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# Curve Crossing for the Reflected Process 

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# Curve Crossing for the Reflected Process* 

Ron Doney ${ }^{\dagger} \quad$ Ross Maller $\ddagger$


#### Abstract

Let $R_{n}=\max _{0 \leq j \leq n} S_{j}-S_{n}$ be a random walk $S_{n}$ reflected in its maximum. We give necessary and sufficient conditions for finiteness of passage times of $R_{n}$ above horizontal or certain curved (power law) boundaries. Necessary and sufficient conditions are also given for the finiteness of the expected passage time of $R_{n}$ above linear and square root boundaries.


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Keywords: Reflected process, rate of growth, renewal theorems, power law boundaries.

[^0]
## 1 Introduction and Preliminary Results

Let $X, X_{1}, X_{2}, \ldots$, be i.i.d. rvs with $\operatorname{cdf} F(\cdot)$ on $\mathbb{R}$, not degenerate at 0 , and

$$
S_{n}=X_{1}+X_{2}+\cdots+X_{n}, \quad S_{0}=0
$$

the corresponding random walk. Denote by

$$
R_{n}=\max _{0 \leq j \leq n} S_{j}-S_{n}, n=0,1,2, \cdots
$$

the random walk reflected in its maximum. Of course $R_{n} \geq 0, n=0,1,2, \ldots$. The reflected process is of fundamental importance in the theory of random walks and is also an object of interest, in itself, in many applied areas. In Section 2 we give necessary and sufficient conditions for the almost sure (a.s.) finiteness of passage times of $R_{n}$ out of power law regions of the form [ $0, r n^{\kappa}$ ], for $r>0, \kappa \geq 0$ (Theorem 2.1), and for the finiteness of expected values of passage times of $R_{n}$ out of linear $(\kappa=1)$ or parabolic $(\kappa=1 / 2)$ regions (Theorem 2.2). To complete the present section, we introduce some notation which will be useful throughout the paper, and state an introductory Proposition 1.1, which gives some basic preliminary properties of $R_{n}$. The section concludes with references to some recent interesting applications of the reflected process.

Let

$$
\begin{equation*}
S_{n}^{*}=\max _{0 \leq j \leq n} S_{j}, n=0,1,2, \cdots \tag{1.1}
\end{equation*}
$$

so that

$$
\begin{align*}
R_{n} & =S_{n}^{*}-S_{n}=\max _{0 \leq j \leq n}\left(S_{j}-S_{n}\right)=\max _{0 \leq j<n}\left(-\sum_{i=j+1}^{n} X_{i}\right) \vee 0 \\
& =\left(-\min _{0 \leq j<n} \sum_{i=j+1}^{n} X_{i}\right) \vee 0 \stackrel{D}{=}\left(-\min _{1 \leq j \leq n} S_{j}\right) \vee 0=-\min _{0 \leq j \leq n} S_{j} . \tag{1.2}
\end{align*}
$$

The equality in distribution can be seen by a time reversal of $\left(X_{1}, \ldots, X_{n}\right)$. The identity (1.2) (equality in distribution for each $n=1,2, \ldots$, but not of processes) is of course well known. Another useful representation is to write $R_{n}$ as the sum of its increments:

$$
\begin{equation*}
R_{n}=\sum_{i=1}^{n} \Delta_{i} \tag{1.3}
\end{equation*}
$$

where, as is easily checked,

$$
\begin{equation*}
\Delta_{i}=R_{i}-R_{i-1}=-X_{i} I\left(X_{i} \leq R_{i-1}\right)-R_{i-1} I\left(X_{i}>R_{i-1}\right), \quad i=1,2, \ldots \tag{1.4}
\end{equation*}
$$

When $E|X|<\infty$ we calculate, with $\mathcal{F}_{i}=\sigma\left(X_{1}, X_{2}, \ldots, X_{i}\right)$ and $\mathcal{F}_{0}$ as the trivial $\sigma$-field:

$$
\begin{equation*}
E\left(\Delta_{i} \mid \mathcal{F}_{i-1}\right)=-\int_{\left(-\infty, R_{i-1}\right]} y d F(y)-R_{i-1} \bar{F}\left(R_{i-1}\right)=-E X+\int_{R_{i-1}}^{\infty} \bar{F}(y) d y \tag{1.5}
\end{equation*}
$$

(where $\bar{F}(y)=1-F(y)$ ). If in addition $E X \leq 0$, we have $E\left(\Delta_{i} \mid \mathcal{F}_{i-1}\right) \geq$ 0 a.s., consequently $R_{n}$ is a submartingale when $E|X|<\infty$ and $E X \leq 0$.

The next proposition lists some of the basic properties (most already known, in some form) of $R_{n}$. If $F(0-)=0$ then $R_{n}=0$ while if $F(0)=1$ then $R_{n}=-S_{n}$, so we exclude these cases in the proposition. We will use "rv" to mean "random variable"; " $\xrightarrow{D}$ "for convergence in distribution; " $\xrightarrow{P}$ for convergence in probability; and " $\stackrel{D}{\text { " }}$ will denote equality in distribution.
Proposition 1.1. Suppose $0<F(0-) \leq F(0)<1$. Then
(a) There is no $x>0$ such that $\lim _{n \rightarrow \infty} P\left(R_{n} \leq x\right)=1$;
(b) $\limsup _{n \rightarrow \infty} R_{n}=+\infty$ a.s.;
(c) $R_{n} \xrightarrow{P} \infty(n \rightarrow \infty) \Longleftrightarrow \liminf _{n \rightarrow \infty} S_{n}=-\infty$ a.s.;
(d)

$$
\begin{aligned}
& P\left(R_{n}=0 \text { i.o. }\right)<1 \Longrightarrow \lim _{n \rightarrow \infty} S_{n}=-\infty \text { a.s. } \\
\Longrightarrow & \lim _{n \rightarrow \infty} R_{n}=+\infty \text { a.s. } \Longrightarrow P\left(R_{n}=0 \text { i.o. }\right)=0 ;
\end{aligned}
$$

(e) $\sum_{n \geq 1} P\left(R_{n} \leq x\right)<\infty$ for some (hence every) $x \geq 0$

$$
\Longleftrightarrow \lim _{n \rightarrow \infty} R_{n}=+\infty \text { a.s.; }
$$

$$
\begin{equation*}
R_{n} \text { is tight as } n \rightarrow \infty \Longrightarrow \lim _{n \rightarrow \infty} S_{n}=+\infty \text { a.s. } \tag{f}
\end{equation*}
$$

$$
\Longrightarrow R_{n} \xrightarrow{D} R \text { for some rv } R \text { with } P(0<R<\infty)=1
$$

Remarks. (i) Part (a) of Proposition 1.1 shows that $R_{n} \xrightarrow{P} 0(n \rightarrow \infty)$ cannot occur. Part (f) shows that $R_{n}$ is stochastically bounded if and only if $S_{n}$ drifts to $+\infty$ a.s. The heuristic explanation is that $\max _{0 \leq j \leq n} S_{j}$ is then close to $S_{n}$ for large $n$, and cancellation results in a finite $R_{n}$. This situation has been well studied in various applications (see, e.g., Bingham et al., 1987, p. 388, Takács 1978), and we will mainly be concerned with the other cases, when $S_{n}$ oscillates or drifts to $-\infty$ a.s., so that $R_{n}$ continues to grow with $n$ (Parts (b)-(e) of Proposition 1.1). Our aim is to estimate its rate of growth in various ways.
(ii) Analytic conditions for $\liminf _{n \rightarrow \infty} S_{n}=-\infty$ a.s. and $\lim _{n \rightarrow \infty} S_{n}=$ $\pm \infty$ a.s. are in Kesten and Maller (1996). See also Proposition 2.1 below.
(iii) We remark that with the obvious modifications all our results apply to the reflected process $r_{n}:=S_{n}-\min _{0 \leq j \leq n} S_{j}$. For a financial application of $r_{n}$, see Glynn and Iglehart (1995). Hansen (2004) has some interesting generalisations and an application to genetics of the maximal sequence $R_{n}^{*}:=$ $\max _{1 \leq j \leq n} R_{j}$. $R_{n}$ is used extensively in modelling; see, e.g., Doney and Maller (2005b), Iglehart (1972), Takács (1978). The first time the reflected process upcrosses a fixed level gives the optimal time to exercise a "Russian" option (Shepp and Shiryaev 1993, 1996, Asmussen et al. 2004). There are many other applications of $R_{n}, R_{n}^{*}$, and $r_{n}$, etc., in Finance and elsewhere.

