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Abstract

Suppose that G is a group, X a subset of G and π a set of natural numbers. The π-product graph $\mathcal{P}_\pi(G, X)$ has X as its vertex set and distinct vertices are joined by an edge if the order of their product is in π. If X is a set of involutions, then $\mathcal{P}_\pi(G, X)$ is called a π-product involution graph. In this paper we study the connectivity and diameters of $\mathcal{P}_\pi(G, X)$ when G is a finite symmetric group and X is a G-conjugacy class of involutions.

1 Introduction

There is a cornucopia of combinatorial and geometric structures which are associated with groups. These range from graphs to posets and topological spaces such as simplicial complexes. An example of the latter type arises in a finite group G where for a prime p dividing the order of G we may define the poset of all non-trivial p-subgroups of G, denoted $\mathcal{S}_p(G)$, ordered by inclusion. This poset has a rich structure, as has its associated order complex $|\mathcal{S}_p(G)|$ known as the *Brown complex*, after being studied by - among others - Brown in his paper [11]. An analogous order complex, called the *Quillen complex*, can be defined for the poset $\mathcal{A}_p(G)$ of all non-trivial elementary abelian p-subgroups of G. Indeed, Quillen showed in [18] that the Brown and Quillen complexes are G-homotopy equivalent. Thévenez and Webb later showed that the complexes consisting of chains of normal series of p-subgroups, and chains of radical p-subgroups are also G-homotopy equivalent to the Brown and Quillen complexes (see [25] and [26] for full details). In the case when G is a group of Lie type, the order complex $|\mathcal{S}_p(G)|$ is the same as the building of G.

The aforementioned subgroup complexes arise in many different areas. Brown was motivated by cohomology and the calculation of cohomology groups for discrete groups - the subject of [12]. The complexes are also closely related to fusion in finite groups and the existence of strongly p-embedded subgroups in G is equivalent to the disconnectedness of $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ (further details of which can be found for example in [1]). It is also possible to build modular representations of G by first defining such representations on stabilisers of the simplices of these complexes, following the constructions of Ronan and Smith in [19], [20], [21] and [22]. A good survey of the versatility of such complexes can be found in [24].

We mention a few graphs among the multitude of such structures that we may associate to a given group G. Let X be a subset of G. The commuting graph $\mathcal{C}(G, X)$ has vertex set X and distinct elements $x, y \in X$ are joined by an edge whenever $xy = yx$. The case when $X = G \setminus Z(G)$, first studied in [9], has been the focus of interest recently - see [10], [15] and [16]. When X is taken to be a G-conjugacy class of involutions, we get the so-called commuting involution graph, the subject of a number of papers (see [2], [3], [4], [5], [14], [17] and [23]).

If π is a set of natural numbers, then the π-product graph $\mathcal{P}_\pi(G, X)$ again has vertex set X, with distinct vertices $x, y \in X$ joined by an edge if the order of xy is in π. In the case when X is a G-conjugacy class of involutions, we note that $\mathcal{P}_{\{2\}}(G, X)$ is just a commuting involution graph. Taking π to be the set of all odd natural numbers and X a G-conjugacy class, $\mathcal{P}_\pi(G, X)$ becomes the local fusion graph $\mathcal{F}(G, X)$ which has featured in [6] and [7].

In the case when X is a set of involutions we refer to $\mathcal{P}_\pi(G, X)$ as a π-product involution graph. It is such graphs when X is a conjugacy class that we consider in this paper for $G = \mathrm{Sym}(n)$, the symmetric group of degree n. We use the standard distance metric on $\mathcal{P}_\pi(G, X)$, which we denote by $d(\cdot, \cdot)$. For $x \in X$ and $i \in \mathbb{N}$ we denote the set of vertices distance i from x in $\mathcal{P}_\pi(G, X)$ by $\Delta_i(x)$. We also denote by $\Omega := \{1, \ldots, n\}$ the underlying set upon which $\mathrm{Sym}(n)$ acts.

We first consider the case when $\pi = \{4\}$. Or, in other words, two distinct involutions $x, y \in X$ are joined by an edge whenever $\langle x, y\rangle \cong \mathrm{Dih}(8)$, the dihedral group of order 8. In considering this, we are in effect looking at a section of the poset $\mathcal{S}_2(\mathrm{Sym}(n))$. Our first result determines when $\mathcal{P}_{\{4\}}(G, X)$ is connected and in such cases, the diameter of $\mathcal{P}_{\{4\}}(G, X)$ is also determined.

Theorem 1.1. *Suppose* $G = \mathrm{Sym}(n)$, $t = (1, 2)\cdots(2m-1, 2m) \in G$, *and let* X *denote the* G*-conjugacy class of* t.

(i) The graph $\mathcal{P}_{\{4\}}(G, X)$ *is disconnected if and only if one of the following holds:*

  (a) $n = 2m + 1$;

  (b) $m = 1$;

  (c) $(n, m) = (4, 2)$ *or* $(6, 3)$.

(ii) If $\mathcal{P}_{\{4\}}(G, X)$ *is connected, then* $\mathrm{Diam}(\mathcal{P}_{\{4\}}(G, X)) = 2$.

In (i)(a) of Theorem 1.1 we observe that $\mathcal{P}_{\{4\}}(G, X)$ consists of n copies of $\mathcal{P}_{\{4\}}(\mathrm{Sym}(2m), Y)$ where Y consists of all involutions of cycle type 2^m. This corresponds to the n possible fixed points of the involutions of X. Cases (i)(b) and (i)(c) result in totally disconnected graphs.

For symmetric groups, the diameters of the connected π-product involution graphs have been determined when $\pi = \{2\}$ - that is the commuting involution graphs - and $\pi = \mathbb{N}_{\mathrm{odd}}$ (=the set of all odd natural numbers) - the local fusion graphs. In the former case the diameter is bounded above by 3 except for three small cases when the diameter is 4. Moreover, the diameter can be 3 infinitely often. In the latter case, the connected local fusion graphs for symmetric groups always have diameter 2. So, from this perspective, $\mathcal{P}_\pi(G, X)$ for $\pi = \{4\}$ and $\pi = \mathbb{N}_{\mathrm{odd}}$ are bed fellows. However, this apparent similarity does not extend to the case that $\pi = \{2^a\}$ for some $a \geq 3$. Indeed, we shall derive the following result.

Theorem 1.2. *Suppose that* $G = \mathrm{Sym}(n)$, $2m = 2^a \leq n$ *for some* $a \geq 3$, $t = (1, 2)(3, 4)\cdots(2m-1, 2m)$ *and* X *is the* G*-conjugacy class of* t. *Then*

(i) $\mathcal{P}_{\{2m\}}(G, X)$ *is connected if and only if* $n \geq 2m + 2$*; and*

(ii) *if* $\mathcal{P}_{\{2m\}}(G, X)$ *is connected, then*

$$\min\{m, \lceil n/2 - m\rceil\} \leq \mathrm{Diam}(\mathcal{P}_{\{2m\}}(G, X)) \leq 2m - 1$$

(where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x). Taking $n = 4m$ in this theorem gives an infinite family of π-product involution graphs whose diameter is unbounded.

Specializing to the case $m = 4$ (so $2m = 8$) we can give precise values for the diameter of $\mathcal{P}_{\{8\}}(G, X)$ in our next theorem.

Theorem 1.3. *Suppose* $G = \mathrm{Sym}(n)$, $t = (1, 2)(3, 4)(5, 6)(7, 8)$ *and let* X *be the* G*-conjugacy class of* t*. Then*

(i) *for* $10 \leq n \leq 14$, $\mathrm{Diam}(\mathcal{P}_{\{8\}}(G, X)) = 3$*; and*

(ii) *for* $n \geq 15$, $\mathrm{Diam}(\mathcal{P}_{\{8\}}(G, X)) = 4$.

An analogous version of Theorem 1.2 also holds for any odd prime power.

Theorem 1.4. *Suppose that* $G = \mathrm{Sym}(n)$, p *is an odd prime and* $q = p^a$ *for some* $a \geq 1$*. Let* $t = (1, 2) \cdots (q - 2, q - 1)$ *and* X *be the* G*-conjugacy class of* t*. Then*

(i) $\mathcal{P}_{\{q\}}(G, X)$ *is connected if and only if* $n \geq q$*; and*

(ii) *if* $\mathcal{P}_{\{q\}}(G, X)$ *is connected, then*

$$\min\{q - 1, n + 1 - q\} \leq \mathrm{Diam}(\mathcal{P}_{\{q\}}(G, X)) \leq q - 1.$$

Our final result combines Theorems 1.2 and 1.4.

