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THE DIRAC OPERATOR AND THE LIMIT-OF-DISCRETE-SERIES FOR THE UNIVERSAL COVER OF $\mathrm{SL}_2(\mathbb{R})$

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ABSTRACT. The unitary principal series of the universal cover of $\mathrm{SL}_2(\mathbb{R})$ admits a limit-of-discrete-series. We show how this representation leads to an explicit K-cycle which generates K_1 of the reduced C^*-algebra.

1. INTRODUCTION

Let G denote the universal cover of $\mathrm{SL}_2(\mathbb{R})$. Then G is a connected Lie group with Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. Let $\mathfrak{A} = C^*_r(G)$ denote the reduced C^*-algebra of G. It is known (see, for example [CEN]) that, in the sense of K-theory of C^*-algebras, we have

$$K_0\,\mathfrak{A} = 0, \qquad K_1\,\mathfrak{A} = \mathbb{Z}. \tag{1}$$

In this article we relate this result to the representation theory of G.

The group G has the following properties:
– G is a 3-dimensional connected Lie group
– G has infinite centre isomorphic to $\mathbb{Z}$
– the maximal compact subgroup of G is trivial
– G is non-linear, i.e. it is not a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$.

The fact that it is a non-linear group with infinite centre places it outside the range of much classical representation theory, due to Harish-Chandra and others. However, the Plancherel formula was established by Pukánszky [P]. In the reduced dual of G, there is one very special representation, which is in the unitary principal series of G *and* is the direct sum of two elements in the discrete series. We will call this representation the *limit-of-discrete-series* of G. This representation factors through the quotient group $\mathrm{SL}_2(\mathbb{R})$, and becomes the well-known limit-of-discrete-series for $\mathrm{SL}_2(\mathbb{R})$.

The limit-of-discrete series for $\mathrm{SL}_2(\mathbb{R})$ is the induced representation

$$\pi := \mathrm{Ind}_B^{\mathrm{SL}_2(\mathbb{R})}\chi \tag{2}$$

where χ is the unique quadratic character

$$\chi : \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix} \mapsto \begin{cases} 1 & \text{when } x > 0 \\ -1 & \text{when } x < 0 \end{cases}$$

of the standard Borel subgroup B of $\mathrm{SL}_2(\mathbb{R})$. The representation π splits as the direct sum of two irreducible representations:

$$\pi = \pi^+ \oplus \pi^-$$

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The representations π^+ and π^- are not in the discrete series; but their characters θ^+, θ^- are not identically zero on the elliptic set – they have this feature in common with the discrete series of $\mathrm{SL}_2(\mathbb{R})$.

The formal Fourier transform $\widehat{D}$ of the Dirac operator D breaks up as the direct sum of multiplication operators on complex Hermitian line bundles. The multiplication is by real-valued scalar functions. With one exception, the scalar functions stay away from 0, i.e. they remain either positive or negative, the corresponding Kasparov triples are degenerate, and they make no contribution to K-theory.

We show in this article how the limit-of-discrete-series for G allows one to construct a certain complex hermitian line bundle L on the real line $\{q : q \in \mathbb{R}\}$ which realises a generator of $KK^1(\mathbb{C}, C_0(\mathbb{R}))$. We define a $\widehat{D}$-invariant complex Hermitian line bundle L as follows.

$$L_q := \begin{cases} \mathbb{C}\begin{pmatrix} f_\ell \\ -f_{-\ell} \end{pmatrix} & \text{for } q \leq 1/4, q = \ell(1-\ell) \\ \\ \mathbb{C}\begin{pmatrix} f_{1/2} \\ -f_{-1/2} \end{pmatrix} & \text{for } q \geq 1/4 \end{cases}$$

For $q \leq 1/4$ the fibre L_q is spanned by a spinor made from the *lowest weight vector* of the discrete series $D(\ell, +)$ and the *highest weight vector* of the discrete series $D(\ell, -)$ with $q = \ell(1-\ell)$. For $q \geq 1/4$ the fibre L_q is spanned by a spinor made from two vectors of weight $1/2$ and $-1/2$. Since $\ell = 1/2$ when $q = 1/4$, these two line bundles can be glued together to form a line bundle L over $\mathbb{R}$. The C_0-sections of L are spinor fields vanishing at infinity.

On these spinor fields, the operator $\widehat{D}$ is multiplication by a function ω_q for which $\omega_q \to \pm\infty$ as $q \to \mp\infty$. This implies that the unbounded Kasparov triple $[C_0(\mathbb{R}, L), 1, \omega]$ represents a generator of $KK^1(\mathbb{C}, C_0(\mathbb{R}))$, and thence a generator of $K_1(\mathfrak{A})$.

In §2, we give the structure theorem for $\mathfrak{A}$ via the compact-operator-valued Fourier transform.

In §3, we describe in detail the construction of the triple $[C_0(\mathbb{R}, L), 1, \omega]$.

In §4, we give, for completeness, a self-contained proof of (1).

2. Reduced C^*-algebra

We begin with the Plancherel formula of Pukánszky [P] for the universal cover of $\mathrm{SL}_2(\mathbb{R})$.