## 2 Passage Times above Power Law Boundaries

We can measure the rate of growth of $R_{n}$ by seeing how quickly it leaves a region. For constants $\kappa>0, r>0$, or $\kappa=0, r \geq 0$, define

$$
\begin{equation*}
\tau_{\kappa}(r)=\min \left\{n \geq 1: R_{n}>r n^{\kappa}\right\} . \tag{2.1}
\end{equation*}
$$

(Here and throughout give the minimum of the empty set the value $+\infty$.) Simply write $\tau(r)$ for $\tau_{0}(r)$. Let $X^{+}=\max (X, 0)$ and $X^{-}=X^{+}-X$ (and similarly for $X_{i}^{+}$and $\left.X_{i}^{-}\right)$. Our main result is:

Theorem 2.1. (a) Suppose $\kappa=0$. We have $\tau_{0}(r)=\tau(r)<\infty$ a.s. for some, hence all, $r \geq 0$, if and only if $F(0-)>0$, and if this is so, then in fact $E\left(e^{\lambda \tau(r)}\right)<\infty$ at least for small enough $\lambda$, for all $r \geq 0$.
(b) Suppose $\kappa>0$. We have $\tau_{\kappa}(r)<\infty$ a.s. for all $r>0$ if and only if
(i) for $\kappa>1$ :

$$
\begin{equation*}
E\left(X^{-}\right)^{1 / \kappa}=\infty \tag{2.2}
\end{equation*}
$$

(ii) for $0<\kappa \leq 1$ :

$$
\begin{equation*}
E\left(X^{-}\right)^{1 / \kappa}=\infty \text { or } \liminf _{n \rightarrow \infty}\left(\frac{S_{n}}{n^{\kappa}}\right)=-\infty \text { a.s. } \tag{2.3}
\end{equation*}
$$

Remarks. (i) If $F(0-)=0$ (so that $X_{i} \geq 0$ a.s.) then $R_{n}=0$ for all $n=1,2, \cdots$, so $\tau_{\kappa}(r)=\infty$ a.s. for all $r>0$, and also Part (b) of Theorem 2.1 cannot occur. If $F(0)=1$ (so that $X_{i} \leq 0$ a.s.) then $R_{n}=-S_{n}$ for all $n=1,2, \cdots$, and $\lim \sup _{n \rightarrow \infty} R_{n} / n^{\kappa}=\infty$ a.s. iff $\lim \inf _{n \rightarrow \infty} S_{n} / n^{\kappa}=-\infty$ a.s. When $\kappa>1$ the latter is equivalent to $E\left(X^{-}\right)^{1 / \kappa}=\infty$ by Theorem 1 of Kesten and Maller (1998). Thus Part (b) (as well as Part (a)) of Theorem 2.1 remains true if $F(0-)=0$ or $F(0)=1$.
(ii) The results of Theorem 2.1 can also be expressed as conditions for $R_{n}=O\left(n^{\kappa}\right)$ a.s., as $n \rightarrow \infty$ (or sometimes for $R_{n}=o\left(n^{\kappa}\right)$ a.s., as can be seen from the proof of the theorem).
(iii) Explicit criteria in terms of the distribution function $F$ of the $X_{i}$ for $\lim \inf _{n \rightarrow \infty} S_{n} / n^{\kappa}=-\infty$ a.s. are known from Kesten and Maller (1998) and Doney and Maller (2005a). Actually, it's more convenient to deal with $\lim \sup _{n \rightarrow \infty} S_{n} / n^{\kappa}=+\infty$ a.s. and then perform a sign reversal. The conditions are listed in Proposition 2.1 below. (Parts (a) and (b) of the proposition are due to Chow and Zhang (1986) and Erickson (1973), respectively.) To state them, we need the integrals

$$
\begin{equation*}
A_{-}(x)=\int_{0}^{x} F(-y) d y(x>0) \text { and } J_{+}=\int_{[0, \infty)} \frac{x d F(x)}{A_{-}(x)} \tag{2.4}
\end{equation*}
$$

Note that $0 \leq A_{-}(x) \leq E X^{-}$. We only need $J_{+}$when $F(0-)>0$, in which case we let $A_{-}(x) / x$ have its limiting value, $F(0-)$, at 0 . We also need the function defined, for $y \geq 0$, when $E X^{-}<\infty$, as

$$
\begin{equation*}
W(y)=\int_{0}^{y} \int_{-\infty}^{-z}|x| F(d x) d z=\int_{0}^{y} \int_{x}^{\infty} F(-z) d z d x+\int_{0}^{y} x F(-x) d x \tag{2.5}
\end{equation*}
$$

Note that $W(y)>0$ for all $y>0$ if $F$ is not concentrated on $[0, \infty)$, thus, certainly if $E\left(X^{-}\right)^{\frac{1}{\kappa}}=\infty$ for some $\kappa>0$. When $W(y)>0$ for all $y>0$ and
$1 / 2<\kappa<1$, define, for $\lambda>0$,

$$
\begin{equation*}
I_{\kappa}(\lambda):=\int_{1}^{\infty} \exp \left\{-\lambda\left(\frac{y^{\frac{2 \kappa-1}{\kappa}}}{W(y)}\right)^{\frac{\kappa}{1-\kappa}}\right\} \frac{d y}{y} \leq \infty \tag{2.6}
\end{equation*}
$$

Proposition 2.1. Assume $0<F(0-) \leq F(0)<1$.
Then $\lim \sup _{n \rightarrow \infty} S_{n} / n^{\kappa}=\infty$ a.s. if and only if:
(a) for $\kappa>1$ :

$$
\begin{equation*}
\int_{[1, \infty)}\left(\frac{x^{\frac{1}{\kappa}}}{1+x^{\frac{1}{\kappa}-1} \int_{0}^{x} F(-z) d z}\right) F(d x)=\infty ; \tag{2.7}
\end{equation*}
$$

(b) for $\kappa=1$ or $\frac{1}{2}<\kappa<1$ and $E|X|=\infty$ :

$$
\begin{equation*}
J_{+}=\infty ; \tag{2.8}
\end{equation*}
$$

(c) when $0 \leq \kappa \leq \frac{1}{2}$ :

$$
\begin{equation*}
J_{+}=\infty \text { or } 0 \leq E X \leq E|X|<\infty ; \tag{2.9}
\end{equation*}
$$

(d) when $\frac{1}{2}<\kappa<1, E|X|<\infty$, and $E X \neq 0$ :

$$
\begin{equation*}
E X>0 \tag{2.10}
\end{equation*}
$$

(e) when $\frac{1}{2}<\kappa<1, E|X|<\infty$, and $E X=0$ :
(i) $E\left(X^{+}\right)^{\frac{1}{\kappa}}=\infty$, or
(ii) $E\left(X^{+}\right)^{\frac{1}{\kappa}}<\infty=E\left(X^{-}\right)^{\frac{1}{\kappa}}$ and $I_{\kappa}(\lambda)=\infty$ for all $\lambda>0$.

