \chi_3,$$

and we have

$$\dim(H^1(\mathrm{Alt}(11), \chi_2)) = \dim(H^1(\mathrm{Alt}(11), \chi_3)) = 0.$$

Therefore we deduce that H must fix a non-zero vector of V, and using Lemma 2.4 for the case $H \cong \mathrm{Sym}(11)$ completes the proof. □

LEMMA 5.11. *Suppose that $H \leq G$ with $\mathrm{Soc}(H) \cong \mathrm{Alt}(12)$. Then H fixes a non-zero vector of V.*

Proof. Suppose that $H \cong \mathrm{Alt}(12)$. Using Table A.41 we again see only one possible character restriction, namely

$$4\chi_1 + 2\chi_2 + \chi_3.$$

Since $\dim(H^1(\mathrm{Alt}(12),\chi_2)) = 1$ and $\dim(H^1(\mathrm{Alt}(12),\chi_3)) = 0$, we deduce that H must fix a non-zero vector of V. Now we apply Lemma 2.4 for the case $H \cong \mathrm{Sym}(11)$ to complete the proof. □

Lemma 5.12. *Suppose that $H \leq G$ with $\mathrm{Soc}(H) \cong L_2(11)$. Then H fixes a non-zero vector of V.*

Proof. If $H \cong L_2(11)$, then Table A.2 implies that the only possible character restriction of H to V is $6\chi_1 + 3\chi_2 + 2\chi_3$. We have $\dim(H^1(L_2(11),\chi_2)) = 2$ and $\dim(H^1(L_2(11),\chi_3)) = 0$, and so H must fix a vector or hyperplane. By Lemma 2.4 any subgroup $L_2(11):2$ of G must also fix a vector or hyperplane. □

Lemma 5.13. *Suppose that $H \leq G$ with $\mathrm{Soc}(H) \cong L_2(13)$. Then H fixes a non-zero vector of V.*

Proof. If $H \cong L_2(13)$, then with Table A.3 to hand we find the only possible character restriction of H to V is $4\chi_1 + 2\chi_2 + 2\chi_3$. Since $\dim(H^1(L_2(13),\chi_2)) = 2$ and $\dim(H^1(L_2(13),\chi_3)) = 0$, we see that H must fix a vector or hyperplane. By Lemma 2.4 any subgroup $L_2(13):2$ of G must also fix a vector or hyperplane. □

Lemma 5.14. *Suppose that $H \leq G$ with $\mathrm{Soc}(H) \cong L_2(17)$. Then H fixes a non-zero vector of V.*

Proof. Using Table A.5 there are four possible character restrictions of H to V.

(i) $8\chi_1 + 2\chi_3 + 2\chi_4$
(ii) $8\chi_1 + 2\chi_2 + 2\chi_4$
(iii) $8\chi_1 + 2\chi_2 + 4\chi_3$
(iv) $8\chi_1 + 4\chi_2 + 2\chi_3$

We have

$$\dim(H^1(L_2(17),\chi_2)) = 1, \dim(H^1(L_2(17),\chi_3)) = 1$$

and

$$\dim(H^1(L_2(17),\chi_4)) = 0,$$

and so H must fix a vector or hyperplane. By Lemma 2.4 any subgroup $L_2(17):2$ of G must also fix a vector or hyperplane. □

Lemma 5.15. *Suppose that $H \leq G$ with $\mathrm{Soc}(H) \cong L_2(25)$. Then H fixes a non-zero vector of V.*

Proof. If $H \cong L_2(25)$, then Table A.7 yields only one possible character restriction of H to V is $4\chi_1 + 2\chi_4$. Now $\dim(H^1(L_2(25),\chi_4)) = 0$, and so H must fix a vector or hyperplane. Since $\mathrm{Aut}(L_2(25)) \sim L_2(25):2^2$, by Lemma 2.4 any subgroup of $\mathrm{Aut}(L_2(25))$ must also fix a vector or hyperplane. □

Lemma 5.16. *Suppose that $H \leq G$ with $\mathrm{Soc}(H) \cong L_3(3)$. Then H fixes a non-zero vector of V.*