Theorem 2.1. *The following representations enter into the Plancherel formula:*

$$\text{Principal series :} \qquad \{(V_{q,\tau}, \pi_{q,\tau}) : q \geq 1/4,\ 0 \leq \tau \leq 1\}, \qquad \Omega = q$$

$$\text{Discrete series :} \qquad (D_{\ell,+}, \omega_{\ell,+}), \quad (D_{\ell,-}, \omega_{\ell,-}), \quad \ell \geq 1/2, \qquad \Omega = \ell(1-\ell)$$

where Ω is the Casimir operator. For every test function f on G, smooth with compact support, we have

$$f(e) = \int_0^\infty \int_0^1 \sigma[\Re \tanh \pi(\sigma + i\tau)]\Theta(\sigma,\tau)(f)d\tau d\sigma + \int_{1/2}^\infty (\ell - 1/2)\Theta(\ell)(f)d\ell$$

where the Harish-Chandra characters are

$$\Theta(\sigma,\tau)(f) = \text{trace} \int_G \pi_{q,\tau}(g)f(g)dg$$

$$\Theta(\ell)(f) = \text{trace} \int_G (\omega_{\ell,+} \oplus \omega_{\ell,-})(g)f(g)dg$$

and $\sigma = \sqrt{q - 1/4}$.

This is a *measure-theoretic* statement. We need a more precise statement in topology.

Note that $\Omega = 1/4$ for each of the following representations;

$$\pi_{1/4,1/2}, \quad \omega_{1/2,+}, \quad \omega_{1/2,-}.$$

In fact the representation $\pi_{1/4,1/2}$ is reducible and

$$V_{1/4,1/2} = D_{1/2,+} \oplus D_{1/2,-}, \quad \pi_{1/4,1/2} = \omega_{1/2,+} \oplus \omega_{1/2,-}$$

see Eqn.(2.4) in [KM, p.40].

We will define the *parameter space* $\mathcal{Z}$ to be the union of the sets

$$\{q \in \mathbb{R} : q \leq 1/4\}$$

$$\{(q,\tau) \in \mathbb{R} \times \mathbb{R} : q \geq 1/4, 0 \leq \tau \leq 1\}$$

with identification of the point $1/4$ in the first set with $(1/4, 1/2)$ in the second, and with identification of $(q, 0)$ with $(q, 1)$ for all $q \geq 1/4$.

The G-Hilbert spaces $V_{q,\tau}$ form a continuous field of Hilbert-spaces over $q \geq 1/4, 0 \leq \tau \leq 1$. We extend this to a continuous field V_* of G-Hilbert-spaces over $\mathcal{Z}$ by defining

$$V_q = D_{\ell,+} \oplus D_{\ell,-}$$

where $q = \ell(1-\ell)$ noting that

$$V_{1/4} = V_{1/4,1/2}, \quad V_{q,0} = V_{q,1} \qquad \forall q \geq 1/4.$$

The Casimir operator on the G-modules V_q and $V_{q,\tau}$ is precisely the multiplication by the parameter q.

We have the following structure theorem.

Theorem 2.2. *Let* $\mathfrak{A}$ *denote the reduced* C^**-algebra* $C^*_r(G)$*. The Fourier transform* $f \mapsto \widehat{f}$ *induces an isomorphism of* $\mathfrak{A}$ *onto the* C^**-algebra*

$$\widehat{\mathfrak{A}} := \{F \in C_0(\mathcal{Z}, \mathfrak{K}(V_*)) : F(q)D_{\ell,+} \subset D_{\ell,+}, F(q)D_{\ell,-} \subset D_{\ell,-} \text{ if } q \leq 1/4\}.$$

Proof. The reduced C^*-algebra is a quotient of the full C^*-algebra $C^*(G)$:

$$1 \to \mathfrak{I} \to C^*(G) \to C^*_r(G) \to 1.$$

The complementary series makes no contribution. The C^*-algebra $\mathfrak{A}$ is a quotient of the full C^*-algebra in [KM] and the primitive ideal space of $\mathfrak{A}$ contains every point in the support of Plancherel measure on $\widehat{G}$ (the unitary dual of G), by Theorem (2.1). $\square$

Note that the Jacobson topology on the primitive ideal spectrum of $\mathfrak{A}$ is exactly right: it has a double point at $q = 1/4 \in \mathcal{Z}$, where the unitary representation $\pi_{1/4,1/2}$ is reducible.

We observe that the algebra $\widehat{\mathfrak{A}}$ is (strongly) Morita equivalent to the algebra

$$\mathfrak{B} := \{F \in C_0(\mathcal{Z}, M_2(\mathbb{C})) : F(q) \text{ is diagonal if } q \leq 1/4\}.$$

One way to see this is to note that $\widehat{\mathfrak{A}} \cong \mathfrak{B} \otimes \mathfrak{K}$ however it is instructive to consider the explicit bimodules yielding the Morita equivalence.

We introduce the notation $\mathbb{C}^2_+$ and $\mathbb{C}^2_-$ for the subspaces $\mathbb{C} \oplus 0$ and $0 \oplus \mathbb{C}$ in $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$. We now form the module $\mathcal{E}$ defined by

$$\{F \in C_0(\mathcal{Z}, V_* \otimes \mathbb{C}^2) : F(q) \in (D_{\ell,+} \otimes \mathbb{C}^2_+) \oplus (D_{\ell,-} \otimes \mathbb{C}^2_-) \text{ if } q = \ell(1-\ell) \leq {}^1\!/\!_4\}.$$

