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L-PACKETS AND DEPTH FOR $\mathrm{SL}_2(K)$ WITH K A LOCAL FUNCTION FIELD OF CHARACTERISTIC 2

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Abstract. Let $\mathcal{G} = \mathrm{SL}_2(K)$ with K a local function field of characteristic 2. We review Artin-Schreier theory for the field K, and show that this leads to a parametrization of certain L-packets in the smooth dual of $\mathcal{G}$. We relate this to a recent geometric conjecture. The L-packets in the principal series are parametrized by quadratic extensions, and the supercuspidal L-packets of cardinality 4 are parametrised by biquadratic extensions. Each supercuspidal packet of cardinality 4 is accompanied by a singleton packet for $\mathrm{SL}_1(D)$. We compute the depths of the irreducible constituents of all these L-packets for $\mathrm{SL}_2(K)$ and its inner form $\mathrm{SL}_1(D)$.

1. Introduction

The special linear group SL_2 has been a mainstay of representation theory for at least 45 years, see [GGPS]. In that book, the authors show how the unitary irreducible representations of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{Q}_p)$ can be woven together in the context of automorphic forms. This comes about in the following way. The classical notion of a cusp form f in the upper half plane leads first to the concept of a cusp form on the adele group of GL_2 over $\mathbb{Q}$, and thence to the idea of an automorphic cuspidal representation π_f of the adele group of GL_2. We recall that the adele group of GL_2 is the restricted product of the local groups $\mathrm{GL}_2(\mathbb{Q}_p)$ where p is a place of $\mathbb{Q}$. If p is infinite then $\mathbb{Q}_p$ is the real field $\mathbb{R}$; if p is finite then $\mathbb{Q}_p$ is the p-adic field. The unitary representation π_f may be expressed as $\otimes\pi_p$ with one local representation for each local group $\mathrm{GL}_2(\mathbb{Q}_p)$. It is this way that the unitary representation theory of groups such as $\mathrm{GL}_2(\mathbb{Q}_p)$ enters into the modern theory of automorphic forms.

Let X be a smooth projective curve over $\mathbb{F}_q$. Denote by F the field $\mathbb{F}_q(X)$ of rational functions on X. For any closed point x of X we denote by F_x the completion of F at x and by $\mathfrak{o}_x$ its ring of integers. If we choose a local coordinate t_x at x (i.e., a rational function on X which vanishes at x to order one), then we obtain isomorphisms $F_x \simeq \mathbb{F}_{q_x}((t_x))$ and $\mathfrak{o}_x \simeq \mathbb{F}_{q_x}[[t_x]]$, where $\mathbb{F}_{q_x}$ is the residue field of x; in general, it is a finite extension of $\mathbb{F}_q$ containing $q_x = q^{deg(x)}$ elements. Thus, we now have a *local function field* attached to each point of X.

With all this in the background, it seems natural to us to study the representation theory of $\mathrm{SL}_2(K)$ with K a local function field. The case when K has characteristic 2 has many special features – and we focus on this case in this article. A local function field K of characteristic 2 is of the form $K = \mathbb{F}_q((t))$, the field of Laurent series with coefficients in $\mathbb{F}_q$, with $q = 2^f$. This example is particularly interesting because there are countably many quadratic extensions of $\mathbb{F}_q((t))$.

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Artin-Schreier theory is a branch of Galois theory, and more specifically is a positive characteristic analogue of Kummer theory, for Galois extensions of degree equal to the characteristic p. Artin and Schreier (1927) introduced Artin-Schreier theory for extensions of prime degree p, and Witt (1936) generalized it to extensions of prime power degree p^n. If K is a field of characteristic p, a prime number, any polynomial of the form

$$X^p - X + \alpha$$

for $\alpha \in K$, is called an Artin-Schreier polynomial. When α does not lie in the subset $\{y \in K \,|\, y = x^p - x \text{ for } x \in K\}$, this polynomial is irreducible in $K[X]$, and its splitting field over K is a cyclic extension of K of degree p. This follows since for any root β, the numbers $\beta + i$, for $1 \leq i \leq p$, form all the root – by Fermat's little theorem – so the splitting field is $K(\beta)$. Conversely, any Galois extension of K of degree p equal to the characteristic of K is the splitting field of an Artin-Schreier polynomial. This can be proved using additive counterparts of the methods involved in Kummer theory, such as Hilbert's theorem 90 and additive Galois cohomology. These extensions are called Artin-Schreier extensions.

For the moment, let F be a local nonarchimedean field with odd residual characteristic. The L-packets for $\mathrm{SL}_2(F)$ are classified in the paper [LR] by Lansky-Rhaguram. They comprise: the principal series L-packets $\xi_E = \{\pi_E^1, \pi_E^2\}$ where E/F is a quadratic extension; the unramified supercuspidal L-packet of cardinality 4; and the supercuspidal L-packets of cardinality 2.

We now revert to the case of a local function field K of characteristic 2. We consider $\mathrm{SL}_2(K)$. Drawing on the accounts in [Da, Th1, Th2], we review Artin-Schreier theory, adapted to the local function field K, with special emphasis on the quadratic extensions of K.

The L-packets in the principal series of $\mathrm{SL}_2(K)$ are parametrized by quadratic extensions, and the supercuspidal L-packets of cardinality 4 are parametrised by biquadratic extensions L/K. There are countably many such supercuspidal L-packets. In this article, we do not consider supercuspidal L-packets of cardinality 2.

The concept of *depth* can be traced back to the concept of *level* of a character. Let χ be a non-trivial character of $K^\times$. The level of χ is the least integer $n \geq 0$ such that χ is trivial on the higher unit group U_K^{n+1}, see [BH, p.12]. The depth of a Langlands parameter ϕ is defined as follows. Let r be a real number, $r \geq 0$, let $\mathrm{Gal}(K_s/K)^r$ be the r-th ramification subgroup of the absolute Galois group of K. Then the depth of ϕ is the smallest number $d(\phi) \geq 0$ such that ϕ is trivial on $\mathrm{Gal}(K_s/K)^r$ for all $r > d(\phi)$.

The *depth* $d(\pi)$ of an irreducible $\mathcal{G}$-representation π was defined by Moy and Prasad [MoPr1, MoPr2] in terms of filtrations $P_{x,r}(r \in \mathbb{R}_{\geq 0})$ of the parahoric subgroups $P_x \subset \mathcal{G}$.

Let $\mathcal{G} = \mathrm{SL}_2(K)$. Let $\mathbf{Irr}(\mathcal{G})$ denote the smooth dual of $\mathcal{G}$. Thanks to a recent article [ABPS1], we have, for every Langlands parameter $\phi \in \Phi(\mathcal{G})$ with L-packet $\Pi_\phi(\mathcal{G}) \subset \mathbf{Irr}(\mathcal{G})$

$$d(\phi) = d(\pi) \quad \text{for all } \pi \in \Pi_\phi(\mathcal{G}). \tag{1}$$

The equation (1) is a big help in the computation of the depth $d(\pi)$. To each biquadratic extension L/K, there is attached a Langlands parameter $\phi = \phi_{L/K}$, and an L-packet Π_ϕ of cardinality 4. The depth of the parameter $\phi_{L/K}$ depends on the extension L/K. More precisely, the numbers $d(\phi)$ depend on the breaks in the upper ramification filtration of the Galois group

$$\mathrm{Gal}(L/K) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

For certain extensions L/K the allowed depths can be any odd number $1, 3, 5, 7, \ldots$. For the other extensions L/K, the allowed depths are $3, 5, 7, 9, \ldots$. Accordingly, the depth of each irreducible supercuspidal representation π in the packet Π_ϕ is given by the formula

$$d(\pi) = 2n + 1 \tag{2}$$

where $n = 0, 1, 2, 3, \ldots$ or $1, 2, 3, 4, \ldots$ depending on L/K. Let D be a central division algebra of dimension 4 over K. The parameter ϕ is relevant for the inner form $\mathrm{SL}_1(D)$, which admits singleton L-packets, and the depths are given by the formula (2).