Proof. From Table A.12 the only possible character restriction is $4\chi_1 + 2\chi_4$. Observing that $\dim(H^1(L_3(3), \chi_4)) = 1$ we see that H must fix a vector or hyperplane of V. Since $\mathrm{Aut}(L_3(3)) \sim L_3(3) : 2$, the result follows using Lemma 2.4. □

LEMMA 5.17. *Suppose that $H \leq G$ with $\mathrm{Soc}(H) \cong U_3(3)$. Then H fixes a non-zero vector of V.*

Proof. Suppose $H \cong U_3(3)$. The information in Table A.19 allows three possible character restictions of H to V, which are

(i) $2\chi_1 + 9\chi_2$
(ii) $4\chi_1 + 4\chi_2 + 2\chi_3$
(iii) $20\chi_1 + 6\chi_2$

Since $\dim(H^1(U_3(3), \chi_2)) = 1$ and $\dim(H^1(U_3(3), \chi_3)) = 0$ we see that in cases (ii) and (iii) H must fix a non-zero vector of V. Suppose then that H is as in case (i). Note that $U_3(3)$ contains a maximal subgroup $L_3(2)$, and as in [**25**] we observe that $U_3(3)$ can be generated by this subgroup along with an element of order 3 which centralises a subgroup Sym(3) of this $L_3(2)$. The character restriction in case (ii) implies that this $L_3(2)$ subgroup contains a Frobenius subgroup of type $7C : 3D$ or $7C : 3E$. From Lemma 4.16 we know that $L_3(2)$ subgroups of the latter type do not exist in G, while in the former case we have constructed representative $L_3(2)$ subgroups in the proof of Lemma 4.15. Using these representatives, and the generation method described above, we check that any such $U_3(3)$ subgroup must fix a vector in V. Since $\mathrm{Aut}(U_3(3)) \sim U_3(3) : 2$, we apply Lemma 2.4 to complete the proof. □

LEMMA 5.18. *Suppose that $H \leq G$ with $\mathrm{Soc}(H) \cong M_{11}$. Then H fixes a non-zero vector of V.*

Proof. Using Table A.43 we have only one possible character restriction, namely $4\chi_1 + 2\chi_2 + \chi_3 + \chi_4$. However

$$\dim(H^1(M_{11}, \chi_2)) = 1, \dim(H^1(M_{11}, \chi_3)) = 0$$

and

$$\dim(H^1(M_{11}, \chi_4)) = 1,$$

whence the result follows. □

LEMMA 5.19. *Suppose that $H \leq G$ with $\mathrm{Soc}(H) \cong M_{12}$. Then H fixes a non-zero vector of V.*

Proof. Using Table A.44, the only possible character restriction is $4\chi_1 + 2\chi_2 + \chi_3 + \chi_4$. Since $\dim(H^1(M_{12}, \chi_2)) = 2$, $\dim(H^1(M_{12}, \chi_3)) = 0$ and $\dim(H^1(M_{12}, \chi_4)) = 1$ we see that H must fix a non-zero vector of V. The result follows from Lemma 2.4, since $\mathrm{Aut}(M_{12}) \sim M_{12} : 2$. □

In Section 4 we have eliminated all the groups H with $F^*(H)$ in List 1(i)-(iii), with the exception of the case $F^*(H) \cong L_2(128)$. All groups H with $F^*(H)$ in List 1(iv) are ruled out in Section 5. As a consequence the proof of Theorem 1.1 is complete.

Appendix A. $\mathbb{F}_2$-*character tables*

Here we list certain irreducible $\mathbb{F}_2$-character values for various groups referred to in the main body of the paper. In each case the information is produced using either [**24**], MAGMA or GAP (or a combination). Note that the characters listed are not necessarily absolutely irreducible. Unless otherwise stated, the notation used in these tables follows [**24**].

TABLE A.1.