Theorem 1.5. *Suppose that* $G = \mathrm{Sym}(n)$*, and* $p_1, \ldots, p_r$ *are distinct primes with* $p_i < p_{i+1}$ *for* $i = 1, \ldots, r - 1$*. Let* $q = p_1^{a_1} \cdots p_r^{a_r}$ *for some* $a_1, \ldots, a_r \geq 1$ *with* $a_1 \geq 2$ *if* $p_1 = 2$ *and set*

$$q_i = \begin{cases} p_i^{a_i} & \text{if } p_i = 2\text{; and} \\ p_i^{a_i} - 1 & \text{otherwise,} \end{cases}$$

and $2m = q_1 \cdots q_r$*. Assuming* $2m \leq n$*, let* $t = (1, 2) \cdots (2m - 1, 2m)$ *and* X *be the* G*-conjugacy class of* t*.*

(i) *The graph* $\mathcal{P}_{\{q\}}(G, X)$ *is connected if and only if*

$$n \geq \begin{cases} 2m + 2 & \text{if } p_1 = 2\text{; and} \\ 2m & \text{otherwise.} \end{cases}$$

(ii) *If* $\mathcal{P}_{\{q\}}(G, X)$ *is connected, then*

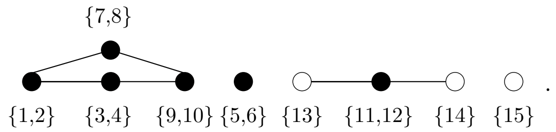
$$\mathrm{Diam}(\mathcal{P}_{\{q\}}(G, X)) \leq \max_i q_i$$

This paper is arranged as follows. In Section 2 we introduce the notion of the *x-graph* of an element of X. These are graphs that encapsulate the $C_G(x)$-orbits of X and were first introduced by Bates, Bundy, Perkins and Rowley in [3]. We present a number of their results, and relate the connected components of an x-graph to the disc $\Delta_1(t)$ for a fixed involution t of X. Section 3 begins by considering combinations of connected components of x-graphs, and we show that Theorem 1.1 holds when restricted to the supports of such components. In particular we consider the case when our conjugacy class consists of elements of full support in Lemma 3.7. We then proceed to give a general proof of Theorem 1.1 at the end of this section. The paper concludes in Section 4 with an analysis of π-product graphs when $\pi \neq \{4\}$. We begin by considering the case when $\pi = \{2^a\}$ for some $a \geq 3$. Calculations of the sizes of discs $\Delta_i(t)$ for certain π-product involution graphs are given and these give a direct proof of Theorem 1.3. This is proceeded by constructive proofs of Theorems 1.2, 1.4 and 1.5. Finally, we consider some smaller symmetric groups and calculate the sizes of discs of the π-product graphs $\mathcal{P}_\pi(G, X)$ when $\pi = \{6\}$ or $\{8\}$.

2 Preliminary Results

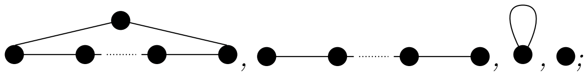
Throughout this paper, we set $G = \mathrm{Sym}(n)$ and consider G as acting on a set of n letters (or points), $\Omega = \{1, \ldots, n\}$. Let $t \in G$ be a fixed involution and let X be the G-conjugacy class of t. For an element $g \in G$, we denote the set of fixed points of g on Ω by $\mathrm{fix}(g)$ and define the *support* of g to be $\mathrm{supp}(g) := \Omega \setminus \mathrm{fix}(g)$.

To study the graph $\mathcal{P}_{\{4\}}(G, X)$, we first introduce another type of graph known as an *x-graph*. Indeed, let $x \in X$. The x-graph corresponding to x, denoted $\mathcal{G}_x$, has vertex set given by the orbits of Ω under $\langle t \rangle$. Two vertices σ, γ are joined in $\mathcal{G}_x$ if there exists $\sigma_0 \in \sigma$ and $\gamma_0 \in \gamma$ such that $\{\sigma_0, \gamma_0\}$ is an orbit of Ω under $\langle x \rangle$. We call the vertices corresponding to transpositions of t *black vertices*, denoted ●, and those corresponding to fixed points of t *white vertices*, denoted ○. As an example, let $n = 15$, $t = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$ and $x = (1,7)(2,3)(4,10)(8,9)(11,13)(12,14)$. The x-graph $\mathcal{G}_x$ is given by

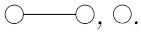


We note that we could swap the roles of t and x to produce another x-graph which we denote by $\mathcal{G}_t^x$. In general the x-graph $\mathcal{G}_x^y$ has vertices the orbits of Ω under $\langle y \rangle$, and edges defined by the orbits of Ω under $\langle x \rangle$.

The concepts of x-graphs were first introduced in [3] as a tool for studying the commuting involution graphs of the symmetric groups. More recently they have also been used in the study of local fusion graphs for the symmetric groups (see [7] for further details). The versatility of x-graphs in calculations arises from the simple observation that each black vertex has valency at most two and each white vertex has valency at most one. Consequently, we may fully determine the possible connected components of a given x-graph.

Lemma 2.1. *Let $x \in X$. The possible connected components of $\mathcal{G}_x$ are* *(i)* ;

(ii) ; *and*

(iii) .

In the subsequent discussion, we will consider x-graphs up to isomorphism. It is implicit that such an isomorphism will preserve vertex colours. We also fix $t = (1, 2) \cdots (2m - 1, 2m) \in G$.

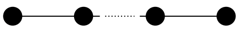
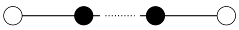
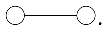
Bates, Bundy, Perkins and Rowley's interest in x-graphs stemmed from the following elementary result.

Lemma 2.2. *(i) Every graph with b black vertices of valency at most two, w white vertices of valency at most one and exactly b edges is the x-graph for some $x \in X$ (with $m = b$ and $n = 2b + w$).*

(ii) Let $x, y \in X$. Then x and y are in the same $C_G(t)$-orbit if and only if $\mathcal{G}_x$ and $\mathcal{G}_y$ are isomorphic graphs.

Proof. See Lemma 2.1 of [3]. □

Part (i) of Lemma 2.2 is of particular interest, as it confirms that when employing a combinatorial approach using the connected components of x-graphs, we must consider all possible connected components given in Lemma 2.1. This approach will be used repeatedly in the proof of Theorem 1.1.

An immediate consequence of the definition of $\mathcal{G}_x$ is that the number of black vertices is equal to the number of edges. Consequently the number of connected components of the form  containing at least one black vertex must be equal to the number of connected components of the form  and .

Lemma 2.1 allows a combinatorial approach to be used when considering conjugate involutions. Indeed, given a connected component C_i, we may define Ω_i to be the union of all vertices of C_i. We may then define the i-part of t, denoted t_i, to be the product of those transpositions of t that occur in $\mathrm{Sym}(\Omega_i)$. We define x_i similarly. By analysing the structure of the connected components given in Lemma 2.1 it is possible to relate the order of tx to the x-graph $\mathcal{G}_x$.

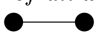
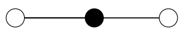
Lemma 2.3. *Suppose that $x \in X$ and that $C_1, \ldots, C_k$ are the connected components of $\mathcal{G}_x$. Denote the number of black vertices, white vertices and cycles in C_i by b_i, w_i and c_i respectively. Then*

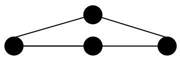
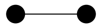
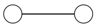
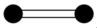
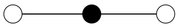
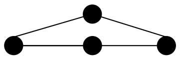
(i) the order of tx is the least common multiple of the orders of $t_i x_i$ (for $i = 1, \ldots, k$); and

(ii) the order of $t_i x_i$ is $(2b_i + w_i)/(1 + c_i)$ for each $i = 1, \ldots, k$.

Proof. See Proposition 2.2 of [3]. □

We have the following immediate corollary to Lemmas 2.1 and 2.3.

Corollary 2.4. *For $\mathcal{P}_{\{4\}}(G, X)$ the disc $\Delta_1(t)$ consists of all $x \in X$ whose x-graphs have at least one connected component of the form* , 

or  *and all other components have the form* ○, ●, , , , ,  *or* .

Proof. The element x lies in $\Delta_1(t)$ precisely when tx has order 4. The result then follows from Lemmas 2.1 and 2.3. □

We conclude this section by noting that we can define an x-graph for any two (not-necessarily conjugate) involutions. This we will do frequently in Section 3. However, in such a situation it is no longer the case that the number of edges of $\mathcal{G}_x$ is equal to the number of black vertices.

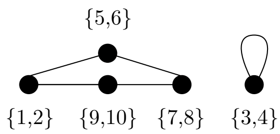
3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Note that for $m \geq 2$ and $t = (1,2)\cdots(2m-1,2m)$, the involution $x = (1,3)(2,4)(5,6)\cdots(2m-1,2m) \in X$ satisfies $d(t,x) \geq 2$. Thus it suffices to prove when $\mathcal{P}_{\{4\}}(G,X)$ is connected, that for all $x \in X$ we have $d(t,x) \leq 2$. To do this we consider pairs or triples of connected components C_i, C_j and C_k of $\mathcal{G}_x$ and the corresponding parts t_i, t_j, t_k, x_i, x_j, x_k of t and x. We then construct an element $y_{ijk} \in \mathrm{Sym}(\mathrm{supp}(t_it_jt_k))$ which is conjugate to $t_it_jt_k$ and such that the x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ have connected components featuring in Corollary 2.4.

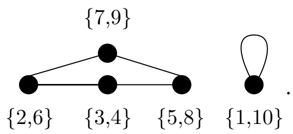
We begin by proving a few preliminary results, dealing with the case $n = 2m$.