This contrasts with the case of $\mathrm{SL}_2(\mathbb{Q}_p)$ with $p > 2$. Here there is a unique biquadratic extension L/K, and a unique tamely ramified parameter $\phi : \mathrm{Gal}(L/K) \to \mathrm{SO}_3(\mathbb{R})$ of depth zero.

We move on to consider the geometric conjecture in [ABPS]. Let $\mathfrak{B}(\mathcal{G})$ denote the Bernstein spectrum of $\mathcal{G}$, let $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$, and let $T^{\mathfrak{s}}, W^{\mathfrak{s}}$ denote the complex torus, finite group, attached by Bernstein to $\mathfrak{s}$. For more details at this point, we refer the reader to [R]. The Bernstein decomposition provides us, inter alia, with the following data: a canonical disjoint union

$$\mathbf{Irr}(\mathcal{G}) = \bigsqcup \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$$

and, for each $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$, a finite-to-one surjective map

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \to T^{\mathfrak{s}}/W^{\mathfrak{s}}$$

onto the quotient variety $T^{\mathfrak{s}}/W^{\mathfrak{s}}$. The geometric conjecture in [ABPS] amounts to a refinement of these statements. The refinement comprises the assertion that we have a *bijection*

$$\mathbf{Irr}(\mathcal{G})^{s} \simeq T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \tag{3}$$

where $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$ is the *extended quotient* of the torus $T^{\mathfrak{s}}$ by the finite group $W^{\mathfrak{s}}$. If the action of $W^{\mathfrak{s}}$ on $T^{\mathfrak{s}}$ is free, then the extended quotient is equal to the ordinary quotient $T^{\mathfrak{s}}/W^{\mathfrak{s}}$. If the action is not free, then the extended quotient is a finite disjoint union of quotient varieties, one of which is the ordinary quotient. The bijection (3) is subject to certain constraints, itemised in [ABPS].

In the case of SL_2, the torus $T^{\mathfrak{s}}$ is of dimension 1, and the finite group $W^{\mathfrak{s}}$ is either 1 or $\mathbb{Z}/2\mathbb{Z}$. So, in this context, the content of the conjecture is rather modest: but a proof is required, and such a proof is duly given in §7.

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2. Artin-Schreier theory

Let K be a local field with positive characteristic p, containing the n-th roots of unity ζ_n. The cyclic extensions of K whose degree n is coprime with p are described by Kummer theory. It is well known that any cyclic extension L/K of degree n, $(n, p) = 1$, is generated by a root α of an irreducible polynomial $x^n - a \in K[x]$. We fix an algebraic closure $\overline{K}$ of K and a separable closure K^s of K in $\overline{K}$. If $\alpha \in K^s$

is a root of $x^n - a$ then $K(\alpha)/K$ is a cyclic extension of degree n and is called a Kummer extension of K.

Artin-Schreier theory aims to describe cyclic extensions of degree equal to or divisible by $ch(K) = p$. It is therefore an analogue of Kummer theory, where the role of the polynomial $x^n - a$ is played by $x^n - x - a$. Essentially, every cyclic extension of K with degree $p = ch(K)$ is generated by a root α of $x^p - x - a \in K[x]$.

Let $\wp$ denote the Artin-Schreier endomorphism of the additive group K^s [Ne]:

$$\wp : K^s \to K^s, \quad x \mapsto x^p - x.$$

Given $a \in K$ denote by $K(\wp^{-1}(a))$ the extension $K(\alpha)$, where $\wp(\alpha) = a$ and $\alpha \in K^s$. We have the following characterization of finite cyclic Artin-Schreier extensions of degree p:

Theorem 2.1. *(i) Given $a \in K$, either $\wp(x) - a \in K[x]$ has one root in K in which case it has all the p roots are in K, or is irreducible.*

(ii) If $\wp(x) - a \in K[x]$ is irreducible then $K(\wp^{-1}(a))/K$ is a cyclic extension of degree p, with $\wp^{-1}(a) \subset K^s$.

(iii) If L/K be a finite cyclic extension of degree p, then $L = K(\wp^{-1}(a))$, for some $a \in K$.

(See [Th1, p.34] for more details.)

We fix now some notation. K is a local field with characteristic $p > 1$ with finite residue field k. The field of constants $k = \mathbb{F}_q$ is a finite extension of $\mathbb{F}_p$, with degree $[k : \mathbb{F}_p] = f$ and $q = p^f$.

Let $\mathfrak{o}$ be the ring of integers in K and denote by $\mathfrak{p} \subset \mathfrak{o}$ the (unique) maximal ideal of $\mathfrak{o}$. This ideal is principal and any generator of $\mathfrak{p}$ is called a uniformizer. A choice of uniformizer $\varpi \in \mathfrak{o}$ determines isomorphisms $K \cong \mathbb{F}_q((\varpi))$, $\mathfrak{o} \cong \mathbb{F}_q[[\varpi]]$ and $\mathfrak{p} = \varpi\mathfrak{o} \cong \varpi\mathbb{F}_q[[\varpi]]$.

A normalized valuation on K will be denoted by ν, so that $\nu(\varpi) = 1$ and $\nu(K) = \mathbb{Z}$. The group of units is denoted by $\mathfrak{o}^\times$.

2.1. **The Artin-Schreier symbol.** Let L/K be a finite Galois extension. Let $N_{L/K}$ be the norm map and denote by $\mathrm{Gal}(L/K)^{ab}$ the abelianization of $\mathrm{Gal}(L/K)$. The reciprocity map is a group isomorphism

$$K^\times / N_{L/K} L^\times \xrightarrow{\simeq} \mathrm{Gal}(L/K)^{ab}. \tag{4}$$

The Artin symbol is obtained by composing the reciprocity map with the canonical morphism $K^\times \to K^\times / N_{L/K} L^\times$

$$b \in K^\times \mapsto (b, L/K) \in \mathrm{Gal}(L/K)^{ab}. \tag{5}$$

From the Artin symbol we obtain a pairing

$$K \times K^\times \longrightarrow \mathbb{Z}/p\mathbb{Z}, (a, b) \mapsto (b, L/K)(\alpha) - \alpha, \tag{6}$$

where $\wp(\alpha) = a$, $\alpha \in K^s$ and $L = K(\alpha)$.

Definition 2.2. *Given $a \in K$ and $b \in K^\times$, the Artin-Schreier symbol is defined by*

$$[a, b) = (b, L/K)(\alpha) - \alpha.$$

The Artin-Schreier symbol is a bilinear map satisfying the following properties, see [Ne, p.341]:

$$[a_1 + a_2, b) = [a_1, b) + [a_2, b); \tag{7}$$

$$[a, b_1 b_2) = [a, b_1) + [a, b_2); \tag{8}$$

$$[a, b) = 0, \forall a \in K \Leftrightarrow b \in N_{L/K} L^\times, L = K(\alpha) \text{ and } \wp(\alpha) = a; \tag{9}$$

$$[a, b) = 0, \forall b \in K^\times \Leftrightarrow a \in \wp(K). \tag{10}$$

2.2. **The groups $K/\wp(K)$ and $K^\times/K^{\times p}$.** In this section we recall some properties of the groups $K/\wp(K)$ and $K^\times/K^{\times p}$ and use them to redefine the pairing (6). Dalawat [Da2, Da] interprets $K/\wp(K)$ and $K^\times/K^{\times p}$ as $\mathbb{F}_p$-spaces. This interpretation will be particularly useful in §4.

Consider the additive group K. By [Da, Proposition 11], the $\mathbb{F}_p$-space $K/\wp(K)$ is countably infinite . Hence, $K/\wp(K)$ is infinite as a group.