This can be equipped with two (pointwise) inner products. Firstly we have a $\mathfrak{B}$-valued inner product, which is to say a pointwise $M_2(\mathbb{C})$-valued inner product satisfying the required diagonality condition. For $F = F_1 \otimes F_2$, $G = G_1 \otimes G_2$ the inner product is defined to be

$$\langle F, G\rangle_{\mathfrak{B}} = \langle F_1(z), G_1(z)\rangle_{V_z} \, F_2(z)\langle G_2(z), - \rangle_{\mathbb{C}^2}.$$

Secondly we have an $\widehat{\mathfrak{A}}$ valued inner product, which pointwise takes values in $\mathfrak{K}(E)$. For $F = F_1 \otimes F_2$, $G = G_1 \otimes G_2$ the inner product is defined to be

$$\langle F, G\rangle_{\widehat{\mathfrak{A}}} = \langle F_2(z), G_2(z)\rangle_{\mathbb{C}^2} \, F_1(z)\langle G_1(z), - \rangle_{V_z}.$$

The Hilbert modules obtained by equipping $\mathcal{E}$ with these two inner products effect a Morita equivalence between the algebras $\widehat{\mathfrak{A}}$ and $\mathfrak{B}$.

We remark that a field of operators on the field V_* of Hilbert spaces can naturally be regarded as an adjointable operator on $\mathcal{E}$ with the $\mathfrak{B}$-valued inner product.

3. The K-cycle

We recall the parameter space $\mathcal{Z}$ from §2 and corresponding continuous field of G-Hilbert-spaces over $\mathcal{Z}$:

$$\{V_q : q \leq 1/4\}$$
$$\{V_{q,\tau} : q \geq 1/4, 0 \leq \tau \leq 1\}$$

Let $\mathfrak{g}$ denote the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$, and let $C(\mathfrak{g})$ denote the Clifford algebra of $\mathfrak{g}$ with respect to the negative definite quadratic form on $\mathfrak{g}$. Let X_0, X_1, X_2 denote an orthonormal basis in $\mathfrak{g}$. Note that the notation in [P, (1.1)] is $l_k = X_k$.

Following the algebraic approach in [HP, Def. 3.1.2] the *Dirac operator* is the element of the algebra $U(\mathfrak{g}) \otimes C(\mathfrak{g})$ given by

$$D = X_0 \otimes c(X_0) + X_1 \otimes c(X_1) + X_2 \otimes c(X_2)$$

where $c(X_k)$ denotes Clifford multiplication by X_k.

Let

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

and set

$$c(X_k) = i\sigma_k, \qquad k = 0, 1, 2$$

Then we have

$$c(X_k)^2 = -1$$

for all $k = 0, 1, 2$.

We have

$$\begin{aligned} D &= i(X_0\sigma_0 + X_1\sigma_1 + X_2\sigma_2) \\ &= i\begin{pmatrix} X_0 & X_1 + iX_2 \\ X_1 - iX_2 & -X_0 \end{pmatrix} \end{aligned}$$

The operator D acts on 2-spinor fields in the following way: the elements of the Lie algebra $\mathfrak{g}$ give rise to *right*-invariant vector fields on G and in this way $X_j, j = 0, 1, 2$ form differential operators on scalar fields. The matrix then acts by differentiating the components of a spinor field.

Viewing a compactly supported 2-spinor field as a pair of scalar valued functions on the group G, it is an element of $C_c(G) \oplus C_c(G) \subseteq \mathfrak{A} \oplus \mathfrak{A}$. In this way the Dirac operator D gives rise to an unbounded adjointable operator on $\mathfrak{A} \oplus \mathfrak{A}$ viewed as a Hilbert module over $\mathfrak{A}$.

Let now π be a unitary representation, in the principal series or the discrete series of G, on a Hilbert space V_π. The infinitesimal generators H_0, H_1, H_2, which act on the Hilbert space V_π, are determined by the following equation [P, p.98]:

$$\exp(-itH_k) = \pi(\exp(tX_k)) \qquad \forall t \in \mathbb{R}, k = 0, 1, 2$$

On each of the Hilbert spaces V_q and $V_{q,\tau}$ we therefore have three self-adjoint operators, namely H_0, H_1 and H_2. These form a field of operators on the field of Hilbert spaces V_*. The spectrum of H_0 is discrete with eigenvalues $m = \ell, \ell+1, \ell+2, \ldots$ and $m = -\ell, \ell-1, -\ell-2, \ldots$ in the case that $q < 1/4$ and with eigenvalues $m \in \tau + \mathbb{Z}$ for $q \geq 1/4$. Each eigenvalue has multiplicity 1 and we let f_m be an orthogonal basis of eigenvectors of H_0 so that

$$H_0 f_m = m f_m.$$

Following [P, p.100], we define

$$H_+ = H_1 + iH_2, \qquad H_- = H_1 - iH_2$$

In addition, we have the following equations

$$\begin{aligned} H_+ f_m &= (q + m(m+1))^{1/2} f_{m+1} \\ H_- f_m &= (q + m(m-1))^{1/2} f_{m-1} \end{aligned}$$

which hold for all m when $q \geq 1/4$ and where the first equation holds for all $m \neq -\ell$, the second for all $m \neq \ell$ when $q < 1/4$. The special cases of $H_+ f_{-\ell}$ and $H_- f_\ell$ are both zero.