$\mathbf{L_2(8)}$	$1A$	$3A$	$7ABC$	$9ABC$
χ_1	1	1	1	1
χ_2	6	−3	−1	0
χ_3	8	−1	1	−1
χ_4	12	3	−2	−3

TABLE A.2.

$\mathbf{L_2(11)}$	$1A$	$3A$	$5A$	$5B$	$11A$	$11B$
χ_1	1	1	1	1	1	1
χ_2	10	−2	0	0	−1	−1
χ_3	10	1	0	0	−1	−1
χ_4	24	0	−1	−1	2	2

TABLE A.3.

$\mathbf{L_2(13)}$	$1A$	$3A$	$7ABC$	$13AB$
χ_1	1	1	1	1
χ_2	12	0	−2	−1
χ_3	14	−1	0	1
χ_4	36	0	1	−3

TABLE A.4.

$\mathbf{L_2(16)}$	$1A$	$3A$	$5A$	$15ABCD$
χ_1	1	1	1	1
χ_2	8	−4	−2	1
χ_3	8	2	3	−3
χ_4	16	4	−4	−1
χ_5	16	1	1	1
χ_6	32	−4	2	−4

TABLE A.5.

$\mathbf{L_2(17)}$	$1A$	$3A$	$9ABC$
χ_1	1	1	1
χ_2	8	−1	−1
χ_3	8	−1	−1
χ_4	16	−2	1
χ_5	48	3	0

Table A.6.

$\mathbf{L_2(19)}$	$1A$	$3A$	$5AB$	$9ABC$
χ_1	1	1	1	1
χ_2	18	0	-2	0
χ_3	20	2	0	-1
χ_4	36	0	1	0
χ_5	60	-3	0	0

Table A.7.

$\mathbf{L_2(25)}$	$1A$	$3A$	$3B$	$3C$	$3D$	$5A$
χ_1	1	1	1	1	1	1
χ_2	26	-1	-1	8	-1	1
χ_3	26	-1	8	-1	-1	1
χ_4	38	11	2	2	2	-2

Table A.8.

$\mathbf{L_2(27)}$	$1A$	$3AB$	$7ABC$
χ_1	1	1	1
χ_2	26	-1	-1

Table A.9.

$\mathbf{L_2(32)}$	$1A$	$3A$	$11ABCDE$
χ_1	1	1	1
χ_2	10	-5	-1
χ_3	20	5	-2
χ_4	20	5	-2
χ_5	32	-1	-1
χ_6	40	-5	-4
χ_7	40	-5	7

Table A.10.

$\mathbf{L_2(32):5}$	$1A$	$3A$	$5A$	$11ABCDE$
χ_1	1	1	1	1
χ_2	4	4	-1	4
χ_2	10	-5	0	-1
χ_3	20	5	0	-2
χ_4	20	5	0	-2
χ_5	32	-1	2	-1
χ_6	40	-5	0	-4
χ_7	40	-5	0	7

TABLE A.11.

$\mathbf{L_3(2)}$	$1A$	$3A$	$7A$	$7B$
χ_1	1	1	1	1
χ_2	3	0	$b7$	$*$
χ_3	3	0	$*$	$b7$
χ_4	8	-1	1	1

TABLE A.12.

$\mathbf{L_3(3)}$	$1A$	$3A$	$3B$	$13ABCD$
χ_1	1	1	1	1
χ_2	12	3	0	-1
χ_3	26	-1	-1	0

TABLE A.13.

$\mathbf{L_3(4)}$	$1A$	$3A$	$5A$	$7A$
χ_1	1	1	1	1
χ_2	9	0	-1	$b7-1$
χ_3	9	0	-1	$**$
χ_4	16	-2	1	2

TABLE A.14.

$\mathbf{L_3(8)}$	$1A$	$3A$	$7IJK$	$9ABC$
χ_1	1	1	1	1
χ_2	9	0	2	3
χ_3	9	0	2	3
χ_4	24	-3	3	6
χ_5	27	0	-1	0
χ_6	27	0	-1	0
χ_7	27	0	-1	0
χ_8	27	0	-1	0
χ_9	27	0	-1	-3
χ_{10}	27	0	-1	-3

Table A.15.