Lemma 3.1. *Let $m \geq 5$, $n = 2m$ and suppose that $x \in X$ is such that $\mathcal{G}_x$ is connected. Then there exists $y \in X$ such that $d(t,y) = d(y,x) = 1$.*

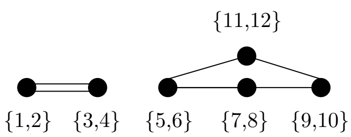
Proof. Without loss of generality we may assume that $x = (1,2m)(2,3)\cdots(2m-2,2m-1)$. If $m = 5$, then taking $y = (1,10)(2,6)(3,4)(5,8)(7,9)$ we see that $\mathcal{G}_y$ and $\mathcal{G}_x^y$ are given respectively by



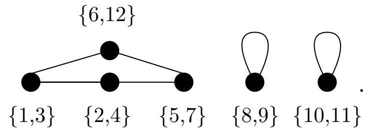
and



If $m = 6$, we take $y = (1,3)(2,4)(5,7)(6,12)(8,9)(10,11)$. Then $\mathcal{G}_y$ and $\mathcal{G}_x^y$ are, respectively



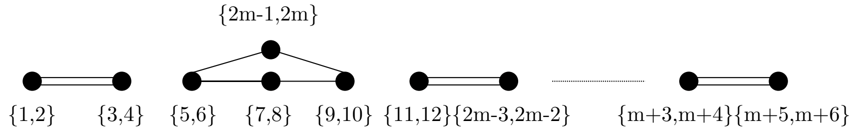
and



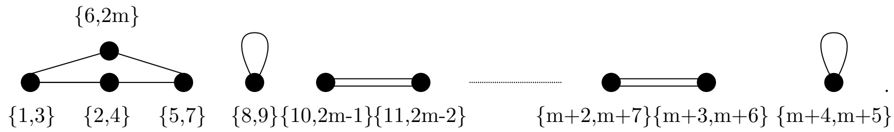
In the general case when $m \geq 7$, we take

$$y = (1,3)(2,4)(5,7)(6,2m)(8,9)(10,2m-1)(11,2m-2)\cdots(m+4,m+5).$$

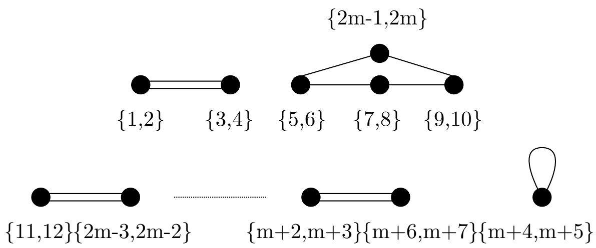
The exact nature of the associated x-graphs is dependent on the parity of m. If m is even, then $\mathcal{G}_y$ is given by



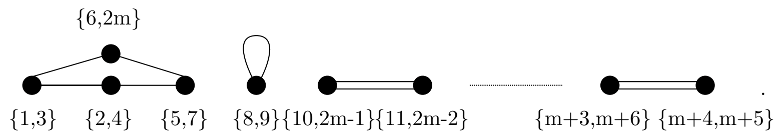
and $\mathcal{G}_x^y$ is given by



If m is odd, the graphs $\mathcal{G}_y$ and $\mathcal{G}_x^y$ are, respectively



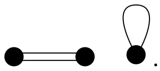
and



In all cases, the given graphs satisfy the conditions of Corollary 2.4, whence $d(t,y) = d(y,x) = 1$. □

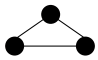
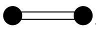
The proof of Lemma 3.1 illustrates a general feature that the actual x-graphs constructed may vary depending on the parity and values of the given parameters (such as the parameter m above). However, in using Corollary 2.4 we are only interested in the connected components of the x-graph. Thus for the sake of brevity, in all future proofs we will only describe the connected components of each x-graph.

Lemma 3.2. *Suppose that $m = 3$, $n = 6$ and $x \in X$. If $\mathcal{G}_x$ is connected, then there exists $y \in X$ such that the x-graphs $\mathcal{G}_y$ and $\mathcal{G}_x^y$ are isomorphic to*



Proof. Without loss of generality, we may assume that $x = (1,6)(2,3)(4,5)$. Then $y = (1,2)(3,6)(4,5)$ is the required element. □

Lemma 3.3. *Let $m = 4$ and $n = 8$. Suppose that $x \in X\setminus\{t\}$ has a disconnected x-graph, $\mathcal{G}_x$. Then there exists $y \in X$ such that $d(t,y) = d(y,x) = 1$.*

Proof. If $\mathcal{G}_x$ has a connected component of the form , then we may assume that $x = (1,6)(2,3)(4,5)(7,8)$. The element $y = (1,8)(2,4)(3,6)(5,7)$ is then the desired y. The other possibilities occur when $\mathcal{G}_x$ has one or two connected components of the form , corresponding respectively to $x = (1,3)(2,4)(5,6)(7,8)$ and $x = (1,3)(2,4)(5,7)(6,8)$. The y satisfying the lemma for both such x is $y = (1,8)(2,3)(4,5)(6,7)$. □

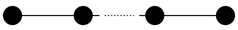
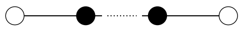
Lemmas 3.1, 3.2 and 3.3 combine to prove Theorem 1.1 in the case when $n = 2m$.

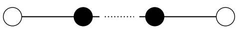
Corollary 3.4. *If $n = 2m$, then Theorem 1.1 holds.*

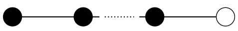
Proof. Let $x \in X$. If $\mathcal{G}_x$ has connected components containing precisely 4 black vertices then we leave the parts of t and x corresponding to such components alone. We then apply Lemma 3.1 to any connected component containing at least 5 black vertices, and Lemma 3.2 to any connected component containing 3 black vertices to obtain the desired result. Otherwise all connected components have at most 3 black vertices. Applying Lemma 3.3 to a collection of components containing a total of 4 black vertices, Lemma 3.2 to any remaining connected components containing 3 black vertices, and leaving all other connected components invariant gives the result. □

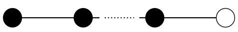
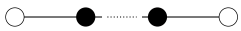
Before presenting the proof of Theorem 1.1 we give a further three intermediate results.

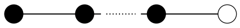
Lemma 3.5. *Let $x \in X$. Suppose that $\mathcal{G}_x$ has connected components C_i and C_j of the given forms. Then there exists $y_{ij} \in H := \mathrm{Sym}(\mathrm{supp}(t_i) \cup \mathrm{supp}(t_j))$ which is H-conjugate to t_it_j and such that the connected components of the x-graphs $\mathcal{G}_{y_{ij}}^{t_it_j}$ and $\mathcal{G}_{x_ix_j}^{y_{ij}}$ satisfy the conditions of Corollary 2.4.*

(i) C_i:  *(with $q \geq 3$ black vertices),*
C_j:  *(with $r \geq 0$ black vertices);*

(ii) C_i:  *(with $q \geq 2$ black vertices), C_j : $\emptyset$;*

(iii) C_i:  *(with $q \geq 2$ black vertices), C_j: ○;*

(iv) C_i:  *(with $q \geq 1$ black vertices),*
C_j:  *(with $r \geq 1$ black vertices); and*

(v) C_i and C_j are both of the form  *(with $q, r \geq 1$ black vertices respectively).*

Proof. For each case, without loss of generality we give explicit formulations of the t_i and x_i. For ease of notation, where parameters q and r have been defined we set $v = 2(q + r)$.

In case (i) assume that $t_i = (1, 2) \cdots (2q-1, 2q)$, $t_j = (2q+1, 2q+2) \cdots (v-1, v)(v + 1)(v + 2)$, and $x_i = (1)(2, 3) \cdots (2q - 2, 2q - 1)(2q)$. We now consider three possibilities. If $r = 0$, then we may assume that $x_j = (2q + 1, 2q + 2)$ and we take

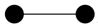
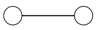
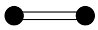
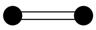
$$y_{ij} := (1, 2q)(2)(2q - 1)(3, 2q - 2) \cdots (q, q + 1)(2q + 1, 2q + 2).$$

If $r = 1$, then taking $x_j = (2q + 1, v + 1)(2q + 2, v + 2)$ we define

$$y_{ij} := (1, 2q)(2)(2q - 1)(3, 2q - 2) \cdots (q, q + 1)(v + 1, v + 2)(2q + 1, 2q + 2).$$

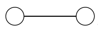
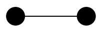
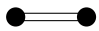
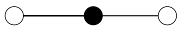
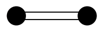
Finally, if $r > 1$, then we assume that $x_j = (2q+1, v+1)(2q+2, 2q+3) \cdots (v-2, v - 1)(v, v + 2)$ and define

$$\begin{aligned} y_{ij} := (1, 2q)(2)(2q - 1)(3, 2q - 2) \cdots (q, q + 1)(v + 1, v + 2) \\ (2q + 1, v) \cdots (2q + r, 2q + r + 1). \end{aligned}$$

We see that the x-graph $\mathcal{G}^{t_i t_j}_{y_{ij}}$ has connected components of the form ,  and - depending on the values of q and r - also  and . Similarly $\mathcal{G}^{y_{ij}}_{x_i x_j}$ has connected components of the form ,  and in some cases also  and  as required.

For (ii) we may set $t_i = (1, 2) \cdots (2q - 1, 2q)(2q + 1)(2q + 2)$ and $x_i = (1, 2q + 2)(2, 3) \cdots (2q, 2q + 1)$. Then the element

$$y_i = (1)(2, 2q - 1)(3, 2q - 2) \cdots (q, q + 1)(2q)(2q + 1, 2q + 2)$$

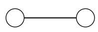
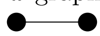
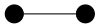
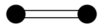
results in the x-graph $\mathcal{G}^{t_i}_{y_i}$ having connected components ,  and in some cases  and  - depending on the value and parity of q. The graph $\mathcal{G}^{y_i}_{x_i}$ has connected components  and possibly  and .

Considering case (iii), we take $t_i = (1, 2) \cdots (2q - 1, 2q)(2q + 1)$, $x_i = (1)(2, 3) \cdots (2q, 2q + 1)$ and $t_j = x_j = (2q + 2)$. If $q = 2$, define

$$y_{ij} = (1, 4)(2)(3)(5, 6),$$

whilst if $q \geq 3$ define

$$y_{ij} = (1, 2q)(2)(2q - 1)(3, 2q - 2) \cdots (q, q + 1)(2q + 1, 2q + 2).$$

Then the permutation y_{ij} gives the desired x-graphs. Indeed, $\mathcal{G}^{t_i t_j}_{y_{ij}}$ has connected components of the form  and , $\mathcal{G}^{y_{ij}}_{x_i x_j}$ has components of the form  and  (with the black vertex omitted if $q = 2$) and both x-graphs may also have connected components of the form  and  depending on the value and parity of q.