Proposition 2.3. *$K/\wp(K)$ is a discrete abelian torsion group.*

Proof. The ring of integers decomposes as a (direct) sum

$$\mathfrak{o} = \mathbb{F}_q + \mathfrak{p}$$

and we have

$$\wp(\mathfrak{o}) = \wp(\mathbb{F}_q) + \wp(\mathfrak{p}).$$

The restriction $\wp : \mathfrak{p} \to \mathfrak{p}$ is an isomorphism, see [Da, Lemma 8]. Hence,

$$\wp(\mathfrak{o}) = \wp(\mathbb{F}_q) + \mathfrak{p}$$

and $\mathfrak{p} \subset \wp(K)$. It follows that $\wp(K)$ is an open subgroup of K and $K/\wp(K)$ is discrete. Since $\wp(K)$ is annihilated by p, $K/\wp(K)$ is a torsion group. □

Now we concentrate on the multiplicative group $K^\times$. For any $n > 0$, let U_n be the kernel of the reduction map from $\mathfrak{o}^\times$ to $(\mathfrak{o}/\mathfrak{p}^n)^\times$. In particular, $U_1 = ker(\mathfrak{o}^\times \to k^\times)$. The U_n are $\mathbb{Z}_p$-modules , because they are commutative pro-p-groups. By [Da2, Proposition 20], the $\mathbb{Z}_p$-module U_1 is not finitely generated. As a consequence, $K^\times/K^{\times p}$ is infinite, see [Da2, Corollary 21]. The next result gives a characterization of the topological group $K^\times/K^{\times p}$.

Proposition 2.4. *$K^\times/K^{\times p}$ is a profinite abelian p-torsion group.*

Proof. There is a canonical isomorphism $K^\times \cong \mathbb{Z} \times \mathfrak{o}^\times$. The group of units is a direct product $\mathfrak{o}^\times \cong \mathbb{F}_q^\times \times U_1$, with $q = p^f$. By [Iw, p.25], the group U_1 is a direct product of countable many copies of the ring of p-adic integers

$$U_1 \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times ... = \prod_{\mathbb{N}} \mathbb{Z}_p.$$

Give $\mathbb{Z}$ the discrete topology and $\mathbb{Z}_p$ the p-adic topology. Then, for the product topology, $K^\times = \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}_p$ is a topological group, locally compact, Hausdorff and totally disconnected.

Now, $K^{\times p}$ decomposes as a product of countable many components

$$K^{\times p} \cong p\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times p\mathbb{Z}_p \times p\mathbb{Z}_p \times ...$$

$$= p\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \prod_{\mathbb{N}} p\mathbb{Z}_p.$$

Note that $p\mathbb{Z}/(q-1)\mathbb{Z} = \mathbb{Z}/(q-1)\mathbb{Z}$, since p and $q-1$ are coprime. Denote by $z = \prod_n z_n$ an element of $\prod_{\mathbb{N}} \mathbb{Z}_p$, where $z_n = \sum_{i=0}^{\infty} a_{i,n} p^i \in \mathbb{Z}_p$, for every n.

The map

$$\varphi : \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}_p \to \mathbb{Z}/p\mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z}$$

defined by

$$(x, y, z) \mapsto (x \mod p), \prod_n pr_0(z_n))$$

where $pr_0(z_n) = a_{0,n}$ is the projection, is clearly a group homomorphism.

Now, $\mathbb{Z}/p\mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z} = \prod_{n=0}^{\infty} \mathbb{Z}/p\mathbb{Z}$ is a topological group for the product topology, where each component $\mathbb{Z}/p\mathbb{Z}$ has the discrete topology. It is compact, Hausdorff and totally disconnected. Therefore, $\prod_{n=0}^{\infty} \mathbb{Z}/p\mathbb{Z}$ is a profinite group.

Since

$$ker\varphi = p\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \prod_{\mathbb{N}} p\mathbb{Z}_p,$$

it follows that there is an isomorphism of topological groups

$$K^{\times}/K^{\times p} \cong \prod_{\mathbb{N}_0} \mathbb{Z}/p\mathbb{Z},$$

where $K^{\times}/K^{\times p}$ is given the quotient topology. Therefore, $K^{\times}/K^{\times p}$ is profinite. □

From propositions 2.3 and 2.4, $K/\wp(K)$ is a discrete abelian group and $K/K^{\times p}$ is an abelian profinite group, both annihilated by $p = ch(K)$. Therefore, Pontryagin duality coincides with $Hom(-, \mathbb{Z}/p\mathbb{Z})$ on both of these groups, see [Th2]. The pairing (6) restricts to a pairing

$$[.,.) : K/\wp(K) \times K^{\times}/K^{\times p} \to \mathbb{Z}/p\mathbb{Z}. \tag{11}$$

which we refer from now on to the **Artin-Schreier pairing**. It follows from (9) and (10), that the pairing is nondegenerate (see also [Th2, Proposition 3.1]). The next result shows that the pairing is perfect.

Proposition 2.5. *The Artin-Schreier symbol induces isomorphisms of topological groups*

$$K^{\times}/K^{\times p} \xrightarrow{\simeq} \mathrm{Hom}(K/\wp(K), \mathbb{Z}/p\mathbb{Z}), bK^{\times p} \mapsto (a + \wp(K) \mapsto [a, b))$$

and

$$K/\wp(K) \xrightarrow{\simeq} \mathrm{Hom}(K^{\times}/K^{\times p}, \mathbb{Z}/p\mathbb{Z}), a + \wp(K) \mapsto (bK^{\times p} \mapsto [a, b))$$

Proof. The result follows by taking $n = 1$ in Proposition 5.1 of [Th2], and from the fact that Pontryagin duality for the groups $K/\wp(K)$ and $K^{\times}/K^{\times p}$ coincide with $Hom(-, \mathbb{Z}/p\mathbb{Z})$ duality. Hence, there is an isomorphism of topological groups between each such group and its bidual. □

Let B be a subgroup of the additive group of K with finite index such that $\wp(K) \subseteq B \subseteq K$. The composite of two finite abelian Galois extensions of exponent p is again a finite abelian Galois extension of exponent p. Therefore, the composite

$$K_B = K(\wp^{-1}(B)) = \prod_{a \in B} K(\wp^{-1}(a))$$

is a finite abelian Galois extension of exponent p. On the other hand, if L/K is a finite abelian Galois extension of exponent p, then $L = K_B$ for some subgroup $\wp(K) \subseteq B \subseteq K$ with finite index.

All such extensions lie in the maximal abelian extension of exponent p, which we denote by $K_p = K(\wp^{-1}(K))$. The extension K_p/K is infinite and Galois. The corresponding Galois group $G_p = Gal(K_p/K)$ is an infinite profinite group and may be identified, under class field theory, with $K^\times/K^{\times p}$, see [Th2, Proposition 5.1]. The case $ch(K) = 2$ leads to $G_2 \cong K^\times/K^{\times 2}$ and will play a fundamental role in the sequel.

3. Quadratic characters

From now on we take K to be a local function field with $ch(K) = 2$. Therefore, K is of the form $\mathbb{F}_q((\varpi))$ with $q = 2^f$.

When $K = \mathbb{F}_q((\varpi))$, we have, according to [Iw, p.25],

$$U_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times ... = \prod_{\mathbb{N}} \mathbb{Z}_2$$

with countably infinite many copies of $\mathbb{Z}_2$, the ring of 2-adic integers.

Artin-Schreier theory provides a way to parametrize all the quadratic extensions of $K = \mathbb{F}_q((\varpi))$. By proposition 2.4, there is a bijection between the set of quadratic extensions of $\mathbb{F}_q((\varpi))$ and the group

$$\mathbb{F}_q((\varpi))^\times/\mathbb{F}_q((\varpi))^{\times 2} \cong \prod_{\mathbb{N}_0} \mathbb{Z}/2\mathbb{Z} = G_2$$

where G_2 is the Galois group of the *maximal abelian extension of exponent* 2. Since G_2 is an infinite profinite group, there are countably many quadratic extensions.