Remarks. (i) If $F(0-)=0$, then $\lim \sup S_{n} / n^{\kappa}=\limsup \left|S_{n}\right| / n^{\kappa}$ a.s. and Theorem 1 of Kesten and Maller (1998) gives the required criterion. If $F(0)=1$, then $\lim \sup S_{n} / n^{\kappa}=\infty$ a.s cannot occur. Thus the assumption $0<F(0-) \leq F(0)<1$ in Proposition 2.1 is not restrictive.
(ii) In general, neither of the two conditions in (2.3) imply each other, as can be seen from a perusal of Proposition 2.1.

Our final result considers the expected value of the passage time of $R_{n}$ above linear and square root boundaries. These are important practical cases. The random walk precursor of Theorem 2.2 (a) is in Gundy and Siegmund (1967).

Theorem 2.2. (a) Suppose $\sigma^{2}=E X^{2}<\infty$ and $E X=0$. Then
(i) $E \tau_{1 / 2}(\sigma r)=\infty$ for $r \geq 1$;
(ii) $E \tau_{1 / 2}(\sigma r)<\infty$ for $r<1$.
(b) Suppose $E|X|<\infty$ and $E X<0$. Then
(i) $E \tau_{1}(r)=\infty$ for $r \geq-E X$;
(ii) $E \tau_{1}(r)<\infty$ for $r<-E X$.

## 3 Lévy Processes

As might be expected, there is a counterpart of Theorem 2.1 relating to the large time behaviour of a Lévy process, and also for the results of Proposition 1.1, with appropriate interpretations. With $\left\{X_{t}\right\}_{t \geq 0}$ denoting such a process (there should be no confusion with the $X$ 's denoting the increments of the random walk in the rest of the paper), let $R_{t}:=\sup _{0 \leq s \leq t} X_{s}-X_{t}, t \geq 0$, denote the process reflected in its maximum, and let $\tau_{\kappa}(r)$ be the first passage time of $R_{t}$ above the curve $r(t+1)^{\kappa}$ for parameter $\kappa \geq 0$. Then we have:
(a) $\tau_{0}(r)<\infty$ for some (hence all) $r \geq 0$ if and only if $X$ is not a subordinator; (b) $\tau_{\kappa}(r)<\infty$ for all $r>0$ if and only if
(1) for $\kappa>1$ : $\quad E\left(X_{1}^{-}\right)^{1 / \kappa}=\infty$;
(2) for $0<\kappa \leq 1: \quad E\left(X_{1}^{-}\right)^{1 / \kappa}=\infty$ or $\liminf _{t \rightarrow \infty} X_{t} / t^{\kappa}=-\infty$ a.s.

We omit the proofs of these, which are not always trivial but follow methodology now well understood from, e.g., Doney (2004), Doney and Maller (2004). In this respect the following representation can be deduced from Doney (2004). Let $\left\{\tau_{n}\right\}_{n=1,2, \ldots}$ be the times at which a jump of $X_{t}$ of size $J_{n}$ with magnitude exceeding 1 occurs. Then

$$
\begin{equation*}
R_{\tau_{n}-}=\sup _{0 \leq s<\tau_{n}} X_{s}-X_{\tau_{n}-}=\left(R_{n}^{(S)}+\tilde{i}_{n}\right)^{+} \tag{3.1}
\end{equation*}
$$

where $R_{n}^{(S)}$ is the discrete time process $\tilde{S}_{n}:=\sup _{\tau_{n-1} \leq s \leq \tau_{n}} X_{s}$, reflected in its maximum; that is, $R_{n}^{(S)}=\max _{1 \leq j \leq n} \tilde{S}_{j}-\tilde{S}_{n}$; and $\tilde{i}_{n}$ is independent of $R_{n}^{(S)}$ and distributed as $\inf _{0 \leq s \leq \tau_{1}} X_{s}$. Furthermore, except for its first increment, $\tilde{S}_{n}$ is a random walk, that is, $\tilde{S}_{n}-\tilde{S}_{n-1}$ are i.i.d. rvs, $i=2,3, \cdots$.

The Lévy process version of Proposition 2.1 is in Doney and Maller (2005a).

Analogous to Parts (d) and (e) of Proposition 1.1, we can show the following:
(d') there is a (random) $t_{0}>0$ such that, for all $t \geq t_{0}, R_{t}>0$ a.s. $\Longleftrightarrow \lim _{t \rightarrow \infty} X_{t}=-\infty$ a.s. $\Longleftrightarrow \lim _{t \rightarrow \infty} R_{t}=+\infty$ a.s.
(e') we have that $\int_{1}^{\infty} P\left(R_{t} \leq x\right) d t<\infty$ for some (hence every) $x \geq 0$ $\Longleftrightarrow \lim _{t \rightarrow \infty} R_{t}=+\infty$ a.s.

There are also results analogous to Theorem 2.2 which we hope to discuss elsewhere.

## 4 Proofs

Proof of Proposition 1.1. Assume $0<F(0-) \leq F(0)<1$.
(a) Suppose there is an $x_{0}>0$ such that $\lim _{n \rightarrow \infty} P\left(R_{n} \leq x_{0}\right)=1$. Since $F(0-)>0$ there are $\varepsilon>0, \delta>0$, such that $P(X \leq-\varepsilon)>\delta$. Choose $K>1$ so that $K \varepsilon>x_{0}$. Suppose $X_{i} \leq-\varepsilon, n+1 \leq i \leq n+K$. Then for $n=1,2, \ldots$,

$$
S_{n+K}=S_{n}+\sum_{i=n+1}^{n+K} X_{i} \leq S_{n}-K \varepsilon<S_{n}-x_{0} \leq S_{n+K}^{*}-x_{0},
$$

so $R_{n+K}>x_{0}$. Thus for $n=1,2, \ldots$,

$$
P\left(R_{n+K}>x_{0}\right) \geq P\left(X_{i} \leq \varepsilon, n+1 \leq i \leq n+K\right) \geq \delta^{K}>0
$$

so $\lim \inf _{n \rightarrow \infty} P\left(R_{n}>x_{0}\right)>0$, giving a contradiction.
(b) By (a) we have $\limsup _{n \rightarrow \infty} P\left(R_{n}>x\right)>0$ for all $x>0$ and this implies the required result by the Hewitt-Savage 0-1 law.
(c) We have

$$
\begin{align*}
R_{n} \xrightarrow{P} \infty & \Longleftrightarrow \lim _{n \rightarrow \infty} P\left(R_{n} \leq x\right)=0 \forall x>0 \\
& \Longleftrightarrow \lim _{n \rightarrow \infty} P\left(\min _{0 \leq j \leq n} S_{j} \geq-x\right)=0  \tag{1.2}\\
& \Longleftrightarrow \min _{0 \leq j \leq n} S_{j} \xrightarrow{P}-\infty \\
& \Longleftrightarrow \min _{0 \leq j \leq n} S_{j} \rightarrow-\infty \text { a.s(since the sequence is monotone) } \\
& \Longleftrightarrow \liminf _{n \rightarrow \infty} S_{n}=-\infty \text { a.s. }
\end{align*}
$$