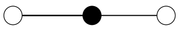
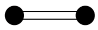
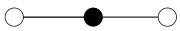
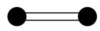
Turning to (iv), if $q = 1$, then without loss of generality we have that $t_i = (1,2)(v+1)$, $t_j = (3,4)\cdots(v-1,v)(v+2)(v+3)$, $x_i = (1)(2,v+1)$ and $x_j = (3,v+2)(4,5)\cdots(v-2,v-1)(v,v+3)$ (take $x_j = (3,6)(4,7)$ if $r = 1$). When $r = 1$, define

$$y_{ij} = (1)(2)(3,5)(4,7)(6).$$

The x-graphs $\mathcal{G}_{y_{ij}}^{t_i t_j}$ and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ are isomorphic to

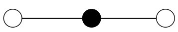
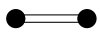
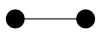
respectively as required. If $r > 1$, then

$$y_{ij} = (1)(2)(3,v-2)\cdots(r+1,r+2)(v-1,v+2)(v,v+1)(v+3)$$

is our desired element. Indeed in this case $\mathcal{G}_{y_{ij}}^{t_i tj}$ has connected components of the form ,  and  in addition to components of the form  and/or  (depending on the value of r), whilst $\mathcal{G}_{x_i x_j}^{y_{ij}}$ has components of the forms  and  in addition to components of the form  and/or  (depending on the value of r).

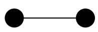
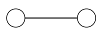
If $q > 1$, then we define $t_i = (1,2)\cdots(2q-1,2q)(v+1)$, $t_j = (2q+1,2q+2)\cdots(v-1,v)(v+2)(v+3)$, $x_i = (1)(2,3)\cdots(2q-2,2q-1)(2q,v+1)$ and $x_j = (2q+1,v+2)(2q+2,2q+3)\cdots(v-2,v-1)(v,v+3)$. Our desired element is then

$$y_{ij} = (1)(2,2q-1)\cdots(q,q+1)(2q)(2q+1,v-2)\cdots$$
$$\cdots(2q+r-1,2q+r)(v-1,v+2)(v,v+1)(v+3).$$

It follows that the connected components of $\mathcal{G}_{y_{ij}}^{t_i t_j}$ and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ have the form  and  and possibly  and , whilst $\mathcal{G}_{y_{ij}}^{t_i t_j}$ has an additional connected component of the form .

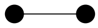
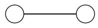
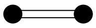
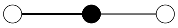
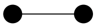
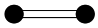
For case (v) we assume without loss of generality that $q \geq r$. We consider the subcases $q = r = 1$, $q > r = 1$, $q = r > 1$ and $q > r > 1$ in turn. If $q = r = 1$, then we take $t_i = (1,2)(5)$, $t_j = (3,4)(6)$, $x_i = (1)(2,5)$ and $x_j = (3)(4,6)$. Defining

$$y_{i,j} = (1,3)(5,6)(2)(4),$$

we see that $\mathcal{G}_{y_{ij}}^{t_i t_j}$ has isomorphism type   and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ has isomorphism type   as required.

If $q > r = 1$, then setting $t_i = (1,2)\cdots(2q-1,2q)(v+1)$, $t_j = (2q+1,2q+2)(v+2)$, $x_i = (1)(2,3)\cdots(2q,v+1)$ and $x_j = (2q+1)(2q+2,v+2)$ we define

$$y_{ij} = (1,2(q-1))(2,2(q-1)-1)\cdots(q-1,q)(2q-1,2q+1)$$
$$(2q)(2q+2)(v+1,v+2).$$

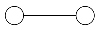
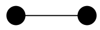
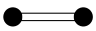
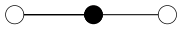
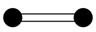
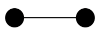
Consequently $\mathcal{G}_{y_{ij}}^{t_i t_j}$ has connected components of the form ,  and  and/or  (depending on the value of q), whilst $\mathcal{G}_{x_i x_j}^{y_{ij}}$ has components of the form ,  and  and/or .

When $r > 1$ we may assume that $t_i = (1,2)\cdots(2q-1,2q)(v+1)$, $t_j = (2q+1,2q+2)\cdots(v-1,v)(v+2)$, $x_i = (1)(2,3)\cdots(2q-2,2q-1)(2q,v+1)$ and $x_j = (2q+1)(2q+2,2q+3)\cdots(v-2,v-1)(v,v+2)$. Define

$$y_{ij} = (1,2q+1)(2,2q+2)\cdots(2q-1,v-1)(2q)(v)(v+1,v+2)$$

if $q = r$ and

$$y_{ij} = (1,2(q-r))(2,2(q-r)-1)\cdots(q-r,q-r+1)$$
$$(2(q-r)+1,2q+1)\cdots(2q-1,v-1)(2q)(v)(v+1,v+2)$$

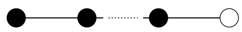
if $q \neq r$. We see that $\mathcal{G}_{y_{ij}}^{t_i t_j}$ has connected components ,  and  in addition to  for some values of q and r. The x-graph $\mathcal{G}_{x_i x_j}^{y_{ij}}$ also has the desired properties having a component of the form , some components of the form  and possibly also ,  and/or  depending on the values of q and r and the parity of $q - r$.

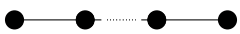
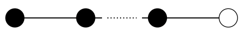
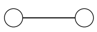
□

We note that in cases (ii) and (iv) above, $t_i t_j$ and $x_i x_j$ have different cycle types. This is a fact which we will utilise in the proof of Theorem 1.1

In a similar vein to Lemma 3.5 we next consider collections of three connected components simultaneously.

Lemma 3.6. *Let $x \in X$. Suppose that $\mathcal{G}_x$ has connected components C_i, C_j and C_k of the given forms. Then there exists $y_{ijk} \in H := \mathrm{Sym}(\mathrm{supp}(t_i) \cup \mathrm{supp}(t_j) \cup \mathrm{supp}(t_k))$ which is H-conjugate to $t_i t_j t_k$ and such that the connected components of the x-graphs $\mathcal{G}_{y_{ijk}}^{t_i t_j t_k}$ and $\mathcal{G}_{x_i x_j x_k}^{y_{ijk}}$ satisfy the conditions of Corollary 2.4:*

(i) C_i, C_j and C_k are each of the form  *(having $q, r, s \geq 1$ black vertices respectively); and*

(ii) C_i:  *(with $q \geq 1$ black vertices),*
C_j:  *(with $r \geq 1$ black vertices), C_k:* .

Proof. We follow the approach of the proof of Lemma 3.5 and construct the appropriate t_i and x_i. We also set $v = 2(q+r)$ and $w = 2(q+r+s)$.

For case (i) we may assume without loss of generality that $q \geq r \geq s \geq 1$ and set

$$t_i = (1,2)\cdots(2q-1,2q)(w+1),\ t_j = (2q+1,2q+2)\cdots(v-1,v)(w+2)$$
$$\text{and}\quad t_k = (v+1,v+2)\cdots(w-1,w)(w+3).$$

We also set

$$\begin{aligned} x_i &= (1)(2,3)\cdots(2q-2,2q-1)(2q,w+1),\\ x_j &= (2q+1)(2q+2,2q+3)\cdots(v-2,v-1)(v,w+2),\ \text{and}\\ x_k &= (v+1)(v+2,v+3)\cdots(w-2,w-1)(w,w+3), \end{aligned}$$

taking $x_i = (1)(2,w+1)$ in the case when $q = 1$, $x_j = (2q+1)(2q+2,w+2)$ when $r = 1$ and $x_k = (v+1)(v+2,w+3)$ when $s = 1$. There are three subcases to consider. If $q = r = s = 1$, then taking $y_{ijk} = (1,4)(5,8)(6,7)(2)(3)(9)$ we see that the x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ are both isomorphic to

○——●——○ ●——● ○,

which has the desired form. If $s = 1$, but $q \neq 1$, we set

$$\begin{aligned} y_{ijk} = (1,w+2)&(2,w+1)(3,2q)(4,2q-1)\cdots(q+1,q+2)\\ &(2q+1,v)\cdots(2q+r,2q+r+1)(v+1)(w)(w+3), \end{aligned}$$

whilst if $s > 1$, we define

$$\begin{aligned} y_{ijk} = (1,w+2)&(2,w+1)(3,2q)(4,2q-1)\cdots(q+1,q+2)\\ &(2q+1,v)\cdots(2q+r,2q+r+1)\\ &(v+2,w-1)\cdots(v+s,v+s+1)(v+1)(w)(w+3). \end{aligned}$$

It follows that the x-graph $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ has connected components of the form ○——●——○ and ○ with further components of the form ● (if $s = 1$), ●——● (if $s > 1$) and ●══● and/or  depending on the values of q, r and s. Meanwhile, $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ has connected components of the form ●——●, ○——○ and ○, in addition to ●══● and/or  depending on the parameters q, r, s.

Finally, we consider case (ii). Assume that $t_i = (1,2)\cdots(2q-1,2q)$, $t_j = (2q+1,2q+2)\cdots(v-1,v)(v+1)$, $t_k = (v+2)(v+3)$, $x_i = (1)(2,3)\cdots(2q-2,2q-1)(2q)$, $x_j = (2q+1)(2q+2,2q+3)\cdots(v,v+1)$ and $x_k = (v+2,v+3)$. If $q = r = 1$, then defining

$$y_{ijk} = (1,3)(2)(4)(5,6)(7)$$

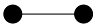
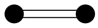
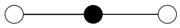
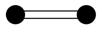
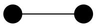
we see that the x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ are of isomorphism type

●——● ○——○ ○ and ○——●——○ ● ○

respectively. Meanwhile, if $q = 1$ and $r > 1$, then setting

$$y_{ijk} = (1)(2,v-3)\cdots(r,r+1)(v-2)(v-1,v+2)(v,v+1)(v+3)$$

results in the x-graph $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ having connected components ,

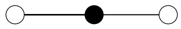
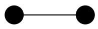
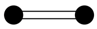
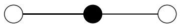
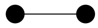
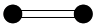
, ○ and possibly  and . Moreover, $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ has connected components ○, ,  and for some values of r also  and/or .

