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TWISTS OF RATIONAL CHEREDNIK ALGEBRAS

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Abstract. The main result of the paper is that braided Cherednik algebras introduced by the first two authors are cocycle twists of rational Cherednik algebras of imprimitive complex reflection groups. This gives a new construction of mystic reflection groups and a new proof that such groups have Artin-Schelter regular rings of quantum polynomial invariants. Furthermore, the main resulr leads to a construction of finite-dimensional representations of braided Cherednik algebras: to each d-dimensional module over a rational Cherednik algebra H the twist construction associates a d^2-dimensional module over its braided counterpart $\overline{H}$.

In this first version of the paper, we give a full proof of the main result and sketch the application to representations of braided Cherednik algebras.

1. Rational Cherednik algebras

1.1. **Complex reflection groups.** Let V be an n-dimensional vector space over $\mathbb{C}$. We denote by V^* the vector space $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ dual to V. If $y \in V^*$, $x \in V$, we write $\langle y, x\rangle$ to denote the evaluation of y on x.

Let G be a finite subgroup of $\mathrm{GL}(V)$. If $g \in G$, $x \in V$, we write $g(x)$ to denote the action of g on x. Note that G also acts on V^*: if $g \in G$, $y \in V^*$, one has $\langle g(y), x\rangle = \langle y, g^{-1}(x)\rangle$ for all $x \in V$.

Recall that a complex reflection on V is an element $s \in \mathrm{GL}(V)$ of finite order such that $(\mathrm{Id} - s)V$ is a one-dimensional subspace of V. Equivalently, the characteristic polynomial of s is $(t-1)^{n-1}(t-\lambda)$ for some root of unity $\lambda \neq 1$. Note that in this case s acts on V^* also as a complex reflection, with characteristic polynomial $(t-1)^{n-1}(t-\lambda^{-1})$.

By definition, a complex reflection group on V is a finite subgroup of $\mathrm{GL}(V)$ generated by complex reflections on V.

Note the direct product construction for complex reflection groups: if $G_1 \subset \mathrm{GL}(V_1)$, $G_2 \subset \mathrm{GL}(V_2)$ are complex reflection groups, then $G_1 \times G_2$ is naturally a complex reflection group on $V_1 \oplus V_2$.

1.2. **Rational Cherednik algebras.** Rational Cherednik algebras are a solution to a particular case of Drinfeld's problem [7]. They were introduced by Etingof and Ginzburg in [8]. We describe their construction as follows.

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Let $G \subset \mathrm{GL}(V)$ be finite. Denote by S the set of complex reflections in G, and let $c\colon S \to \mathbb{C}$, $s \mapsto c_s$ be a function such that

$$c_{gsg^{-1}} = c_s \qquad \text{for all } g \in G,\ s \in S.$$

Define the bilinear map

$$\kappa_c\colon V^* \times V \to \mathbb{C}G, \qquad \kappa(y, x) = \langle y, x\rangle 1 + \sum_{s\in S} c_s \langle y, (\mathrm{Id} - s)x\rangle s.$$

The rational Cherednik algebra $H_c(G)$ of the group G with parameter c is defined by its presentation:

- generators: V, $\mathbb{C}G$, V^*;
- relations:
  $xx' - x'x = yy' - y'y = 0;$
  $gx = g(x)g,\ yg = g \cdot g^{-1}(y),$
  $yx - xy = \kappa_c(y, x)$
  for all $x, x' \in V$, $y, y' \in V^*$, $g \in G$.

Remark 1.1. The case $c = 0$ leads to the algebra $H_0(G) = \mathcal{A}(V) \rtimes G$, the semidirect product of the Weyl algebra $\mathcal{A}(V)$ of V, which is acted upon by G, with the group G. This algebra has no finite-dimensional modules. However, $H_c(G)$ may have finite-dimensional modules for some special values of the parameter c.

Remark 1.2. It is usually assumed that G is a complex reflection group: this case is sufficiently general because one can for virtually all purposes replace $H_c(G)$ by its subalgebra generated by V, $\mathbb{C}G'$ and V^* where G' is the subgroup of G generated by S.

The following is a crucial property of the rational Cherednik algebra. The notation $\mathbb{Z}_{\geq 0}$ refers to the set of non-negative integers.

Theorem 1.3 (PBW type theorem, [8]). *Let $x_1, \dots, x_n$ be a basis of V, and $y_1, \dots, y_n$ be a basis of the dual space V^*. As a $\mathbb{C}$-vector space, the algebra $H_c(G)$ has basis*

$$\{x_1^{k_1} \dots x_n^{k_n} g y_1^{l_1} \dots y_n^{l_n} \mid g \in G,\ k_1, \dots, k_n, l_1, \dots, l_n \in \mathbb{Z}_{\geq 0}\}.$$

In other words, $H_c \cong \mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}G \otimes \mathbb{C}[y_1, \dots, y_n]$ as vector spaces.

Remark 1.4. It is easy to see from the defining relations of $H_c(G)$ that the "monomials" of the form $x_1^{k_1} \dots x_n^{k_n} g y_1^{l_1} \dots y_n^{l_n}$ are a spanning set of $H_c(G)$. To show this, one does not need the precise formula for $\kappa_c(y, x)$: the condition that $\kappa_c(y, x) \in \mathbb{C}G$ is enough. Linear independence is more intricate and does require $\kappa_c(y, x)$ to be of the above special form.

2. Mystic reflection groups

2.1. **The groups $G(m,p,n)$ and $\mu(G(m,p,n))$.** We fix a basis $\{x_1,\dots,x_n\}$ of the space V and the dual basis $\{y_1,\dots,y_n\}$ of V^*. The group $\mathrm{GL}(V)$ is then identified with the group $\mathrm{GL}_n(\mathbb{C})$ of $n\times n$ invertible matrices. Let

$$\mathbb{S}_n=\{g\in\mathrm{GL}(V)\mid \forall i\ g(x_i)\in\{x_1,\dots,x_n\}\},\qquad \mathbb{T}_n=\{t\in\mathrm{GL}(V)\mid \forall i\ t(x_i)\in\mathbb{C}x_i\}$$

be the group of permutation matrices, respectively, the group of diagonal matrices. It is easy to see that

$$\mathbb{S}_n\cap\mathbb{T}_n=\{\mathrm{Id}\},\qquad \mathbb{S}_n \text{ normalises } \mathbb{T}_n \text{ within } \mathrm{GL}_n(\mathbb{C}),\qquad \text{so that } \mathbb{S}_n\ltimes\mathbb{T}_n\subset\mathrm{GL}_n(\mathbb{C}).$$

