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THE SPECTRUM OF THE DIRAC OPERATOR FOR THE UNIVERSAL COVER OF $\mathrm{SL}_2(\mathbb{R})$

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Abstract. Using representation theory, we compute the spectrum of the Dirac operator on the universal covering group of $\mathrm{SL}_2(\mathbb{R})$, exhibiting it as the generator of $KK^1(\mathbb{C}, \mathfrak{A})$, where $\mathfrak{A}$ is the reduced C^*-algebra of the group. This yields a new and direct computation of the K-theory of $\mathfrak{A}$. A fundamental role is played by the limit-of-discrete-series representation, which is the frontier between the discrete and the principal series of the group.

1. Introduction

Let G denote the universal cover of $\mathrm{SL}_2(\mathbb{R})$ and let $\mathfrak{A} = C^*_r(G)$ be the reduced C^*-algebra of G. The group G is a non-linear group with infinite centre which places it outside the range of much classical representation theory of Harish-Chandra et al and in particular the K-theory of $\mathfrak{A}$ is known only through the deep results on the Connes-Kasparov conjecture due to Chabert, Echterhoff and Nest [CEN]. In this article, we give a direct computation of the K-theory of $\mathfrak{A}$ using the Plancherel formula for G, established by Pukánszky [P], studying the algebra via the Fourier transform. Moreover this also enables us to compute the spectrum of the Dirac operator on G, thereby establishing that this is the generator of $KK^1(\mathbb{C}, \mathfrak{A})$.

We use the Fourier transform to identify $\mathfrak{A}$ as an algebra of operator-valued functions on a parameter space built from the discrete series and the principal series of G. Here there is an analogy with the representation theory of $\mathrm{SL}_2(\mathbb{R})$. This has a representation, the limit-of-discrete-series, which is a reducible representation in the principal series of $\mathrm{SL}_2(\mathbb{R})$ sharing properties with representations from the discrete series. Now, for G, the analogue of the discrete series is a field of pairs of irreducible representations parametrised by the interval $(-\infty, 1/4)$, whose limit at $1/4$ is again a reducible representation in the principal series. This limit representation we call the *limit-of-discrete-series* for G. The principal series is parametrised by a cylinder $S^1 \times [1/4, \infty)$ and it is the attaching of the two spaces at the limit-of-discrete-series that yields the generator in K-theory. We remark that the Casimir operator for G, which acts as a scalar on each irreducible representation, gives the real parameter in $(-\infty, 1/4)$ for the discrete series and $[1/4, \infty)$ for the principal series.

As one might expect, the Dirac operator D, defined on spinor fields on G, provides an interesting element of the K-theory of $\mathfrak{A}$. We examine the spectrum of D by identifying its Fourier transform $\widehat{D}$ and in particular we

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investigate how the spectrum changes as the Casimir parameter varies over $(-\infty, \infty)$.

The operator $\widehat{D}$ can be thought of as a field of operators on the tensor product of the field of representations of G with the field of spinors. The idea is to decompose the tensor product field into $\widehat{D}$-invariant pieces. We have one infinite dimensional field on which $\widehat{D}$ has a spectral gap and thus is trivial at the level of KK-theory. This leaves a one-dimensional field for which the spectrum varies from minus infinity to infinity with the Casimir parameter. This establishes that D gives a non-trivial element in K-theory, and a homotopy argument then shows that the Dirac class generates $KK^1(\mathbb{C}, \mathfrak{A})$.

2. Representation theory and the Fourier transform

We begin with the Plancherel formula of Pukánszky [P] for the universal cover of $\mathrm{SL}_2(\mathbb{R})$.

Theorem 2.1. *The following representations enter into the Plancherel formula:*

$$\begin{array}{lll} \text{Principal series}: & \{(V_{q,\tau}, \pi_{q,\tau}) : q \geq 1/4,\ 0 \leq \tau \leq 1\}, & \Omega = q \\ \text{Discrete series}: & \{(W_{\ell,\pm}, \omega_{\ell,\pm}) :\ \ell \geq 1/2\}, & \Omega = \ell(1-\ell) \end{array}$$

where Ω is the Casimir operator. For every test function f on G, smooth with compact support, we have

$$f(e) = \int_0^\infty \int_0^1 \sigma[\Re \tanh \pi(\sigma + i\tau)]\Theta(\sigma,\tau)(f) d\tau d\sigma + \int_{1/2}^\infty (\ell - 1/2)\Theta(\ell)(f) d\ell$$

where the Harish-Chandra characters are

$$\Theta(\sigma,\tau)(f) = \operatorname{trace} \int_G \pi_{q,\tau}(g) f(g) dg$$

$$\Theta(\ell)(f) = \operatorname{trace} \int_G (\omega_{\ell,+} \oplus \omega_{\ell,-})(g) f(g) dg$$

and $\sigma = \sqrt{q - 1/4}$.

This is a *measure-theoretic* statement. We need a more precise statement in topology.

Note that $\Omega = 1/4$ for each of the following representations;

$$\pi_{1/4,1/2}, \quad \omega_{1/2,+}, \quad \omega_{1/2,-}.$$

In fact the representation $\pi_{1/4,1/2}$ is reducible and

$$V_{1/4,1/2} = W_{1/2,+} \oplus W_{1/2,-}, \quad \pi_{1/4,1/2} = \omega_{1/2,+} \oplus \omega_{1/2,-}$$

see Eqn.(2.4) in [KM, p.40].