$\mathbf{L_3(8):3}$	$1A$	$3A$	$7IJK$	$9ABC$	$3B$
χ_1	1	1	1	1	1
χ_2	2	2	2	2	−1
χ_3	9	0	2	3	0
χ_4	9	0	2	3	0
χ_5	24	−3	3	6	0
χ_6	27	0	−1	0	0
χ_7	27	0	−1	0	0
χ_8	27	0	−1	0	0
χ_9	27	0	−1	0	0
χ_{10}	27	0	−1	−3	3
χ_{11}	27	0	−1	−3	3
χ_{12}	54	0	−2	−6	−3
χ_{13}	54	0	−2	−6	−3

Table A.16.

$\mathbf{L_4(3)}$	$1A$	$3A$	$3B$	$3C$	$3D$	$5A$	$9A$	$9B$
χ_1	1	1	1	1	1	1	1	1
χ_2	26	−1	−1	8	−1	1	2	−1
χ_3	26	−1	8	−1	−1	1	−1	2
χ_4	38	11	2	2	2	−2	−1	−1

Table A.17.

$\mathbf{L_4(4)}$	$1A$	$3AB$	$3C$	$3D$	$5AB$	$5CD$	$5E$	$7A$	$7B$	$9AB$
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	8	−1	−4	2	−2	3	−2	1	1	−1
χ_3	8	−1	−4	2	−2	3	−2	1	1	−1
χ_4	12	−6	6	0	7	2	2	−2	−2	0
χ_5	16	7	4	1	−4	1	1	a	a'	1
χ_6	16	7	4	1	−4	1	1	a'	a	1
χ_7	28	10	4	−2	8	3	−2	0	0	−2
χ_8	32	−13	8	2	−8	2	2	4	4	−1
χ_9	36	9	9	0	11	−4	1	1	1	0
χ_{10}	40	4	−8	−2	−10	5	0	−2	−2	1
χ_{11}	40	4	−8	−2	−10	5	0	−2	−2	1
χ_{12}	48	3	−12	0	−2	−2	−2	−1	−1	0
χ_{13}	48	3	−12	0	−2	−2	−2	−1	−1	0

where $a = \omega^4 + \omega^2 + \omega - 1$ for a 7-th root of unity ω, and a' is the complex conjugate of a.

TABLE A.18.

$\mathbf{U_3(4)}$	$1A$	$3A$	$5ABCD$	$5EF$	$13ABCD$	$15ABCD$
χ_1	1	1	1	1	1	1
χ_2	12	0	−3	2	−1	0
χ_3	16	−2	6	1	3	3
χ_4	36	0	1	−4	−3	−5

TABLE A.19.

$\mathbf{U_3(3)}$	$1A$	$3A$	$3B$	$7A$	$7B$
χ_1	1	1	1	1	1
χ_2	6	−3	0	−1	−1
χ_3	14	5	−1	0	0
χ_4	32	−4	−1	$-b7$	**
χ_5	32	−4	−1	**	$-b7$

TABLE A.20.

$\mathbf{U_3(8)}$	$1A$	$3AB$	$3C$	$7ABC$	$9ABC$
χ_1	1	1	1	1	1
χ_2	24	6	−3	3	6
χ_3	54	−9	0	−2	0
χ_4	54	9	0	−2	−6

TABLE A.21.

$\mathbf{U_3(8):3_1}$	$1A$	$3AB$	$3C$	$7ABC$	$9ABC$
χ_1	1	1	1	1	1
χ_2	2	2	2	2	2
χ_3	24	6	−3	3	6
χ_4	54	−9	0	−2	0
χ_5	54	9	0	−2	−6
χ_6	54	9	0	−2	−6
χ_7	54	9	0	−2	−6

TABLE A.22.