Let

$$\mathcal{C}'\subseteq\mathcal{C}\subset\mathbb{C}^\times$$

be two finite multiplicative subgroups of $\mathbb{C}^\times$. Note that $\mathcal{C}$, respectively $\mathcal{C}'$, is equal to the cyclic group of mth, respectively $(\frac{m}{p})$th, roots of unity, where $m=|\mathcal{C}|$, $\frac{m}{p}=|\mathcal{C}'|$. The following subgroups of $\mathbb{S}_n\ltimes\mathbb{T}_n$ are complex reflection groups on V that are part of the Shephard-Todd classification [**?**]:

$$G(m,p,n)=\{wt\in\mathbb{S}_n\ltimes\mathcal{C}^n\mid \det t\in\mathcal{C}'\},$$

where $\mathcal{C}^n$ denotes the group of diagonal matrices where all diagonal entries are from $\mathcal{C}$. The set of complex reflections in $G(m,p,n)$ is

$$S=\{s_{ij}^{(\epsilon)}:1\le i<j\le n,\ \epsilon\in\mathcal{C}\}\cup\{t_i^{(\zeta)}:1\le i\le n,\ \zeta\in\mathcal{C}'\setminus\{1\}\},$$

where $t_i^{(\epsilon)}\in\mathbb{T}_n$, $s_{ij}^{(\epsilon)}\in\mathbb{S}_n\ltimes\mathbb{T}_n$ are defined for $\epsilon\in\mathbb{C}^\times$ by

$$t_i^{(\epsilon)}(x_k)=\begin{cases}\epsilon x_i, & k=i,\\ x_k, & k\ne i,\end{cases}\qquad s_{ij}^{(\epsilon)}(x_k)=\begin{cases}x_k, & k\notin\{i,j\},\\ \epsilon x_j, & k=i,\\ \epsilon^{-1}x_i, & k=j.\end{cases}$$

If $n\ge 3$, the complex reflections $s_{ij}^{(\epsilon)}$, $\epsilon\in\mathcal{C}$, are involutions and form a single conjugacy class in $G(m,p,n)$. To each $\zeta\in\mathcal{C}'\setminus\{1\}$ there corresponds a separate conjugacy class $\{t_i^{(\zeta)}:1\le i\le n\}$.

Of special interest to us in the present paper will be the groups

$$W_{\mathcal{C},\mathcal{C}'}:=\mu(G(m,p,n))=\{wt\in\mathbb{S}_n\ltimes\mathcal{C}^n\mid \det wt\in\mathcal{C}'\},$$

defined if $m=|\mathcal{C}|$ is even. These groups were found in [2] and independently in [10] and are studied in more detail in [4]; they are a family of *mystic reflection groups*. Recall from [4] that the group $W_{\mathcal{C},\mathcal{C}'}$ coincides with $G(m,p,n)$ if $\frac{m}{p}$ is even; is isomorphic but not equal to $G(m,p,n)$ if $\frac{m}{p}$ and n afre odd; and is not isomorphic to $G(m,p,n)$ if $\frac{m}{p}$ is odd and n is even. However, the group algebras $\mathbb{C}W_{\mathcal{C},\mathcal{C}'}$ and $\mathbb{C}G(m,p,n)$ are always isomorphic as algebras.

Elements of the following subset of $W_{\mathcal{C},\mathcal{C}'}$ are termed *mystic reflections*:

$$\underline{S}:=\{\sigma_{ij}^{(\epsilon)}:1\le i\le j\le n,\ \epsilon\in\mathcal{C}\}\cup\{t_i^{(\zeta)}:1\le i\le n,\ \zeta\in\mathcal{C}'\setminus\{1\}\},$$

where $t_i^{(\zeta)}$ are as above, and $\sigma_{ij}^{(\epsilon)} \in \mathbb{S}_n \ltimes \mathbb{T}_n$ is defined by

$$\sigma_{ij}^{(\epsilon)}(x_k) = \begin{cases} x_k, & k \notin \{i,j\}, \\ \epsilon x_j, & k = i, \\ -\epsilon^{-1} x_i, & k = j. \end{cases}$$

Observe that the $\sigma_{ij}^{(\epsilon)}$ are elements of $\mathrm{GL}(V)$ of order 4 with characteristic polynomial $(x-1)^{n-2}(x^2+1)$. If $n \geq 3$, they form a single conjugacy class in $W_{\mathcal{C},\mathcal{C}'}$. As before, $\{t_i^{(\zeta)} : 1 \leq i \leq n\}$ is a conjugacy class for each $\zeta \in \mathcal{C}' \setminus \{1\}$. Finally, the set $\underline{S}$ generates $W_{\mathcal{C},\mathcal{C}'}$, see [2, 4].

2.2. **The algebra $H_c(G(m,p,n))$.** The rational Cherednik algebra $H_c(G(m,p,n))$ is a particular case of $H_c(G)$ above. Hence it has the following presentation:

- generators: $x_1, \ldots, x_n$; $g \in G(m,p,n)$; $y_1, \ldots, y_n$;
- relations:
  $x_i x_j - x_j x_i = y_i y_j - y_j y_i = 0$;
  $g x_i g^{-1} = g(x_i)$, $g y_i g^{-1} = g(y_i)$;
  $y_i x_j - x_j y_i = c_1 \sum_{\epsilon \in \mathcal{C}} \epsilon s_{ij}^{(\epsilon)}$;
  $y_i x_i - x_i y_i = 1 - c_1 \sum_{k \neq i} \sum_{\epsilon \in \mathcal{C}} s_{ij}^{(\epsilon)} - \sum_{\zeta \in \mathcal{C}' \setminus \{1\}} c_\zeta t_i^{(\zeta)}$,
  for all $1 \leq i,j \leq n$, $i \neq j$, $g \in G(m,p,n)$.

Here c_1 and c_ζ, $\zeta \in \mathcal{C}'$, are arbitrary complex parameters: c_1 denotes the value of the function $c\colon S \to \mathbb{C}$ on the conjugacy class of $s_{ij}^{(\epsilon)}$, and c_ζ the value of c at $t_i^{(\zeta)}$.

2.3. **The braided Cherednik algebra $\underline{H}_c$.** Consider the group $W_{\mathcal{C},\mathcal{C}'}$ and its set $\underline{S}$ of mystic reflections as above. Let $c\colon \underline{S} \to \mathbb{C}$ be a function invariant with respect to conjugation in $W_{\mathcal{C},\mathcal{C}'}$. In [2], an associative algebra $\underline{H}_c(W_{\mathcal{C},\mathcal{C}'})$ was defined by the following presentation, where we use underlined generators $\underline{x}_i$, respectively $\underline{y}_i$, to stress that they anticommute:

- generators: $\underline{x}_1, \ldots, \underline{x}_n$; $g \in W_{\mathcal{C},\mathcal{C}'}$; $\underline{y}_1, \ldots, \underline{y}_n$;
- relations:
  $\underline{x}_i \underline{x}_j + \underline{x}_j \underline{x}_i = \underline{y}_i \underline{y}_j + \underline{y}_j \underline{y}_i = 0$;
  $g \underline{x}_i g^{-1} = g(\underline{x}_i)$, $g \underline{y}_i g^{-1} = g(\underline{y}_i)$;
  $\underline{y}_i \underline{x}_j + \underline{x}_j \underline{y}_i = c_1 \sum_{\epsilon \in \mathcal{C}} \epsilon \sigma_{ij}^{(\epsilon)}$;
  $\underline{y}_i \underline{x}_i - \underline{x}_i \underline{y}_i = 1 + c_1 \sum_{k \neq i} \sum_{\epsilon \in \mathcal{C}} \sigma_{ij}^{(\epsilon)} + \sum_{\zeta \in \mathcal{C}' \setminus \{1\}} c_\zeta t_i^{(\zeta)}$,
  for all $1 \leq i,j \leq n$, $i \neq j$, $g \in W_{\mathcal{C},\mathcal{C}'}$.