To each quadratic extension $K(\alpha)/K$, with $\alpha^2 - \alpha = a$, we associate the Artin-Schreier symbol

$$[a, .) : K^\times/K^{\times 2} \to \mathbb{Z}/2\mathbb{Z}.$$

Now, let φ denote the isomorphism $\mathbb{Z}/2\mathbb{Z} \cong \mu_2(\mathbb{C}) = \{\pm 1\}$ with the group of roots of unity. We obtain, by composing with the Artin-Schreier symbol, a unique multiplicative quadratic character

$$\chi_a : K^\times \to \mathbb{C}^\times,\ \chi_a = \varphi([a, .)) \tag{12}$$

Proposition 2.5 shows that every quadratic character of $\mathbb{F}_q((\varpi))^\times$ arises in this way.

Example 3.1. *The unramified quadratic extension of K is $K(\wp^{-1}(\mathfrak{o}))$, see* [Da] *proposition* 12*. According to Dalawat, the group $K/\wp(K)$ may be regarded as an $\mathbb{F}_2$-space and the image of $\mathfrak{o}$ under the canonical surjection $K \to K/\wp(K)$ is an $\mathbb{F}_2$-line, i.e., isomorphic to $\mathbb{F}_2$. Since $\wp_{|\mathfrak{p}} : \mathfrak{p} \to \mathfrak{p}$ is an isomorphism, the image of $\mathfrak{p}$ in $K/\wp(K)$ is $\{0\}$, see lemma* 8 *in* [Da]*. Now, choose any $a_0 \in \mathfrak{o}\backslash\mathfrak{p}$ such that the image of a_0 in $\mathfrak{o}/\mathfrak{p}$ has nonzero trace in $\mathbb{F}_2$, see* [Da, Proposition 9]*. The* *quadratic character* $\chi_{a_0} = \varphi([a_0, .))$ *associated with* $K(\wp^{-1}(\mathfrak{o}))$ *via class field theory is precisely the unramified character* $(n \mapsto (-1)^n)$ *from above. Note that any other choice* $b_0 \in \mathfrak{o}\backslash\mathfrak{p}$*, with* $a_0 \neq b_0$*, gives the same unique unramified character, since there is only one nontrivial coset* $a_0 + \wp(K)$ *for* $a_0 \in \mathfrak{o}\backslash\mathfrak{p}$*.*

Let $\mathcal{G}$ denote $\mathrm{SL}_2(K)$, let $\mathcal{B}$ be the standard Borel subgroup of $\mathcal{G}$, let $\mathcal{T}$ be the diagonal subgroup of $\mathcal{G}$. Let χ be a character of $\mathcal{T}$. Then, χ inflates to a character of $\mathcal{B}$. Denote by $\pi(\chi)$ the (unitarily) induced representation $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi)$. The representation space $V(\chi)$ of $\pi(\chi)$ consists of locally constant complex valued functions $f : \mathcal{G} \to \mathbb{C}$ such that, for every $a \in K^\times$, $b \in K$ and $g \in \mathcal{G}$, we have

$$f\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = |a|\chi(a)f(g)$$

The action of $\mathcal{G}$ on $V(\chi)$ is by right translation. The representations $(\pi(\chi), V(\chi))$ are called (unitary) principal series of $\mathcal{G} = SL_2(K)$.

Let χ be a quadratic character of $K^\times$. The reducibility of the induced representation $\mathrm{Ind}_B^G(\chi)$ is well known in zero characteristic. Casselman proved that the same result holds in characteristic 2 and any other positive characteristic p.

Theorem 3.2. [Ca, Ca2] *The representation* $\pi(\chi) = \mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi)$ *is reducible if, and only if,* χ *is either* $|.|^{\pm}$ *or a nontrivial quadratic character of* $K^\times$*.*

For a proof see [Ca, Theorems 1.7, 1.9] and [Ca2, §9].

From now on, χ will be a quadratic character. It is a classical result that the unitary principal series for GL_2 are irreducible. For a representation of GL_2 parabolically induced by $1 \otimes \chi$, Clifford theory tells us that the dimension of the intertwining algebra of its restriction to SL_2 is 2. This is exactly the induced representation of SL_2 by χ:

$$\mathrm{Ind}_{\widetilde{B}}^{\mathrm{GL}_2(K)}(1 \otimes \chi)_{|\mathrm{SL}(2,K)} \xrightarrow{\simeq} \mathrm{Ind}_B^{\mathrm{SL}_2(K)}(\chi)$$

where $\widetilde{B}$ denotes the standard Borel subgroup of $\mathrm{GL}_2(K)$. This leads to reducibility of the induced representation $\mathrm{Ind}_B^G(\chi)$ into two inequivalent constituents. Thanks to M. Tadic for helpful comments at this point.

The two irreducible constituents

$$\pi(\chi) = \mathrm{Ind}_B^G(\chi) = \pi(\chi)^+ \oplus \pi_(\chi)^- \tag{13}$$

define an L-packet $\{\pi(\chi)^+, \pi(\chi)^-\}$ for SL_2.

4. Biquadratic extensions of $\mathbb{F}_q((\varpi))$

Quadratic extensions L/K are obtained by adjoining an $\mathbb{F}_2$-line $D \subset K/\wp(K)$. Therefore, $L = K(\wp^{-1}(D)) = K(\alpha)$ where $D = span\{a + \wp(K)\}$, with $\alpha^2 - \alpha = a$. In particular, if $a_0 \in \mathfrak{o}\backslash\mathfrak{p}$ such that the image of a_0 in $\mathfrak{o}/\mathfrak{p}$ has nonzero trace in $\mathbb{F}_2$, the $\mathbb{F}_2$-line $V_0 = span\{a_0 + \wp(K)\}$ contains all the cosets $a_i + \wp(K)$ where a_i is an integer and so $K(\wp^{-1}(\mathfrak{o})) = K(\wp^{-1}(V_0)) = K(\alpha_0)$ where $\alpha_0^2 - \alpha_0 = a_0$ gives the unramified quadratic extension.

Biquadratic extensions are computed the same way, by considering $\mathbb{F}_2$-planes $W = span\{a + \wp(K), b + \wp(K)\} \subset K/\wp(K)$. Therefore, if $a + \wp(K)$ and $b + \wp(K)$ are $\mathbb{F}_2$-linearly independent then $K(\wp^{-1}(W)) := K(\alpha, \beta)$ is biquadratic, where $\alpha^2 - \alpha = a$ and $\beta^2 - \beta = b$, $\alpha, \beta \in K^s$. Therefore, $K(\alpha, \beta)/K$ is biquadratic if $b - a \notin \wp(K)$.

A biquadratic extension containing the line V_0 is of the form $K(\alpha_0, \beta)/K$. There are countably many quadratic extensions L_0/K containing the unramified quadratic extension. They have ramification index $e(L_0/K) = 2$. And there are countably many biquadratic extensions L/K which do not contain the unramified quadratic extension. They have ramification index $e(L/K) = 4$.

So, there is a plentiful supply of biquadratic extensions $K(\alpha, \beta)/K$.

4.1. **Ramification.** The space $K/\wp(K)$ comes with a filtration

$$0 \subset_1 V_0 \subset_f V_1 = V_2 \subset_f V_3 = V_4 \subset_f \ldots \subset K/\wp(K) \tag{14}$$

where V_0 is the image of $\mathfrak{o}_K$ and V_i $(i > 0)$ is the image of $\mathfrak{p}^{-i}$ under the canonical surjection $K \to K/\wp(K)$. For $K = \mathbb{F}_q((\varpi))$ and $i > 0$, each inclusion $V_{2i} \subset_f V_{2i+1}$ is a sub-$\mathbb{F}_2$-space of codimension f. The $\mathbb{F}_2$-dimension of V_n is

$$dim_{\mathbb{F}_2} V_n = 1 + \lceil n/2 \rceil f, \tag{15}$$

for every $n \in \mathbb{N}$, where $\lceil x \rceil$ is the smallest integer bigger than x.

