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MILDNESS AND THE DENSITY OF RATIONAL POINTS ON CERTAIN TRANSCENDENTAL CURVES

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ABSTRACT. We use a result due to Rolin, Speissegger and Wilkie to show that definable sets in certain o-minimal structures admit definable parameterizations by mild maps. We then use this parameterization to prove a result on the density of rational points on curves defined by restricted Pfaffian functions.

1. INTRODUCTION

The main result of this note is a generalization of some results of Pila ([9]) to a wider collection of curves. Before stating the result, we need some definitions. A sequence $f_1, \ldots, f_r : U \to \mathbb{R}$ of analytic functions on an open set $U \subseteq \mathbb{R}^n$ is said to be a *Pfaffian chain* of *order* r and *degree* α if there are polynomials $P_{i,j} \in \mathbb{R}[X_1, \ldots, X_{n+j}]$ of degree at most α such that

$$df_j = \sum_{i=1}^{n} P_{i,j}(\bar{x}, f_1(\bar{x}), \ldots, f_j(\bar{x}))dx_i, \text{ for } j = 1, ..., r.$$

Given such a chain, we say that a function $f : U \to \mathbb{R}$ is *Pfaffian* of *order* r and *degree* (α, β) with chain $f_1, \ldots, f_r$, if there is a polynomial $P \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ of degree at most β such that $f(\bar{x}) = P(\bar{x}, f_1(\bar{x}), \ldots, f_r(\bar{x}))$.

Let $U \subseteq \mathbb{R}^n$ be an open set containing $[0, 1]^n$. To every function $f : U \to \mathbb{R}$, we associate a new function $\hat{f} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\hat{f}(\bar{x}) = \begin{cases} f(\bar{x}) & \text{if } \bar{x} \in [0,1]^n, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\mathbb{R}_{\mathrm{an}}$ is the expansion of the real ordered field by all functions of the form $\hat{f}$, where $f : U \to \mathbb{R}$ is analytic, $[0, 1]^n \subseteq U$ and $n \geq 1$. We let $\mathbb{R}_{\mathrm{resPfaff}}$ be the reduct of this structure given by the same description, but with the word 'analytic' replaced by 'Pfaffian'.

For $q \in \mathbb{Q}$, the *height* of q is $H(q) = \max\{|a|, b\}$, where $q = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \geq 1$ and $\gcd(a, b) = 1$. The height of $\bar{q} \in \mathbb{Q}^n$, again written $H(\bar{q})$, is defined as the

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maximum of the heights of the coordinates of $\bar{q}$. For a set $X \subseteq \mathbb{R}^n$ and $H \geq 1$, we let

$$X(\mathbb{Q}, H) = \{\bar{q} \in X \cap \mathbb{Q}^n : H(\bar{q}) \leq H\}.$$

A transcendental function $f : \mathbb{R}^n \to \mathbb{R}$ is one that does not satisfy any non-zero polynomial equation $P(y, x_1, \dots, x_n) = 0$, for $P \in \mathbb{R}[Y, X_1, \dots, X_n]$.

Proposition 1.1. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a transcendental analytic function definable in $\mathbb{R}_{resPfaff}$, and let $X = graph(f)$. Then there exist $c > 0$ and $\gamma > 0$ such that for $H \geq 3$*

$$\# X(\mathbb{Q}, H) \leq c(\log H)^{\gamma}.$$

When f is Pfaffian, and not assumed to be definable in $\mathbb{R}_{\text{resPfaff}}$, this result is due to Pila ([9]). The extra generality here, as far as functions definable in $\mathbb{R}_{\text{resPfaff}}$ are considered, is to include functions implicitly defined by restricted Pfaffian functions.