The key property of the algebra $\underline{H}_c(W_{\mathcal{C},\mathcal{C}'})$ proved in [2] is the following PBW type theorem analogous to Theorem 1.3. Here and below, $S_{-\mathbf{1}}(V)$ denotes the algebra generated by $x_1, \ldots, x_n$ subject to the relations $x_i x_j + x_j x_i = 0$ whenever $i \neq j$, and $S_{-\mathbf{1}}(V^*)$ is the algebra generated by $y_1, \ldots, y_n$ subject to the relations $y_i y_j + y_j y_i = 0$ whenever $i \neq j$.

Theorem 2.1 ([2]). *The algebra $\underline{H}_c$ has $\mathbb{C}$-basis*

$$\{x_1^{k_1}\dots x_n^{k_n} g y_1^{l_1}\dots y_n^{l_n} \mid g\in W_{\mathcal{C},\mathcal{C}'},\ k_1,\dots,k_n,l_1,\dots,l_n\in\mathbb{Z}_{\geq 0}\},$$

hence $\underline{H}_c \cong S_{-\mathbf{1}}(V)\otimes \mathbb{C}W_{\mathcal{C},\mathcal{C}'}\otimes S_{-\mathbf{1}}(V^*)$ as vector spaces. □

3. Main result

We will now prove the following

Theorem 3.1. *If m is even, the braided Cherednik algebra $\underline{H}_c$ is a cocycle twist of the rational Cherednik algebra $H_c(G(m,p,n))$. More precisely, there exists a covariant action of the abelian group $T = T(2,1,n) = G(2,1,n)\cap \mathbb{T}_n$ on the rational Cherednik algebra $H_c(G(m,p,n))$ and a cocycle $\mathcal{F}\in\mathbb{C}T\otimes\mathbb{C}T$ such that the new product $\star$ on $H_c(G(m,p,n))$ obtained by twisting the original product by $\mathcal{F}$ yields an algebra isomorphic to $\underline{H}_c$.*

3.1. **The cocycle $\mathcal{F}$.** Observe that the group $T=T(2,1,n)$ is generated by $t_i^{(-1)}$, $i=1,\dots,n$, and is isomorphic to the direct product of n copies of the group of order 2. Define

$$\mathcal{F} = \prod_{1\leq j<i\leq n} f(-1,i,j),$$

where

$$f\left(-1,i,j\right) = \frac{1}{2}\left(1\otimes 1 + t_i^{(-1)}\otimes 1 + 1\otimes t_j^{(-1)} - t_i^{(-1)}\otimes t_j^{(-1)}\right).$$

The following fact can be checked directly:

Proposition 3.2. *The elements $f(-1,i,j)$ and $\mathcal{F}$ are involutions and quasitriangular structures for the Hopf algebra $\mathbb{C}T\otimes\mathbb{C}T$.* □

Corollary 3.3. *$\mathcal{F}$ is a counital cocycle for $\mathbb{C}T$.* □

The other ingredient needed to twist the associative product on $H_c(G(m,p,n))$ is the action of T on $H_c(G(m,p,n))$ given in the following

Proposition 3.4. *There is a covariant action $\triangleright$ of the group T on the algebra $H_c(G(m,p,n))$ given by*

$$t_i^{(-1)}\triangleright g = t_i^{(-1)} g t_i^{(-1)}, \qquad t_i^{(-1)}\triangleright x_j = t_i^{(-1)}(x_j), \qquad t_i^{(-1)}\triangleright y_j = t_i^{(-1)}(y_j)$$

for $1\leq i,j\leq n$, $g\in G(m,p,n)$.

Proof. Recall that m is even. The subgroup $G(m,p,n)$ of $\mathrm{GL}(V)$ is invariant under the conjugation by $t_i^{(-1)}$: indeed, if $w\in\mathbb{S}_n$ and $t\in T(m,p,n)$, one has $t_i^{(-1)}(wt)t_i^{(-1)} = (wt)t_i^{(-1)}t_j^{(-1)}$ where j is the image of the indexd i under the permutation w. Note that $t_i^{(-1)}t_j^{(-1)}\in T(m,p,n)$ for even m. One then checks that the defining relations of $H_c(G(m,p,n))$ are preserved by the action of $t_i^{(-1)}$. □

3.2. **The twisted associative product $\star$.** According to [11, Section 2], the new associative product $\star$ on $H_c(G(m,p,n))$ is defined as

$$a \star b = \cdot(\mathcal{F}^{-1} \triangleright (a \otimes b)),$$

where $\cdot\colon H_c(G(m,p,n)) \otimes H_c(G(m,p,n)) \to H_c(G(m,p,n))$ is the original product on the rational Cherednik algebra. Recall that $\mathcal{F}^{-1} = \mathcal{F}$.

3.3. **A refinement of the main theorem.** We will actually prove the following refinement of Theorem 3.1, where an isomorphism ϕ between the two algebras is specified explicitly.

Theorem 3.5. *There exists an isomorphism*

$$\phi\colon \underline{H}_c \to (H_c(G(m,p,n)), \star)$$

of associative algebras, such that

$$\phi(\underline{x}_i) = x_i, \qquad \phi(\underline{y}_i) = y_i, \qquad \phi(\sigma_{ij}^{(\epsilon)} = -s_{ij}^{-\epsilon}, \qquad \phi(t_i^{(\zeta)}) = t_i^{(\zeta)}$$

for all $i, j = 1, \dots, n$, $\epsilon \in \mathcal{C}$, $\zeta \in \mathcal{C}'$.

The proof of Theorem 3.5 will consist of several steps outlined below.

3.4. **Commutation relations between the $\underline{x}_i$ and between the $\underline{y}_j$.** First of all we need to check that the map ϕ extends from the generators to a well-defined algebra homomorphism on $\underline{H}_c$. This will be achieved by showing that the defining relations of $\underline{H}_c$ are satisfied by the elements $\phi(\underline{x}_i)$, $\phi(\underline{y}_i)$ and $\phi(g)$, $g \in W_{\mathcal{C},\mathcal{C}'}$ of the algebra $(H_c(G(m,p,n)), \star)$.

The following fact will help us to write an expression for the product $\star$ in $H_c(G(m,p,n))$ in terms of the untwisted product:

Lemma 3.6. *(1) Let i, j be indices such that $1 \le j < i \le n$. Suppose that $a, b \in H_c(G(m,p,n))$ are such that $t_i^{(-1)} \triangleright a = a$ or $t_j^{(-1)} \triangleright b = b$. Then $f(-1,i,j) \triangleright (a \otimes b) = a \otimes b$.*

(2) If $a, b \in H_c(G(m,p,n))$ are such that $t_i^{(-1)} \triangleright a = a$ or $t_j^{(-1)} \triangleright b = b$ for all pairs i, j where $1 \le j < i \le n$, then $a \star b = ab$.