$\mathbf{U_3(8):3_2}$	$1A$	$3AB$	$3C$	$7ABC$	$9ABC$
χ_1	1	1	1	1	1
χ_2	2	2	2	2	2
χ_3	24	6	−3	3	6
χ_4	48	12	−6	6	12
χ_5	54	−9	0	−2	−6
χ_6	54	−9	0	−2	−6
χ_7	54	−9	0	−2	−6

Table A.23.

$\mathbf{U_3(8).3_3}$	$1A$	$3AB$	$3C$	$7ABC$	$9ABC$
χ_1	1	1	1	1	1
χ_2	2	2	2	2	2
χ_3	24	6	−3	3	6
χ_4	54	−9	0	−2	−6

Table A.24.

$\mathbf{U_3(8).3^2}$	$1A$	$3AB$	$3C$	$7ABC$	$9ABC$
χ_1	1	1	1	1	1
χ_2	2	2	2	2	2
χ_3	2	2	2	2	2
χ_4	2	2	2	2	2
χ_5	2	2	2	2	2
χ_6	24	6	−3	3	6
χ_7	48	12	−6	6	12
χ_8	54	−9	0	−2	−6
χ_9	54	−9	0	−2	−6
χ_{10}	54	−9	0	−2	−6

Table A.25.

$\mathbf{U_4(2)}$	$1A$	$3A$	$3B$	$3C$	$3D$	$5A$	$9A$	$9B$
χ_1	1	1	1	1	1	1	1	1
χ_2	6	−3	−3	3	0	1	0	0
χ_3	8	−1	−1	−4	2	−2	−1	−1
χ_4	14	5	5	2	−1	−1	−1	−1
χ_5	40	4	4	−8	−2	0	1	1
χ_6	64	−8	−8	4	−2	−1	1	1

Table A.26.

$\mathbf{U_4(3)}$	$1A$	$3A$	$3B$	$3C$	$3D$	$5A$	$7A$	$7B$
χ_1	1	1	1	1	1	1	1	1
χ_2	20	−7	2	2	2	0	−1	−1
χ_3	34	7	7	−2	−2	−1	−1	−1
χ_4	34	7	2	7	2	−1	−1	−1

TABLE A.27.

$\mathbf{Sp_4(4)}$	$1A$	$3A$	$3B$	$5A$	$5B$	$5C$	$5D$	$5E$
χ_1	1	1	1	1	1	1	1	1
χ_2	8	−4	2	−2	−2	3	3	−2
χ_3	8	2	−4	3	3	−2	−2	−2
χ_4	16	4	1	−4	−4	1	1	1
χ_5	16	1	4	1	1	−4	−4	1
χ_6	32	−4	−4	2	2	−8	−8	2
χ_7	32	−4	−4	−8	−8	2	2	2

TABLE A.28.

$\mathbf{Sp_4(4).4}$	$1A$	$3A$	$3B$	$5A$	$5B$	$5C$	$5D$	$5E$
χ_1	1	1	1	1	1	1	1	1
χ_2	16	−2	−2	1	1	1	1	−4
χ_3	32	5	5	−3	−3	−3	−3	2
χ_4	64	−8	−8	−6	−6	−6	−6	4

TABLE A.29.

$\mathbf{Sp_6(2)}$	$1A$	$3A$	$3B$	$3C$	$5A$	$7A$
χ_1	1	1	1	1	1	1
χ_2	6	3	−3	0	1	−1
χ_3	8	−4	−1	2	−2	1
χ_4	14	2	5	−1	−1	0
χ_5	48	−12	3	0	−2	−1

TABLE A.30.

$\mathbf{Sz(8)}$	$1A$	$5A$	$7ABC$	$13ABC$
χ_1	1	1	1	1
χ_2	12	−3	−2	−1
χ_3	48	3	−1	−4

TABLE A.31.

$\mathbf{G_2(3)}$	$1A$	$3A$	$3C$
χ_1	1	1	1
χ_2	14	5	−4

TABLE A.32.

$\mathbf{G_2(4)}$	$1A$	$3A$	$3B$	$5A$	$5B$	$5C$	$5D$	$7A$	$13A$	$13B$
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	12	−6	0	2	2	−3	−3	−2	−1	−1
χ_3	28	10	−2	3	3	3	3	0	2	2
χ_4	36	9	0	−4	−4	1	1	1	−3	−3

Table A.33.