The proof of the proposition is a modification of Pila's proof in [8]. To this end, we need a parameterization result which, although a simple consequence of a result due to Rolin, Speissegger and Wilkie ([11]), may be of some independent interest. We need two further definitions, the first of which is due to Pila ([10]). We use the following multi-index notation: for any $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$, we define the modulus $|\alpha| := \alpha_1 + \dots + \alpha_k$, the factorial $\alpha! := \alpha_1! \cdot \ldots \cdot \alpha_k!$ and the differential operator

$$D^{\alpha} := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_k^{\alpha_k}}.$$

Definitions 1.2. Let $A > 0$, $C \geq 0$. A C^{∞} function $\phi : (0,1)^k \to (0,1)$ is said to be (A, C)-*mild* if

$$|D^{\alpha} \phi(\bar{x})| \leq \alpha!(A|\alpha|^C)^{|\alpha|}$$

for all $\alpha \in \mathbb{N}^k$, all $\bar{x} \in (0,1)^k$ (where $0^0 = 1$). We say that a map $\Phi : (0,1)^k \to (0,1)^n$ is (A, C)-*mild* if each of its coordinate functions is (A, C)-mild.

Definitions 1.3. Fix an o-minimal structure $\tilde{\mathbb{R}}$ expanding the real field, and let $X \subseteq \mathbb{R}^n$ be definable. A *parameterization* of X is a finite set $\mathcal{S}$ of definable maps $\Phi_1, \dots, \Phi_l : (0,1)^{\dim X} \to \mathbb{R}^n$ such that $X = \bigcup \operatorname{Im}(\Phi_i)$. A parameterization is said to be (A, C)-*mild* if each of the parameterizing maps is (A, C)-mild. We say that $\tilde{\mathbb{R}}$ admits C-*mild parameterization* if for every definable set $X \subseteq (0,1)^n$ there is an (A, C)-mild parameterization of X, for some A.

Example 1.4. For a compact box $B \subseteq \mathbb{R}^n$, suppose that $f = (f_1, \dots, f_m) : B \to \mathbb{R}^m$ extends to an analytic function in a neighborhood of B. Then there exist (for example, by [6, 2.2.10]) positive constants A and K such that

$$|D^{\alpha} f_i(x)| \leq \alpha! K A^{|\alpha|}$$

for all $x \in B$, $\alpha \in \mathbb{N}^n$, and $i \in \{1, \dots, m\}$. If $B = [0,1]^n$ and $f((0,1)^n) \subseteq (0,1)^m$, then by making A larger we may take $K = 1$, in which case the graph of $f|_{(0,1)^n}$ has an $(A, 0)$-mild parameterization consisting of one map, namely $\Phi : (0,1)^n \to (0,1)^{n+m}$ defined by $\Phi(\bar{x}) = (\bar{x}, f(\bar{x}))$.

Proposition 1.5. *Any reduct of $\mathbb{R}_{an}$ expanding the real ordered field admits 0-mild parameterization.*

We remark on the relationship between the notion of a mild function and that of a Gevrey function. In [4], van den Dries and Speissegger consider $\mathbb{R}_{\mathcal{G}}$, the expansion of the real ordered field by the class of Gevrey functions $\mathcal{G}$, which is a certain family of real-valued C^∞ functions on the sets $[0, R] = \prod_{i=1}^n [0, R_i]$, for each $n \in \mathbb{N}$ and $R_1, \ldots, R_n > 0$, which are analytic on $(0, R] = \prod_{i=1}^n (0, R_i]$. For each n-ary function $f : [0, R] \to \mathbb{R}$ in $\mathcal{G}$ there exist constants $A, B > 0$ and $\kappa \in (0, 1]$ such that

$$|D^\alpha f(\overline{x})| \leq \alpha! A B^{|\alpha|} |\alpha|^{\kappa|\alpha|}$$