We will define the *parameter space* $\mathcal{Z}$ to be the union of the sets

$$\{q \in \mathbb{R} : q \leq 1/4\}$$

$$\{(q,\tau) \in \mathbb{R} \times \mathbb{R} : q \geq 1/4, 0 \leq \tau \leq 1\}$$

with identification of the point $1/4$ in the first set with $(1/4, 1/2)$ in the second, and with identification of $(q, 0)$ with $(q, 1)$ for all $q \geq 1/4$.

The G-Hilbert spaces $V_{q,\tau}$ form a continuous field of Hilbert-spaces over $q \geq 1/4, 0 \leq \tau \leq 1$. We extend this to a continuous field V_* of G-Hilbert-spaces over $\mathcal{Z}$ by defining

$$V_q = W_{\ell,+} \oplus W_{\ell,-}$$

where $q = \ell(1-\ell)$ noting that

$$V_{1/4} = V_{1/4,1/2}, \quad V_{q,0} = V_{q,1} \qquad \forall q \geq 1/4.$$

The Casimir operator on the G-modules V_q and $V_{q,\tau}$ is precisely the multiplication by the parameter q.

We have the following structure theorem.

Theorem 2.2. *Let $\mathfrak{A}$ denote the reduced C^*-algebra $C^*_r(G)$. The Fourier transform $f \mapsto \widehat{f}$ induces an isomorphism of $\mathfrak{A}$ onto the C^*-algebra*

$$\widehat{\mathfrak{A}} := \{F \in C_0(\mathcal{Z}, \mathfrak{K}(V_*)) : F(q)W_{\ell,+} \subset ds_{\ell,+}, F(q)W_{\ell,-} \subset W_{\ell,-} \text{ if } q \leq 1/4\}.$$

Proof. The reduced C^*-algebra is a quotient of the full C^*-algebra $C^*(G)$:

$$1 \to \mathfrak{I} \to C^*(G) \to C^*_r(G) \to 1.$$

The complementary series makes no contribution. The C^*-algebra $\mathfrak{A}$ is a quotient of the full C^*-algebra in [KM] and the primitive ideal space of $\mathfrak{A}$ contains every point in the support of Plancherel measure on $\widehat{G}$ (the unitary dual of G), by Theorem (2.1). □

Note that the Jacobson topology on the primitive ideal spectrum of $\mathfrak{A}$ is exactly right: it has a double point at $q = 1/4 \in \mathcal{Z}$, where the unitary representation $\pi_{1/4,1/2}$ is reducible.

We observe that the algebra $\widehat{\mathfrak{A}}$ is (strongly) Morita equivalent to the algebra

$$\mathfrak{B} := \{F \in C_0(\mathcal{Z}, M_2(\mathbb{C})) : F(q) \text{ is diagonal if } q \leq 1/4\}.$$

One way to see this is to note that $\widehat{\mathfrak{A}} \cong \mathfrak{B} \otimes \mathfrak{K}$ however it is instructive to consider the explicit bimodules yielding the Morita equivalence.

We introduce the notation $\mathbb{C}^2_+$ and $\mathbb{C}^2_-$ for the subspaces $\mathbb{C} \oplus 0$ and $0 \oplus \mathbb{C}$ in $\mathbb{C}^2$. We now form the submodule $\mathcal{E}$ of $C_0(\mathcal{Z}, V_* \otimes \mathbb{C}^2)$ consisting of all functions F such that for $q = \ell(1-\ell) \leq 1/4$, we have

$$F(q) \in (W_{\ell,+} \otimes \mathbb{C}^2_+) \oplus (W_{\ell,-} \otimes \mathbb{C}^2_-).$$

The module $\mathcal{E}$ can be equipped with two (pointwise) inner products. Firstly we have a $\mathfrak{B}$-valued inner product, which is to say a pointwise $M_2(\mathbb{C})$-valued inner product satisfying the required diagonality condition. For $F = F_1 \otimes F_2$, $G = G_1 \otimes G_2$ the inner product is defined to be

$$\langle F, G\rangle_{\mathfrak{B}} = \langle F_1(z), G_1(z)\rangle_{V_z} \, F_2(z)\langle G_2(z), - \rangle_{\mathbb{C}^2}.$$

Secondly we have an $\widehat{\mathfrak{A}}$ valued inner product, which pointwise takes values in $\mathfrak{K}(E)$. For $F = F_1 \otimes F_2$, $G = G_1 \otimes G_2$ the inner product is defined to be

$$\langle F, G\rangle_{\widehat{\mathfrak{A}}} = \langle F_2(z), G_2(z)\rangle_{\mathbb{C}^2} \, F_1(z)\langle G_1(z), - \rangle_{V_z}.$$

The Hilbert modules obtained by equipping $\mathcal{E}$ with these two inner products effect a Morita equivalence between the algebras $\widehat{\mathfrak{A}}$ and $\mathfrak{B}$.

We remark that a field of operators on the field V_* of Hilbert spaces can naturally be regarded as an adjointable operator on $\mathcal{E}$ with the $\mathfrak{B}$-valued inner product.