$\mathbf{G_2(8)}$	$1A$	$3A$	$3B$	$7ABC$	$7DEF$
χ_1	1	1	1	1	1
χ_2	18	-9	0	4	-3
χ_3	42	15	-3	7	7

Table A.34.

$\mathbf{Alt(5)}$	$1A$	$3A$	$5AB$
χ_1	1	1	1
χ_2	4	-2	-1
χ_3	4	1	-1

Table A.35.

$\mathbf{Alt(6)}$	$1A$	$3A$	$3B$	$5AB$
χ_1	1	1	1	1
χ_2	4	1	-2	-1
χ_3	4	-2	1	-1
χ_4	16	-2	-2	1

Table A.36.

$\mathbf{Alt(7)}$	$1A$	$3A$	$3B$	$5A$	$7A$	$7B$
χ_1	1	1	1	1	1	1
χ_2	4	-2	1	-1	$b7$	$**$
χ_3	4	-2	1	-1	$**$	$b7$
χ_4	6	3	0	1	-1	-1
χ_5	14	2	-1	-1	0	0
χ_6	20	-4	-1	0	-1	-1

Table A.37.

$\mathbf{Alt(8)}$	$1A$	$3A$	$3B$	$5A$	$7A$	$7B$	$15A$	$15B$
χ_1	1	1	1	1	1	1	1	1
χ_2	4	-2	1	-1	$-b7$	$**$	$-b15$	$**$
χ_3	4	-2	1	-1	$**$	$-b7$	$**$	$-b15$
χ_4	6	3	0	1	-1	-1	-2	-2
χ_5	14	2	-1	-1	0	0	2	2
χ_6	20	-4	-1	0	-1	-1	$-1+b15$	$**$
χ_7	20	-4	-1	0	-1	-1	$**$	$-1+b15$

Table A.38.

Alt(9)	$1A$	$3A$	$3B$	$3C$	$5A$	$7A$	$9A$	$9B$	$15A$	$15B$
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	8	5	-1	2	3	1	-1	-1	0	0
χ_3	8	-4	-1	2	-2	1	2	-1	1	1
χ_4	8	-4	-4	2	-2	1	-1	2	1	1
χ_5	20	-4	2	-1	0	-1	-1	-1	$-1+b15$	$**$
χ_6	20	-4	2	-1	0	-1	-1	-1	$**$	$-1+b15$
χ_7	26	8	-1	-1	1	-2	-1	-1	-2	-2
χ_8	48	6	3	0	-2	-1	0	0	1	1

Table A.39.

Alt(10)	$1A$	$3A$	$3B$	$3C$	$5A$	$5B$	$7A$	$9A$	$9B$
χ_1	1	1	1	1	1	1	1	1	1
χ_2	8	5	2	-1	3	-2	1	-1	-1
χ_3	16	-8	4	-2	-4	1	2	1	1
χ_4	26	8	-1	-1	1	1	-2	-1	-1
χ_5	48	6	0	3	-2	-2	-1	0	0

Table A.40.

Alt(11)	$1A$	$3A$	$3B$	$3C$	$5A$	$5B$	$7A$	$9A$
χ_1	1	1	1	1	1	1	1	1
χ_2	10	7	4	1	5	0	3	1
χ_3	32	-16	8	-4	-8	2	4	2
χ_4	44	20	5	-1	9	-1	2	-1

Table A.41.

Alt(12)	$1A$	$3A$	$3B$	$3C$	$3D$	$5A$	$5B$	$7A$	$9A$	$9BC*2$
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	10	7	4	-2	1	5	0	3	1	-2
χ_3	32	-16	8	2	-4	-8	2	4	2	-1
χ_4	44	20	5	2	-1	9	-1	2	-1	2

Table A.42.

Alt(13)	$1A$	$3A$	$3B$	$3C$	$3D$	$5A$	$5B$
χ_1	1	1	1	1	1	1	1
χ_2	12	9	6	3	0	7	2

TABLE A.43.