Proof. (1) follows from properties of a quasitriangular structure [11]. (2) is immediate from (1) and the definition of $\star$. □

The first step in proving Theorem 3.5 will be to show that

$$\phi(\underline{x}_r) \star \phi(\underline{x}_s) = -\phi(\underline{x}_s) \star \phi(\underline{x}_r) \qquad \text{whenever } 1 \le r < s \le n. \tag{1}$$

Note that the condition $t_i^{(-1)} \triangleright x_r \neq x_r$ *and* $t_j^{(-1)} \triangleright x_s \neq x_s$ is satisfied for, and only for, $(i,j)=(r,s)$. Then by Lemma 3.6, $x_r \star x_s = x_r x_s$ given that $r<s$, and

$$\begin{aligned}\mathcal{F} \triangleright (x_s \otimes x_r) = f(-1,s,r) \triangleright (x_s \otimes x_r) &= \frac{1}{2}\left(1\otimes 1 + t_s^{(-1)} \otimes 1 + 1 \otimes t_r^{(-1)} - t_s^{(-1)} \otimes t_r^{(-1)}\right) \triangleright (x_s \otimes x_r) \\ &= \frac{1}{2}\left(x_s \otimes x_r - x_s \otimes x_r - x_s \otimes x_r - x_s \otimes x_r\right) \\ &= -x_s \otimes x_r.\end{aligned}$$

It follows that $x_s \star x_r = -x_s x_r$. Relation (1) is proved.

In the same way it is proved that

$$\phi(\underline{y}_r) \star \phi(\underline{y}_s) = -\phi(\underline{y}_s) \star \phi(\underline{y}_r) \qquad \text{whenever } 1 \leq r < s \leq n. \tag{2}$$

In the same way one shows that

$$x_i \star y_j = \begin{cases} x_i y_j & \text{if } i<j, \\ -x_i y_j & \text{if } i>j,\end{cases}$$

and similarly for $y_i \star x_j$.

3.5. **The main commutator relation between $\underline{x}_i$ and $\underline{y}_j$.** We will now check the following relation, obtained by applying ϕ to both sides of the main commutator relation in $\underline{H}_c$ for $i \neq j$:

$$\phi(\underline{y}_i) \star \phi(\underline{x}_j) + \phi(\underline{x}_j) \star \phi(\underline{y}_i) = \sum_{\epsilon \in \mathcal{C}} \epsilon \phi(\sigma_{ij}^{(\epsilon)}).$$

Note that since $\sigma_{ij}^{(\epsilon)} = \sigma_{ji}^{(-\epsilon^{-1})}$, we can define $\phi(\sigma_{ij}^{(\epsilon)}) = \phi(\sigma_{ji}^{(-\epsilon^{-1})}) = -s_{ji}^{(\epsilon^{-1})}$ for $i>j$. First we will show the $i<j$ case:

$$\begin{aligned}\phi(\underline{y}_i) \star \phi(\underline{x}_j) + \phi(\underline{x}_j) \star \phi(\underline{y}_i) &= y_i \star x_j + x_j \star y_i = y_i x_j - x_j y_i \\ &= (x_j y_i + c_1 \sum_{\epsilon \in \mathcal{C}} \epsilon s_{ij}^{(\epsilon)}) - x_j y_i \\ &= c_1 \sum_{\epsilon \in \mathcal{C}} \epsilon s_{ij}^{(\epsilon)} = c_1 \sum_{\epsilon \in \mathcal{C}} (-\epsilon) s_{ij}^{(-\epsilon)},\end{aligned}$$

where the last step is valid since $-1 \in \mathcal{C}$. The last expression is $c_1 \sum_{\epsilon \in \mathcal{C}} \epsilon \phi(\sigma_{ij}^{(\epsilon)})$ as required. The $i>j$ case is completely analogous. The maion commutator relation for $i \neq j$ is thus satisfied.

3.6. **The main commutator relation between $\underline{x}_i$ and $\underline{y}_i$.** We will now check the case where $i=j$. We need to show that

$$\phi(\underline{y}_i) \star \phi(\underline{x}_i) - \phi(\underline{x}_i) \star \phi(\underline{y}_i) = 1 + c_1 \sum_{j \neq i} \sum_{\epsilon \in \mathcal{C}} \phi(\sigma_{ij}^{(\epsilon)}) + \sum_{\zeta \in \mathcal{C}' \setminus \{1\}} c_\zeta \phi(t_i^{(\zeta)}),$$

but this is easy to deduce from the definition of ϕ and the observation that

$$\phi(\underline{y}_i) \star \phi(\underline{x}_i) - \phi(\underline{x}_i) \star \phi(\underline{y}_i) = y_i x_i - x_i y_i.$$

3.7. **The semidirect product relations.** We will now check the following relations:

$$\phi(\sigma_{ij}^{(\epsilon)}) \star \phi(\underline{x}_k) = \phi(\sigma_{ij}^{(\epsilon)}(\underline{x}_k)) \star \phi(\sigma_{ij}^{(\epsilon)}), \qquad \phi(t_i^{(\zeta)}) \star \phi(\underline{x}_k) = \phi(t_i^{(\zeta)}(\underline{x}_k)) \star \phi(t_i^{(\zeta)}).$$

Note that $t_i^{(\zeta)}$ is invariant under the action of $t_j^{(-1)}$ because these elements of $\mathrm{GL}(V)$ commute. The latter semidirect product relation therefore holds by virtue of the semidirect product relations in $H_c(G(m,p,n))$ and Lemma 3.6.

We now show the former relation, written as

$$\phi(\sigma_{ab}^{(\epsilon)}) \star \phi(\underline{x}_k) = \phi(\sigma_{ab}^{(\epsilon)}(\underline{x}_k)) \star \phi(\sigma_{ab}^{(\epsilon)}).$$

One can show that

$$f(-1,i,j) \triangleright (-s_{ab}^{(-\epsilon)} \otimes x_k) = \begin{cases} -s_{ab}^{(\epsilon)} \otimes x_k & \text{if } i \in \{a,b\},\ j = k \\ -s_{ab}^{(-\epsilon)} \otimes x_k & \text{otherwise,} \end{cases}$$

which implies

(i) $1 \le k < a \implies \mathcal{F} \triangleright (-s_{ab}^{(-\epsilon)} \otimes x_k) = -s_{ab}^{(-\epsilon)} \otimes x_k$
(ii) $k = a \implies \mathcal{F} \triangleright (-s_{ab}^{(-\epsilon)} \otimes x_a) = -s_{ab}^{(\epsilon)} \otimes x_a$
(iii) $a < k < b \implies \mathcal{F} \triangleright (-s_{ab}^{(-\epsilon)} \otimes x_k) = -s_{ab}^{(\epsilon)} \otimes x_k$
(iv) $k = b \implies \mathcal{F} \triangleright (-s_{ab}^{(-\epsilon)} \otimes x_b) = -s_{ab}^{(-\epsilon)} \otimes x_b$
(v) $b < k \le n \implies \mathcal{F} \triangleright (-s_{ab}^{(-\epsilon)} \otimes x_k) = -s_{ab}^{(-\epsilon)} \otimes x_k$