3. The Dirac operator

Let $\mathfrak{g}$ denote the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$, and let $C(\mathfrak{g})$ denote the Clifford algebra of $\mathfrak{g}$ with respect to the negative definite quadratic form on $\mathfrak{g}$. Let X_0, X_1, X_2 denote an orthonormal basis in $\mathfrak{g}$. Note that the notation in [P, (1.1)] is $l_k = X_k$.

Following the algebraic approach in [HP, Def. 3.1.2] the *Dirac operator* is the element of the algebra $U(\mathfrak{g}) \otimes C(\mathfrak{g})$ given by

$$D = X_0 \otimes c(X_0) + X_1 \otimes c(X_1) + X_2 \otimes c(X_2)$$

where $c(X_k)$ denotes Clifford multiplication by X_k.

Let

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

and set

$$c(X_k) = i\sigma_k, \qquad k = 0, 1, 2$$

Then we have

$$c(X_k)^2 = -1$$

for all $k = 0, 1, 2$.

We have

$$\begin{aligned} D &= i(X_0\sigma_0 + X_1\sigma_1 + X_2\sigma_2) \\ &= i \begin{pmatrix} X_0 & X_1 + iX_2 \\ X_1 - iX_2 & -X_0 \end{pmatrix} \end{aligned}$$

The operator D acts on 2-spinor fields in the following way: the elements of the Lie algebra $\mathfrak{g}$ give rise to *right*-invariant vector fields on G and in this way $X_j, j = 0, 1, 2$ form differential operators on scalar fields. The matrix then acts by differentiating the components of a spinor field.

Viewing a compactly supported 2-spinor field as a pair of scalar valued functions on the group G, it is an element of $C_c(G) \oplus C_c(G) \subseteq \mathfrak{A} \oplus \mathfrak{A}$. In this way the Dirac operator D gives rise to an unbounded adjointable operator on $\mathfrak{A} \oplus \mathfrak{A}$ viewed as a Hilbert module over $\mathfrak{A}$.

Let now π be a unitary representation, in the principal series or the discrete series of G, on a Hilbert space V_π. The infinitesimal generators H_0, H_1, H_2, which act on the Hilbert space V_π, are determined by the following equation [P, p.98]:

$$\exp(-itH_k) = \pi(\exp(tX_k)) \qquad \forall t \in \mathbb{R}, k = 0, 1, 2.$$

Let us now recall the parameter space $\mathcal{Z}$ from §2 and corresponding continuous field of G-Hilbert-spaces over $\mathcal{Z}$:

$$\{V_q : q \leq 1/4\}, \qquad \{V_{q,\tau} : q \geq 1/4, 0 \leq \tau \leq 1\}.$$

On each of the Hilbert spaces V_q and $V_{q,\tau}$ we therefore have three self-adjoint operators, namely H_0, H_1 and H_2. These form a field of operators on the field of Hilbert spaces V_*. The spectrum of H_0 is discrete with eigenvalues $m = \ell, \ell+1, \ell+2, \ldots$ and $m = -\ell, \ell-1, -\ell-2, \ldots$ in the case that $q < 1/4$ and with eigenvalues $m \in \tau + \mathbb{Z}$ for $q \geq 1/4$. Each eigenvalue has multiplicity 1 and we let f_m be an orthogonal basis of eigenvectors of H_0 so that

$$H_0 f_m = m f_m.$$

Following [P, p.100], we define

$$H_+ = H_1 + iH_2, \qquad H_- = H_1 - iH_2$$

In addition, we have the following equations

$$H_+ f_m = (q + m(m+1))^{1/2} f_{m+1}$$
$$H_- f_m = (q + m(m-1))^{1/2} f_{m-1}$$

which hold for all m when $q \geq 1/4$ and where the first equation holds for all $m \neq -\ell$, the second for all $m \neq \ell$ when $q < 1/4$. The special cases of $H_+ f_{-\ell}$ and $H_- f_\ell$ are both zero.

By analogy with the Dirac operator D above, we construct a field of self-adjoint operators

$$\mathbb{H} = \begin{pmatrix} H_0 & H_1 + iH_2 \\ H_1 - iH_2 & -H_0 \end{pmatrix} \tag{1}$$

on the field of Hilbert spaces $V_* \oplus V_*$. Since the algebra $\widehat{\mathfrak{A}}$ consists of fields of compact operators on V_*, the operator $\mathbb{H}$ can also be thought of as acting on $\widehat{\mathfrak{A}} \oplus \widehat{\mathfrak{A}}$ by composition. (One must additionally note that the operators $H_j, j = 0, 1, 2$, respect the decomposition of V_q as $W_{\ell,+} \oplus W_{\ell,-}$ for $q \leq 1/4$.)

Since the operator D acts on $\mathfrak{A} \oplus \mathfrak{A}$ and we have an isomorphism from $\mathfrak{A}$ to $\widehat{\mathfrak{A}}$ given by the Fourier transform (Theorem 2.2), we obtain an operator $\widehat{D}$ on $\widehat{\mathfrak{A}} \oplus \widehat{\mathfrak{A}}$. We will show that

$$\widehat{D} = -\mathbb{H}.$$

To demonstrate this it suffices to show that the isomorphism $\mathfrak{A} \cong \widehat{\mathfrak{A}}$ takes the differential operator iX_j to $-H_j$ for each j.