By analogy with the Dirac operator D above, we construct a field of self-adjoint operators

$$\mathbb{H} = \begin{pmatrix} H_0 & H_1 + iH_2 \\ H_1 - iH_2 & -H_0 \end{pmatrix} \tag{3}$$

on the field of Hilbert spaces $V_* \oplus V_*$. Since the algebra $\widehat{\mathfrak{A}}$ consists of fields of compact operators on V_*, the operator $\mathbb{H}$ can also be thought of as acting on $\widehat{\mathfrak{A}} \oplus \widehat{\mathfrak{A}}$ by composition. (One must additionally note that the operators $H_j, j = 0, 1, 2$ respect the decomposition of V_q as $D_{\ell,+} \oplus D_{\ell,-}$ for $q \leq 1/4$.)

Since the operator D acts on $\mathfrak{A} \oplus \mathfrak{A}$ and we have an isomorphism from $\mathfrak{A}$ to $\widehat{\mathfrak{A}}$ given by the Fourier transform (Theorem 2.2), we obtain an operator $\widehat{D}$ on $\widehat{\mathfrak{A}} \oplus \widehat{\mathfrak{A}}$. We will show that

$$\widehat{D} = -\mathbb{H}.$$

To demonstrate this it suffices to show that the isomorphism $\mathfrak{A} \cong \widehat{\mathfrak{A}}$ takes the differential operator iX_j to $-H_j$ for each j.

For f a smooth compactly supported function on G the Fourier transform of f is defined by

$$\hat{f}(\pi) = \int f(g)\pi(g)\, dg.$$

Since H_j is the infinitesimal generator of the 1-parameter group $\exp(-itH_j) = \pi(\exp(tX_j))$ we have $H_j = i\frac{d}{dt}\pi(\exp(tX_j))|_{t=0}$. Hence

$$\begin{aligned}
-H_j\hat{f}(\pi) &= -\int f(g)H_j\pi(g)\, dg \\
&= -\int f(g) i\frac{d}{dt}\pi(\exp(tX_j))|_{t=0}\pi(g)\, dg \\
&= -i\frac{d}{dt}\int f(g)\pi(\exp(tX_j)g)\, dg|_{t=0} \\
&= -i\frac{d}{dt}\int f(\exp(-tX_j)g')\pi(g')\, dg'|_{t=0} \quad \text{by left invariance of } dg \\
&= -i\int \frac{d}{dt}f(\exp(-tX_j)g')|_{t=0}\pi(g')\, dg' \\
&= i\int X_j(f)\pi(g')\, dg' \\
&= i\widehat{X_j(f)}(\pi)
\end{aligned}$$

since X_j is a right-invariant vector field. This establishes that the Fourier transform takes iX_j to $-H_j$ and hence that $\widehat{D} = -\mathbb{H}$.

We will therefore study the operator $\mathbb{H}$ in detail. Recall that $\mathbb{H}$ is defined on the field $V_* \oplus V_*$, and a crucial observation at this point is the emergence of two dimensional invariant subspaces for $\mathbb{H}$ at each point of the parameter space $\mathcal{Z}$.

Each such subspace E_m is spanned by a pair of vectors $\begin{pmatrix} f_m \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ f_{m-1} \end{pmatrix}$, where m is in the set $\tau + \mathbb{Z}$ for $q \geq 1/4$ and $m = \ell+1, \ell+2, \ldots$ or $-\ell, -\ell-1, \ldots$ for $q < 1/4$. We have the following equations

$$\mathbb{H}\begin{pmatrix} f_m \\ 0 \end{pmatrix} = \begin{pmatrix} H_0 & H_+ \\ H_- & -H_0 \end{pmatrix}\begin{pmatrix} f_m \\ 0 \end{pmatrix} = \begin{pmatrix} mf_m \\ (q + m(m-1))^{1/2} f_{m-1} \end{pmatrix}$$

and

$$\mathbb{H}\begin{pmatrix} 0 \\ f_{m-1} \end{pmatrix} = \begin{pmatrix} H_0 & H_+ \\ H_- & -H_0 \end{pmatrix}\begin{pmatrix} 0 \\ f_{m-1} \end{pmatrix} = \begin{pmatrix} (q + m(m-1))^{1/2} f_m \\ -(m-1)f_{m-1} \end{pmatrix}.$$

With respect to this basis, the operator $\mathbb{H}$ is given by the following symmetric matrix

$$\begin{pmatrix} m & (q+m(m-1))^{1/2} \\ (q+m(m-1))^{1/2} & -(m-1) \end{pmatrix}.$$

This symmetric matrix has the following eigenvalues

$$\lambda = \frac{1}{2} \pm \sqrt{1/4 + q + 2m(m-1)}.$$

In the case that $q \geq 1/4$ the subspaces E_m for $m \in \tau + \mathbb{Z}$ span the whole of $V_{q,\tau}$. However, for $q < 1/4$ there are a further two 1-dimensional subspaces spanned by the vectors

$$\begin{pmatrix} f_\ell \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ f_{-\ell} \end{pmatrix}$$

These subspaces are invariant since $H_-(f_\ell) = 0$ and $H_+(f_{-\ell}) = 0$.