One therefore has

$$\phi(\sigma_{ab}^{(\epsilon)}) \star \phi(\underline{x}_k) = \begin{cases} -\epsilon x_b s_{ab}^{(\epsilon)} & \text{if } k = a \\ -x_k s_{ab}^{(\epsilon)} & \text{if } a < k < b \\ \epsilon^{-1} x_a s_{ab}^{(-\epsilon)} & \text{if } k = b \\ -x_k s_{ab}^{(-\epsilon)} & \text{otherwise.} \end{cases}$$

The same method shows that

$$\phi(\sigma_{ab}^{(\epsilon)}(\underline{x}_k)) \star \phi(\sigma_{ab}^{(\epsilon)}) = \begin{cases} -\epsilon x_b s_{ab}^{(\epsilon)} & \text{if } k = a \\ -x_k s_{ab}^{(\epsilon)} & \text{if } a < k < b \\ \epsilon^{-1} x_a s_{ab}^{(-\epsilon)} & \text{if } k = b \\ -x_k s_{ab}^{(-\epsilon)} & \text{otherwise.} \end{cases}$$

and so, comparing these with the results of the LHS, we have that:

$$\phi(\sigma_{ab}^{(\epsilon)}) \star \phi(\underline{x}_k) = \phi(\sigma_{ab}^{(\epsilon)}(\underline{x}_k)) \star \phi(\sigma_{ab}^{(\epsilon)})$$

for $a < b$. The case $a > b$ is similar.

3.8. **The relations between group elements.** We must also check the relations between the elements of the group. Recall that $\phi(\sigma_{ij}^{(\epsilon)}) = -s_{ij}^{(-\epsilon)}$ for $i < j$ therefore $\phi(\sigma_{ij}^{(\epsilon)}) = \phi(\sigma_{ji}^{(-\epsilon^{-1})}) = -s_{ji}^{(\epsilon^{-1})}$ for $i > j$). First:

$$\phi(\sigma_{ab}) \star\text{-commutes with } \phi(\sigma_{cd}) \text{ if } \{a,b\} \cap \{c,d\} = \varnothing$$

where $\sigma_{ab} = \sigma_{ab}^{(1)}$.

First consider $a < b$, $c < d$. To begin, we will consider what the star product looks like, then we'll work through the required cases:

$$\phi(\sigma_{ab}) \star \phi(\sigma_{cd}) = m(\mathcal{F} \triangleright (-s_{ab}^{(-1)} \otimes -s_{cd}^{(-1)})) = m(\mathcal{F} \triangleright (s_{ab}^{(-1)} \otimes s_{cd}^{(-1)}))$$

and now consider the action of $f(-1,i,j)$:

$$\frac{1}{2}\left(1 \otimes 1 + t_i^{(-1)} \otimes 1 + 1 \otimes t_j^{(-1)} - t_i^{(-1)} \otimes t_j^{(-1)}\right) \triangleright (s_{ab}^{(-1)} \otimes s_{cd}^{(-1)})$$

for $j < i$.

The only case that needs to be checked is $i \in \{a, b\}$ *and* $j \in \{c, d\}$. Here:

$$\begin{aligned}&\frac{1}{2}\left(1 \otimes 1 + t_i^{(-1)} \otimes 1 + 1 \otimes t_j^{(-1)} - t_i^{(-1)} \otimes t_j^{(-1)}\right) \triangleright (s_{ab}^{(-1)} \otimes s_{cd}^{(-1)})\\ &= \frac{1}{2}(s_{ab}^{(-1)} \otimes s_{cd}^{(-1)} + s_{ab} \otimes s_{cd}^{(-1)} + s_{ab}^{(-1)} \otimes s_{cd} - s_{ab} \otimes s_{cd}) \ (*)\end{aligned}$$

Now consider the choices for a, b, c, d. There are actually six cases, which are listed below. Also given in the list is the total number of $f(-1,i,j)$ which act non-trivially (recalling that $a < b$, $c < d$ and $j < i$):

(i) $a < b < c < d$: Here, there *all* possible $f(-1,i,j)$ act trivially.
(ii) $c < d < a < b$: Here, exactly four $f(-1,i,j)$ act non-trivially, as there are two choices for i and two choices for j.
(iii) $a < c < b < d$: Here, exactly one $f(-1,i,j)$ acts non-trivially, namely $f(-1,b,c)$.
(iv) $c < a < d < b$: Here, exactly three $f(-1,i,j)$ act non-trivially, as there are two choices for i when $j = a$ and just one choice for i, namely $i = b$ when $j = d$.
(v) $a < c < d < b$: Here, exactly two $f(-1,i,j)$ act non-trivially, as there are two choices for j when $i = b$.
(vi) $c < a < b < d$: Here, exactly two $f(-1,i,j)$ act non-trivially, as there are two choices for i when $j = c$.

Observe the order in which these cases are written; cases (i) and (ii), (iii) and (iv), (v) and (vi) are paired up. By this I mean that if in $\phi(\sigma_{ab}) \star \phi(\sigma_{cd})$ the a, b, c, d correspond to case (i), (iii) or (v), then the product $\phi(\sigma_{cd}) \star \phi(\sigma_{ab})$ will correspond to case (ii), (iv) or (vi) respectively, and vice versa. This is seen by looking at the order of the indices in the cases above. Note that there are only precisely two evaluations for $f(-1,i,j) \triangleright (s_{ab}^{(-1)} \otimes s_{cd}^{(-1)})$. One of them is trivial, and the other is $(*)$. Now, instead of considering the cases (i)-(vi) in turn, we will keep looking at non-trivial actions of $f(-1,i,j)$ until all the cases are accounted for. So, we know what happens when exactly one $f(-1,i,j)$ acts non-trivially, this is just $(*)$. If a second $f(-1,i,j)$ acts non-trivially, we obtain:

$$\begin{aligned}
&f(-1,i,j)\triangleright\frac{1}{2}(s_{ab}^{(-1)}\otimes s_{cd}^{(-1)}+s_{ab}\otimes s_{cd}^{(-1)}+s_{ab}^{(-1)}\otimes s_{cd}-s_{ab}\otimes s_{cd})\\
&=\frac{1}{4}[(1\otimes 1)\triangleright(s_{ab}^{(-1)}\otimes s_{cd}^{(-1)}+s_{ab}\otimes s_{cd}^{(-1)}+s_{ab}^{(-1)}\otimes s_{cd}-s_{ab}\otimes s_{cd})\\
&+(t_i^{(-1)}\otimes 1)\triangleright(s_{ab}^{(-1)}\otimes s_{cd}^{(-1)}+s_{ab}\otimes s_{cd}^{(-1)}+s_{ab}^{(-1)}\otimes s_{cd}-s_{ab}\otimes s_{cd})\\
&+(1\otimes t_j^{(-1)})\triangleright(s_{ab}^{(-1)}\otimes s_{cd}^{(-1)}+s_{ab}\otimes s_{cd}^{(-1)}+s_{ab}^{(-1)}\otimes s_{cd}-s_{ab}\otimes s_{cd})\\
&-(t_i^{(-1)}\otimes t_j^{(-1)})\triangleright(s_{ab}^{(-1)}\otimes s_{cd}^{(-1)}+s_{ab}\otimes s_{cd}^{(-1)}+s_{ab}^{(-1)}\otimes s_{cd}-s_{ab}\otimes s_{cd})]
\end{aligned}$$