$\mathbf{M_{11}}$	$1A$	$3A$	$5A$	$11AB$
χ_1	1	1	1	1
χ_2	10	-1	0	-1
χ_3	32	-4	2	-1
χ_4	44	-1	-1	0

TABLE A.44.

$\mathbf{M_{12}}$	$1A$	$3A$	$3B$	$5A$	$11AB$
χ_1	1	1	1	1	1
χ_2	10	1	-2	0	-1
χ_3	32	-4	2	2	-1
χ_4	44	-1	2	-1	1

TABLE A.45.

$\mathbf{J_2}$	$1A$	$3A$	$3B$	$5AB$	$5CD$	$7A$
χ_1	1	1	1	1	1	1
χ_2	12	-6	0	2	-3	-2
χ_3	28	10	-2	3	3	0
χ_4	36	9	0	-4	1	1

References

1. J. L. Alperin: *Projective modules for $SL(2,2n)$*, J. Pure Appl. Algebra 15 (1979), no. 3, 219–234.
2. M. Aschbacher: *Finite group theory*, Second edition, Cambridge Studies in Advanced Mathematics, 10. Cambridge University Press, Cambridge, 2000. xii+304 pp.
3. M. Aschbacher: *Some multilinear forms with large isometry groups*, Geometries and groups (Noordwijkerhout, 1986). Geom. Dedicata 25 (1988), no. 1–3, 417–465.
4. M. Aschbacher: *The 27-dimensional module for E_6*, I. Invent. Math. 89 (1987), no. 1, 159–195.
5. M. Aschbacher: *On the maximal subgroups of the finite classical groups*, Invent. Math. 76 (1984), no. 3, 469–514.
6. M. Aschbacher, G. M. Seitz: *Involutions in Chevalley groups over fields of even order*, Nagoya Math. J. 63 (1976), 1–91.
7. C. Bates: *The Maximal Subgroups of $E_7(2)$ and Other Topics*, PhD thesis, The University of Manchester (2006).
8. C. Bates, P. Rowley: *Centralizers of real elements in finite groups*, Arch. Math. (Basel) 85 (2005), no. 6, 485–489.
9. D. J. Benson: *Modular representation theory. New trends and methods*, Second printing of the 1984 original. Lecture Notes in Mathematics, 1081. Springer-Verlag, Berlin, 2006. xii+231 pp.
10. D. J. Benson: *The Loewy structure of the projective indecomposable modules for A_8 in characteristic* 2, Comm. Algebra 11 (1983), no. 13, 1395–1432.
11. W. Bosma, J. Cannon, C. Playoust: *The Magma algebra system. I. The user language'*, J. Symbolic Comput., 24, no 3-4 (1997), 235–265.
12. J. N. Bray: *An improved method for generating the centralizer of an involution*, Arch. Math. (Basel) 74 (2000), no. 4, 241–245.
13. R. W. Carter: *Simple groups of Lie type*, Reprint of the 1972 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley and Sons, Inc., New York, 1989. x+335 pp.
14. E. Cline, B. Parshall, L. Scott: *Cohomology of finite groups of Lie type. I*, Inst. Hautes Études Sci. Publ. Math. No. 45 (1975), 169–191.
15. J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson: *Atlas of Finite Groups*, Clarendon, Oxford (1985).
16. B. N. Cooperstein: *The fifty-six-dimensional module for E_7. I. A four form for E_7*, J. Algebra 173 (1995), no. 2, 361–389.

17. D. I. Deriziotis, M. W. Liebeck: *Centralizers of semisimple elements in finite twisted groups of Lie type*, J. London Math. Soc. (2) 31 (1985), no. 1, 48–54.

18. L. E. Dickson: *Linear groups, with an exposition of the Galois field theory*, Teubner (1901), reprinted Dover (1958).

19. K. Fleischmann, I. Janiszczak: *The semisimple conjugacy classes of finte groups of Lie-type E_6 and E_7*, Comm. Alg. 21(1) (1993), 93–161.