Note that something very special occurs in the limit-of-discrete series when $q = 1/4$ and $\tau = 1/2$. Here, when $m = 1/2$ we have

$$q + m(m-1) = 0$$

and so in this case the operator matrix $\begin{pmatrix} H_0 & H_+ \\ H_- & -H_0 \end{pmatrix}$ restricted to the 2-dimensional subspace spanned by $\begin{pmatrix} f_{1/2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ f_{-1/2} \end{pmatrix}$ is the diagonal matrix

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

For $q < 1/4$, we have the following eigenvector equations

$$\begin{pmatrix} H_0 & H_+ \\ H_- & -H_0 \end{pmatrix} \begin{pmatrix} f_l \\ 0 \end{pmatrix} = l \begin{pmatrix} f_l \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} H_0 & H_+ \\ H_- & -H_0 \end{pmatrix} \begin{pmatrix} 0 \\ f_{-l} \end{pmatrix} = l \begin{pmatrix} 0 \\ f_{-l} \end{pmatrix}.$$

With respect to the basis given by the vectors $\begin{pmatrix} f_l \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ f_{-l} \end{pmatrix}$ our operator matrix has the diagonal form $\begin{pmatrix} l & 0 \\ 0 & l \end{pmatrix}$, so we see that as q approaches $1/4$ and consequently ℓ approaches $1/2$ this matches up with the limit-of-discrete series case of $q = 1/4, \tau = 1/2$, justifying the terminology.

As noted above, the operator $\mathbb{H}$ can be thought of either as an operator on $\widehat{\mathfrak{A}} \oplus \widehat{\mathfrak{A}}$ or an operator on $V_* \oplus V_*$. Taking the latter view, the operator acts on the space $C_0(\mathcal{Z}, V_* \oplus V_*)$ of continuous sections of $V_* \oplus V_*$ vanishing at infinity, which is a Hilbert module over $C_0(\mathcal{Z})$. Indeed there is an isomorphism of Hilbert modules

$$(\widehat{\mathfrak{A}} \oplus \widehat{\mathfrak{A}}) \otimes_{\widehat{\mathfrak{A}}} C_0(\mathcal{Z}, V_*) \cong C_0(\mathcal{Z}, V_* \oplus V_*)$$

and viewing $\mathbb{H}$ as an operator on $\widehat{\mathfrak{A}} \oplus \widehat{\mathfrak{A}}$, the corresponding operator on $C_0(\mathcal{Z}, V_* \oplus V_*)$ is given by $\mathbb{H} \otimes 1$.

The algebra $\widehat{\mathfrak{A}}$ is contained in $C_0(\mathcal{Z},\mathfrak{K}(V_*))$ (or $C_0(\mathcal{Z},\mathfrak{K})$ for short) and can factorise the tensor product further as

$$(\widehat{\mathfrak{A}}\oplus\widehat{\mathfrak{A}})\otimes_{\widehat{\mathfrak{A}}} C_0(\mathcal{Z},\mathfrak{K})\otimes_{C_0(\mathcal{Z},\mathfrak{K})} C_0(\mathcal{Z},V_*)$$

Hence at the level of KK-theory viewing $\mathbb{H}$ as an operator on sections of $V_*\oplus V_*$ corresponds to the composition of the forgetful inclusion of $\widehat{\mathfrak{A}}$ into $C_0(\mathcal{Z},\mathfrak{K})$ with the Morita equivalence from $C_0(\mathcal{Z},\mathfrak{K})$ to $C_0(\mathcal{Z})$ (which is implemented by the module $C_0(\mathcal{Z},V_*)$).

We now consider the restriction of the field of Hilbert spaces to the copy of the real line inside $\mathcal{Z}$ given by the union

$$\{q\in\mathbb{R}: q\le 1/4\}\cup\{(q,1/2)\in\mathbb{R}\times\mathbb{R}: q\ge 1/4\},$$

which we regard as parametrized by q. Correspondingly we restrict $\mathbb{H}$ to the Hilbert spaces over this line. Passing from $\mathbb{H}$ viewed as an operator on an $\widehat{\mathfrak{A}}$-Hilbert module to $\mathbb{H}|_{\mathbb{R}}$ viewed as an operator on a $C_0(\mathbb{R})$-Hilbert module corresponds to the composition

$$\widehat{\mathfrak{A}}\hookrightarrow C_0(\mathcal{Z},\mathfrak{K})\underset{\text{m.e.}}{\sim} C_0(\mathcal{Z})\twoheadrightarrow C_0(\mathbb{R}).$$

We will see in Section 4 that this composition induces an isomorphism at the level of KK-theory. In this section we will identify the class $[C_0(\mathbb{R},V_*|_{\mathbb{R}}),1,\mathbb{H}|_{\mathbb{R}}]$ in $KK^1(\mathbb{C},C_0(\mathbb{R}))$.

Remark 3.1. *The half-line $\{q\in\mathbb{R}: q\ge 1/4\}$ has the following significance in representation theory. The corresponding unitary representations $(V_{q,1/2},\pi_{q,1/2})$ all factor through $\mathrm{SL}_2(\mathbb{R})$ and constitute the odd principal series π_q of $\mathrm{SL}_2(\mathbb{R})$. In particular, the representation $\pi_{1/4,1/2}$ is the limit-of-discrete series for $\mathrm{SL}_2(\mathbb{R})$. It is the direct sum of two irreducible representations whose characters θ_+ and θ_- do not vanish on the elliptic set. In this respect, they resemble representations in the discrete series. So the term* limit-of-discrete-series *for $\pi_{1/4,1/2}$ is surely apt.*

We now attempt to glue together some one-dimensional eigenspaces to form a complex hermitian line bundle L over $\mathbb{R}$. Take $q\ge 1/4$ and consider the subspace $E_{1/2}$. The restriction $\mathbb{H}|_{E_{1/2}}$ is given by the matrix

$$\begin{pmatrix} 1/2 & \sqrt{q-1/4}\\ \sqrt{q-1/4} & 1/2\end{pmatrix}$$

from which we readily see that the vector $\begin{pmatrix} f_{1/2}\\ -f_{-1/2}\end{pmatrix}$ is an eigenvector of $\mathbb{H}$ with eigenvalue $1/2-\sqrt{q-1/4}$. Note that the eigenvalue tends to $1/2$ as $q\to 1/4^+$.