$$\begin{aligned}
&=\frac{1}{4}[s_{ab}^{(-1)}\otimes s_{cd}^{(-1)}+s_{ab}\otimes s_{cd}^{(-1)}+s_{ab}^{(-1)}\otimes s_{cd}-s_{ab}\otimes s_{cd}\\
&+s_{ab}\otimes s_{cd}^{(-1)}+s_{ab}^{(-1)}\otimes s_{cd}^{(-1)}+s_{ab}\otimes s_{cd}-s_{ab}^{(-1)}\otimes s_{cd}\\
&+s_{ab}^{(-1)}\otimes s_{cd}+s_{ab}\otimes s_{cd}+s_{ab}^{(-1)}\otimes s_{cd}^{(-1)}-s_{ab}\otimes s_{cd}^{(-1)}\\
&-s_{ab}\otimes s_{cd}-s_{ab}^{(-1)}\otimes s_{cd}-s_{ab}\otimes s_{cd}^{(-1)}+s_{ab}^{(-1)}\otimes s_{cd}^{(-1)}]\\
&=s_{ab}^{(-1)}\otimes s_{cd}^{(-1)}
\end{aligned}$$

after cancelling. This actually completes the proof of this relation, once we observe that case (i) produces a trivial action and case (ii) produces a trivial action, as in case (ii), there are four non-trivial actions of $f(-1,i,j)$, but we just showed that if *two* lots of $f(-1,i,j)$ act non-trivially, they act trivially, therefore after all four of them act, it will result in a trivial action. Similarly, cases (v) and (vi) produce a trivial action. Finally, cases (iii) and (vi) produce the same non-trivial action, as by the same reasoning, the first two $f(-1,i,j)$ in case (iv) will act trivially and thus the third $f(-1,i,j)$ will act non-trivially.

Now we must consider the cases where $a>b$ and $c<d$, and, $a>b$ and $c>d$. Note that for $a>b$ we have that:

$$\phi(\sigma_{ab})=\phi(\sigma_{ba}^{(-(1)^{-1})})=\phi(\sigma_{ba}^{(-1)})=-s_{ba}^{(1)}=-s_{ba}=-s_{ab}.$$

Because the only difference is that the upper index is negated, and that a non-trivial action of the $t_i^{(-1)}$ is essentially just to negate the upper index, the calculations for the next two cases are essentially the same, and are therefore omitted.

The next relation we're required to check is that:

$$\phi(\sigma_{ab})\star\phi(\sigma_{ab})=t_a^{(-1)}t_b^{(-1)}$$

Let $a,b\in\{1,\dots,n\}$, $a<b$. Then:

$$\phi(\sigma_{ab})\star\phi(\sigma_{ab})=m(\mathcal{F}\triangleright(s_{ab}^{(-1)}\otimes s_{ab}^{(-1)})).$$

So, now consider the action of $f(-1,i,j)$ on $s_{ab}\otimes s_{ab}$. Like before, it's clear that this action is only non-trivial when $i\in\{a,b\}$ and $j\in\{a,b\}$, and since $j<i$ the only possible case of this is that $j=a$

and $i = b$. So here we have:

$$\begin{aligned} m(\mathcal{F} \triangleright (s_{ab} \otimes s_{ab})) &= m(f(-1,b,a) \triangleright (s_{ab}^{(-1)} \otimes s_{ab}^{(-1)}) \\ &= m\left(\frac{1}{2}(1 \otimes 1 + t_b^{(-1)} \otimes 1 + 1 \otimes t_a^{(-1)} - t_b^{(-1)} \otimes t_a^{(-1)}) \triangleright (s_{ab}^{(-1)} \otimes s_{ab}^{(-1)})\right) \\ &= m\left(\frac{1}{2}(s_{ab}^{(-1)} \otimes s_{ab}^{(-1)} + s_{ab} \otimes s_{ab}^{(-1)} + s_{ab}^{(-1)} \otimes s_{ab} - s_{ab} \otimes s_{ab})\right) \\ &= \frac{1}{2}\left(s_{ab}^{(-1)} s_{ab}^{(-1)} + s_{ab} s_{ab}^{(-1)} + s_{ab}^{(-1)} s_{ab} - s_{ab} s_{ab}\right) \\ &= s_{ab} s_{ab}^{(-1)}. \end{aligned}$$

In the final steps we used that the $s_{ab}^{(\epsilon)}$ are involutions and that $s_{ab}^{(-1)} s_{ab} = s_{ab} s_{ab}^{(-1)}$. Note that in general we know that $s_{ab}^{(\epsilon)} = s_{ab} t_a^{(\epsilon)} t_b^{(\epsilon^{-1})}$, therefore we conclude that for $a < b$:

$$\phi(\sigma_{ab}) \star \phi(\sigma_{ab}) = s_{ab} s_{ab}^{(-1)} = s_{ab} s_{ab} t_a^{(-1)} t_b^{(-1)} = t_a^{(-1)} t_b^{(-1)}$$

as required.

Now let $a, b \in \{1, \ldots, n\}$, $a > b$. Then:

$$\phi(\sigma_{ab}) \star \phi(\sigma_{ab}) = m(\mathcal{F} \triangleright (s_{ab} \otimes s_{ab})).$$

So, now consider the action of $f(-1,i,j)$ on $s_{ab} \otimes s_{ab}$. Again, we immediately observe that this action is only non-trivial when $i \in \{a,b\}$ and $j \in \{a,b\}$, and since $j < i$ the only possible case of this is when $j = b$ and $i = a$. Now we have:

$$\begin{aligned} m(\mathcal{F} \triangleright (s_{ab} \otimes s_{ab})) &= m(f(-1,a,b) \triangleright (s_{ab} \otimes s_{ab}) \\ &= m\left(\frac{1}{2}(1 \otimes 1 + t_a^{(-1)} \otimes 1 + 1 \otimes t_b^{(-1)} - t_a^{(-1)} \otimes t_b^{(-1)}) \triangleright (s_{ab} \otimes s_{ab})\right) \\ &= m\left(\frac{1}{2}(s_{ab} \otimes s_{ab} + s_{ab}^{(-1)} \otimes s_{ab} + s_{ab} \otimes s_{ab}^{(-1)} - s_{ab}^{(-1)} \otimes s_{ab}^{(-1)})\right) \\ &= \frac{1}{2}\left(s_{ab} s_{ab} + s_{ab}^{(-1)} s_{ab} + s_{ab} s_{ab}^{(-1)} - s_{ab}^{(-1)} s_{ab}^{(-1)}\right) \\ &= s_{ab} s_{ab}^{(-1)}. \end{aligned}$$

This is just what we had above, so we know $s_{ab} s_{ab}^{(-1)} = t_a^{(-1)} t_b^{(-1)}$ and therefore, for $a > b$:

$$\phi(\sigma_{ab}) \star \phi(\sigma_{ab}) = s_{ab} s_{ab}^{(-1)} = t_a^{(-1)} t_b^{(-1)}$$

as required.