Let L/K denote a Galois extension with Galois group G. For each $i \geq -1$ we define the i^{th}-ramification subgroup of G (in the lower numbering) to be:

$$G_i = \{\sigma \in G : \sigma(x) - x \in \mathfrak{p}_L^{i+1}, \forall x \in \mathfrak{o}_L\}.$$

An integer t is a *break* for the filtration $\{G_i\}_{i \geq -1}$ if $G_t \neq G_{t+1}$. The study of ramification groups $\{G_i\}_{i \geq -1}$ is equivalent to the study of breaks of the filtration.

There is another decreasing filtration with upper numbering $\{G^i\}_{i \geq -1}$ and defined by the *Hasse-Herbrand function* $\psi = \psi_{L/K}$:

$$G^u = G_{\psi(u)}.$$

In particular, $G^{-1} = G_{-1} = G$ and $G^0 = G_0$, since $\psi(0) = 0$.

Let $G_2 = Gal(K_2/K)$ be the Galois group of the maximal abelian extension of exponent 2, $K_2 = K(\wp^{-1}(K))$. Since $G_2 \cong K^\times/K^{\times 2}$ (proposition 2.4), the pairing $K^\times/K^{\times 2} \times K/\wp(K) \to \mathbb{Z}/2\mathbb{Z}$ from (11) coincides with the pairing $G_2 \times K/\wp(K) \to \mathbb{Z}/2\mathbb{Z}$.

The profinite group G_2 comes equipped with a ramification filtration $(G_2^u)_{u \geq -1}$ in the upper numbering, see [Da, p.409]. For $u \geq 0$, we have an orthogonal relation [Da, Proposition 17]

$$(G_2^u)^\perp = \overline{\mathfrak{p}^{-\lceil u \rceil + 1}} = V_{\lceil u \rceil - 1} \tag{16}$$

under the pairing $G_2 \times K/\wp(K) \to \mathbb{Z}/2\mathbb{Z}$.

Since the upper filtration is more suitable for quotients, we will compute the upper breaks. By using the Hasse-Herbrand function it is then possible to compute the lower breaks in order to obtain the lower ramification filtration.

According to [Da, Proposition 17], the positive breaks in the filtration $(G^v)_v$ occur precisely at integers prime to p. So, for $ch(K) = 2$, the positive breaks will occur at odd integers. The lower numbering breaks are also integers. If G is cyclic of prime order, then there is a unique break for any decreasing filtration $(G^v)_v$ (see [Da], Proposition 14). In general, the number of breaks depends on the possible filtration of the Galois group.

Given a plane $W \subset K/\wp(K)$, the filtration (14) $(V_i)_i$ on $K/\wp(K)$ induces a filtration $(W_i)_i$ on W, where $W_i = W \cap V_i$. There are three possibilities for the filtration breaks on a plane and we will consider each case individually.

Case 1 : W contains the line V_0, i.e. $L_0 = K(\wp^{-1}(W))$ contains the unramified quadratic extension $K(\wp^{-1}(V_0)) = K(\alpha_0)$ of K. The extension has residue degree $f(L_0/K) = 2$ and ramification index $e(L_0/K) = 2$. In this case, there is an integer $t > 0$, necessarily odd, such that the filtration $(W_i)_i$ looks like

$$0 \subset_1 W_0 = W_{t-1} \subset_1 W_t = W.$$

By the orthogonality relation (16), the upper ramification filtration on $G = Gal(L_0/K)$ looks like

$$\{1\} = ... = G^{t+1} \subset_1 G^t = ... = G^0 \subset_1 G^{-1} = G$$

Therefore, the upper ramification breaks occur at -1 and t.

The number of such W is equal to the number of planes in V_t containing the line V_0 but but not contained in the subspace V_{t-1}. This number can be computed and equals the number of biquadratic extensions of K containing the unramified quadratic extensions and with a pair of upper ramification breaks $(-1, t)$, $t > 0$ and odd. Here is an example.

Example 4.1. *The number of biquadratic extensions containing the unramified quadratic extension and with a pair of upper ramification breaks* $(-1, 1)$ *is equal to the number of planes in an* $1 + f$*-dimensional* $\mathbb{F}_2$*-space, containing the line* V_0*. There are precisely*

$$1 + 2 + 2^2 + ... + 2^{f-1} = \frac{1 - 2^f}{1 - 2} = q - 1$$

of such biquadratic extensions.

Case 2.1 : W does not contains the line V_0 and the induced filtration on the plane W looks like

$$0 = W_{t-1} \subset_2 W_t = W$$

for some integer t, necessarily odd.

The number of such W is equal to the number of planes in V_t whose intersection with V_{t-1} is $\{0\}$. Note that, there are no such planes when $f = 1$. So, for $K = \mathbb{F}_2((\varpi))$, **case 2.1** does not occur.

Suppose $f > 1$. By the orthogonality relation, the upper ramification ramification filtration on $G = Gal(L/K)$ looks like

$$\{1\} = ... = G^{t+1} \subset_2 G^t = ... = G^{-1} = G$$

Therefore, there is a single upper ramification break occurring at $t > 0$ and is necessarily odd.

For $f = 1$ there is no such biquadratic extension. For $f > 1$, the number of these biquadratic extensions equals the number of planes W contained in an $\mathbb{F}_2$-space of dimension $1+fi$, $t = 2i-1$, which are transverse to a given codimension-f $\mathbb{F}_2$-space.

Case 2.2 : W does not contains the line V_0 and the induced filtration on the plane W looks like

$$0 = W_{t_1-1} \subset_1 W_{t_1} = W_{t_2-1} \subset_1 W_{t_2} = W$$

for some integers t_1 and t_2, necessarily odd, with $0 < t_1 < t_2$.

The orthogonality relation for this case implies that the upper ramification filtration on $G = Gal(L/K)$ looks like

$$\{1\} = ... = G^{t_2+1} \subset_1 G^{t_2} = ... = G^{t_1+1} \subset_1 G^{t_1} = ... = G$$

The upper ramification breaks occur at odd integers t_1 and t_2.

There is only a finite number of such biquadratic extensions, for a given pair of upper breaks (t_1, t_2).

5. Langlands parameter

We have the following canonical homomorphism:

$$\mathbf{W}_K \to \mathbf{W}_K^{ab} \simeq K^\times \to K^\times/K^{\times 2}.$$

According to §2, we also have

$$K^\times/K^{\times 2} \simeq \prod \mathbb{Z}/2\mathbb{Z}$$

the product over countably many copies of $\mathbb{Z}/2\mathbb{Z}$. Using the countable axiom of choice, we choose two copies of $\mathbb{Z}/2\mathbb{Z}$. This creates a homomorphism

$$\mathbf{W}_K \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

There are countably many such homomorphisms.

Following [We], denote by α, β, γ the images in $\mathrm{PSL}_2(\mathbb{C})$ of the elements

$$z_\alpha = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad z_\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z_\gamma = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

in $\mathrm{SL}_2(\mathbb{C})$.

Note that $z_\alpha, z_\beta, z_\gamma \in \mathrm{SU}_2(\mathbb{C})$ so that

$$\alpha, \beta, \gamma \in \mathrm{PSU}_2(\mathbb{C}) = \mathrm{SO}_3(\mathbb{R}).$$

Denote by J the group generated by α, β, γ:

$$J := \{\epsilon, \alpha, \beta, \gamma\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

The group J is unique up to conjugacy in $G = \mathrm{PSL}_2(\mathbb{C})$.