If $q = 2$, then we take

$$y_{ijk} = (1, v+1)(2, v+2)(3)(4)(2q+1, v)\cdots(2q+r, 2q+r+1)(v+3),$$

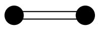
whilst if $q > 2$ we take

$$\begin{aligned} y_{ijk} = (1, v+1)(2, v+2)(3)(4, 2q-1)\cdots(q+1, q+2)(2q) \\ (2q+1, v)\cdots(2q+r, 2q+r+1)(v+3). \end{aligned}$$

Consequently, the x-graph $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ has connected components  and ○ and possibly also ●, ,  and . Meanwhile, $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ has connected components ,  and ○ and in some cases also  and/or .

□

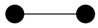
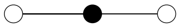
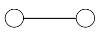
Lemma 3.7. *Suppose that $m \geq 2$, $n \geq 7$ with $n \neq 2m+1$ and let $x \in X \setminus \{t\}$ be such that $\mathrm{fix}(t) = \mathrm{fix}(x)$. Then there exists $y \in X$ such that $d(t, y) = d(y, x) = 1$.*

Proof. By considering $t, x \in \mathrm{Sym}(\mathrm{supp}(t))$ and appealing to Corollary 3.4 we may assume that $m = 2$ or 3 and hence that $|\,\mathrm{fix}(t)| \geq 2$. If $\mathcal{G}_x$ contains a connected component of the form , then without loss of generality we have

$$t = (1,2)(3,4)(5,6)(7)\cdots(n) \quad \text{and} \quad x = (1,3)(2,4)(5,6)(7)\cdots(n)$$

(where the transposition $(5,6)$ is replaced by $(5)(6)$ if $m = 2$). We take $y = (1,4)(2)(3)(5,6)(7)\cdots(n-2)(n-1,n)$ (again replacing $(5,6)$ by $(5)(6)$ if $m = 2$). If $m = 3$ and $\mathcal{G}_x$ contains a cycle of three black vertices, then we may assume that

$$t = (1,2)(3,4)(5,6)(7)\cdots(n) \quad \text{and} \quad x = (1,6)(2,3)(4,5)(7)\cdots(n).$$

In this case, we set $y = (1)(2,3)(4)(5,6)(7,8)(9)\cdots(n)$. In all cases we have that $\mathcal{G}_y$ has one connected component of the form , $\mathcal{G}_x^y$ has one connected component of the form  and all other connected components of these x-graphs are of the form ○, ●,  and . Thus $d(t, y) = d(y, x) = 1$ by Corollary 2.4. □

We are now in a position to prove Theorem 1.1 in the general case. For $x \in X$ we proceed by considering collections of connected components $\{C_i\}_{i \in I}$ of $\mathcal{G}_x$ for some set I, and then finding an element $y_I \in \mathrm{Sym}(\cup_{i \in I}\mathrm{supp}(C_i))$ that

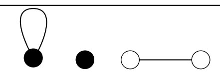
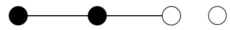
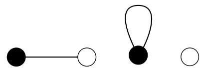
Representative $x \in X \setminus (\Delta_1(t) \cup \{t\})$	x-graph, $\mathcal{G}_x$	Representative $y \in \Delta_1(t) \cap \Delta_1(x)$
$(1,2)(5,6)$		$(3,6)(4,5)$
$(2,5)(4,6)$		$(1,3)(5,6)$
$(2,3)(4,5)$		$(1,4)(5,6)$
$(2,5)(3,4)$		$(1,5)(2,6)$
$(1,3)(2,4)$		$(1,4)(5,6)$

Table 1: Representatives of $X \setminus (\Delta_1(t) \cup \{t\})$ and their corresponding neighbours in $\Delta_1(t) \cap \Delta_1(x)$ for $n = 6$, $m = 2$.

is conjugate to $t_I := \prod_{i \in I} t_i$ such that the connected components of $\mathcal{G}_{y_I}^{t_I}$ and $\mathcal{G}_{x_I}^{y_I}$ satisfy the conditions of Corollary 2.4 (where $x_I := \prod_{i \in I} x_i$). The product of all such y_I will then be our desired element of X.

Proof of Theorem 1.1:
Let $x \in X$.

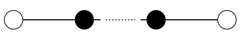
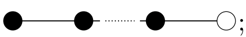
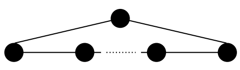
(i) First assume that $n = 2m + 1$. We observe that the product of two elements of X that fix distinct elements of Ω cannot have order 4. Thus $\mathcal{P}_{\{4\}}(G, X)$ consists of n copies of the $\{4\}$-product involution graph $\mathcal{P}_{\{4\}}(G, Y)$, where Y is the conjugacy class of $\mathrm{Sym}(2m)$ consisting of elements of cycle type 2^m.

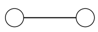
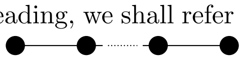
If $m = 1$, then $\mathcal{P}_{\{4\}}(G, X)$ is clearly totally disconnected.

In the case that $(n, m) = (4, 2)$ (respectively $(n, m) = 6, 3)$), then any x-graph will contain 2 (respectively 3) black vertices and 2 (respectively 3) edges. It follows from Corollary 2.4 that $\mathcal{P}_{\{4\}}(G, X)$ is totally disconnected.

(ii) Assume that $m \neq 1$ and $(n, m) \neq (4, 2), (6, 3)$ or $(2m + 1, m)$. We first consider the case that $(n, m) = (6, 2)$. If distinct involutions $t = (1, 2)(3, 4)$ and x of cycle type 2^2 do not have product of order 4, then the reader may check that the x-graph $\mathcal{G}_x$ will be isomorphic to one given in Table 1 and that the given element y satisfies $d(t, y) = d(y, x) = 1$. Hence $\mathcal{P}_{\{4\}}(G, X)$ is connected and $\mathrm{Diam}(\mathcal{P}_{\{4\}}(G, X)) = 2$.

By Corollary 3.4, as $n \neq 2m + 1$, we may assume that $|\mathrm{fix}(t)| \geq 2$, and so $\mathcal{G}_x$ contains at least 2 white vertices. Moreover, by Lemma 3.7 we only need to consider the case when $\mathrm{fix}(t) \neq \mathrm{fix}(x)$. Let α, β, γ and δ denote the number of connected components (containing at least 1 black vertex and 1 edge) of $\mathcal{G}_x$ of the form

α: ;
β: ;
γ: ; and
δ: ,

and let ϵ denote the number of connected components of the form . For ease of reading, we shall refer to components of type α instead of components of the form . Similarly for β, γ, δ and ϵ. Note that $\alpha \leq \beta + \epsilon$, and as $\mathrm{fix}(t) \neq \mathrm{fix}(x)$ it follows that β, γ and ϵ are not all zero.

If $\gamma \geq 2$, then partitioning the components of type γ into pairs or triples we obtain a suitable y_I from Lemmas 3.5(v) and 3.6(i). Indexing the remaining connected components by J, a suitable y_J such that $t_J y_J$ and $x_J y_J$ have orders 1, 2 or 4 may be constructed using Lemmas 3.1, 3.2, and 3.5(i),(ii). In the forthcoming cases, when referring to the construction of y_J, it will be implicit that $t_J y_J$ and $x_J y_J$ have orders 1, 2 or 4.

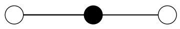
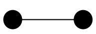
If $\gamma = 1$ and $\beta \neq 0$, then we pair the unique component of type γ with one of type β to obtain an element y_I via Lemma 3.5(iv). An element y_J for the remaining components follows from Lemmas 3.1, 3.2 and 3.5(i),(ii).

If $\gamma = 1$, $\beta = 0$ and $\alpha \geq 1$, then $\epsilon \geq 1$. Hence we may use Lemma 3.6(ii) to construct the element y_I and Lemmas 3.1, 3.2 and 3.5(i) to obtain a suitable y_J.

If $\gamma = 1$ and $\alpha = \beta = 0$, then there must be a connected component of $\mathcal{G}_x$ consisting of a single vertex. Assume first that the connected component of type γ contains at least two black vertices. If there is an isolated white vertex in $\mathcal{G}_x$, then the existence of y_I follows from Lemma 3.5(iii). Conversely, if there is an isolated black vertex, then - as the number of black vertices equals the number of edges - there must be a connected component of type ϵ. Applying Lemma 3.6(ii) to this connected component, the connected component of type γ and an isolated black vertex results in our element y_I. Applying Lemmas 3.1 and 3.2 to our remaining components as appropriate gives our desired element y_J.

Now assume that our connected component of type γ contains precisely one black vertex. If all other white vertices are isolated, then $\mathcal{G}_x$ contains a connected component consisting of a cycle of $u \geq 1$ black vertices. We may consider one such cycle, an isolated white vertex and the connected component of type γ to correspond to those components indexed by I. Thus without loss of generality

$$t_I = (1,2)\cdots(2u-1,2u)(2u+1,2u+2)(2u+3)(2u+4); \quad \text{and}$$
$$x_I = (1,2u)(2,3)\cdots(2u-2,2u-1)(2u+1)(2u+2,2u+3)(2u+4),$$

unless $u = 1$ in which case we let $t_I = (1,2)(3,4)(5)(6)$ and $x_I = (1,2)(3)(4,5)(6)$. Taking $y_I = (1)(2u)(2,2u-1)\cdots(u,u+1)(2u+1,2u+3)(2u+2,2u+4)$ (or $y_I = (1)(2)(3,5)(4,6)$ if $u = 1$) it follows that $\mathcal{G}_{y_I}^{t_I}$ has one connected component of the form , $\mathcal{G}_{x_I}^{y_I}$ has one connected component of the form  and the remaining components of these x-graphs are of the form , , ,  and , thus satisfying the conditions of Corollary 2.4.