Finally we must check that $\phi(\sigma_{ab}) \star \phi(\sigma_{bc}) = \phi(\sigma_{bc}) \star \phi(\sigma_{ca})$. Once again we find ourselves with six cases to check. The cases are listed below, again along with the number of $f(-1,i,j)$ which act non-trivially on the given case. Note that the above conclusion holds that $f(-1,i,j)$ acts in only precisely two ways, trivially, or non-trivially, and the non-trivial action looks like $(*)$. Also note that the $t_i^{(-1)}$'s in $f(-1,i,j)$ only ever hit the left leg of the expression in question, and that the $t_j^{(-1)}$'s in $f(-1,i,j)$ only ever hit the right leg. So now, recall that $f(-1,i,j)$ only acts non-trivially when i is one of the subscripts in the left leg, *and* when j is one of the subscripts in the right leg, otherwise it acts trivially. Armed with this, here are the cases:

(i) $a < b < c$: Here we are checking that

$$m(\mathcal{F} \triangleright (s_{ab}^{(-1)} \otimes s_{bc}^{(-1)})) = m(\mathcal{F} \triangleright (s_{bc}^{(-1)} \otimes s_{ac})).$$

On the LHS we see we have a trivial action as $j < i$ so we cannot ever have $j \in \{b, c\}$ and $i \in \{a, b\}$. Similarly, on the RHS, there are two $f(-1, i, j)$ which act non-trivially, when $j = a$ and $i = b$ or c, so this gives a trivial action. So now we have that the LHS is equal to:

$$\begin{aligned} m(s_{ab}^{(-1)} \otimes s_{bc}^{(-1)}) &= s_{ab}^{(-1)} s_{bc}^{(-1)} \\ &= s_{ab} t_a^{(-1)} t_b^{(-1)} s_{bc} t_b^{(-1)} t_c^{(-1)} \\ &= s_{ab} s_{bc} t_a^{(-1)} t_c^{(-1)} t_b^{(-1)} t_c^{(-1)} \\ &= s_{ab} s_{bc} t_a^{(-1)} t_b^{(-1)}, \end{aligned}$$

and the RHS is equal to:

$$\begin{aligned} m(s_{bc}^{(-1)} \otimes s_{ac}) &= s_{bc}^{(-1)} s_{ac} \\ &= s_{bc} t_b^{(-1)} t_c^{(-1)} s_{ac} \\ &= s_{bc} s_{ac} t_a^{(-1)} t_b^{(-1)}, \end{aligned}$$

and we can see that these expressions are the same, since as cycles $(ab)(bc) = (abc) = (bc)(ac)$.

(ii) $c < a < b$: Here we are checking that:

$$m(\mathcal{F} \triangleright (s_{ab}^{(-1)} \otimes s_{bc})) = m(\mathcal{F} \triangleright (s_{bc} \otimes s_{ca}^{(-1)})).$$

Here, on both sides, exactly two $f(-1, i, j)$ act non-trivially, namely $i = b$ and $j = c, a$, therefore we have a trivial action on both sides. Hence the LHS is:

$$\begin{aligned} m(s_{ab}^{(-1)} \otimes s_{bc}) &= s_{ab}^{(-1)} s_{bc} \\ &= s_{ab} t_a^{(-1)} t_b^{(-1)} s_{bc} \\ &= s_{ab} s_{bc} t_a^{(-1)} t_c^{(-1)}. \end{aligned}$$

The RHS is:

$$\begin{aligned} m(s_{bc} \otimes s_{ac}) &= s_{bc} s_{ca}^{(-1)} \\ &= s_{bc} s_{ac} t_a^{(-1)} t_c^{(-1)} \end{aligned}$$

and these expressions are equal because, as in part (i), as cycles $(ab)(bc) = (abc) = (bc)(ac)$.

(iii) $b < c < a$: Here we're checking:

$$m(\mathcal{F} \triangleright (s_{ab} \otimes s_{bc}^{(-1)})) = m(\mathcal{F} \triangleright (s_{bc}^{(-1)} \otimes s_{ca}^{(-1)})).$$

Once again, on the LHS, exactly two $f(-1, i, j)$ act non-trivially, namely $i = a$ and $j = b, c$, so overall $\mathcal{F}$ will act trivially. On the RHS there are no cases of $f(-1, i, j)$ acting non-trivially, as $j < i$ and therefore no choices for $i \in \{b, c\}$ and $j \in \{a, c\}$, so we have a trivial action here as well. Hence the LHS becomes:

$$\begin{aligned} m(s_{ab} \otimes s_{bc}^{(-1)}) &= s_{ab} s_{bc}^{(-1)} \\ &= s_{ab} s_{bc} t_b^{(-1)} t_c^{(-1)}. \end{aligned}$$

The RHS becomes:

$$\begin{aligned} m(s_{bc}^{(-1)} \otimes s_{ca}^{(-1)}) &= s_{bc}^{(-1)} s_{ca}^{(-1)} \\ &= s_{bc} t_b^{(-1)} t_c^{(-1)} s_{ca} t_a^{(-1)} t_c^{(-1)} \\ &= s_{bc} s_{ca} t_b^{(-1)} t_a^{(-1)} t_a^{(-1)} t_c^{(-1)} \\ &= s_{bc} s_{ca} t_b^{(-1)} t_c^{(-1)}, \end{aligned}$$

and again, these expressions are equal. The next three cases are a little trickier.

(iv) $a < c < b$: Here we have to check that:

$$m(\mathcal{F} \triangleright (s_{ab}^{(-1)} \otimes s_{bc})) = m(\mathcal{F} \triangleright (s_{bc} \otimes s_{ca})).$$

Now, on the left, there is precisely one $f(-1, i, j)$ which acts non-trivially, namely $f(-1, b, c)$. On the right, there are three $f(-1, i, j)$ which act non-trivially — $j = a$, $i = b, c$, and, $j = c$, $i = b$. Both of these result in a non-trivial action on both sides (as in the previous relation, three non-trivial actions results in a non-trivial action). Hence, the LHS becomes (using $(*)$):

$$\begin{aligned} m(\mathcal{F} \triangleright (s_{ab}^{(-1)} \otimes s_{bc})) &= \frac{1}{2} m(s_{ab}^{(-1)} \otimes s_{bc} + s_{ab} \otimes s_{bc} + s_{ab}^{(-1)} \otimes s_{bc}^{(-1)} - s_{ab} \otimes s_{bc}^{(-1)}) \\ &= \frac{1}{2}(s_{ab}^{(-1)} s_{bc} + s_{ab} s_{bc} + s_{ab}^{(-1)} s_{bc}^{(-1)} - s_{ab} s_{bc}^{(-1)}). \end{aligned}$$