(d) Let $P\left(R_{n}=0\right.$ i.o. $)<1$ and suppose $S_{n}$ does not drift to $-\infty$ a.s. Then $\lim \sup _{n \rightarrow \infty} S_{n}=+\infty$ a.s. and there are infinitely many ascending ladder times a.s., i.e., $S_{n}>S_{n-1}^{*}$ i.o. a.s. Thus $S_{n}^{*}=S_{n}$ i.o. a.s., i.e., $R_{n}=$ 0 i.o. a.s., a contradiction. Hence $\lim _{n \rightarrow \infty} S_{n}=-\infty$ a.s. Conversely, suppose $\lim _{n \rightarrow \infty} S_{n}=-\infty$ a.s., and suppose $\lim _{\inf }^{n \rightarrow \infty}{ }^{2} R_{n}<\infty$ a.s., so there is an $a>0$ such that $R_{n} \leq a$ i.o. a.s. Then $S_{n}^{*}-S_{n} \leq a$ i.o. a.s., so $S_{n}>S_{n}^{*}-a$ i.o. a.s. But this is impossible when $S_{n}$ drifts to $-\infty$ a.s.
(e) Suppose $\sum_{n} P\left(R_{n} \leq x\right)<\infty$ for some $x \geq 0$. Then we have $\sum_{n} P\left(R_{n}=0\right)<\infty$, so by the Borel-Cantelli lemma, $P\left(R_{n}=0\right.$ i.o. $)=0$. Hence $\lim _{n \rightarrow \infty} R_{n}=+\infty$ a.s. and $\lim _{n \rightarrow \infty} S_{n}=-\infty$ a.s., by Part (d). Conversely, suppose $\lim _{n \rightarrow \infty} S_{n}=-\infty$ a.s. Then by Theorem 2.1 of Kesten and Maller (1996) (interchanging + and - in their result, that is, applying their result to the random walk $\tilde{S}_{n}=\sum_{i=1}^{n}\left(-X_{i}\right)$ ), for every $x \geq 0$

$$
\infty>\sum_{n \geq 1} P\left(\min _{0 \leq j \leq n} S_{j} \geq-x\right)=\sum_{n \geq 1} P\left(R_{n} \leq x\right)
$$

(f) Note that

$$
\begin{aligned}
R_{n} \text { tight } & \Longleftrightarrow \lim _{x \rightarrow \infty} \limsup _{n \rightarrow \infty} P\left(R_{n}>x\right)=0 \\
& \Longleftrightarrow \lim _{x \rightarrow \infty} \limsup _{n \rightarrow \infty} P\left(\min _{0 \leq j \leq n} S_{j}<-x\right)=0(\text { by }(1.2)) \\
& \Longleftrightarrow \lim _{x \rightarrow \infty} P\left(\min _{j \geq 0} S_{j}<-x\right)=0 \\
& \Longleftrightarrow \min _{j \geq 0} S_{j} \text { is a finite rv }(>-\infty \text { a.s. }) \\
& \Longleftrightarrow \lim _{n \rightarrow \infty} S_{n}=+\infty \text { a.s. }
\end{aligned}
$$

If these are true then clearly $R_{n} \xrightarrow{D} R:=-\min _{j \geq 0} S_{j}$, and $P(R=0)=0$ since $X_{1}$ is not degenerate at 0 .

Proof of Theorem 2.1. (a) Take an $r \geq 0$. If $F(0-)=0$ then $R_{n}=0$ a.s. and $\tau(r)=\tau_{0}(r)=\infty$ a.s. Conversely, suppose $F(0-)>0$. Clearly, the reflected process crosses the barrier $r \geq 0$ before $-S$ does, and this latter time has finite mean, because it is the time of the first visit to the positive half-line of a random walk with positive drift. Thus, indeed, $\tau(r)<\infty$ a.s.

To see that $E\left(e^{\lambda \tau(r)}\right)<\infty$ at least for small enough $\lambda$, write, for $r \geq$ $0, n=1,2, \ldots$,

$$
\begin{equation*}
\{\tau(r)>n\}=\left\{\max _{1 \leq j \leq n} \max _{0 \leq k \leq j-1}\left(-\sum_{i=k+1}^{j} X_{i}\right) \vee 0 \leq r\right\} . \tag{4.1}
\end{equation*}
$$

Choose $\varepsilon>0, \delta \in(0,1), K \geq 1$, so that $F(-\varepsilon-) \geq \delta$ and $K \varepsilon>r$. Then $F((-r / K)-) \geq F(-\varepsilon-) \geq \delta$, so

$$
\begin{equation*}
P\left(S_{K}<-r\right) \geq P^{K}\left(X_{1}<-r / K\right)=F^{K}((-r / K)-) \geq \delta^{K}>0 . \tag{4.2}
\end{equation*}
$$

Consider the event $\{\tau(r)>K n\}$, for $n=1,2, \ldots$, and note that the event on the right hand side of (4.1), with $n$ replaced by $K n$, includes the event

$$
\begin{equation*}
\left\{-\sum_{i=(\ell-1) K+1}^{\ell K} X_{i} \leq r, 1 \leq \ell \leq n\right\} \tag{4.3}
\end{equation*}
$$

(Just take the terms corresponding to $j=\ell K, k=(\ell-1) K, 1 \leq \ell \leq n$, from the maxima in (4.1), with $n$ replaced by $K n$ ). The sums in (4.3) are i.i.d., each with the distribution of $S_{K}$, so by (4.2)

$$
P\{\tau(r)>K n\} \leq P^{n}\left(-S_{K} \leq r\right) \leq\left(1-\delta^{K}\right)^{n}
$$

Thus $E\left(e^{\lambda \tau(r)}\right)<\infty$ if $\lambda<-\log \left(1-\delta^{K}\right)$.
(b) We first prove the forward direction for both parts.
(b) (i) Keep $\kappa>1$, and suppose $\tau_{\kappa}(r)<\infty$ a.s. for all $r>0$. If $E\left(X^{-}\right)^{1 / \kappa}<\infty$, the Marcinkiewicz-Zygmund law (e.g., Chow and Teicher 1988, p. 125) gives

$$
\lim _{n \rightarrow \infty}\left(\frac{\sum_{i=1}^{n} X_{i}^{-}}{n^{\kappa}}\right)=0 \text { a.s. }
$$

But if this is so then

$$
\begin{aligned}
& R_{n}=\max _{0 \leq j \leq n} S_{j}-S_{n}=\max _{0 \leq j<n}\left(-\sum_{i=j+1}^{n} X_{i}\right) \vee 0 \\
& \leq \max _{0 \leq j<n}\left(\sum_{i=j+1}^{n} X_{i}^{-}\right)=\sum_{i=1}^{n} X_{i}^{-}=o\left(n^{\kappa}\right) \text { a.s. }
\end{aligned}
$$

giving a contradiction. Thus the forward direction of Part (b) (i) is proved.
(b) (ii) Keep $0 \leq \kappa \leq 1$. Let $T_{n}$ be the strict increasing ladder times of $S_{n}$, i.e., $T_{0}=0$ and

$$
\begin{equation*}
T_{n}=\min \left\{j \geq 1: S_{T_{n-1}+j}>S_{T_{n-1}}\right\}, \quad n=1,2, \ldots \tag{4.4}
\end{equation*}
$$

If $T_{n-1}<\infty$, define the depth of an excursion of $S_{n}$ below the maximum as

$$
\begin{equation*}
D_{n}=\max _{T_{n-1} \leq j<T_{n}}\left(-\sum_{i=T_{n-1}+1}^{j} X_{i}\right), \quad n=1,2, \ldots \tag{4.5}
\end{equation*}
$$

(In (4.5), and throughout, we make the convention that $\sum_{i=a}^{b}=0$ when $b<a$.) The rv $D_{n}$ measures the height of an excursion of $R_{n}$ away from 0 ; we have $R_{T_{n}}=0, n=1,2$, and

$$
\begin{equation*}
\max _{T_{n-1}^{\leq j<T_{n}}} R_{j}=D_{n}, \quad n=1,2, \ldots \tag{4.6}
\end{equation*}
$$

(If two ladder times $T_{n-1}, T_{n}$ occur at consecutive integers, so that $R_{T_{n-1}}=$ $R_{T_{n}}=0$, (4.5) gives $D_{n}=0$, agreeing with (4.6), and formally registering that the depth of the nonexistent excursion is 0 .)

Lemma 4.1. Keep $0<\kappa \leq 1$ and suppose $\lim _{n \rightarrow \infty} S_{n}=+\infty$ a.s. Then $E\left(X^{-}\right)^{1 / \kappa}<\infty$ if and only if $E\left(D_{1}^{1 / \kappa}\right)<\infty$.