Conversely, if there exists a white vertex that is not isolated, then it will be in a component of type ϵ. Again, as the number of edges and black vertices must be equal, there exists an isolated black vertex. We take the connected component of type γ along with one of type ϵ and an isolated black vertex to be those indexed by I. The existence of y_I then follows from Lemma 3.6(ii). Finally applying Lemmas 3.1 and 3.2 as appropriate to the remaining connected components, we obtain an element y_J as required.

If $\gamma = 0$, but $\beta \neq 0$, then consider the connected components of type α. If there exist connected components of type α containing at least 3 black vertices, then we may pair these up with connected components of type β and ϵ and apply Lemma 3.5(i) to obtain our element y_I. If all connected components of type α contain at most 2 black vertices, then we simply apply Lemma 3.5(ii) (if required) to the connected components of type β to obtain y_I. Finally, applying Lemmas 3.1, 3.2 and 3.5(ii) to the remaining connected components ensures the existence of y_J.

If $\beta = \gamma = 0$ and $\alpha \neq 0$, then we apply Lemma 3.5(i) to the connected components of type α and ϵ (if required) to obtain y_I and Lemmas 3.1 and 3.2 to the remaining connected components to find a suitable y_J.

If $\alpha = \beta = \gamma = 0$, then $\epsilon \geq 1$ as by assumption $\mathrm{fix}(t) \neq \mathrm{fix}(x)$. As the number of edges of $\mathcal{G}_x$ equals the number of black vertices, there exists an isolated black vertex. Moreover, as $m \geq 2$, there are two possible cases. If every black vertex is isolated, then there exists m connected components of type ϵ. Take two such components and two isolated black vertices as the components corresponding to our indexing set I. Without loss of generality, we may assume that the parts of t and x corresponding to I are

$$t_I = (1,2)(3,4)(5)(6)(7)(8) \quad \text{and} \quad x_I = (1)(2)(3)(4)(5,6)(7,8).$$

Let $y_I = (1,5)(2,6)(3)(4)(7)(8)$. Then the connected components of $\mathcal{G}_{y_I}^{t_I}$ and $\mathcal{G}_{x_I}^{y_I}$ satisfy the conditions of Corollary 2.4.

Conversely, if there is only one isolated black vertex, then there exists a connected component which is a cycle of $u \geq 1$ black vertices. Thus taking the components indexed by I to be an isolated black vertex, a cycle of $u \geq 1$ black vertices and a component of type ϵ, we may take t_I and x_I to be

$$t_I = (1,2)\cdots(2u-1,2u)(2u+1,2u+2)(2u+3)(2u+4); \quad \text{and}$$
$$x_I = (1,2u)(2,3)\cdots(2u-2,2u-1)(2u+1)(2u+2)(2u+3,2u+4),$$

or $t_I = (1,2)(3,4)(5)(6)$ and $x_I = (1,2)(3)(4)(5,6)$ if $u = 1$. Taking $y_I = (1)(2u)(2,2u-1)\cdots(u,u+1)(2u+1,2u+3)(2u+2,2u+4)$ (or $y_I = (1)(2)(3,5)(4,6)$ if $u = 1$), we see that the connected components of $\mathcal{G}_{y_I}^{t_I}$ and $\mathcal{G}_{x_I}^{y_I}$ satisfy the conditions of Corollary 2.4. Finally, applying Lemmas 3.1 and 3.2 to the remaining cycles of black vertices gives the desired y_J.

Since all possible x-graphs have been analysed, this completes the proof of Theorem 1.1. □

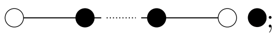
4 The cases $\pi \neq \{4\}$:

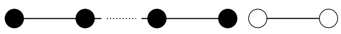
We illustrate the exceptional nature of $\mathcal{P}_{\{4\}}(G,X)$ with a brief exploration of other π-product involution graphs. We begin by considering the case that $\pi = \{2m\}$ and $2m = 2^a$ for some $a \geq 3$. The simplest such case arises when $a = 3$. Thus for $G = \mathrm{Sym}(n)$, we consider the G-conjugacy class of $t = (1,2)(3,4)(5,6)(7,8)$, which we denote by X. As $\mathrm{supp}(t)$ has size 8, it suffices to consider $8 \leq n \leq 16$. We calculate the sizes of the discs $\Delta_i(t)$ of $\mathcal{P}_{\{8\}}(G,X)$ using the computer algebra package MAGMA (see [8] and [13]). Theorem 1.3 is an immediate consequence of our calculations, which are summarised in Table 2. The situation observed is indicative of the situation which arises when $\pi = \{2m\}$ and $2m = 2^a$ for some $a \geq 3$. This leads to the formulation of Theorem 1.2, a proof of which we now give.

n	$\|\Delta_1(t)\|$	$\|\Delta_2(t)\|$	$\|\Delta_3(t)\|$	$\|\Delta_4(t)\|$	$\|X\|$	$\mathrm{Diam}(\mathcal{P}_{\{8\}}(G,X))$
8					105	Totally Disconnected
9					945	Totally Disconnected
10	384	4308	32		4725	3
11	1152	16076	96		17325	3
12	2304	49382	288		51975	3
13	3840	123974	7320		135135	3
14	5760	267014	42540		315315	3
15	8064	512630	154140	840	675675	4
16	10752	902012	431760	6825	1351350	4

Table 2: The sizes of the discs $\Delta_i(t)$ for $\mathcal{P}_{\{8\}}(G,X)$, where $G := \mathrm{Sym}(n)$ and X is the G-conjugacy class of $t := (1,2)(3,4)(5,6)(7,8)$.

Proof of Theorem 1.2:
(i) We note that as the support of t has size $2m$, any x in X - the G-conjugacy class of t - will have an x-graph, $\mathcal{G}_x$, containing m black vertices. If the order of tx is $2m$, then $\mathcal{G}_x$ must be of the form

(1) ; or

(2) ,

(where additional isolated white vertices may be present). In both cases, we see that $|\mathrm{supp}(t) \cap \mathrm{supp}(x)| = 2m-2$. It follows that if $n = 2m$ or $2m+1$, then $\mathcal{P}_{\{2m\}}(G,X)$ is totally disconnected. Thus assume that $n \geq 2m+2$. Denote the m transpositions that x is comprised of by $x_1, \dots, x_m$, where $\min \mathrm{supp}(x_i) \leq \min \mathrm{supp}(x_{i+1})$ for all $i = 1, \dots, m-1$. We will construct elements $y_i \in X$ for $i = 1, \dots, m$ such that

$$y_i = x_1 \cdots x_i w_{i+1} \cdots w_m \tag{4.1}$$

for some transpositions $w_{i+1}, \dots, w_m$ and such that y_i is connected to t in $\mathcal{P}_{\{2m\}}(G,X)$. At each stage, if y_i involves the transposition x_{i+1} we are done, so we will assume that this is not the case.

Assume first that $|\mathrm{supp}(x_1) \cap \mathrm{supp}(t)| = 2$. Then without loss we may assume that $x_1 = (1,3)$. Define

$$y_1 = (1,3)(4,5)(6,7) \cdots (2m-2, 2m-1)(2m+1, 2m+2).$$

Thus $\mathcal{G}_{y_1}$ has the form (2). If $|\mathrm{supp}(x_1) \cap \mathrm{supp}(t)| = 1$, then we may assume that $x_1 = (1, 2m+1)$. Taking

$$y_1 = (1, 2m+1)(2,3)(4,5) \cdots (2m-4, 2m-3)(2m-2, 2m+2),$$

it follows that $\mathcal{G}_{y_1}$ is isomorphic to (1). Finally, if $|\mathrm{supp}(x_1) \cap \mathrm{supp}(t)| = 0$, then we assume that $x_1 = (2m+1, 2m+2)$. The element

$$y_1 = (2,3)(4,5) \cdots (2m-2, 2m-1)(2m+1, 2m+2)$$

results in an x-graph $\mathcal{G}_{y_1}$ of type (2). In all cases we see that t and y_1 are adjacent - and hence connected - vertices of $\mathcal{P}_{\{2m\}}(G,X)$.

Suppose that for some $1 \leq i < m$ an element y_i of the form (4.1) exists with y_i connected to t in $\mathcal{P}_{\{2m\}}(G, X)$. Note that y_i fixes at least two elements of Ω as $n \geq 2m + 2$. We denote these elements by f_1, f_2. Define

$$x_{j,1} := \min \mathrm{supp}(x_j) \quad \text{and} \quad x_{j,2} := \max \mathrm{supp}(x_j). \tag{4.2}$$

for $1 \leq j \leq i$ and

$$w_{j,1} := \min \mathrm{supp}(w_j) \quad \text{and} \quad w_{j,2} := \max \mathrm{supp}(w_j). \tag{4.3}$$

for $i + 1 \leq j \leq m$. Set $\alpha = x_{i+1,1}$, $\beta = x_{i+1,2}$ and $w = w_{i+1} \cdots w_m$. Thus $x_{i+1} = (\alpha, \beta)$.

We follow an analogous approach to that used to define y_1. If $|\mathrm{supp}(x_{i+1}) \cap \mathrm{supp}(w)| = 2$, then without loss we have that $w_{i+1,1} = \alpha$ and $w_{i+2,1} = \beta$. First we construct an element $z_{i+1} \in X$ given by

$$\begin{aligned} z_{i+1} = (x_{1,1}, w_{m,2})(x_{1,2}, x_{2,1})(x_{2,2}, x_{3,1}) \cdots (x_{i-1,2}, x_{i,1}) \\ (x_{i,2}, \alpha)(w_{i+2,2}, w_{i+3,1}) \cdots (w_{m-1,2}, w_{m,1})(f_1, f_2). \end{aligned}$$

Consequently $\mathcal{G}^{y_i}_{z_{i+1}}$ has the form given in (2). The element

$$y_{i+1} = x_1 x_2 \cdots x_{i+1}(w_{i+1,2}, w_{i+2,2}) w_{i+3} \cdots w_m$$

results in an x-graph $\mathcal{G}^{z_{i+1}}_{y_{i+1}}$ of isomorphism type (1). We deduce that y_{i+1} is connected to y_i and hence to t, and that $d(y_i, y_{i+1}) \leq 2$.