for all $\overline{x} \in [0, R]$ and $\alpha \in \mathbb{N}^n$ (see [4, 2.6]). It follows that $\mathbb{R}_{\mathcal{G}}$ is definably equivalent to an expansion of the real ordered field by a family of functions, each of which is (B, κ)-mild for some $B > 0$ and $\kappa \in (0, 1]$. It is therefore natural to ask whether $\mathbb{R}_{\mathcal{G}}$ admits 1-mild parameterization. To the best of our knowledge, this question is open and does not follow from the methods of this paper. The proof of Proposition 1.5 considers a set $X \subseteq (0, 1)^n$ definable in some fixed reduct of $\mathbb{R}_{\text{an}}$, and uses [11] to construct a parameterization $\Phi_1, \ldots, \Phi_l : (0, 1)^{\dim X} \to (0, 1)^n$ of X such that the definable maps $\Phi_1, \ldots, \Phi_l$ all extend to (definable) analytic functions on a neighborhood of $[0, 1]^{\dim X}$, from which Proposition 1.5 follows using Example 1.4. In contrast, [4] relies on the model completeness construction in [3], and therefore represents a set $X \subset (0, 1)^n$ definable in $\mathbb{R}_{\mathcal{G}}$ as a finite union of projections of manifolds which are zero sets of Gevrey functions, but which are not themselves graphs of Gevrey functions. The question of whether such manifolds have 1-mild parameterizations appears to be open.

2. $\mathcal{C}$-parameterization

In this section we observe that the results in [11] imply a parameterization result. So, we work in the setting of [11], and fix, for every compact box $B \subseteq \mathbb{R}^n$ and every $n \in \mathbb{N}$, an $\mathbb{R}$-algebra $\mathcal{C}_B$ of functions $f : B \to \mathbb{R}$ such that the following hold.

- ($\mathcal{C}_1$) Each of the projection functions $\langle x_1, \ldots, x_n \rangle \mapsto x_i$, restricted to B, is in $\mathcal{C}_B$, and for every function $f \in \mathcal{C}_B$ the restriction of f to the interior of B is C^∞.
- ($\mathcal{C}_2$) If $B' \subseteq \mathbb{R}^m$ is a compact box and $g_1, \ldots, g_n \in \mathcal{C}_{B'}$ are such that $g(B') \subseteq B$, where $g = \langle g_1, \ldots, g_n \rangle$, then for every $f \in \mathcal{C}_B$, the composition $f \circ g$ is in $\mathcal{C}_{B'}$.
- ($\mathcal{C}_3$) For every compact box $B' \subseteq B$ and function $f \in \mathcal{C}_B$, the restriction of f to B' is in $\mathcal{C}_{B'}$. For every $f \in \mathcal{C}_B$ there is a compact box $B' \subseteq \mathbb{R}^n$, the interior of which contains B, and a function $g \in \mathcal{C}_{B'}$ such that $g|_B = f$.
- ($\mathcal{C}_4$) For every $f \in \mathcal{C}_B$ and $i = 1, \ldots, n$, the partial derivative $\frac{\partial f}{\partial x_i}$ is in $\mathcal{C}_B$.

Note that the partial derivatives in ($\mathcal{C}_4$) exist by ($\mathcal{C}_1$) and ($\mathcal{C}_3$). Since we shall not need the precise statements of the remaining assumptions, we only state rough versions of them. The full details can be found in [11].

- ($\mathcal{C}_5$) For each $n \geq 1$ and each box $B \in \mathbb{R}^n$ containing the origin, the collection of germs at the origin of functions in $\mathcal{C}_B$ forms a quasianalytic class.
- ($\mathcal{C}_6$) This collection of germs is closed under extraction of implicit functions.
- ($\mathcal{C}_7$) This collection of germs is closed under monomial division.

The example which will interest us is as follows. Suppose that $\tilde{\mathbb{R}}$ is a polynomially bounded o-minimal expansion of the real field. For each compact box, let $\mathcal{C}_B$ be the collection of definable functions $f : B \to \mathbb{R}$ which admit a definable C^∞ extension to some open set containing B. By well known properties of o-minimal structures ([2],[7]) these algebras satisfy the above requirements. In particular, if $\tilde{\mathbb{R}}$ is a reduct of $\mathbb{R}_{\text{an}}$, then each function f in $\mathcal{C}_B$ is the restriction to B of an analytic function defined in a neighborhood of B, as in Example 1.4.

We now recall some further definitions from [11]. Given a polyradius $\bar{r} = \langle r_1, \ldots, r_n \rangle \in (0, \infty)^n$ we let $I_{\bar{r}} = \prod(-r_i, r_i)$ and let $\bar{I}_{\bar{r}}$ be the topological closure of $I_{\bar{r}}$. Write $\mathcal{C}_{n,\bar{r}}$ for $\mathcal{C}_{\bar{I}_{\bar{r}}}$.