Proof. Assume $\lim _{n \rightarrow \infty} S_{n}=+\infty$ a.s. Then $T_{n}<\infty$ a.s. for all $n$ and in fact $E T_{1}<\infty$. Thus the $D_{n}$ are well defined. Since $S_{j} \leq 0,0 \leq j<T_{1}$, we have

$$
D_{1}=\max _{0 \leq j<T_{1}}\left(-S_{j}\right) \geq-S_{1}=S_{1}^{-}=X_{1}^{-},
$$

and one direction of the lemma is obvious. Conversely,

$$
D_{1} \leq \max _{1 \leq j<T_{1}}\left(\sum_{i=1}^{j} X_{i}^{-}\right)=\sum_{i=1}^{T_{1}-1} X_{i}^{-}=F_{1}, \text { say. }
$$

Now for $0<\kappa \leq 1, E\left(X^{-}\right)^{1 / \kappa}<\infty$ and $\lim _{n \rightarrow \infty} S_{n}=+\infty$ a.s. imply $E T_{1}^{1 / \kappa}<\infty$ (Kesten and Maller, 1996, Theorem 2.1), so we can apply Theorem I.5.2 of Gut (1988, p. 22) to get $E F_{1}^{1 / \kappa}<\infty$ and hence $E D_{1}^{1 / \kappa}<\infty$.

Lemma 4.2. Keep $\kappa>0$. If $E D_{1}^{1 / \kappa}<\infty$ and $\lim _{n \rightarrow \infty} S_{n}=+\infty$ a.s. then $\lim _{n \rightarrow \infty} R_{n} / n^{\kappa}=0$ a.s., and so $P\left(\tau_{\kappa}(r)=\infty\right)>0$ for all large $r$.

Proof. Again with $T_{n}$ as the increasing ladder times of $S_{n}$,

$$
\begin{align*}
& \max _{j \geq T_{n}}\left(\frac{R_{j}}{j^{\kappa}}\right)=\max _{m>n} \max _{T_{m-1} \leq j<T_{m}}\left(\frac{R_{j}}{j^{\kappa}}\right) \\
& \leq \max _{m>n}\left(\frac{\max _{T_{m-1} \leq j<T_{m}} R_{j}}{T_{m-1}^{\kappa}}\right)=\max _{m>n}\left(\frac{D_{m}}{T_{m-1}^{\kappa}}\right) . \tag{4.7}
\end{align*}
$$

Since $\lim _{n \rightarrow \infty} S_{n}=+\infty$ a.s., we have $E T_{1}<\infty$, and thus $\lim _{m \rightarrow \infty} T_{m} / m=$ $E T_{1}$ a.s. The $D_{m}$ are i.i.d., and with $E D_{1}^{1 / \kappa}<\infty$, by hypothesis, so we have $\lim _{m \rightarrow \infty}\left(D_{m} / m^{\kappa}\right)=0$ a.s. Thus the righthand side of (4.7) tends to 0 a.s. as $n \rightarrow \infty$, giving $\lim _{n \rightarrow \infty} R_{n} / n^{\kappa}=0$ a.s.

We can now complete the proof of the forward direction of Part (b) (ii) of Theorem 2.1. We have $0<\kappa \leq 1$ and $\tau_{\kappa}(r)<\infty$ for all $r>0$, and must prove that (2.3) holds.

If $E\left(X^{-}\right)^{1 / \kappa}=\infty$ then (2.3) holds, so suppose $E\left(X^{-}\right)^{1 / \kappa}<\infty$. Then by Lemmas 4.1 and 4.2 we cannot have $\lim _{n \rightarrow \infty} S_{n}=+\infty$ a.s., consequently $\lim \inf _{n \rightarrow \infty} S_{n}=-\infty$ a.s. By (2.9) with $\kappa=0$ (and interchanging + and - ) this means

$$
\begin{equation*}
J_{-}:=\int_{[0, \infty)}\left(\frac{x}{A_{+}(x)}\right)|d F(-x)|=\infty \text { or } 0 \leq-E X \leq E|X|<\infty, \tag{4.8}
\end{equation*}
$$

where $A_{+}(x)=\int_{0}^{x}(1-F(y)) d y$ for $x>0$. Now suppose $0<\kappa \leq 1 / 2$. Then by (2.9) (interchanging $+/-$ ), we have

$$
\begin{equation*}
\liminf _{n \rightarrow \infty}\left(\frac{S_{n}}{n^{\kappa}}\right)=-\infty \text { a.s., } \tag{4.9}
\end{equation*}
$$

so (2.3) holds.
Next consider $1 / 2<\kappa \leq 1$. We still have (4.8). If $E|X|=\infty$ then $J_{-}=\infty$ by (4.8), and then (4.9) holds by (2.8) (interchanging + and - ). If $\kappa=1$ we can finish here because $E|X|<\infty$ cannot occur. If it did, we would have, a.s. as $n \rightarrow \infty$,

$$
\frac{R_{n}}{n}=\frac{S_{n}^{*}}{n}-\frac{S_{n}}{n} \rightarrow(E X)^{-}
$$

Thus if $E X=0$ then $P\left(\tau_{1}(r)=\infty\right)>0$ for all $r>0$ while if $E X<0$ then $P\left(\tau_{1}(r)=\infty\right)>0$ for all $r>|E X|$. Either is a contradiction.

Finally, consider $1 / 2<\kappa<1$ and $E|X|<\infty$. Then $E X \leq 0$ by (4.8). If $E X<0$ then $\lim _{n \rightarrow \infty} S_{n} / n=E X<0$ a.s., so (4.9) and hence (2.3) holds. It remains to consider the case $E X=0$.

The next lemma allows us to deal with this. To state it, recall the definitions of the functions $W(y)$ and the integral $I_{\kappa}(\lambda)$ in (2.5) and (2.6), respectively.

We also need the functions, for $x>0$,

$$
\nu_{+}(x)=\int_{[0, x]} y d F(y) \text { and } \nu_{-}(x)=-\int_{[-x, 0]} y d F(y)
$$

so that $\nu(x)=\nu_{+}(x)-\nu_{-}(x)=E(X I(|X| \leq x))$.
Lemma 4.3. Keep $1 / 2<\kappa<1$. Suppose that $E|X|<\infty, E X=0$, $E\left(X^{-}\right)^{1 / \kappa}<\infty=E\left(X^{+}\right)^{1 / \kappa}$, and, for some $\lambda>0, I_{\kappa}(\lambda)<\infty$. Then

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left(\frac{R_{n}}{n^{\kappa}}\right) \leq 9 \cdot 2^{\kappa}\left(6 \lambda 2^{\kappa}\right)^{1-\kappa} \text { a.s. } \tag{4.10}
\end{equation*}
$$

Proof. Fix $1 / 2<\kappa<1$, suppose $E|X|<\infty, E X=0, E\left(X^{-}\right)^{1 / \kappa}<\infty=$ $E\left(X^{+}\right)^{1 / \kappa}$, and, for some $\lambda>0, I_{\kappa}(\lambda)<\infty$. We can write

$$
\begin{equation*}
R_{n}=\max _{0 \leq j \leq n}\left(-\sum_{i=j+1}^{n} X_{i}\right)=\max _{0 \leq j \leq n}\left(\sum_{i=j+1}^{n} X_{i}^{-}-\sum_{i=j+1}^{n} X_{i}^{+}\right) . \tag{4.11}
\end{equation*}
$$

Let $D>0$ and note that

$$
\begin{aligned}
\sum_{i=j+1}^{n} X_{i}^{+} & \geq D \sum_{i=j+1}^{n} I\left(X_{i}^{+}>D\right)+\sum_{i=j+1}^{n} X_{i}^{+} I\left(X_{i}^{+} \leq D\right) \\
& =(n-j) D-D \sum_{i=j+1}^{n} I\left(X_{i}^{+} \leq D\right)+\sum_{i=j+1}^{n} X_{i}^{+} I\left(X_{i}^{+} \leq D\right)
\end{aligned}
$$