The pre-image of J in $\mathrm{SL}_2(\mathbb{C})$ is the group $\{\pm 1, \pm z_\alpha, \pm z_\beta, \pm z_\gamma\}$ and is isomorphic to the group U_8 of unit quaternions $\{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$.

The centralizer and normalizer of J are given by

$$C_G(J) = J, \quad N_G(J) = O$$

where $O \simeq S_4$ the symmetric group on 4 letters. The quotient $O/J \simeq \mathrm{GL}_2(\mathbb{Z}/2)$ is the full automorphism group of J.

Each biquadratic extension L/K determines a Langlands parameter

$$\phi : \mathrm{Gal}(L/K) \to \mathrm{SO}_3(\mathbb{R}) \subset \mathrm{SO}_3(\mathbb{C}) \tag{17}$$

Define

$$S_\phi = C_{\mathrm{PSL}_2(\mathbb{C})}(\mathrm{im}\,\phi) \tag{18}$$

Then we have $S_\phi = J$, since $C_G(J) = J$, and whose conjugacy class depends only on L, since $O/J = \mathrm{Aut}(J)$.

Define the new group

$$\mathcal{S}_\phi = C_{\mathrm{SL}_2(\mathbb{C})}(\mathrm{im}\,\phi)$$

To align with the notation in [ABPS2], replace $\phi^\sharp$ in [ABPS2] by ϕ in the present article. We have the short exact sequence

$$1 \to \mathcal{Z}_\phi \to \mathcal{S}_\phi \to S_\phi \to 1$$

with $\mathcal{Z}_\phi = \mathbb{Z}/2\mathbb{Z}$.

Let D be a central division algebra of dimension 4 over K, and let Nrd denote the reduced norm on $D^\times$. Define

$$\mathrm{SL}_1(D) = \{x \in D^\times : \mathrm{Nrd}(x) = 1\}.$$

Then $\mathrm{SL}_1(D)$ is an inner form of $\mathrm{SL}_2(K)$. In the local Langlands correspondence [ABPS2] for the inner forms of SL_2, the L-parameter ϕ is enhanced by elements $\rho \in \mathbf{Irr}(\mathcal{S}_\phi)$. Now the group $\mathcal{S}_\phi \simeq U_8$ admits four characters $\rho_1, \rho_2, \rho_3, \rho_4$ and one irreducible representation ρ_0 of degree 2.

The parameter ϕ creates a big packet with five elements, which are allocated to $\mathrm{SL}_2(K)$ or $\mathrm{SL}_1(D)$ according to central characters. So ϕ assigns an L-packet Π_ϕ to $\mathrm{SL}_2(K)$ with 4 elements, and a singleton packet to the inner form $\mathrm{SL}_1(D)$. None of these packets contains the Steinberg representation of $\mathrm{SL}_2(K)$ and so each Π_φ is a supercuspidal L-packet with 4 elements.

To be explicit: ϕ assigns to $\mathrm{SL}_2(K)$ the supercuspidal packet

$$\{\pi(\phi,\rho_1), \pi(\phi,\rho_2), \pi(\phi,\rho_3), \pi(\phi,\rho_4)\}$$

and to $\mathrm{SL}_1(D)$ the singleton packet

$$\{\pi(\phi,\rho_0)\}$$

and this phenomenon occurs countably many times.

Each supercuspidal packet of four elements is the *JL-transfer* of the singleton packet, in the following sense: the irreducible supercuspidal representation θ of $\mathrm{GL}_2(K)$ which yields the 4-packet upon restriction to $\mathrm{SL}_2(K)$ is the image in the JL-correspondence of the irreducible smooth representation ψ of $\mathrm{GL}_1(D)$ which yields two copies of $\pi(\phi,\rho_0)$ upon restriction to $\mathrm{SL}_1(D)$:

$$\theta = JL(\psi).$$

Each parameter $\phi : \mathbf{W}_K \to \mathrm{PGL}_2(\mathbb{C})$ lifts to a Galois representation

$$\phi : \mathbf{W}_K \to \mathrm{GL}_2(\mathbb{C}).$$

This representation is *triply imprimitive*, as in [We]. Let $\mathfrak{T}(\phi)$ be the group of characters χ of $\mathbf{W}_K$ such that $\chi \otimes \phi \simeq \phi$. Then $\mathfrak{T}(\phi)$ is non-cyclic of order 4.

6. Depth

Let L/K be a biquadratic extension. We fix an algebraic closure $\overline{K}$ of K such that $L \subset \overline{K}$. From the inclusion $L \subset \overline{K}$, there is a natural surjection

$$\pi_{L/K} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(L/K)$$

Let K^{ur} be the maximal unramified extension of K in $\overline{K}$ and let K^{ab} be the maximal abelian extension of K in $\overline{K}$. We have a commutative diagram, where the horizontal maps are the canonical maps and the vertical maps are the natural projections

$$\begin{array}{ccccccccc}
1 & \longrightarrow & I_{\overline{K}/K} & \xrightarrow{\iota_1} & Gal(\overline{K}/K) & \xrightarrow{p_1} & Gal(K^{ur}/K) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle id} & & \\
1 & \longrightarrow & I_{K^{ab}/K} & \xrightarrow{\iota_2} & Gal(K^{ab}/K) & \xrightarrow{p_2} & Gal(K^{ur}/K) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \beta} & & \\
1 & \longrightarrow & \mathfrak{I}_{L/K} & \xrightarrow{\iota_3} & Gal(L/K) & \xrightarrow{p_3} & Gal(L \cap K^{ur}/K) & \longrightarrow & 1
\end{array}$$

In the above notation, we have $\pi_{L/K} = \pi_2 \circ \pi_1$.

Let

$$...\mathfrak{I}^{(2)} \subset \mathfrak{I}^{(1)} \subset \mathfrak{I}^{(0)} \subset G = \mathrm{Gal}(L/K) \tag{19}$$

be the filtration of the relative inertia subgroup $\mathfrak{I}^{(0)} = \mathfrak{I}_{L/K}$ of $\mathrm{Gal}(L/K)$, $\mathfrak{I}^{(1)}$ is the wild inertia subgroup, and so on... Note that $\mathfrak{I}^{(r)}$ is the restriction of the filtration G^r of $G = Gal(L/K)$ to the subgroup $\mathfrak{I}_{L/K}$, i.e, $\mathfrak{I}^{(r)} = \iota_3(G^r)$.

Let

$$...I^{(2)} \subset I^{(1)} \subset I^{(0)} \subset G = \mathrm{Gal}(\overline{K}/K) \tag{20}$$

be the filtration of the absolute inertia subgroup $I^{(0)} = I_{\overline{K}/K}$ of $\mathrm{Gal}(K^s/K)$, $I^{(1)}$ is the wild inertia subgroup, and so on...

Lemma 6.1. *We have*

$$(\forall r)\ \pi_{L/K} I^{(r)} = \mathfrak{I}^{(r)}$$

Proof. This follows immediately from the above diagram. Here, we identify $I^{(r)}$ with $\iota_1(I^{(r)})$ and $\mathfrak{I}^{(r)}$ with $\iota_3(\mathfrak{I}^{(r)})$.

□

Lemma 6.2. *Let L/K be a biquadratic extension, let ϕ be the Langlands parameter (17), $\phi = \alpha \circ \pi_{L/K}$ with $\alpha : \mathrm{Gal}(L/K) \to \mathrm{SO}_3(\mathbb{R})$. Then we have $d(\phi) = r - 1$ where r is the least integer for which $\mathfrak{I}^{(r)} = 1$.*

Proof. The depth of a Langlands parameter ϕ is easy to define. For $r \in \mathbb{R} \geq 0$ let $\mathrm{Gal}(F_s/F)^r$ be the r-th ramification subgroup of the absolute Galois group of F. Then the depth of ϕ is the smallest number $d(\phi) \geq 0$ such that ϕ is trivial on $\mathrm{Gal}(F_s/F)^r$ for all $r > d(\phi)$.