For f a smooth compactly supported function on G the Fourier transform of f is defined by

$$\hat{f}(\pi) = \int f(g)\pi(g)\, dg.$$

Since H_j is the infinitesimal generator of the 1-parameter group $\exp(-itH_j) = \pi(\exp(tX_j))$ we have $H_j = i\frac{d}{dt}\pi(\exp(tX_j))|_{t=0}$. Hence

$$\begin{aligned}
-H_j\hat{f}(\pi) &= -\int f(g)H_j\pi(g)\,dg \\
&= -\int f(g)i\frac{d}{dt}\pi(\exp(tX_j))|_{t=0}\pi(g)\,dg \\
&= -i\frac{d}{dt}\int f(g)\pi(\exp(tX_j)g)\,dg|_{t=0} \\
&= -i\frac{d}{dt}\int f(\exp(-tX_j)g')\pi(g')\,dg'|_{t=0} \quad \text{by left invariance of } dg \\
&= -i\int \frac{d}{dt}f(\exp(-tX_j)g')|_{t=0}\pi(g')\,dg' \\
&= i\int X_j(f)\pi(g')\,dg' \\
&= i\widehat{X_j(f)}(\pi)
\end{aligned}$$

since X_j is a right-invariant vector field. This establishes that the Fourier transform takes iX_j to $-H_j$ and hence that $\widehat{D} = -\mathbb{H}$.

We will therefore study the operator $\mathbb{H}$ in detail. Recall that $\mathbb{H}$ is defined on the field $V_* \oplus V_*$, and a crucial observation at this point is the emergence of two dimensional invariant subspaces for $\mathbb{H}$ at each point of the parameter space $\mathcal{Z}$.

Each such subspace E_m is spanned by a pair of vectors $\begin{pmatrix} f_m \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ f_{m-1} \end{pmatrix}$, where m is in the set $\tau+\mathbb{Z}$ for $q \geq 1/4$ and $m = \ell+1, \ell+2, \dots$ or $-\ell, -\ell-1, \dots$ for $q < 1/4$. We have the following equations

$$\mathbb{H}\begin{pmatrix} f_m \\ 0 \end{pmatrix} = \begin{pmatrix} H_0 & H_+ \\ H_- & -H_0 \end{pmatrix}\begin{pmatrix} f_m \\ 0 \end{pmatrix} = \begin{pmatrix} mf_m \\ (q+m(m-1))^{1/2}f_{m-1} \end{pmatrix}$$

and

$$\mathbb{H}\begin{pmatrix} 0 \\ f_{m-1} \end{pmatrix} = \begin{pmatrix} H_0 & H_+ \\ H_- & -H_0 \end{pmatrix}\begin{pmatrix} 0 \\ f_{m-1} \end{pmatrix} = \begin{pmatrix} (q+m(m-1))^{1/2}f_m \\ -(m-1)f_{m-1} \end{pmatrix}.$$

With respect to this basis, the operator $\mathbb{H}$ is given by the following symmetric matrix

$$\begin{pmatrix} m & (q+m(m-1))^{1/2} \\ (q+m(m-1))^{1/2} & -(m-1) \end{pmatrix}.$$

This symmetric matrix has the following eigenvalues

$$\lambda = \frac{1}{2} \pm \sqrt{1/4 + q + 2m(m-1)}.$$

In the case that $q \geq 1/4$ the subspaces E_m for $m \in \tau+\mathbb{Z}$ span the whole of $V_{q,\tau}$. However, for $q < 1/4$ there are a further two 1-dimensional subspaces spanned by the vectors

$$\begin{pmatrix} f_\ell \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ f_{-\ell} \end{pmatrix}$$

These subspaces are invariant since $H_-(f_\ell) = 0$ and $H_+(f_{-\ell}) = 0$.

Note that something very special occurs in the limit-of-discrete series when $q = 1/4$ and $\tau = 1/2$. Here, when $m = 1/2$ we have

$$q + m(m-1) = 0$$

and so in this case the operator matrix $\begin{pmatrix} H_0 & H_+ \\ H_- & -H_0 \end{pmatrix}$ restricted to the 2-dimensional subspace spanned by $\begin{pmatrix} f_{1/2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ f_{-1/2} \end{pmatrix}$ is the diagonal matrix

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

For $q < 1/4$, we have the following eigenvector equations

$$\begin{pmatrix} H_0 & H_+ \\ H_- & -H_0 \end{pmatrix} \begin{pmatrix} f_l \\ 0 \end{pmatrix} = l \begin{pmatrix} f_l \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} H_0 & H_+ \\ H_- & -H_0 \end{pmatrix} \begin{pmatrix} 0 \\ f_{-l} \end{pmatrix} = l \begin{pmatrix} 0 \\ f_{-l} \end{pmatrix}.$$

With respect to the basis given by the vectors $\begin{pmatrix} f_l \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ f_{-l} \end{pmatrix}$ our operator matrix has the diagonal form $\begin{pmatrix} l & 0 \\ 0 & l \end{pmatrix}$, so we see that as q approaches $1/4$ and consequently ℓ approaches $1/2$ this matches up with the limit-of-discrete series case of $q = 1/4, \tau = 1/2$, justifying the terminology.