In the case that $|\mathrm{supp}(x_{i+1}) \cap \mathrm{supp}(w)| = 1$, we may assume that $w_{i+1,1} = \alpha$ and that $\beta = f_1 \in \mathrm{fix}(y_i)$. Taking

$$\begin{aligned} z_{i+1} = (\beta, x_{2,1})(x_{2,2}, x_{3,1}) \cdots (x_{i-1,2}, x_{i,1})(x_{i,2}, w_{m,2}) \\ (w_{i+1,2}, w_{i+2,1}) \cdots (w_{m-1,2}, w_{m,1})(\alpha, f_2) \end{aligned}$$

and

$$y_{i+1} = x_1 \cdots x_{i+1} w_{i+2} \cdots w_m$$

we see that $\mathcal{G}^{y_i}_{z_{i+1}}$ is of isomorphism type (1), whilst $\mathcal{G}^{z_{i+1}}_{y_{i+1}}$ is of isomorphism type (2). Thus $d(y_i, y_{i+1}) \leq 2$ and y_{i+1} is connected to t.

The final possibility is that $|\mathrm{supp}(x_{i+1}) \cap \mathrm{supp}(w)| = 0$. Consequently, defining

$$\begin{aligned} z_{i+1} = (x_{1,1}, x_{i+1,1})(x_{1,2}, x_{2,1}) \cdots (x_{i-1,2}, x_{i,1})(x_{i,2}, w_{m,2}) \\ (x_{i+1,2}, w_{i+2,1})(w_{i+2,2}, w_{i+3,1}) \cdots (w_{m-1,2}, w_{m,1}) \end{aligned}$$

and

$$y_{i+1} = x_1 \cdots x_{i+1} w_{i+1} w_{i+3} \cdots w_{m-1}$$

we obtain x-graphs $\mathcal{G}^{y_i}_{z_{i+1}}$ and $\mathcal{G}^{z_{i+1}}_{y_{i+1}}$ of types (1) and (2) respectively. We conclude that $d(y_i, y_{i+1}) \leq 2$, and hence y_{i+1} is connected to t in $\mathcal{P}_{\{2m\}}(G, X)$ as required.

(ii) If $\mathcal{P}_{\{2m\}}(G, X)$ is connected, then $n \geq 2m+2$ by (i). Moreover, the above argument shows that $d(t, y_1) \leq 1$ and for $1 \leq i \leq m - 1$ we have $d(y_i, y_{i+1}) \leq 2$.

Thus as $x = y_m$, we conclude that $\mathrm{Diam}(\mathcal{P}_{\{2m\}}(G,X)) \leq 2m-1$. For the lower bound, we note that $x \in \Delta_1(t)$ precisely when $\mathcal{G}_x$ is of type (1) or (2). In particular $|\mathrm{supp}(t) \cap \mathrm{supp}(x)| = 2m-2$. Arguing iteratively we deduce that if $d(t,x) \leq s$, then $|\mathrm{supp}(t) \cap \mathrm{supp}(x)| \geq 2m-2s$. Since X contains an involution x satisfying $|\mathrm{supp}(t) \cap \mathrm{supp}(x)| = \max\{0, 4m-n\}$ we deduce that

$$\mathrm{Diam}(\mathcal{P}_{\{2m\}}(G,X)) \geq \min\{m, \lceil n/2 - m \rceil\}$$

as required. □

We note that it is also possible to define $y_1 \in X$ of the form (4.1) which is connected to t by a path of length 2 in $\mathcal{P}_{\{2m\}}(G,X)$. Indeed, if $x_1 = (1,3)$, we set

$$\begin{aligned} z_1 =& (2,5)(4,2m)(6,7)\cdots(2m-2,2m-1)(2m+1,2m+2); \text{ and} \\ y_1 =& (1,3)(4,2m+1)(5,6)(7,8)\cdots(2m-1,2m). \end{aligned}$$

If $x_1 = (1, 2m+1)$, then define

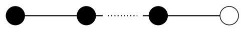
$$\begin{aligned} z_1 =& (2,3)(4,5)\cdots(2m-2,2m-1)(2m+1,2m+2); \text{ and} \\ y_1 =& (1,2m+1)(2,2m+2)(3,4)\cdots(2m-5,2m-4)(2m-3,2m). \end{aligned}$$

Finally, if $x_1 = (2m+1, 2m+2)$, then take

$$\begin{aligned} z_1 =& (1,2m+1)(2,3)(4,5)\cdots(2m-4,2m-3)(2m-2,2m+2); \text{ and} \\ y_1 =& (3,4)(5,6)\cdots(2m-1,2m)(2m+1,2m+2). \end{aligned}$$

In each case, the x-graphs $\mathcal{G}_{z_1}$ and $\mathcal{G}_{y_1}^{z_1}$ are of isomorphism type (1) or (2) as required.

We now consider Theorem 1.4. A non-constructive proof of the connectivity of $\mathcal{P}_{\{q\}}(G,X)$ using Jordan's theorem is contained in the proof of [7, Theorem 4.1]. Here we give a constructive proof in a similar vein to the proof of Theorem 1.2 above.

Proof of Theorem 1.4:
(i) We first note that the elements of the disc $\Delta_1(t)$ are precisely those elements $x \in X$ whose x-graph, $\mathcal{G}_x$, is of isomorphism type . Consequently, $\mathcal{P}_{\{q\}}(G,X)$ is totally disconnected if $n = q-1$. Thus assume that $n \geq q$ and hence that $|\mathrm{fix}(t)| \geq 1$.

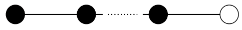
We proceed as in the proof of Theorem 1.2 and set $2m = q-1$. Let $x \in X$ be given and denote the transpositions of x as $x = x_1 x_2 \cdots x_m$. As in the proof of Theorem 1.2, we will construct elements $y_i \in X$ for $i = 1, \dots m$ such that

$$y_i = x_1 \cdots x_i w_{i+1} \cdots w_m \tag{4.4}$$

for some transpositions w_j and such that y_i is connected to t in $\mathcal{P}_{\{q\}}(G,X)$. Mirroring the situation of the proof of Theorem 1.2, we may assume that x_{i+1} is not a transposition of y_i. We continue to use the notation $x_{j,1}, x_{j,2}$ and $w_{j,1}, w_{j,2}$ previously introduced in (4.2) and (4.3) respectively. For convenience we define $y_0 := t$ and for each y_i we consider the cases $\mathrm{supp}(y_i) = \mathrm{supp}(x)$ and $\mathrm{supp}(y_i) \neq \mathrm{supp}(x)$ separately.

Assume that $\text{supp}(t) = \text{supp}(x)$. Thus $|\text{supp}(t) \cap \text{supp}(x_1)| = 2$ and without loss of generality we may take $x_1 = (1, 3)$. Define z_1 and y_1 by

$$\begin{aligned} z_1 =& (2,3)(4,5)\cdots(2m, 2m+1); \text{ and} \\ y_1 =& x_1(2,4)(5,6)\cdots(2m-1, 2m). \end{aligned}$$

The x-graphs $\mathcal{G}_{z_1}$ and $\mathcal{G}_{y_1}^{z_1}$ are both of isomorphism type  and hence t and y_1 are connected in $\mathcal{P}_{\{q\}}(G, X)$ with $d(t, y_1) \leq 2$.

When $\text{supp}(t) \neq \text{supp}(x)$ we consider three subcases. If $|\text{supp}(t) \cap \text{supp}(x_1)| = 2$, then without loss we have $x_1 = (1, 3)$ and we take

$$y_1 = x_1(4,5)(6,7)\cdots(2m, 2m+1).$$

If $|\text{supp}(t) \cap \text{supp}(x_1)| = 1$, then we may assume that $x_1 = (1, 2m+1)$ and thus take

$$y_1 = x_1(2,3)(4,5)\cdots(2m-2, 2m-1).$$

In both cases, $\mathcal{G}_{y_1}$ is of the required form. Consequently t and y_1 are adjacent in $\mathcal{P}_{\{q\}}(G, X)$. Finally, suppose that $|\text{supp}(t) \cap \text{supp}(x_1)| = 0$ and hence that $x_1 = (2m+1, 2m+2)$. Defining

$$\begin{aligned} z_1 =& (1, 2m+1)(2,3)\cdots(2m-2, 2m-1); \text{ and} \\ y_1 =& (x_1)(1,2)(3,4)\cdots(2m-1, 2m-2), \end{aligned}$$

we have that $\mathcal{G}_{z_1}$ and $\mathcal{G}_{y_1}^{z_1}$ are of the aforementioned isomorphism type, and so t and y_1 are connected in $\mathcal{P}_{\{q\}}(G, X)$. Moreover, $d(t, y_1) \leq 2$.