Now for $q<1/4$ we see that $\begin{pmatrix} f_{\ell}\\ -f_{-\ell}\end{pmatrix}$ is an eigenvector of $\mathbb{H}$ with eigenvalue ℓ tending to $1/2$ as $q\to 1/4^-$.

We can thus define a $\mathbb{H}$-invariant complex Hermitian line bundle L as follows.

$$\text{Define } L_q := \begin{cases} \mathbb{C}\begin{pmatrix} f_\ell \\ -f_{-\ell} \end{pmatrix} & \text{for } q \leq 1/4, q = \ell(1-\ell) \\ \\ \mathbb{C}\begin{pmatrix} f_{1/2} \\ -f_{-1/2} \end{pmatrix} & \text{for } q \geq 1/4 \end{cases}$$

On this line-bundle the field of operators $\mathbb{H}$ is simply multiplication by the function

$$\omega(q) := \begin{cases} \ell = \frac{1}{2} + \sqrt{1/4 - q} & \text{for } q \leq 1/4 \\ \frac{1}{2} - \sqrt{q - 1/4} & \text{for } q \geq 1/4. \end{cases}$$

In particular we note that $\omega(q)$ tends to $\pm\infty$ as $q \to \mp\infty$. The field of operators induces an operator on the Hilbert module $C_0(\mathbb{R}, L)$ of C_0-sections of the bundle, and this operator is multiplication by ω. This means that the unbounded Kasparov triple $[C_0(\mathbb{R}, L), 1, \omega]$ represents the generator of $KK^1(\mathbb{C}, C_0(\mathbb{R}))$.

Theorem 3.2. *The unbounded Kasparov triple* $[C_0(\mathbb{R}, L), 1, \omega]$ *represents the generator of* $KK^1(\mathbb{C}, C_0(\mathbb{R}))$.

Similarly we have a $\mathbb{H}$-invariant line bundle M with

$$M_q := \begin{cases} \mathbb{C}\begin{pmatrix} f_\ell \\ f_{-\ell} \end{pmatrix} & \text{for } q \leq 1/4, q = \ell(1-\ell) \\ \\ \mathbb{C}\begin{pmatrix} f_{1/2} \\ f_{-1/2} \end{pmatrix} & \text{for } q \geq 1/4 \end{cases}$$

on which the operator $\mathbb{H}$ is multiplication by the function

$$\varepsilon(q) := \begin{cases} \ell = \frac{1}{2} + \sqrt{1/4 - q} & \text{for } q \leq 1/4 \\ \frac{1}{2} + \sqrt{q - 1/4} & \text{for } q \geq 1/4. \end{cases}$$

In this case we see that $\varepsilon(q) \to +\infty$ as $q \to \pm\infty$ from which we see that the corresponding Kasparov triple $[C_0(\mathbb{R}, M), 1, \varepsilon]$ in $KK^1(\mathbb{C}, C_0(\mathbb{R}))$ represents the zero element of KK^1.

We now examine how the remaining 2-dimensional subspaces E_m match up at $q = 1/4$. For $q \geq 1/4$ we have $m = 1/2 + k$ where $k \in \mathbb{Z}$ and we exclude the case $k = 0$ which we have already considered. Now for each $k > 0$ we take the 2-dimensional bundle $N^{(k)}$ whose fibres are $E_{1/2+k}$ for $q \geq 1/4$ and which are $E_{\ell+k}$ for $q \leq 1/4$. These agree at $q = 1/4$ since $\ell = 1/2$ at this point.

For $k < 0$ we take the 2-dimensional bundle $N^{(k)}$ whose fibres are $E_{1/2+k}$ for $q \geq 1/4$ and which are $E_{-\ell+1+k}$ for $q \leq 1/4$. We note that when $\ell = 1/2$ we obtain $E_{-\ell+1+k} = E_{1/2+k}$. Thus for each $k \neq 0$ we can view m as a continuous function of q defined by

$$m(q) := \begin{cases} \ell + k = \frac{1}{2} + \sqrt{1/4 - q} + k & \text{for } q \le 1/4, k = 1, 2, \ldots \\ -\ell + 1 + k = \frac{1}{2} - \sqrt{1/4 - q} + k & \text{for } q \le 1/4, k = -1, -2, \ldots \\ \frac{1}{2} + k & \text{for } q \ge 1/4, k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