The RHS becomes:

$$\begin{aligned} m(\mathcal{F} \triangleright (s_{bc} \otimes s_{ca})) &= \frac{1}{2} m(s_{bc} \otimes s_{ca} + s_{bc}^{(-1)} \otimes s_{ca} + s_{bc} \otimes s_{ca}^{(-1)} - s_{bc}^{(-1)} \otimes s_{ca}^{(-1)}) \\ &= \frac{1}{2}(s_{bc} s_{ca} + s_{bc}^{(-1)} s_{ca} + s_{bc} s_{ca}^{(-1)} - s_{bc}^{(-1)} s_{ca}^{(-1)}). \end{aligned}$$

Now observe that:

$$\begin{aligned} s_{ab}^{(-1)} s_{bc} &= s_{bc} s_{ca}^{(-1)} & \qquad s_{ab} s_{bc} &= s_{bc} s_{ca} \\ s_{ab}^{(-1)} s_{bc}^{(-1)} &= s_{bc}^{(-1)} s_{ca} & \qquad s_{ab} s_{bc}^{(-1)} &= s_{bc}^{(-1)} s_{ca}^{(-1)} \end{aligned}$$

hence the expressions are the same.

(v) $b < a < c$: Here we're checking that:

$$m(\mathcal{F} \triangleright (s_{ab} \otimes s_{bc}^{(-1)})) = m(\mathcal{F} \triangleright (s_{bc}^{(-1)} \otimes s_{ca})).$$

On both sides there is precisely one $f(-1, i, j)$ which acts non-trivially: $f(-1, a, b)$ on the left and $f(-1, c, a)$ on the right. Hence, like in case (iv), $\mathcal{F}$ acts non-trivially and we obtain, on the LHS:

$$\begin{aligned} m(\mathcal{F} \triangleright (s_{ab} \otimes s_{bc}^{(-1)})) &= \frac{1}{2} m(s_{ab} \otimes s_{bc}^{(-1)} + s_{ab}^{(-1)} \otimes s_{bc}^{(-1)} + s_{ab} \otimes s_{bc} - s_{ab}^{(-1)} \otimes s_{bc}) \\ &= \frac{1}{2}(s_{ab} s_{bc}^{(-1)} + s_{ab}^{(-1)} s_{bc}^{(-1)} + s_{ab} s_{bc} - s_{ab}^{(-1)} s_{bc}); \end{aligned}$$

on the RHS:

$$\begin{aligned} m(\mathcal{F} \triangleright (s_{bc}^{(-1)} \otimes s_{ca})) &= m(s_{bc}^{(-1)} \otimes s_{ca} + s_{bc} \otimes s_{ca} + s_{bc}^{(-1)} \otimes s_{ca}^{(-1)} - s_{bc} \otimes s_{ca}^{(-1)}) \\ &= s_{bc}^{(-1)} s_{ca} + s_{bc} s_{ca} + s_{bc}^{(-1)} s_{ca}^{(-1)} - s_{bc} s_{ca}^{(-1)} \end{aligned}$$

and again observe that:

$$s_{ab} s_{bc}^{(-1)} = s_{bc}^{(-1)} s_{ca}^{(-1)} \qquad s_{ab}^{(-1)} s_{bc}^{(-1)} = s_{bc}^{(-1)} s_{ca}$$

$$s_{ab} s_{bc} = s_{bc} s_{ca} \qquad s_{ab}^{(-1)} s_{bc} = s_{bc} s_{ca}^{(-1)}$$

and so these expressions are the same.

(vi) $c < b < a$: Finally, we must check that:

$$m(\mathcal{F} \triangleright (s_{ab} \otimes s_{bc})) = m(\mathcal{F} \triangleright (s_{bc} \otimes s_{ca}^{(-1)})).$$

Here, precisely three $f(-1, i, j)$ act non-trivially on the left ($f(-1, b, c)$, $f(-1, a, c)$ and $f(-1, a, b)$) and exactly one $f(-1, i, j)$ acts non-trivially on the right, namely $f(-1, b, c)$, so we have non-trivial actions on both sides. Hence the LHS gives:

$$\begin{aligned} m(\mathcal{F} \triangleright (s_{ab} \otimes s_{bc})) &= m(s_{ab} \otimes s_{bc} + s_{ab}^{(-1)} \otimes s_{bc} + s_{ab} \otimes s_{bc}^{(-1)} - s_{ab}^{(-1)} \otimes s_{bc}^{(-1)}) \\ &= s_{ab} s_{bc} + s_{ab}^{(-1)} s_{bc} + s_{ab} s_{bc}^{(-1)} - s_{ab}^{(-1)} s_{bc}^{(-1)}. \end{aligned}$$

The RHS gives:

$$\begin{aligned} m(\mathcal{F} \triangleright (s_{bc} \otimes s_{ca}^{(-1)})) &= m(s_{bc} \otimes s_{ca}^{(-1)} + s_{bc}^{(-1)} \otimes s_{ca}^{(-1)} + s_{bc} \otimes s_{ca} - s_{bc}^{(-1)} \otimes s_{ca}) \\ &= s_{bc} s_{ca}^{(-1)} + s_{bc}^{(-1)} s_{ca}^{(-1)} + s_{bc} s_{ca} - s_{bc}^{(-1)} s_{ca}. \end{aligned}$$

Finally, from the same observations as we made in part (v), these expressions are equal.

Hence the relation holds for this choice of $\phi(\sigma_{ab})$, thus:

$$\phi(\sigma_{ab}) \star \phi(\sigma_{bc}) = \phi(\sigma_{bc}) \star \phi(\sigma_{ca})$$

for all $a, b, c \in \{1, \ldots, n\}$, $a \neq b \neq c$, $a \neq c$.

3.9. **Conclusion.** Hence all the relations are satisfied and ϕ is a well-defined algebra homomorphism. To see that it is a bijection, the PBW-type theorems state that, as $\mathbb{C}$-vector spaces:

- $H_c(G(m,p,n))$ has basis $\{x_1^{k_1} \dots x_n^{k_n} g y_1^{l_1} \dots y_n^{l_n} : g \in G(m,p,n), k_i, l_i \in \mathbb{N} \cup \{0\}\}$
- $\underline{H}_c(\mu(G(m,p,n)))$ has basis $\{\underline{x}_1^{k_1} \dots \underline{x}_n^{k_n} g \underline{y}_1^{l_1} \dots \underline{y}_n^{l_n} : g \in W_{\mathcal{C},\mathcal{C}'}, k_i, l_i \in \mathbb{N} \cup \{0\}\}$

and now, due to the way we defined it, ϕ maps a basis to a basis, and is therefore bijective. Therefore, ϕ is an algebra isomorphism and so $(H_c(G(m,p,n)), \star) \cong \underline{H}_c(G(m,p,n))$. This completes the proof.

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