Note that α is *injective*. Therefore

$$\phi(I^{(r)}) = 1 \iff (\alpha \circ \pi_{L/K})I^{(r)}) = 1 \iff \alpha(\mathfrak{I}^{(r)}) = 1 \iff \mathfrak{I}^{(r)} = 1.$$

□

For example, the parameter ϕ has depth zero if it is *tamely ramified*, i.e. the least integer r for which $\mathfrak{I}^{(r)} = 1$ is $r = 1$. The relative wild inertia group is 1, but the relative inertia group is not 1.

Case 1: There are two ramification breaks occurring at -1 and some odd integer $t > 0$:

$$\{1\} = ... = \mathfrak{I}^{(t+1)} \subset \mathfrak{I}^{(t)} = ...\mathfrak{I}^{(0)} = \mathfrak{I}_{L/K} \subset \mathrm{Gal}(L/K), \quad d(\varphi) = t$$

The allowed depths are $1, 3, 5, 7, \ldots$.

Case 2.1**:** One single ramification break occurs at some odd integer $t > 0$::

$$\{1\} = ... = \mathfrak{I}^{(t+1)} \subset \mathfrak{I}^{(t)} = ... = \mathfrak{I}^{(0)} = \mathfrak{I}_{L/K} = \mathrm{Gal}(L/K); \quad d(\varphi) = t$$

The allowed depths are $1, 3, 5, 7, \ldots$.

Case 2.2**:** There are two ramification breaks occurring at some odd integers $t_1 < t_2$

$$\{1\} = ... = \mathfrak{I}^{(t_2+1)} \subset \mathfrak{I}^{(t_2)} = ... = \mathfrak{I}^{(t_1+1)} \subset \mathfrak{I}^{(t_1)} = ... = \mathfrak{I}^{(0)} = \mathfrak{I}_{L/K} = \mathrm{Gal}(L/K); \quad d(\varphi) = t_2$$

The allowed depths are $3, 5, 7, 9, \ldots$.
(In the above, $\mathfrak{I}^{(0)} = \mathfrak{I}_{L/K}$)

Theorem 6.3. *Let L/K be a biquadratic extension, let ϕ be the Langlands parameter (17). For every $\pi \in \Pi_\phi(\mathrm{SL}_2(K))$ and $\pi \in \Pi_\phi(\mathrm{SL}_1(D))$ there is an equality of depths:*

$$d(\pi) = d(\phi).$$

The depth of each element in the L-packet Π_ϕ is given by the largest break in the ramification of the Galois group $\mathrm{Gal}(L/K)$. The allowed depths are $1, 3, 5, 7, \ldots$ except in Case 2.2, when the allowed depths are $3, 5, 7, \ldots$.

Proof. This follows from Lemma (6.2), the above computations, and Theorem 3.4 in [ABPS1]. □

This contrasts with the case of $\mathrm{SL}_2(\mathbb{Q}_p)$ with $p > 2$. Here there is a unique biquadratic extension L/K, and a unique tamely ramified parameter $\phi : \mathrm{Gal}(L/K) \to \mathrm{SO}_3(\mathbb{R})$ of depth zero.

6.1. **Quadratic extensions.** Let E/K be a quadratic extension. There are two kinds: the unramified one $E_0 = K(\alpha_0)$ and countably many totally (and wildly) ramified $E = K(\alpha)$.

Theorem 6.4. *For the unramified principal series L-packet $\{\pi_E^1, \pi_E^2\}$, we have*

$$d(\pi_E^1) = d(\pi_E^2) = -1.$$

For the ramified principal series L-packet $\{\pi_E^1, \pi_E^2\}$, we have

$$d(\pi_E^1) = d(\pi_E^2) = n$$

with $n = 1, 2, 3, 4, \ldots$.

Proof. Case 1: E_0/K unramified. Then, $f(E_0/K) = 2$. In this case, we have $G_0 = \{1\}$, and $G_0 = G^0 = \mathfrak{I}_{E_0/K}$. There is only one ramification break at $t = 0$ and the filtration of $G = \mathrm{Gal}(E_0/K)$ in the upper numbering is

$$\{1\} = G^0 \subset G^{-1} = G = \mathbb{Z}/2\mathbb{Z}.$$

The filtration on the relative inertia $\mathfrak{I}^{(t)}$ is

$$\{1\} = \mathfrak{I}_{L_0/K} \subset G = \mathbb{Z}/2\mathbb{Z}$$

with only one break at $t = 0$. Negative depth, as expected.

Case 2: E/K is totally ramified. Then, $e(E/K) = 2$, which is divisible by the residue degree, so the extension is wildly ramified. In this case, there is one break at some $t \geq 1$. This is because of wild ramification, since $G^1 = \{1\}$ if and only if the extension is tamely ramified. The filtration of G in the upper numbering is

$$\{1\} = G^{t+1} \subset G^t = ... = G^0 = G = \mathbb{Z}/2\mathbb{Z}$$

The filtration on the relative inertia $\mathfrak{I}^{(r)}$ is

$$\{1\} = \mathfrak{I}^{(t+1)} \subset \mathfrak{I}^{(t)} = ... = G = \mathbb{Z}/2\mathbb{Z}$$

with only one break at $t \geq 1$. □

7. A commutative triangle

In this section we confirm part of the geometric conjecture in [ABPS] for $\mathrm{SL}_2(\mathbb{F}_q((\varpi)))$. We begin by recalling the underlying ideas of the conjecture.

Let $\mathcal{G}$ be the group of K-points of a connected reductive group over a nonarchimedean local field K. The Bernstein decomposition provides us, inner alia, with the following data: a canonical disjoint union

$$\mathbf{Irr}(\mathcal{G}) = \bigsqcup \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$$

and, for each $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$, a finite-to-one surjective map

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \to T^{\mathfrak{s}}/W^{\mathfrak{s}}$$

The geometric conjecture in [ABPS] amounts to a refinement of these statements. The refinement comprises the assertion that we have a *bijection*

$$\mathbf{Irr}(\mathcal{G})^s \simeq (T^{\mathfrak{s}}//W^{\mathfrak{s}})_2$$

where $(T^{\mathfrak{s}}//W^{\mathfrak{s}})_2$ is the *extended quotient of the second kind* of the torus $T^{\mathfrak{s}}$ by the finite group $W^{\mathfrak{s}}$. This bijection is subject to certain constraints, itemised in [ABPS].

We proceed to define the extended quotient of the second kind. Let W be a finite group and let X be a complex affine algebraic variety. Suppose that W is acting on X as automorphisms of X. Define

$$\widetilde{X}_2 := \{(x, \tau) : \tau \in \mathbf{Irr}(W_x)\}.$$

Then W acts on $\widetilde{X}_2$:

$$\alpha(x, \tau) = (\alpha \cdot x, \alpha_* \tau).$$

Definition 7.1. *The extended quotient of the second kind is defined as*

$$(X//W)_2 := \widetilde{X}_2/W.$$

Thus the extended quotient of the second kind is the ordinary quotient for the action of W on $\widetilde{X}_2$.

We recall that (G, T) are the complex dual groups of $(\mathcal{G}, \mathcal{T})$, so that $G = \mathrm{PSL}_2(\mathbb{C})$. Let $\mathbf{W}_K$ denote the Weil group of K. If ϕ is an L-parameter

$$\mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \to G$$

then an *enhanced Langlands parameter* is a pair (ϕ, ρ) where ϕ is a parameter and $\rho \in \mathbf{Irr}(S_\phi)$.