Recall that the eigenvalues of the restriction of $\mathbb{H}$ to E_m are given by

$$\lambda^{\pm} = \frac{1}{2} \pm \sqrt{1/4 + q + 2m(m-1)}.$$

Writing $2m(m-1)$ as $2(m-1/2)^2 - 1/2$ we see that

$$2m(q)(m(q)-1) := \begin{cases} 2(\sqrt{1/4 - q} + k)^2 - 1/2 & \text{for } q \le 1/4, k = 1, 2, \ldots \\ 2(-\sqrt{1/4 - q} + k)^2 - 1/2 & \text{for } q \le 1/4, k = -1, -2, \ldots \\ 2k^2 - 1/2 & \text{for } q \ge 1/4, k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

$$= \begin{cases} 2(1/4 - q + k^2 + 2k\sqrt{1/4 - q}) - 1/2 & \text{for } q \le 1/4, k = 1, 2, \ldots \\ 2(1/4 - q + k^2 - 2k\sqrt{1/4 - q}) - 1/2 & \text{for } q \le 1/4, k = -1, -2, \ldots \\ 2k^2 - 1/2 & \text{for } q \ge 1/4, k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

$$= \begin{cases} 2(-q + k^2 + 2k\sqrt{1/4 - q}) & \text{for } q \le 1/4, k = 1, 2, \ldots \\ 2(-q + k^2 - 2k\sqrt{1/4 - q}) & \text{for } q \le 1/4, k = -1, -2, \ldots \\ 2k^2 - 1/2 & \text{for } q \ge 1/4, k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

$$1/4 + q + 2m(m-1) = \begin{cases} 1/4 - q + 2(k^2 + 2k\sqrt{1/4 - q}) & \text{for } q \le 1/4, k = 1, 2, \ldots \\ 1/4 - q + 2(k^2 - 2k\sqrt{1/4 - q}) & \text{for } q \le 1/4, k = -1, -2, \ldots \\ q - 1/4 + 2k^2 & \text{for } q \ge 1/4, k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

In particular we see that the discriminant $1/4 + q + 2m(m-1)$ is always at least 2 so that the eigenvalues must always be distinct, indeed they are respectively $\ge \frac{1}{2} + \sqrt{2}$ and $\le \frac{1}{2} - \sqrt{2}$. Thus the bundles of positive and negative eigenspaces within $N^{(k)}$ define $\mathbb{H}$-invariant line bundles which we denote $N^{(k)\pm}$. Moreover in the positive case the eigenvalues tend to $+\infty$ as $q \to \pm\infty$ and in the negative case eigenvalues tend to $-\infty$ as $q \to \pm\infty$.

This establishes the following result.

Theorem 3.3. *Each individual line bundle thus gives a Kasparov triple $[C_0(\mathbb{R}, N^{(k)\pm}), 1, \lambda^{\pm}]$ (where we view the eigenvalue $\lambda^{\pm}$ as a function of q) representing the zero element of KK^1.*

Finally, we have

Theorem 3.4. *The Kasparov triple $[C_0(\mathbb{R}, V_*|_{\mathbb{R}}), 1, \mathbb{H}|_{\mathbb{R}}]$ generates $KK^1(\mathbb{C}, C_0(\mathbb{R})))$.*

Proof. We have seen that the bundle E over $\mathbb{R}$ can be decomposed as the direct sum of L, M and $N^{(k)\pm}$ for $k \in \mathbb{Z} \setminus \{0\}$ and that the operator $\mathbb{H}$ respects this decomposition.

Now restricting $\mathbb{H}$ to an operator on sections of

$$\bigoplus_{k \in \mathbb{Z} \setminus \{0\}} N^{(k)+}$$

the operator acts by multiplication by $\lambda^+ = \frac{1}{2} + \sqrt{1/4 + q + 2m(m-1)}$. The above formulas for the discriminant show that λ^+ tends to infinity as

$k \to \pm\infty$ and also as $q \to \pm\infty$. Hence the corresponding bounded operator $\mathbb{H}(1+\mathbb{H}^2)^{-1/2}$ is a compact perturbation of the identity operator 1. The corresponding bounded Kasparov triple is thus degenerate.

Similarly restricting $\mathbb{H}$ to an operator on sections of

$$M \oplus \bigoplus_{k\in\mathbb{Z}\setminus\{0\}} N^{(k)-}$$

the operator acts by multiplication by $\lambda^- = \frac{1}{2} - \sqrt{1/4 + q + 2m(m-1)}$. Since λ^- tends to minus infinity as $k \to \pm\infty$ and as $q \to \pm\infty$ the corresponding bounded operator $\mathbb{H}(1+\mathbb{H}^2)^{-1/2}$ is a compact perturbation of -1, and again the Kasparov triple is degenerate.

We conclude that neither of these restrictions contribute to the K-theory and thus

$$[C_0(\mathbb{R}, V_*|_{\mathbb{R}}), 1, \mathbb{H}] = [C_0(\mathbb{R}, L), 1, \omega]$$

which is the generator of $KK^1(\mathbb{C}, C_0(\mathbb{R}))$. $\square$

4. K-theory

Let

$$A = \{z \in \mathbb{C}; |z| \leq 1\}, \qquad B = [1,2]$$

and let

$$\mathcal{Y} = A \cup B.$$

The coordinate change

$$\begin{aligned} (q,\tau) &\mapsto -(q+3/4)^{-1}e^{2\pi i\tau}, && \text{for } q \geq 1/4,\ 0 \leq \tau \leq 1 \\ q &\mapsto 2 - (5/4 - q), && \text{for } q \leq 1/4 \end{aligned}$$

respects the identifications in the construction of $\mathcal{Z}$, and transforms the locally compact parameter space $\mathcal{Z}$ into a dense subspace $\mathcal{U}$ of the compact parameter space $\mathcal{Y}$. Explicitly $\mathcal{U}$ is the union of the punctured disc with the half-open interval $[1,2)$.