As noted above, the operator $\mathbb{H}$ can be thought of either as an operator on $\widehat{\mathfrak{A}} \oplus \widehat{\mathfrak{A}}$ or an operator on $V_* \oplus V_*$. Taking the latter view, the operator acts on the space $C_0(\mathcal{Z}, V_* \oplus V_*)$ of continuous sections of $V_* \oplus V_*$ vanishing at infinity, which is a Hilbert module over $C_0(\mathcal{Z})$. Indeed there is an isomorphism of Hilbert modules

$$(\widehat{\mathfrak{A}} \oplus \widehat{\mathfrak{A}}) \otimes_{\widehat{\mathfrak{A}}} C_0(\mathcal{Z}, V_*) \cong C_0(\mathcal{Z}, V_* \oplus V_*)$$

and viewing $\mathbb{H}$ as an operator on $\widehat{\mathfrak{A}} \oplus \widehat{\mathfrak{A}}$, the corresponding operator on $C_0(\mathcal{Z}, V_* \oplus V_*)$ is given by $\mathbb{H} \otimes 1$.

The algebra $\widehat{\mathfrak{A}}$ is contained in $C_0(\mathcal{Z}, \mathfrak{K}(V_*))$ (or $C_0(\mathcal{Z}, \mathfrak{K})$ for short) and can factorise the tensor product further as

$$(\widehat{\mathfrak{A}} \oplus \widehat{\mathfrak{A}}) \otimes_{\widehat{\mathfrak{A}}} C_0(\mathcal{Z}, \mathfrak{K}) \otimes_{C_0(\mathcal{Z}, \mathfrak{K})} C_0(\mathcal{Z}, V_*)$$

Hence at the level of KK-theory viewing $\mathbb{H}$ as an operator on sections of $V_* \oplus V_*$ corresponds to the composition of the forgetful inclusion of $\widehat{\mathfrak{A}}$ into $C_0(\mathcal{Z}, \mathfrak{K})$ with the Morita equivalence from $C_0(\mathcal{Z}, \mathfrak{K})$ to $C_0(\mathcal{Z})$ (which is implemented by the module $C_0(\mathcal{Z}, V_*)$).

We now consider the restriction of the field of Hilbert spaces to the copy of the real line inside $\mathcal{Z}$ given by the union

$$\{q \in \mathbb{R} : q \leq 1/4\} \cup \{(q, 1/2) \in \mathbb{R} \times \mathbb{R} : q \geq 1/4\},$$

which we regard as parametrized by q. Correspondingly we restrict $\mathbb{H}$ to the Hilbert spaces over this line. Passing from $\mathbb{H}$ viewed as an operator on an

$\widehat{\mathfrak{A}}$-Hilbert module to $\mathbb{H}|_{\mathbb{R}}$ viewed as an operator on a $C_0(\mathbb{R})$-Hilbert module corresponds to the composition

$$\widehat{\mathfrak{A}} \hookrightarrow C_0(\mathcal{Z}, \mathfrak{K}) \underset{\text{M.E.}}{\sim} C_0(\mathcal{Z}) \twoheadrightarrow C_0(\mathbb{R}).$$

We will see in Section 4 that this composition induces an isomorphism at the level of KK-theory. In this section we will identify the class $[C_0(\mathbb{R}, V_*|_{\mathbb{R}}), 1, \mathbb{H}|_{\mathbb{R}}]$ in $KK^1(\mathbb{C}, C_0(\mathbb{R}))$.

Remark 3.1. *The half-line $\{q \in \mathbb{R} : q \geq 1/4\}$ has the following significance in representation theory. The corresponding unitary representations $(V_{q,1/2}, \pi_{q,1/2})$ all factor through $\mathrm{SL}_2(\mathbb{R})$ and constitute the odd principal series π_q of $\mathrm{SL}_2(\mathbb{R})$. In particular, the representation $\pi_{1/4,1/2}$ is the limit-of-discrete series for $\mathrm{SL}_2(\mathbb{R})$. It is the direct sum of two irreducible representations whose characters θ_+ and θ_- do not vanish on the elliptic set. In this respect, they resemble representations in the discrete series. So the term* limit-of-discrete-series *for $\pi_{1/4,1/2}$ is surely apt.*

We now attempt to glue together some one-dimensional eigenspaces to form a complex hermitian line bundle L over $\mathbb{R}$. Take $q \geq 1/4$ and consider the subspace $E_{1/2}$. The restriction $\mathbb{H}|_{E_{1/2}}$ is given by the matrix

$$\begin{pmatrix} 1/2 & \sqrt{q-1/4} \\ \sqrt{q-1/4} & 1/2 \end{pmatrix}$$

from which we readily see that the vector $\begin{pmatrix} f_{1/2} \\ -f_{-1/2} \end{pmatrix}$ is an eigenvector of $\mathbb{H}$ with eigenvalue $1/2 - \sqrt{q-1/4}$. Note that the eigenvalue tends to $1/2$ as $q \to 1/4^+$.

Now for $q < 1/4$ we see that $\begin{pmatrix} f_\ell \\ -f_{-\ell} \end{pmatrix}$ is an eigenvector of $\mathbb{H}$ with eigenvalue ℓ tending to $1/2$ as $q \to 1/4^-$.