Definition 2.1. A set $A \subseteq \mathbb{R}^n$ is called a *basic $\mathcal{C}$-set* if there are $\bar{r} \in (0, \infty)^n$ and $f, g_1, \ldots, g_k \in \mathcal{C}_{n,\bar{r}}$ such that

$$A = \{\bar{x} \in I_{\bar{r}} : f(\bar{x}) = 0, g_1(\bar{x}) > 0, \ldots, g_k(\bar{x}) > 0\}.$$

A finite union of basic $\mathcal{C}$-sets is called a *$\mathcal{C}$-set.* A set $A \subseteq \mathbb{R}^n$ is called *$\mathcal{C}$-semianalytic* if for every $\bar{a} \in \mathbb{R}^n$ there is an $\bar{r} \in (0, \infty)^n$ such that

$$(A - \bar{a}) \cap I_{\bar{r}}$$

is a $\mathcal{C}$-set. If A is also a manifold, we call A a $\mathcal{C}$-semianalytic manifold.

Given $m \leq n$ and an injective $\lambda : \{1, \ldots, m\} \to \{1, \ldots, n\}$, we write $\pi_\lambda : \mathbb{R}^n \to \mathbb{R}^m$ for the projection $\bar{x} \mapsto \langle x_{\lambda(1)}, \ldots, x_{\lambda(m)} \rangle$.

Definition 2.2. Let $\bar{r} \in (0, \infty)^n$. A set $M \subseteq I_{\bar{r}}$ is said to be *$\mathcal{C}$-trivial* if one of the following holds:

(i) $M = \{\bar{x} \in I_{\bar{r}} : x_1 \square_1 0, \ldots, x_n \square_n 0\}$, where $\square_i \in \{<, =, >\}$ for each i;
(ii) there exist a permutation λ of $\{1, \ldots, n\}$, a $\mathcal{C}$-trivial $N \subseteq I_{\bar{s}}$ and a $g \in \mathcal{C}_{n-1,\bar{s}}$, where $\bar{s} = \langle r_{\lambda(1)}, \ldots, r_{\lambda(n-1)} \rangle$, such that $g(I_{\bar{s}}) \subseteq (-r_{\lambda(n)}, r_{\lambda(n)})$ and $\pi_\lambda(M) = \text{graph}(g|_N)$.

Note that $\mathcal{C}$-trivial sets are necessarily manifolds; we shall refer to them as $\mathcal{C}$-trivial manifolds. A $\mathcal{C}$-seminanalytic manifold $M \subseteq \mathbb{R}^n$ is called *trivial* if there exist $\bar{a} \in \mathbb{R}^n$ and a $\mathcal{C}$-trivial manifold $N \subseteq \mathbb{R}^n$ such that $M = N + \bar{a}$.

We need two results from [11].

Fact 2.3. *(*[11, 4.7]*) Suppose that $A \subseteq \mathbb{R}^n$ is a bounded $\mathcal{C}$-semianalytic set and that $k \leq n$. Then there are trivial $\mathcal{C}$-semianalytic manifolds $N_i \subseteq \mathbb{R}^{n_i}$ for some $n_i \geq n, i = 1, \ldots J$, such that*

$$\pi_k(A) = \pi_k(N_1) \cup \cdots \cup \pi_k(N_J)$$

where $\pi_k|_{N_i}$ is an immersion, for each i. (Here, π_k is projection onto the first k coordinates.)

Let $\mathbb{R}_{\mathcal{C}}$ be the expansion of the real ordered field by all functions $\hat{f}$, for $f \in \mathcal{C}_{n,\bar{r}}, n \in \mathbb{N}, \bar{r} \in (0, \infty)^n$, where $\hat{f}(\bar{x}) = f(\bar{x})$ on $\bar{I}_{\bar{r}}$ and $\hat{f}(\bar{x}) = 0$ on $\mathbb{R}^n \setminus \bar{I}_{\bar{r}}$.