Then some algebra (recall that $E X^{-}=E X^{+}$) shows that

$$
\begin{align*}
\sum_{i=j+1}^{n} X_{i}^{-}-\sum_{i=j+1}^{n} X_{i}^{+} & \leq \sum_{i=j+1}^{n}\left(X_{i}^{-}-E X^{-}\right)-D \sum_{i=j+1}^{n}\left(I\left(X_{i}^{+} \leq D\right)-F(D)\right) \\
& -\sum_{i=j+1}^{n}\left(X_{i}^{+} I\left(X_{i}^{+} \leq D\right)-\nu_{+}(D)\right)+n \int_{D}^{\infty} \bar{F}(y) d y \tag{4.12}
\end{align*}
$$

We will choose $D$ as follows. Note that $W(x)>0$ for all $x>0$ (because $\left.E\left(X^{+}\right)^{1 / \kappa}=\infty\right)$ and $\lim _{x \rightarrow \infty}(W(x) / x)=0$. Given $\delta>0, x>x_{0}:=$ $\left(\delta / E X^{+}\right)^{1 /(1-\kappa)}$, define $D(x)=D(x, \delta)$ by

$$
D(x)=\inf \left\{y>0: \frac{W(y)}{y} \leq \frac{\delta}{x^{1-\kappa}}\right\} .
$$

Then $0<D(x)<\infty$ for $x>x_{0}, \lim _{x \rightarrow \infty} D(x)=\infty$, and $D(x)$ satisfies

$$
\begin{equation*}
\frac{x^{1-\kappa} W(D(x))}{D(x)}=\delta \tag{4.13}
\end{equation*}
$$

Now take $k \geq 1$ and $1 \leq n \leq 2^{k}$ and let

$$
\begin{align*}
A_{n} & =\sum_{i=1}^{n}\left(X_{i}^{-}-E X^{-}\right)  \tag{4.14}\\
B_{n k} & =D\left(2^{k}\right) \sum_{i=1}^{n}\left(I\left(X_{i}^{+} \leq D\left(2^{k}\right)\right)-F\left(D\left(2^{k}\right)\right)\right)  \tag{4.15}\\
C_{n k} & =\sum_{i=1}^{n}\left(X_{i}^{+} I\left(X_{i}^{+} \leq D\left(2^{k}\right)\right)-\nu_{+}\left(D\left(2^{k}\right)\right)\right) \tag{4.16}
\end{align*}
$$

Then from (4.11) and (4.12)

$$
\begin{gathered}
R_{n} \leq\left|A_{n}\right|+\max _{1 \leq j \leq n}\left|A_{j}\right|+D\left(2^{k}\right)\left(\left|B_{n k}\right|+\max _{1 \leq j \leq n}\left|B_{j k}\right|\right) \\
+\left|C_{n k}\right|+\max _{1 \leq j \leq n}\left|C_{j k}\right|+n \int_{D\left(2^{k}\right)}^{\infty} \bar{F}(y) d y
\end{gathered}
$$

so

$$
\begin{align*}
\max _{1 \leq n \leq 2^{k}} R_{n} \leq & 2 \max _{1 \leq n \leq 2^{k}}\left|A_{n}\right|+2 D\left(2^{k}\right) \max _{1 \leq n \leq 2^{k}}\left|B_{n k}\right| \\
& +2 \max _{1 \leq n \leq 2^{k}}\left|C_{n k}\right|+2^{k} \int_{D\left(2^{k}\right)}^{\infty} \bar{F}(y) d y \tag{4.17}
\end{align*}
$$

The last term on the righthand side of (4.17) is, by (4.13) and the definition of $W(x)$, not larger than $\delta 2^{\kappa k}$. We will show that the other terms on the righthand side of (4.17) are $o\left(2^{\kappa k}\right)$ a.s., as $k \rightarrow \infty$.

We need some properties of $D(x)$. Differentiation using the implicit function theorem gives

$$
\begin{equation*}
D^{\prime}(x)=\frac{(1-\kappa) \delta D^{2}(x)}{x^{2-\kappa} \int_{0}^{D(x)} y \bar{F}(y) d y}, x>x_{0} . \tag{4.18}
\end{equation*}
$$

Hence $D(\cdot)$ is strictly increasing and so has a unique increasing inverse $D^{\leftarrow}(x)$ satisfying, for large $x, x \geq x_{1}$, say,

$$
\begin{equation*}
D^{\leftarrow}(x)=\left(\frac{\delta x}{W(x)}\right)^{1 /(1-\kappa)} \tag{4.19}
\end{equation*}
$$

Our next step is to show that, under our assumption that $I_{\kappa}(\lambda)<\infty$, we have

$$
\begin{equation*}
\lim _{x \rightarrow \infty} x^{1-2 \kappa} W(D(x))=0 . \tag{4.20}
\end{equation*}
$$

To see this, write

$$
I_{\kappa}(\lambda)=\int_{1}^{\infty} e^{-\left(y^{q} / h(y)\right)} d y / y
$$

where $q=(2 \kappa-1) /(1-\kappa)>0$ and $h(x)=(W(x))^{\kappa /(1-\kappa)}$ is an increasing function. (In fact differentiation shows that $W(x)$ is increasing and concave.) Now $I_{\kappa}(\lambda)<\infty$ implies

$$
\sum_{n \geq 1}(\log 2) e^{-2^{(n+1) q} / h\left(2^{n}\right)} \leq \sum_{n \geq 1} \int_{2^{n}}^{2^{n+1}} e^{-y^{q} / h(y)} d y / y<\infty,
$$

thus $\lim _{n \rightarrow \infty} h\left(2^{n}\right) / 2^{(n+1) q}=0$ and so $\lim _{n \rightarrow \infty} h\left(2^{n}\right) / 2^{(n-1) q}=0$. Given $x>0$ choose $n(x)$ so that $2^{n-1} \leq x<2^{n}$. Then

$$
\frac{h(x)}{x^{q}} \leq \frac{h\left(2^{n}\right)}{2^{(n-1) q}} \rightarrow 0, \text { as } x \rightarrow \infty,
$$

thus

$$
\lim _{x \rightarrow \infty} \frac{(W(x))^{\kappa /(1-\kappa)}}{x^{(2 \kappa-1) /(1-\kappa)}}=0 .
$$

Substituting $x=\left(D^{\leftarrow}(x)\right)^{1-\kappa} W(x) / \delta$ from (4.19) gives

$$
\lim _{x \rightarrow \infty} \frac{\delta^{(2 \kappa-1) /(1-\kappa)} W(x)}{\left(D^{\leftarrow}(x)\right)^{2 \kappa-1}}=0
$$

or, equivalently, (4.20) holds, as required.
Now consider first the $C_{n k}$ term in (4.17). By (4.16), $C_{n k}$ is, for each $k$ and $n \leq 2^{k}$, the sum of $n$ i.i.d. mean 0 rvs with variance

$$
\begin{align*}
\operatorname{Var}\left(C_{n k}\right) & =n \operatorname{Var}\left(X_{i}^{+} I\left(X_{i}^{+} \leq D\left(2^{k}\right)\right)\right) \leq n U_{+}\left(D\left(2^{k}\right)\right) \\
& \leq 2^{k} U_{+}\left(D\left(2^{k}\right)\right) \leq 2^{k} W\left(D\left(2^{k}\right)\right)=o\left(2^{2 \kappa k}\right) \tag{4.21}
\end{align*}
$$

where the last estimate follows from (4.20). The inequality $\mid$ median $(Y) \mid \leq$ $\sqrt{2 \operatorname{Var} Y}$ is valid for any mean zero rv, so we have from (4.21)

$$
\max _{1 \leq n \leq 2^{k}} \mid \operatorname{median}\left(\sum_{i=n}^{2^{k}}\left(X_{i}^{+} I\left(X_{i}^{+} \leq D\left(2^{k}\right)\right)-\nu_{+}\left(D\left(2^{k}\right)\right)\right) \mid=o\left(2^{\kappa k}\right),\right.
$$

so by a version of Lévy's inequality (e.g., Chow and Teicher 1988, p. 71), for large enough $k$,

$$
\begin{equation*}
P\left(\max _{1 \leq n \leq 2^{k}}\left|C_{n k}\right|>2 \delta 2^{\kappa k}\right) \leq 2 P\left(\left|C_{2^{k} k}\right|>\delta 2^{\kappa k}\right) \tag{4.22}
\end{equation*}
$$