We can thus define a $\mathbb{H}$-invariant complex hermitian line bundle L as follows.

$$\text{Define } L_q := \begin{cases} \mathbb{C}\begin{pmatrix} f_\ell \\ -f_{-\ell} \end{pmatrix} & \text{for } q \leq 1/4, q = \ell(1-\ell) \\ \\ \mathbb{C}\begin{pmatrix} f_{1/2} \\ -f_{-1/2} \end{pmatrix} & \text{for } q \geq 1/4 \end{cases}$$

On this line-bundle the field of operators $\mathbb{H}$ is simply multiplication by the function

$$\omega(q) := \begin{cases} \ell = \frac{1}{2} + \sqrt{1/4 - q} & \text{for } q \leq 1/4 \\ \frac{1}{2} - \sqrt{q - 1/4} & \text{for } q \geq 1/4. \end{cases}$$

In particular we note that $\omega(q)$ tends to $\pm\infty$ as $q \to \mp\infty$. The field of operators induces an operator on the Hilbert module $C_0(\mathbb{R}, L)$ of C_0-sections of the bundle, and this operator is multiplication by ω. This means that the unbounded Kasparov triple $[C_0(\mathbb{R}, L), 1, \omega]$ represents the generator of $KK^1(\mathbb{C}, C_0(\mathbb{R}))$.

Theorem 3.2. *The unbounded Kasparov triple* $[C_0(\mathbb{R}, L), 1, \omega]$ *represents the generator of* $KK^1(\mathbb{C}, C_0(\mathbb{R}))$.

Similarly we have a $\mathbb{H}$-invariant line bundle M with

$$M_q := \begin{cases} \mathbb{C}\begin{pmatrix} f_\ell \\ f_{-\ell} \end{pmatrix} & \text{for } q \leq 1/4, q = \ell(1-\ell) \\ \\ \mathbb{C}\begin{pmatrix} f_{1/2} \\ f_{-1/2} \end{pmatrix} & \text{for } q \geq 1/4 \end{cases}$$

on which the operator $\mathbb{H}$ is multiplication by the function

$$\varepsilon(q) := \begin{cases} \ell = \frac{1}{2} + \sqrt{1/4 - q} & \text{for } q \leq 1/4 \\ \frac{1}{2} + \sqrt{q - 1/4} & \text{for } q \geq 1/4. \end{cases}$$

In this case we see that $\varepsilon(q) \to +\infty$ as $q \to \pm\infty$ from which we see that the corresponding Kasparov triple $[C_0(\mathbb{R}, M), 1, \varepsilon]$ in $KK^1(\mathbb{C}, C_0(\mathbb{R}))$ represents the zero element of KK^1.

We now examine how the remaining 2-dimensional subspaces E_m match up at $q = 1/4$. For $q \geq 1/4$ we have $m = 1/2 + k$ where $k \in \mathbb{Z}$ and we exclude the case $k = 0$ which we have already considered. Now for each $k > 0$ we take the 2-dimensional bundle $N^{(k)}$ whose fibres are $E_{1/2+k}$ for $q \geq 1/4$ and which are $E_{\ell+k}$ for $q \leq 1/4$. These agree at $q = 1/4$ since $\ell = 1/2$ at this point.

For $k < 0$ we take the 2-dimensional bundle $N^{(k)}$ whose fibres are $E_{1/2+k}$ for $q \geq 1/4$ and which are $E_{-\ell+1+k}$ for $q \leq 1/4$. We note that when $\ell = 1/2$ we obtain $E_{-\ell+1+k} = E_{1/2+k}$. Thus for each $k \neq 0$ we can view m as a continuous function of q defined by

$$m(q) := \begin{cases} \ell + k = \frac{1}{2} + \sqrt{1/4 - q} + k & \text{for } q \leq 1/4, k = 1, 2, \dots \\ -\ell + 1 + k = \frac{1}{2} - \sqrt{1/4 - q} + k & \text{for } q \leq 1/4, k = -1, -2, \dots \\ \frac{1}{2} + k & \text{for } q \geq 1/4, k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

Recall that the eigenvalues of the restriction of $\mathbb{H}$ to E_m are given by

$$\lambda^\pm = \frac{1}{2} \pm \sqrt{1/4 + q + 2m(m-1)}.$$

Writing $2m(m-1)$ as $2(m-1/2)^2 - 1/2$ we see that

$$2m(q)(m(q)-1) := \begin{cases} 2(\sqrt{1/4 - q} + k)^2 - 1/2 & \text{for } q \leq 1/4, k = 1, 2, \dots \\ 2(-\sqrt{1/4 - q} + k)^2 - 1/2 & \text{for } q \leq 1/4, k = -1, -2, \dots \\ 2k^2 - 1/2 & \text{for } q \geq 1/4, k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

$$= \begin{cases} 2(1/4 - q + k^2 + 2k\sqrt{1/4 - q}) - 1/2 & \text{for } q \leq 1/4, k = 1, 2, \dots \\ 2(1/4 - q + k^2 - 2k\sqrt{1/4 - q}) - 1/2 & \text{for } q \leq 1/4, k = -1, -2, \dots \\ 2k^2 - 1/2 & \text{for } q \geq 1/4, k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