Fact 2.4. *(*[11, 5.2 and 5.4]*) The structure $\mathbb{R}_{\mathcal{C}}$ is o-minimal, model complete and polynomially bounded.*

We now use these results to prove a parameterization result. We work in the structure $\mathbb{R}_{\mathcal{C}}$.

Definition 2.5. Let $X \subseteq \mathbb{R}^n$ be definable. A *$\mathcal{C}$-parameterization* of X is a finite set $\mathcal{S}$ of maps $\Phi_1, \dots, \Phi_l$ whose coordinate functions are in $\mathcal{C}_{[0,1]^{\dim X}}$ such that $\{\Phi_i|_{(0,1)^{\dim X}} : i = 1, \dots, l\}$ is a parameterization of X.

Example 2.6. Let $\bar{r} \in (0, \infty)^n$. Let $M = \{\bar{x} \in I_{\bar{r}} : x_1 \square_1 0, \dots, x_n \square_n 0\}$, where $\square_i \in \{<, =, >\}$ for each i. Let $\lambda_1, \dots, \lambda_m$ be, in order, the indices for which $\square_i$ is either $<$ or $>$. For each i, define the map $\phi_i : (0,1)^m \to \mathbb{R}$ by

$$\phi_i(\bar{x}) = \begin{cases} -r_j x_j & \text{if } i = \lambda_j \text{ and } \square_i \text{ is } <, \\ r_j x_j & \text{if } i = \lambda_j \text{ and } \square_i \text{ is } >, \\ 0 & \text{otherwise.} \end{cases}$$

We now see that M has a $\mathcal{C}$-parameterization consisting of one map, namely $\Phi : (0,1)^m \to \mathbb{R}^n$ given by $\Phi(\bar{x}) := (\phi_1(\bar{x}), \dots, \phi_n(\bar{x}))$.

Now we easily have the following, by induction on n.

Lemma 2.7. *Suppose that $M \subseteq \mathbb{R}^n$ is a $\mathcal{C}$-trivial manifold. Then there is a $\mathcal{C}$-parameterization $\mathcal{S}$ of M with $\#\mathcal{S} = 1$.*

Proposition 2.8. *Suppose that $X \subseteq \mathbb{R}^n$ is a bounded definable set. Then X has a $\mathcal{C}$-parameterization.*

Proof. By model completeness, there is an $m \geq 0$ and a quantifier-free definable set $A \subseteq \mathbb{R}^{n+m}$ such that $X = \pi(A)$. Using the fact that $\mathbb{R}_{\mathcal{C}}$ is an expansion of the real field, we may assume that A is bounded and that A is $\mathcal{C}$-semianalytic. By Fact 2.3,

$$X = \pi(N_1) \cup \cdots \cup \pi(N_k)$$

for some $\mathcal{C}$-trivial manifolds $N_1, \dots, N_k$, where each $\pi|_{N_i}$ is an immersion. Thus $\dim(X) = \max\{\dim(N_1), \dots, \dim(N_k)\}$. A $\mathcal{C}$-parameterization of X can be constructed by composing the functions in the $\mathcal{C}$-parameterizations of each of the N_i with the projections π, and then trivially extending any of these functions to $(0,1)^{\dim X}$ if their domain is $(0,1)^{\dim N_i}$ with $\dim N_i < \dim(X)$. □

Note that Proposition 1.5 follows immediately from applying Proposition 2.8 to the given reduct of $\mathbb{R}_{\text{an}}$ and then using Example 1.4.

3. Curves

We now prove Proposition 1.1. In fact, we prove a result about the number of points in a fixed number field $k \subseteq \mathbb{R}$ of degree l. We use the absolute multiplicative height H on k, which agrees with the height on $\mathbb{Q}$ given in the introduction (for the definition of H, see [1]). For $X \subseteq \mathbb{R}^n$ and $H \geq 1$, we let $X(k, H) = X \cap \{\bar{a} \in k^n : H(\bar{a}) \leq H\}$. The following is a special case of [10, Corollary 3.3].