Theorem 7.2. *Let $\mathcal{G} = \mathrm{SL}_2(K)$ with $K = \mathbb{F}_q((\varpi)))$. Let $\mathfrak{s} = [\mathcal{T}, \chi]_G$ be a point in the Bernstein spectrum for the principal series of $\mathcal{G}$. Let $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ be the corresponding Bernstein component in $\mathbf{Irr}(\mathcal{G})$. Then there is a commutative triangle of natural bijections*

$$\begin{array}{ccccc} & & (T^{\mathfrak{s}}//W^{\mathfrak{s}})_2 & & \\ & \swarrow & & \searrow & \\ \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} & & \longrightarrow & & \mathfrak{L}(G)^{\mathfrak{s}} \end{array}$$

where $\mathfrak{L}(G)^{\mathfrak{s}}$ denotes the equivalence classes of enhanced parameters attached to $\mathfrak{s}$.

Proof. We recall that $T^{\mathfrak{s}} = \{\psi\chi : \psi \in \Psi(\mathcal{T})\}$ where $\Psi(\mathcal{T})$ is the group of all unramified quasicharacters of $\mathcal{T}$. With $\lambda \in T^{\mathfrak{s}}$, we define the parameter $\phi(\lambda)$ as follows:

$$\phi(\lambda) : W_K \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C}), \quad (w\Phi_K^n, Y) \mapsto \begin{pmatrix} \lambda(\varpi)^n & 0 \\ 0 & 1 \end{pmatrix}_*$$

where A_* is the image in $\mathrm{PSL}_2(\mathbb{C})$ of $A \in \mathrm{SL}_2(\mathbb{C})$, $Y \in \mathrm{SL}_2(\mathbb{C})$, $w \in I_K$ the inertia group, and Φ_K is a geometric Frobenius. Define, as in §3,

$$\pi(\lambda) := \mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\lambda).$$

Case 1. $\lambda^2 \neq 1$. Send the pair $(\lambda, 1) \in T^{\mathfrak{s}}//W^{\mathfrak{s}}$ to $\pi(\lambda) \in \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ (via the left slanted arrow) and to $\phi(\lambda) \in \mathfrak{L}(G)^{\mathfrak{s}}$ (via the right slanted arrow).

Case 2. Let $\lambda^2 = 1, \lambda \neq 1$. Let $\phi = \phi(\lambda)$. To compute S_ϕ, let $1, w$ be representatives of the Weyl group $W = W(G)$. Then we have

$$C_G(\mathrm{im}\,\phi) = T \sqcup wT$$

So ϕ is a non-discrete parameter, and we have

$$S_\phi \simeq \mathbb{Z}/2\mathbb{Z}.$$

We have two enhanced parameters, namely $(\phi, 1)$ and (ϕ, ϵ) where ϵ is the non-trivial character of $\mathbb{Z}/2\mathbb{Z}$.

Since $\lambda^2 = 1$, there is a point of reducibility. We send

$$(\lambda, 1) \mapsto \pi(\lambda)^+, \ (\lambda, \epsilon) \mapsto \pi(\lambda)^-$$

via the left slanted arrow, and

$$(\lambda, 1) \mapsto (\phi(\lambda), 1), \ (\lambda, \epsilon) \mapsto (\phi(\lambda), \epsilon)$$

via the right slanted arrow. Note that this *includes* the case when λ is the unramified quadratic character of $K^\times$.

Case 3. Let $\lambda = 1$. The *principal parameter*

$$\phi_0 : \mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}(2, \mathbb{C}).$$

is a discrete parameter for which $S_{\phi_0} = 1$. In the local Langlands correspondence for $\mathcal{G}$, the enhanced parameter $(\phi_0, 1)$ corresponds to the Steinberg representation St of $\mathrm{SL}_2(K)$. Note also that, when $\phi = \phi(1)$, we have $S_\phi = 1$. We send

$$(1, 1) \mapsto \pi(1), \quad (1, \epsilon) \mapsto \mathrm{St}$$

via the left slanted arrow and

$$(1, 1) \mapsto (\phi(1), 1), \quad (1, \epsilon) \mapsto (\phi_0, 1)$$

via the right slanted arrow. This establishes that the geometric conjecture in [ABPS] is valid for $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$. □

Let L/K be a quadratic extension of K. Let λ be the quadratic character which is trivial on $N_{L/K}L^\times$. Then λ factors through $\mathrm{Gal}(L/K) \simeq K^\times/N_{L/K}L^\times \simeq \mathbb{Z}/2\mathbb{Z}$ and $\phi(\lambda)$ factors through $\mathrm{Gal}(L/K) \times \mathrm{SL}_2(\mathbb{C})$. The parameters $\phi(\lambda)$ serve as parameters for the L-packets in the principal series of $\mathrm{SL}_2(K)$.

It follows from §3 that, when $K = \mathbb{F}_q((\varpi))$, there are countably many L-packets in the principal series of $\mathrm{SL}_2(K)$.

7.1. **The tempered dual.** If we insist, in the definition of $T^{\mathfrak{s}}$, that the unramified character ψ shall be unitary, then we obtain a copy $\mathbb{T}^{\mathfrak{s}}$ of the circle $\mathbb{T}$. We then obtain a compact version of the commutative triangle, in which the tempered dual $\mathbf{Irr}^{\mathrm{temp}}(\mathcal{G})^{\mathfrak{s}}$ determined by $\mathfrak{s}$ occurs on the left, and the bounded enhanced parameters $\mathfrak{L}^b(G)^{\mathfrak{s}}$ determined by $\mathfrak{s}$ occur on the right. We now isolate the bijective map

$$(\mathbb{T}^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2 \to \mathbf{Irr}^{\mathrm{temp}}(\mathcal{G})^{\mathfrak{s}} \tag{21}$$

and restrict ourselves to the case where $\mathbb{T}^{\mathfrak{s}}$ contains two *ramified* quadratic characters. Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, $W := \mathbb{Z}/2\mathbb{Z}$. We then have $T^{\mathfrak{s}} = \mathbb{T}$, $W^{\mathfrak{s}} = W$ and the generator of W acts on $\mathbb{T}$ sending z to z^{-1}.

The left-hand-side and the right-hand-side of the map (21) each has its own natural topology, as we proceed to explain.

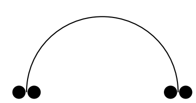
The topology on $(\mathbb{T}/\!/W)_2$ comes about as follows. Let

$$\mathbf{Prim}(C(\mathbb{T}) \rtimes W)$$

denote the primitive ideal space of the noncommutative C^*-algebra $C(\mathbb{T}) \rtimes W$. By the classical Mackey theory for semidirect products, we have a canonical bijection

$$\mathbf{Prim}(C(\mathbb{T}) \rtimes W) \simeq (\mathbb{T}/\!/W)_2. \tag{22}$$

The primitive ideal space on the left-hand side of (22) admits the Jacobson topology. So the right-hand side of (22) acquires, by transport of structure, a compact non-Hausdorff topology. The following picture is intended to portray this topology.



The reduced C^*-algebra of $\mathcal{G}$ is liminal, and its primitive ideal space is in canonical bijection with the tempered dual of $\mathcal{G}$. Transporting the Jacobson topology on the primitive ideal space, we obtain a locally compact topology on the tempered dual of $\mathcal{G}$, see [Dix, 3.1.1, 4.4.1, 18.3.2]. This makes $\mathbf{Irr}^{\mathrm{temp}}(\mathcal{G})^{\mathfrak{s}}$ into a compact space, in the induced topology.

We conjecture that these two topologies make (21) into a homeomorphism. This is a strengthening of the geometric conjecture [ABPS]. In that case, the double-points in the picture arise precisely when the corresponding (parabolically) induced representation has two irreducible constituents. This conjecture is true for $\mathrm{SL}_2(\mathbb{Q}_p)$ with $p > 2$, see [P, Lemma 1]. While in conjectural mode, we mention the following point: the standard Borel subgroup of $\mathrm{SL}_2(K)$ admits countably many ramified quadratic characters and so, following the construction in [ChP], the geometric conjecture predicts that tetrahedra of reducibility will occur countably many times; however, the

R-group machinery is not, to our knowledge, available in positive characteristic, so this remains conjectural.

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