$$= \begin{cases} 2(-q + k^2 + 2k\sqrt{1/4 - q}) & \text{for } q \le 1/4, k = 1, 2, \dots \\ 2(-q + k^2 - 2k\sqrt{1/4 - q}) & \text{for } q \le 1/4, k = -1, -2, \dots \\ 2k^2 - 1/2 & \text{for } q \ge 1/4, k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

$$1/4 + q + 2m(m-1) = \begin{cases} 1/4 - q + 2(k^2 + 2k\sqrt{1/4 - q}) & \text{for } q \le 1/4, k = 1, 2, \dots \\ 1/4 - q + 2(k^2 - 2k\sqrt{1/4 - q}) & \text{for } q \le 1/4, k = -1, -2, \dots \\ q - 1/4 + 2k^2 & \text{for } q \ge 1/4, k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

In particular we see that the discriminant $1/4 + q + 2m(m-1)$ is always at least 2 so that the eigenvalues must always be distinct, indeed they are respectively $\ge \frac{1}{2} + \sqrt{2}$ and $\le \frac{1}{2} - \sqrt{2}$. Thus the bundles of positive and negative eigenspaces within $N^{(k)}$ define $\mathbb{H}$-invariant line bundles which we denote $N^{(k)\pm}$. Moreover in the positive case the eigenvalues tend to $+\infty$ as $q \to \pm\infty$ and in the negative case eigenvalues tend to $-\infty$ as $q \to \pm\infty$.

This establishes the following result.

Theorem 3.3. *Each individual line bundle thus gives a Kasparov triple* $[C_0(\mathbb{R}, N^{(k)\pm}), 1, \lambda^\pm]$ *(where we view the eigenvalue* $\lambda^\pm$ *as a function of* q*) representing the zero element of* KK^1*.*

Finally, we have

Theorem 3.4. *The Kasparov triple* $[C_0(\mathbb{R}, V_*|_{\mathbb{R}}), 1, \mathbb{H}|_{\mathbb{R}}]$ *generates* $KK^1(\mathbb{C}, C_0(\mathbb{R})))$.

Proof. We have seen that the bundle E over $\mathbb{R}$ can be decomposed as the direct sum of L, M and $N^{(k)\pm}$ for $k \in \mathbb{Z} \setminus \{0\}$ and that the operator $\mathbb{H}$ respects this decomposition.

Now restricting $\mathbb{H}$ to an operator on sections of

$$\bigoplus_{k \in \mathbb{Z} \setminus \{0\}} N^{(k)+}$$

the operator acts by multiplication by $\lambda^+ = \frac{1}{2} + \sqrt{1/4 + q + 2m(m-1)}$. The above formulas for the discriminant show that λ^+ tends to infinity as $k \to \pm\infty$ and also as $q \to \pm\infty$. Hence the corresponding bounded operator $\mathbb{H}(1 + \mathbb{H}^2)^{-1/2}$ is a compact perturbation of the identity operator 1. The corresponding bounded Kasparov triple is thus degenerate.

Similarly restricting $\mathbb{H}$ to an operator on sections of

$$M \oplus \bigoplus_{k \in \mathbb{Z} \setminus \{0\}} N^{(k)-}$$

the operator acts by multiplication by $\lambda^- = \frac{1}{2} - \sqrt{1/4 + q + 2m(m-1)}$. Since λ^- tends to minus infinity as $k \to \pm\infty$ and as $q \to \pm\infty$ the corresponding bounded operator $\mathbb{H}(1 + \mathbb{H}^2)^{-1/2}$ is a compact perturbation of -1, and again the Kasparov triple is degenerate.

We conclude that neither of these restrictions contribute to the K-theory and thus

$$[C_0(\mathbb{R}, V_*|_{\mathbb{R}}), 1, \mathbb{H}] = [C_0(\mathbb{R}, L), 1, \omega]$$

which is the generator of $KK^1(\mathbb{C}, C_0(\mathbb{R}))$. □

4. Computing the K-theory

In this section we compute the K-theory of $\mathfrak{A}$ and establish that the Dirac class is a generator of KK^1. We begin with a convenient reparametrisation of the space $\mathcal{Z}$.

Let $\mathcal{Y}$ be the union of the unit disc $A = \{z \in \mathbb{C} : |z| \leq 1\}$ and the interval $B = [1, 2]$.

The coordinate change

$$\begin{aligned} (q, \tau) &\mapsto -(q + 3/4)^{-1} e^{2\pi i \tau}, && \text{for } q \geq 1/4,\ 0 \leq \tau \leq 1 \\ q &\mapsto 2 - (5/4 - q), && \text{for } q \leq 1/4 \end{aligned}$$

respects the identifications in the construction of $\mathcal{Z}$, and transforms the locally compact parameter space $\mathcal{Z}$ into a dense subspace $\mathcal{U}$ of the compact parameter space $\mathcal{Y}$. Explicitly $\mathcal{U}$ is the union of the punctured disc with the half-open interval $[1, 2)$, and the limit-of-discrete-series corresponds to the point 1.

Lemma 4.1. *The reduced C^*-algebra $\mathfrak{A}$ is strongly Morita equivalent to the C^*-algebra $\mathfrak{D}$ of all 2×2-matrix-valued functions on the compact Hausdorff space $\mathcal{Y}$ which are diagonal on B, and vanish at* 0 *and* 2.