Also, the summands of $C_{n k}$ are bounded by $2 D\left(2^{k}\right)$. So by Bernstein's inequality (Chow and Teicher 1988, p. 111), the last probability is bounded by

$$
\begin{equation*}
2 \exp \left(\frac{-\delta^{2} 2^{2 \kappa k}}{2\left(2^{k} W\left(D\left(2^{k}\right)\right)+2 D\left(2^{k}\right) \delta 2^{\kappa k}\right)}\right)=2 \exp \left(\frac{-\delta 2^{\kappa k}}{6 D\left(2^{k}\right)}\right) \tag{4.23}
\end{equation*}
$$

where we used (4.13) to substitute for $W\left(D\left(2^{k}\right)\right)$. Adding over $k$ we find that

$$
\begin{aligned}
\sum_{k \geq 1} e^{-\delta 2^{\kappa k} / 6 D\left(2^{k}\right)} & \leq 2 \sum_{k \geq 1} \int_{2^{k}}^{2^{k+1}} e^{-\delta y^{k} / 6 \cdot 2^{\kappa} D(y)} d y / y \\
& =2 \int_{2}^{\infty} e^{-\lambda y^{\kappa} /\left(\delta^{\kappa /(1-\kappa)} D(y)\right)} d y / y
\end{aligned}
$$

where in the last we chose $\delta$ so that $\delta=6 \cdot 2^{\kappa} \lambda / \delta^{\kappa /(1-\kappa)}$, i.e., $\delta=\left(6 \lambda 2^{\kappa}\right)^{1-\kappa}$. Now change variable to get the last integral as

$$
\begin{equation*}
\int_{D(2)}^{\infty} e^{-\lambda\left(D^{\leftarrow}(z)\right)^{\kappa} /\left(\delta^{\kappa /(1-\kappa)} z\right)} \frac{d z}{D^{\prime}\left(D^{\leftarrow}(z)\right) D^{\leftarrow}(z)} \tag{4.24}
\end{equation*}
$$

In view of (4.19) the exponent here is

$$
\frac{-\lambda z^{\kappa /(1-\kappa)}}{z(W(z))^{\kappa /(1-\kappa)}}=\frac{-\lambda z^{(2 \kappa-1) /(1-\kappa)}}{(W(z))^{\kappa /(1-\kappa)}}
$$

as required in (2.6). Also, by (4.18) and (4.19),

$$
\begin{aligned}
D^{\prime}\left(D^{\leftarrow}(z)\right) D^{\leftarrow}(z) & =\frac{(1-\kappa) \delta z^{2}}{\left(D^{\leftarrow}(z)\right)^{1-\kappa} \int_{0}^{z} y \bar{F}(y) d y} \\
& =\frac{(1-\kappa) z W(z)}{\int_{0}^{z} y \bar{F}(y) d y} \\
& \geq(1-\kappa) z
\end{aligned}
$$

where the last follows because $W(z) \geq \int_{0}^{z} y \bar{F}(y) d y$; see (2.5). As a result of these two calculations the integral in (4.24) is bounded by a multiple of $I_{\kappa}(\lambda)$. Going back to (4.22) we thus have by the Borel-Cantelli lemma

$$
\begin{equation*}
\limsup _{k \rightarrow \infty}\left(\frac{\max _{1 \leq n \leq 2^{k}}\left|C_{n k}\right|}{2^{\kappa k}}\right) \leq 2 \delta=2\left(6 \lambda 2^{\kappa}\right)^{1-\kappa} \text { a.s. } \tag{4.25}
\end{equation*}
$$

Next we have to deal with the ( $B$ ) term in (4.17). For each $k \geq 1$ and $1 \leq n \leq 2^{k}, B_{n k} / D\left(2^{k}\right)$ is a sum of i.i.d. mean zero rvs bounded by 2 (see (4.15)), and we can calculate

$$
\begin{aligned}
\operatorname{Var}\left(B_{n k} / D\left(2^{k}\right)\right) & =\sum_{i=1}^{n} F\left(D\left(2^{k}\right)\right) \bar{F}\left(D\left(2^{k}\right)\right) \\
& \leq 2^{k} \bar{F}\left(D\left(2^{k}\right)\right) \leq 2^{k} U_{+}\left(D\left(2^{k}\right)\right) / D^{2}\left(2^{k}\right) \\
& \leq 2^{k} W\left(D\left(2^{k}\right)\right) / D^{2}\left(2^{k}\right)=\delta 2^{\kappa k} / D\left(2^{k}\right)
\end{aligned}
$$

using (4.13) for the last equality. Thus by a similar argument as for the (C) term, involving Lévy's and Bernstein's inequalities,

$$
\begin{aligned}
P\left(\max _{1 \leq n \leq 2^{k}}\left|B_{n k}\right|>2 \delta 2^{\kappa \delta}\right) & \leq 2 P\left(\left|B_{2^{k} k}\right| / D\left(2^{k}\right)>\delta 2^{\kappa k} / D\left(2^{k}\right)\right) \\
& \leq 2 \exp \left(\frac{-\delta^{2} 2^{2 \kappa k} / D^{2}\left(2^{k}\right)}{2\left(\delta 2^{\kappa k} / D\left(2^{k}\right)+2 \delta 2^{\kappa k} / D\left(2^{k}\right)\right)}\right) \\
& =2 \exp \left(\frac{-\delta 2^{\kappa k}}{6 D\left(2^{k}\right)}\right) .
\end{aligned}
$$

This is the same bound as in (4.23) and the same argument leading to (4.25) gives

$$
\begin{equation*}
\limsup _{k \rightarrow \infty}\left(\frac{\max _{1 \leq n \leq 2^{k}}\left|B_{n k}\right|}{2^{\kappa k}}\right) \leq 2\left(6 \lambda 2^{\kappa}\right)^{1-\kappa} \text { a.s. } \tag{4.26}
\end{equation*}
$$

Finally, for the $(A)$ term in (4.17), we simply use the MarcinkiewiczZygmund law to get $A_{n}=o\left(n^{\kappa}\right)$ a.s., since $E\left(X^{-}\right)^{1 / \kappa}<\infty$. So

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left(\frac{\max _{1 \leq n \leq 2^{k}}\left|A_{n}\right|}{2^{\kappa k}}\right)=0 \text { a.s. } \tag{4.27}
\end{equation*}
$$

Putting (4.25)-(4.27) into (4.17) gives

$$
\limsup _{k \rightarrow \infty}\left(\frac{\max _{1 \leq n \leq 2^{k}} R_{n}}{2^{\kappa k}}\right) \leq 8\left(6 \lambda 2^{\kappa}\right)^{1-\kappa}+\delta=9\left(6 \lambda 2^{\kappa}\right)^{1-\kappa} \quad \text { a.s. }
$$

If $m$ is large choose $k(m)$ so that $2^{k-1} \leq m<2^{k}$. Then

$$
\frac{R_{m}}{m^{\kappa}} \leq 2^{\kappa} \frac{\max _{1 \leq n \leq 2^{k}} R_{n}}{2^{\kappa m}} \leq 2^{\kappa} 9\left(6 \lambda 2^{\kappa}\right)^{1-\kappa}+o(1) \text { a.s. }
$$

which proves (4.10).
Corollary 4.1 (Corollary to Lemma 4.3). Keep $1 / 2<\kappa<1, E|X|<\infty$, $E X=0$. Then $\limsup { }_{n \rightarrow \infty} R_{n} / n^{\kappa}=\infty$ a.s. if and only if $E\left(X^{-}\right)^{1 / \kappa}=\infty$ or $E\left(X^{-}\right)^{1 / \kappa}<\infty=E\left(X^{+}\right)^{1 / \kappa}$ and $I_{\kappa}(\lambda)=\infty \forall \lambda>0$.