Lemma 4.1. *The reduced C^*-algebra $\mathfrak{A}$ is strongly Morita equivalent to the C^*-algebra $\mathfrak{D}$ of all 2×2-matrix-valued functions on the compact Hausdorff space Y which are diagonal on B, and vanish at 0 and 2:*

$$\mathfrak{D} := \{F \in C(\mathcal{Y}, M_2(\mathbb{C})) : F(y) \text{ is diagonal on } B, F(0) = 0 = F(2)\}$$

Proof. By Theorem 2.2 we have an isomorphism $\mathfrak{A} \cong \widehat{\mathfrak{A}}$ given by the Fourier transform and we have already seen that $\widehat{\mathfrak{A}}$ is Morita equivalent to $\mathfrak{B}$. The algebra $\mathfrak{B}$ is isomorphic to $\mathfrak{D}$ via the above change of coordinates. $\square$

We now compute the K-theory. Define a new C^*-algebra as follows:

$$\mathfrak{C} := \{F \in C(\mathcal{Y}, M_2(\mathbb{C})) : F(y) \text{ is diagonal on } B, F(2) = 0\}.$$

The map

$$\mathfrak{C} \to M_2(\mathbb{C}), \qquad F \mapsto F(0)$$

then fits into an exact sequence of C^*-algebras

$$1 \to \mathfrak{D} \to \mathfrak{C} \to M_2(\mathbb{C}) \to 0.$$

This yields the six-term exact sequence

$$\begin{array}{ccccc} K_0(\mathfrak{D}) & \longrightarrow & K_0(\mathfrak{C}) & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(\mathfrak{C}) & \longleftarrow & K_1(\mathfrak{D}) \end{array} \tag{4}$$

Note that $\mathcal{Y} = A \cup B$ is a contractible space. The following homotopy is well-adapted to the C^*-algebra $\mathfrak{C}$. Given $z = x+iy \in \mathcal{Y}$ define h_t as follows:

$$h_t(x+iy) = \begin{cases} x + (1-2t)iy & 0 \leq t \leq 1/2 \\ x + (2-x)(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

This is a homotopy equivalence from $\mathcal{Y}$ to the point $\{2\}$. Given a function $F \in \mathfrak{C}$, the composition $F(h_t(x+iy))$ also lies in $\mathfrak{C}$ since if $x+iy = x \in B$ then $h_t(x+iy)$ also lies in B, and hence $F(h_t(x+iy))$ is diagonal as required.

Thus $F \mapsto F \circ h_t$ induces a homotopy equivalence from $\mathfrak{C}$ to the zero C^*-algebra $\mathfrak{O}$:

$$\mathfrak{C} \sim_h \mathfrak{O}$$

i.e. $\mathfrak{C}$ is a contractible C^*-algebra, and hence the connecting maps in (4) are isomorphisms.

Since K-theory is an invariant of strong Morita equivalence, we have the following result.

Theorem 4.2. *Let $\mathfrak{A}$ denote the reduced C^*-algebra of the universal cover of* $\mathrm{SL}_2(\mathbb{R})$*. Then*

$$K_0(\mathfrak{A}) = 0, \qquad K_1(\mathfrak{A}) = \mathbb{Z}$$

In Section 3 we considered the composition

$$\widehat{\mathfrak{A}} \hookrightarrow C_0(\mathcal{Z}, \mathfrak{K}) \underset{\text{m.e.}}{\sim} C_0(\mathcal{Z}) \twoheadrightarrow C_0(\mathbb{R}). \tag{5}$$

At the level of K-theory this agrees with the composition

$$\mathfrak{D} \hookrightarrow C_0(\mathcal{U}, M_2(\mathbb{C})) \underset{\text{m.e.}}{\sim} C_0(\mathcal{U}) \twoheadrightarrow C_0((0,2)).$$

which includes into the composition

$$\mathfrak{C} \hookrightarrow C_0(\mathcal{Y} \setminus \{2\}, M_2(\mathbb{C})) \underset{\text{m.e.}}{\sim} C_0(\mathcal{Y} \setminus \{2\}) \twoheadrightarrow C_0([0,2)).$$

It follows that the 6-term exact sequence (4) maps commutatively to the 6-term exact sequence

$$\begin{array}{ccccc} K_0(C_0((0,2))) & \longrightarrow & K_0(C_0([0,2))) & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(C_0([0,2))) & \longleftarrow & K_1(C_0((0,2))) \end{array}$$

from which we conclude that the map from $K_*(\mathfrak{D})$ to $K_*(C_0((0,2)))$ is an isomorphism. Correspondingly $K_*(\widehat{\mathfrak{A}}) \to K_*(C_0(\mathbb{R}))$ is an isomorphism.

In Section 3 we showed that (5) takes the Kasparov triple $[\widehat{\mathfrak{A}} \oplus \widehat{\mathfrak{A}}, 1, \mathbb{H}]$ to a generator of $KK^1(\mathbb{C}, C_0(\mathbb{R}))$. We thus conclude the following result.

Theorem 4.3. *Let $\mathfrak{A}$ denote the reduced C^*-algebra of the universal cover of* $\mathrm{SL}_2(\mathbb{R})$ *and let $\widehat{\mathfrak{A}} \cong \mathfrak{A}$ denote its Fourier transform. Then $[\widehat{\mathfrak{A}} \oplus \widehat{\mathfrak{A}}, 1, \mathbb{H}]$ is a generator of $KK^1(\mathbb{C}, \widehat{\mathfrak{A}})$ and hence $[\mathfrak{A} \oplus \mathfrak{A}, 1, D]$ is a generator of $KK^1(\mathbb{C}, \mathfrak{A})$.*

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