20. E. Galois: Letter to A. Chevallier, translated by A. R. Singh and C. S. Yogonanda, `http://link.springer.com/content/pdf/%2410.1007%24%2FBF02834265.pdf`

21. The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.12; 2008. (http://www.gap-system.org)

22. D. Gorenstein: *Finite groups*, Second edition. Chelsea Publishing Co., New York (1980).

23. R. W. Hartley: *Determination of the ternary collineation groups whose coefficients lie in the $GF(2^n)$*, Ann. of Math. 27 (1925), 140–158.

24. C. Jansen, K. Lux, R. Parker, R. Wilson: *An atlas of Brauer characters*, Appendix 2 by T. Breuer and S. Norton. London Mathematical Society Monographs. New Series, 11. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995. xviii+327 pp.

25. P. B. Kleidman, R. A. Wilson: *The maximal subgroups of $E_6(2)$ and $\mathrm{Aut}(E_6(2))$*, Proc. London Math. Soc. (3) 60 (1990), no. 2, 266–294.

26. R. Lawther: *Sublattices generated by root differences*, J. Algebra 412 (2014), 255–263.

27. M. W. Liebeck, J. Saxl: *On the orders of maximal subgroups of the finite exceptional groups of Lie type*, Proc. London Math. Soc. (3) 55 (1987), no. 2, 299–330.

28. M. W. Liebeck, J. Saxl, G. M. Seitz: *Subgroups of maximal rank in finite exceptional groups of Lie type*, Proc. London Math. Soc. (3) 65 (1992), no. 2, 297–325.

29. M. W. Liebeck, J. Saxl, D. M. Testerman: *Simple subgroups of large rank in groups of Lie type*, Proc. London Math. Soc. (3) 72 (1996), no. 2, 425–457.

30. M. W. Liebeck, G. M. Seitz: *Maximal subgroups of large rank in exceptional groups of Lie type*, J. London Math. Soc. (2) 71 (2005), no. 2, 345–361.

31. M. W. Liebeck, G. M. Seitz: *A survey of maximal subgroups of exceptional groups of Lie type. Groups, Combinatorics and Geometry: Durham, 2001*, World Scientific (2003).

32. M. W. Liebeck, G. M. Seitz: *On finite subgroups of exceptional algebraic groups*, J. reine angew. Math. 515 (1999), 25–72.

33. M. W. Liebeck, G. M. Seitz: *On the subgroup structure of exceptional groups of Lie type*, Trans. Amer. Math. Soc. 350 (1998), no. 9, 3409–3482.

34. M. W. Liebeck, G. M. Seitz: *Maximal subgroups of exceptional groups of Lie type, finite and algebraic*, Geom. Dedicata 36 (1990), 353–387.

35. F. Lübeck: *Conjugacy Classes and Character Degrees of $E_7(2)$*, `http://www.math.rwthaachen.de/~Frank.Luebeck/chev/E72.html`

36. F. Lübeck: *Small degree representations of finite Chevalley groups in defining characteristic*, LMS J. Comput. Math. 4 (2001), 135–169 (electronic).

37. H. H. Mitchell: *Determination of the ordinary and modular ternary linear groups*, Trans. Amer. Math. Soc. 12 (1911), 207–242.

38. C. Parker, J. Saxl: *Two maximal subgroups of $E_8(2)$*, Israel J. Math. 153 (2006), 307–318.

39. G. M. Seitz: *Maximal subgroups of finite exceptional groups*, Groups and geometries (Siena, 1996), 155–161, Trends Math., Birkhuser, Basel, 1998.

40. R. A. Wilson: *The Finite Simple Groups*, Graduate Texts in Mathematics, 251. Springer-Verlag London, Ltd., London, 2009. xvi+298 pp.

41. R. Wilson, P. Walsh, J. Tripp, I. Suleiman, R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray, R. Abbot: *Atlas of Finite Group Representations*, version 3, `http://brauer.maths.qmul.ac.uk/Atlas/v3/`