Fact 3.1. *Suppose that $X \subseteq (0,1)^2$ is definable in $\mathbb{R}_{an}$ with dimension 1 and that $\mathcal{S}$ is an $(A, 0)$-mild parameterization of X. Then there is an absolute constant c_0 such that $X(k, H)$ is contained in a union of at most*

$$\#\mathcal{S} \cdot c_0^l \cdot A^{2(1+o(1))}$$

intersections of X with algebraic curves of degree $\lfloor l \cdot \log H \rfloor$. Here the $1+o(1)$ is taken as $H \to \infty$ with absolute implied constant, and $\lfloor \cdot \rfloor$ denotes integer part.

Given a function $F : \mathbb{R}^m \to \mathbb{R}$, we let $V(F) = \{\bar{x} \in \mathbb{R}^m : F(\bar{x}) = 0\}$.

Lemma 3.2. *Suppose that $f : (a,b) \to (0,1)$, with $(a,b) \subseteq (0,1)$, is a transcendental analytic function definable in $\mathbb{R}_{resPfaff}$. Suppose further that graph(f) $= \pi(V(F))$, where $F : \mathbb{R}^{2+n} \to \mathbb{R}$ is a Pfaffian function of order r and degree (α, β), and π is the projection onto the first two coordinates. If $P : \mathbb{R}^2 \to \mathbb{R}$ is a nonzero polynomial of degree d then*

$$\#(\mathit{graph}(f) \cap V(P)) \leq 2^{r(r+1)/2+1}(n+2)^r(\alpha + 2d')^{n+r+2} \tag{1}$$

where $d' = \max\{d, \beta\}$.

Proof. Let $\tilde{P} : \mathbb{R}^{2+n} \to \mathbb{R}$ be given by $\tilde{P}(x,y,\bar{z}) = P(x,y)$. Then graph$(f) \cap V(P) = \pi(V(F) \cap V(\tilde{P}))$. The number of points in graph$(f) \cap V(P)$ is thus bounded by the number of connected components of $V(F) \cap V(\tilde{P})$ (there are only finitely many points in graph$(f) \cap V(P)$, as we have assumed that f is transcendental). By Khovanskii's theorem (as presented in [5, 3.3]) there are at most

$$2^{r(r-1)/2+1}d'(\alpha + 2d' - 1)^{n+1}((2(n+2)-1)(\alpha + d') - 2n - 2)^r$$

such components, and clearly this is less than the right hand side of (1). □

Proposition 3.3. *Suppose that $f : (a,b) \to (0,1)$, with $(a,b) \subseteq (0,1)$, is a transcendental analytic function definable in $\mathbb{R}_{resPfaff}$ and let $X = \mathit{graph}(f)$. Then there are $c, \gamma > 0$ such that (for $H \geq e$)*

$$\#X(k,H) \leq c(\log H)^\gamma.$$

Proof. By model completeness of $\mathbb{R}_{\mathrm{resPfaff}}$ (see [12]), we may suppose that $X = \pi(V(F))$ for some Pfaffian function $F : \mathbb{R}^{2+n} \to \mathbb{R}$ and some $n \geq 0$. Suppose that F is of order r and degree (α, β). By Proposition 1.5 we can take an $(A, 0)$-mild parameterization $\mathcal{S}$ of X, for some A. Combining Fact 3.1 with Lemma 3.2 (with $d = \lfloor l \log H \rfloor$), we have

$$\begin{aligned} \#X(k,H) &\leq \#\mathcal{S} \cdot c_0^l \cdot A^{2(1+o(1))} 2^{r(r+1)/2+1}(n+2)^r(\alpha + 2\max\{\beta, d\})^{n+r+2} \\ &\leq c(\log H)^\gamma \end{aligned}$$

where $\gamma = n + r + 2$. □

The collection of points of a number field k of height at most H is preserved under the inversions $x \to \pm x^{\pm 1}$. Therefore, in counting such points on the graph of a transcendental analytic function $f : \mathbb{R} \to \mathbb{R}$, we may instead consider the graphs of a finite collection of transcendental analytic functions, each defined on a subinterval of $(0,1)$, together with a finite collection of points in $\mathbb{R}^n$. Proposition 1.1 then follows by repeated application of Proposition 3.3.

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