Now suppose that a y_i of the form (4.4) has been defined for some $i < m$ with y_i connected to t in $\mathcal{P}_{\{q\}}(G, X)$. First assume that $\text{supp}(y_i) = \text{supp}(x)$ and that $\alpha \in \text{fix}(y_i) \cap \text{fix}(x)$. Without loss we may assume that $x_{i+1,1} = w_{i+1,1}$ and $x_{i+1,2} = w_{i+2,1}$. Define

$$\begin{aligned} z_{i+1} =& (x_{1,2}, x_{2,1})(x_{2,2}, x_{3,1})\cdots(x_{i-1,2}, x_{i,1})(x_{i,2}, w_{i+1,1}) \\ & (w_{i+1,2}, w_{i+2,1})\cdots(w_{m-1,2}, w_{m,1})(w_{m,2}, \alpha); \text{ and} \\ y_{i+1} =& x_1 x_2 \cdots x_{i+1}(w_{i+1,2}, w_{i+2,2}) w_{i+3}\cdots w_m. \end{aligned}$$

We have that the x-graphs $\mathcal{G}_{z_{i+1}}^{y_i}$ and $\mathcal{G}_{y_{i+1}}^{z_{i+1}}$ have the required form and hence $d(y_i, y_{i+1}) \leq 2$.

It remains to consider the case that $\text{supp}(y_i) \neq \text{supp}(x)$. Let $f_i \in \text{fix}(y_i) \setminus \text{fix}(x)$ and $f_x \in \text{fix}(x) \setminus \text{fix}(y_i)$. In the case when $|\text{supp}(y_i) \cap \text{supp}(x_{i+1})| = 2$, assume that $x_{i+1} = (w_{i+1,1}, w_{i+2,1})$ and set

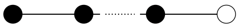
$$\begin{aligned} z_{i+1} =& (x_{1,1}, w_{m,2})(x_{1,2}, x_{2,1})\cdots(x_{i-1,2}, x_{i,1})(x_{i,2}, w_{i+1,1})(w_{i+2,1}, f_i) \\ & (w_{i+2,2}, w_{i+3,1})\cdots(w_{m-1,2}, w_{m,1}); \text{ and} \\ y_{i+1} =& x_1 x_2 \cdots x_{i+1}(w_{i+1,2}, w_{i+2,2}) w_{i+3}\cdots w_m. \end{aligned}$$

If $|\text{supp}(y_i) \cap \text{supp}(x_{i+1})| = 1$, then without loss we have $x_{i+1} = (w_{i+1,1}, f_x)$. Hence we define

$$\begin{aligned} z_{i+1} =& (x_{1,1}, w_{m,2})(x_{1,2}, x_{2,1})\cdots(x_{i-1,2}, x_{i,1})(x_{i,2}, x_{i+1,1}) \\ & (w_{i+2,1}, f_x)(w_{i+2,2}, w_{i+3,1})\cdots(w_{m-1,2}, w_{m,1}); \text{ and} \\ y_{i+1} =& x_1 x_2 \cdots x_{i+1} w_{i+2}\cdots w_{m-1}(w_{m,1}, w_{i+1,2}). \end{aligned}$$

Finally, if x_{i+1} and y_i are disjoint we set

$$\begin{aligned} z_{i+1} =& (x_{1,1}, x_{i+1,1})(x_{1,2}, x_{2,1}) \cdots (x_{i-1,2}, x_{i,1})(x_{i,2}, w_{i+1,1}) \\ & (w_{i+1,2}, w_{i+2,1}) \cdots (w_{m-1,2}, w_{m,1}); \text{ and} \\ y_{i+1} =& x_1 x_2 \cdots x_{i+1} w_{i+1} \cdots w_{m-1}. \end{aligned}$$

For each pair (z_{i+1}, y_{i+1}) the x-graphs $\mathcal{G}_{z_{i+1}}^{y_i}$ and $\mathcal{G}_{y_{i+1}}^{z_{i+1}}$ have isomorphism type . Consequently $d(y_i, y_{i+1}) \leq 2$ and the elements t and y_{i+1} are connected in $\mathcal{P}_{\{q\}}(G, X)$.

(ii) By part (i) we have that $\operatorname{Diam} \mathcal{P}_{\{q\}}(G, x) \leq q - 1$. For the lower bound, we note that for $x \in X$ to be adjacent to t we have $|\operatorname{supp}(t) \cap \operatorname{supp}(x)| = |\operatorname{supp}(t)| - 1$. Taking $x \in X$ such that $|\operatorname{supp}(t) \cap \operatorname{supp}(x)|$ is minimal we have that $|\operatorname{supp}(t) \cap \operatorname{supp}(x)| = \max\{0, 2q - 2 - n\}$ and hence $d(t, x) \geq \min\{q - 1, n + 1 - q\}$ as required. □

Following a similar approach to that used when $p = 2$, we define $y_1 \in X$ of the form (4.4) such that there is a path in $\mathcal{P}_{\{q\}}(G, X)$ from t to y_1 of length 2. These paths will be used in the proof of Theorem 1.5. Such paths were defined in the above proof except when $\operatorname{supp}(t) \neq \operatorname{supp}(x)$ and $|\operatorname{supp}(t) \cap \operatorname{supp}(x_1)| = 1$ or 2. In the latter case, we may assume without loss that $x_1 = (1, 3)$ and define

$$\begin{aligned} z_1 =& (2, 3)(4, 5) \cdots (2m, 2m + 1); \text{ and} \\ y_1 =& (1, 3)(2, 4)(5, 6) \cdots (2m - 1, 2m). \end{aligned}$$

In the former case, we assume that $x_1 = (1, 2m + 1)$ and set

$$\begin{aligned} z_1 =& (2, 3)(4, 5) \cdots (2m, 2m + 1); \text{ and} \\ y_1 =& (1, 2m + 1)(3, 4)(5, 6) \cdots (2m - 1, 2m). \end{aligned}$$

Both cases give rise to x-graphs $\mathcal{G}_{z_1}$ and $\mathcal{G}_{y_1}^{z_1}$ of the required form to show that there is a path from t to y_1 of length 2 in $\mathcal{P}_{\{q\}}(G, X)$.

The proofs of Theorems 1.2 and 1.4 are utilised in the proof of Theorem 1.5.

Proof of Theorem 1.5:
The x-graph $\mathcal{G}_x$ of any $x \in \Delta_1(t)$ must consist of a connected component of isomorphism type  containing $q_i/2$ black vertices for each $i = 2, \ldots, r$. In addition, there will be connected components of types (1) or (2) from the proof of Theorem 1.2 if $p_1 = 2$, or a component of type  containing $q_1/2$ black vertices if $p_1 \neq 2$. It follows that

$$|\operatorname{fix}(t)| \geq \begin{cases} r + 1 & \text{if } p_1 = 2; \text{ and} \\ r & \text{otherwise.} \end{cases} \tag{4.5}$$

We conclude that if

$$n \geq \begin{cases} 2m + 2 & \text{if } p_1 = 2; \text{ and} \\ 2m & \text{otherwise} \end{cases} \tag{4.6}$$

does not hold, then $\mathcal{P}_{\{q\}}(G, X)$ is totally disconnected. Thus assume that (4.6) holds.

Denote the the transpositions of x by

$$x = x_1^{(1)} \cdots x_{q_1/2}^{(1)} x_1^{(2)} \cdots x_{q_2/2}^{(2)} \cdots x_1^{(r)} \cdots x_{q_r/2}^{(r)}.$$

Moreover, if $\ell = \max\{q_i\}$, define $x_{q_j/2+1}^{(j)} = \cdots = x_{q_\ell/2}^{(j)} = 1$ for all $j = 1, \ldots, r$ where these permutations are not already defined.

Using the proofs of Theorems 1.1, 1.2 and 1.4 and the comments following the proofs, we may construct $y_i \in X$ for $i = 0, \ldots, \ell/2$ with $y_0 := t$ satisfying

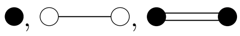
(i) there is a path in $\mathcal{P}_{\{q\}}(G, X)$ from y_i to y_{i+1} of length 2 for $i = 0, \ldots, \ell/2 - 1$; and

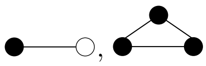
(ii) y_i contains the transpositions $x_1^{(j)}, \ldots, x_i^{(j)}$ for $j = 1, \ldots, r$.

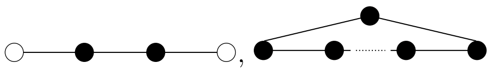
This is allowable as (4.5) holds. The result now follows immediately. $\square$

We briefly consider an example of a case when consists of a composite number. The smallest such situation arises when $\pi = \{6\}$. If $t \in G$ is an involution and X is the G-conjugacy class of t, then any $x \in X$ will be adjacent to t in $\mathcal{P}_{\{6\}}(G, X)$ if the connected components of $\mathcal{G}_x$ consist of components of the form

(i) ;

(ii) ;

(iii) ; and

(iv)  (with 6 black vertices).

Moreover, either one component is of isomorphism type (iv), or there exists at least one component of type (ii) and one component of type (iii).

Finally, Table 3 gives the sizes of the discs $\Delta_i(t)$ of $\mathcal{P}_{\{6\}}(G, X)$ for the symmetric groups $G := \mathrm{Sym}(n)$ $(6 \leq n \leq 10)$, when X is the G-conjugacy class of an involution $t \in G$.

n	m	$\|\Delta_1(t)\|$	$\|\Delta_2(t)\|$	$\|\Delta_3(t)\|$	$\|X\|$	Diam($\mathcal{P}_{\{6\}}(G, X)$)
6	2				45	Totally Disconnected
	3				15	Totally Disconnected
7	2	12	38	54	105	3
	3	12	60	32	105	3
8	2	48	158	3	210	3
	3	72	347		420	2
	4				105	Totally Disconnected
9	2	120	242	15	378	3
	3	216	1043		1260	2
	4	48	836	60	945	3
10	2	240	389		630	2
	3	624	2525		3150	2
	4	416	4308		4725	2
	5	160	784		945	2

Table 3: The sizes of the discs $\Delta_i(t)$ for $\mathcal{P}_{\{6\}}(G, X)$, where X is the G-conjugacy class of $t = (1, 2) \cdots (2m - 1, 2m) \in G := \mathrm{Sym}(n)$.

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