Proof. By Theorem 2.2 we have an isomorphism $\mathfrak{A} \cong \widehat{\mathfrak{A}}$ given by the Fourier transform and we have already seen that $\widehat{\mathfrak{A}}$ is Morita equivalent to $\mathfrak{B}$. The algebra $\mathfrak{B}$ is isomorphic to $\mathfrak{D}$ via the above change of coordinates. □

We now compute the K-theory. Define a new C^*-algebra as follows:

$$\mathfrak{C} := \{F \in C(\mathcal{Y}, M_2(\mathbb{C})) : F(y) \text{ is diagonal on } B, F(2) = 0\}.$$

The map

$$\mathfrak{C} \to M_2(\mathbb{C}), \qquad F \mapsto F(0)$$

then fits into an exact sequence of C^*-algebras

$$1 \to \mathfrak{D} \to \mathfrak{C} \to M_2(\mathbb{C}) \to 0.$$

This yields the six-term exact sequence

$$\begin{array}{ccccc} K_0(\mathfrak{D}) & \longrightarrow & K_0(\mathfrak{C}) & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(\mathfrak{C}) & \longleftarrow & K_1(\mathfrak{D}) \end{array} \tag{2}$$

Note that $\mathcal{Y}$ is a contractible space. The following homotopy is well-adapted to the C^*-algebra $\mathfrak{C}$. Given $z = x + iy \in \mathcal{Y}$ define h_t as follows:

$$h_t(x + iy) = \begin{cases} x + (1 - 2t)iy & 0 \leq t \leq 1/2 \\ x + (2 - x)(2t - 1) & 1/2 \leq t \leq 1 \end{cases}$$

This is a homotopy equivalence from $\mathcal{Y}$ to the point $\{2\}$. Given a function $F \in \mathfrak{C}$, the composition $F(h_t(x + iy))$ also lies in $\mathfrak{C}$ since if $x + iy = x \in B$ then $h_t(x+iy)$ also lies in B, and hence $F(h_t(x+iy))$ is diagonal as required.

Thus $F \mapsto F \circ h_t$ induces a homotopy equivalence from $\mathfrak{C}$ to the zero C^*-algebra, and hence the connecting maps in (2) are isomorphisms.

Since K-theory is an invariant of strong Morita equivalence, we have the following result.

Theorem 4.2. *Let $\mathfrak{A}$ denote the reduced C^*-algebra of the universal cover of* $\mathrm{SL}_2(\mathbb{R})$*. Then*

$$K_0(\mathfrak{A}) = 0, \qquad K_1(\mathfrak{A}) = \mathbb{Z}.$$

Now recall that in Section 3 we considered the composition

$$\widehat{\mathfrak{A}} \hookrightarrow C_0(\mathcal{Z}, \mathfrak{K}) \underset{\text{M.E.}}{\sim} C_0(\mathcal{Z}) \twoheadrightarrow C_0(\mathbb{R}). \tag{3}$$

At the level of K-theory this agrees with the composition

$$\mathfrak{D} \hookrightarrow C_0(\mathcal{U}, M_2(\mathbb{C})) \underset{\text{M.E.}}{\sim} C_0(\mathcal{U}) \twoheadrightarrow C_0((0,2)).$$

which includes into the composition

$$\mathfrak{C} \hookrightarrow C_0(\mathcal{Y} \setminus \{2\}, M_2(\mathbb{C})) \underset{\text{M.E.}}{\sim} C_0(\mathcal{Y} \setminus \{2\}) \twoheadrightarrow C_0([0,2)).$$

It follows that the 6-term exact sequence (2) maps commutatively to the 6-term exact sequence

$$\begin{array}{ccccc} K_0(C_0((0,2))) & \longrightarrow & K_0(C_0([0,2))) & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(C_0([0,2))) & \longleftarrow & K_1(C_0((0,2))) \end{array}$$

from which we conclude that the map from $K_*(\mathfrak{D})$ to $K_*(C_0((0,2)))$ is an isomorphism. Correspondingly $K_*(\widehat{\mathfrak{A}}) \to K_*(C_0(\mathbb{R}))$ is an isomorphism.

In Section 3 we showed that (3) takes the Kasparov triple $[\widehat{\mathfrak{A}} \oplus \widehat{\mathfrak{A}}, 1, \mathbb{H}]$ to a generator of $KK^1(\mathbb{C}, C_0(\mathbb{R}))$. We thus conclude the following result.

Theorem 4.3. *Let $\mathfrak{A}$ denote the reduced C^*-algebra of the universal cover of* $\mathrm{SL}_2(\mathbb{R})$ *and let $\widehat{\mathfrak{A}} \cong \mathfrak{A}$ denote its Fourier transform. Then $[\widehat{\mathfrak{A}} \oplus \widehat{\mathfrak{A}}, 1, \mathbb{H}]$ is a generator of $KK^1(\mathbb{C}, \widehat{\mathfrak{A}})$ and hence $[\mathfrak{A} \oplus \mathfrak{A}, 1, D]$ is a generator of $KK^1(\mathbb{C}, \mathfrak{A})$.*

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