Proof. Keep $1 / 2<\kappa<1, E|X|<\infty, E X=0$. Suppose $\limsup _{n}\left(R_{n} / n^{\kappa}\right)=$ $\infty$ a.s.. If $E\left(X^{-}\right)^{1 / \kappa}<\infty$ and $E\left(X^{+}\right)^{1 / \kappa}<\infty$, that is, $E|X|^{1 / \kappa}<\infty$, we get $\lim _{n \rightarrow \infty} R_{n} / n^{\kappa}=0$ a.s. from the Marcinkiewicz-Zygmund law. If $E\left(X^{-}\right)^{1 / \kappa}<\infty=E\left(X^{+}\right)^{1 / \kappa}$, we have $I_{\kappa}(\lambda)=\infty \forall \lambda>0$ by Lemma 4.3.

Conversely, suppose $E\left(X^{-}\right)^{1 / \kappa}=\infty$. Then by Theorem 2(f) of Kesten and Maller (1998), we have $\lim \sup _{n \rightarrow \infty}\left(-S_{n} / n^{\kappa}\right)=+\infty$ a.s., and since $R_{n}=\max _{0 \leq j \leq n} S_{j}-S_{n} \geq-S_{n}$ this gives $\limsup { }_{n \rightarrow \infty} R_{n} / n^{\kappa}=+\infty$ a.s. Alternatively, suppose $E\left(X^{-}\right)^{1 / \kappa}<\infty=E\left(X^{+}\right)^{1 / \kappa}$ and $I_{\kappa}(\lambda)=\infty \forall \lambda$. Then by Theorem 1 of Doney and Maller [7], $\lim _{\sup }^{n \rightarrow \infty}\left(-S_{n} / n^{\kappa}\right)=\infty$ a.s. so again $\lim \sup _{n \rightarrow \infty} R_{n} / n^{\kappa}=+\infty$ a.s.

Finally we complete the proof of the forward direction in (2.3) by noting that the conditions of the Corollary to Lemma 2.3 are equivalent to (4.9) by Corollary 2 of [7] (interchanging $+/-$ ).

For the converse part of Theorem 2.1(b), note first that, by its definition, for $r>0, \kappa>0, n=1,2, \ldots$,

$$
\begin{aligned}
\left\{\tau_{\kappa}(r)>n\right\} & =\left\{\max _{0 \leq k \leq j} S_{k}-S_{j} \leq r j^{\kappa}, 1 \leq j \leq n\right\} \\
& \subseteq\left\{-X_{j} \leq r j^{\kappa}, 1 \leq j \leq n\right\}
\end{aligned}
$$

the last following just by taking the term for $k=j-1$ from the maximum. So

$$
P\left(\tau_{\kappa}(r)>n\right) \leq \prod_{j=1}^{n} P\left(X_{1}>-r j^{\kappa}\right) \leq \exp \left(-\sum_{j=1}^{n} P\left(X_{1} \leq-r j^{\kappa}\right)\right)
$$

Thus if $\sum_{n \geq 1} P\left(X_{1} \leq-r j^{\kappa}\right)=\infty$, or, equivalently, $E\left(X^{-}\right)^{1 / \kappa}=\infty$, then $P\left(\tau_{\kappa}(r)<\infty\right)=1$.

Next,

$$
\limsup _{n \rightarrow \infty}\left(\frac{R_{n}}{n^{\kappa}}\right)=\limsup _{n \rightarrow \infty}\left(\frac{S_{n}^{*}-S_{n}}{n^{\kappa}}\right) \geq-\liminf _{n \rightarrow \infty}\left(\frac{S_{n}}{n^{\kappa}}\right),
$$

so the second condition in (2.3) also implies $P\left(\tau_{\kappa}(r)<\infty\right)=1$ for $r>0$.
This completes the proof of Theorem 2.1.
Proof of Theorem 2.2. (a) For the square root boundary, assume $E X^{2}<\infty$ and $E X=0$.
(i) Introduce the function

$$
\begin{aligned}
\phi(x) & =2\left\{\int_{x}^{\infty} y \bar{F}(y) d y-x \int_{x}^{\infty} \bar{F}(y) d y\right\} \\
& =2 \int_{0}^{\infty} y \bar{F}(y+x) d y
\end{aligned}
$$

and define

$$
Z_{n}=R_{n}^{2}-n \sigma^{2}+\sum_{1}^{n} \phi\left(R_{i-1}\right), Z_{0}=0
$$

Using (1.5), and similarly calculating

$$
\begin{aligned}
E\left(\Delta_{i}^{2} \mid \mathcal{F}_{i-1}\right) & =\int_{\left(-\infty, R_{i-1}\right]} y^{2} d F(y)+R_{i-1}^{2} \bar{F}\left(R_{i-1}\right) \\
& =\sigma^{2}-2 \int_{R_{i-1}}^{\infty} y \bar{F}(y) d y
\end{aligned}
$$

we also have

$$
Z_{n}=R_{n}^{2}-\sum_{1}^{n} E\left(\Delta_{i}^{2} \mid \mathcal{F}_{i-1}\right)-2 \sum_{1}^{n} R_{i-1} E\left(\Delta_{i} \mid \mathcal{F}_{i-1}\right)
$$

From this it is easy to check that $Z$ is a martingale. Now fix $r>0$ and $m>0$ and write $\tau$ for $\tau_{1 / 2}(\sigma r)$ and $\tau^{m}=m \wedge \tau$. This is a stopping time, so $E Z_{\tau^{m}}=0$, and thus we get

$$
\begin{equation*}
\sigma^{2} E \tau^{m}=E R_{\tau^{m}}^{2}+E \sum_{1}^{\tau^{m}} \phi\left(R_{i-1}\right) \geq E R_{\tau^{m}}^{2}+\phi(0) \tag{4.28}
\end{equation*}
$$

Suppose now that $E \tau<\infty$. By monotone convergence, then, $E \tau^{m} \rightarrow E \tau$ as $m \rightarrow \infty$, while

$$
\liminf _{m \rightarrow \infty} E R_{\tau^{m}}^{2} \geq E R_{\tau}^{2} \geq r^{2} E \tau
$$

by Fatou's lemma. Thus we can let $m \rightarrow \infty$ in (4.28) to get $\sigma^{2}\left(1-r^{2}\right) E \tau \geq$ $\phi(0)>0$. This is impossible if $r \geq 1$, so in this case we must have $E \tau=\infty$.
(ii) We now take $0<r<1$, assume $E \tau=\infty$, and establish a contradiction. Assume the truth of the following statement:
for any $\varepsilon>0$ there is an $m_{\varepsilon}$ such that

$$
\begin{equation*}
E \sum_{1}^{\tau^{m}} \phi\left(R_{i-1}\right) \leq \varepsilon E \tau^{m} \text { for all } m \geq m_{\varepsilon} \tag{4.29}
\end{equation*}
$$

Note that $R_{\tau^{m}}=R_{\tau^{m}-1}+\Delta_{\tau^{m}} \leq \sigma r \sqrt{\tau^{m}}+\Delta_{\tau^{m}}$, and choose $\varepsilon=\frac{\sigma^{2}}{2}\left(1-r^{2}\right)$. Then for any $m \geq m_{\varepsilon}$ we have, using $E Z_{\tau^{m}}=0$, and (4.29),

$$
\begin{aligned}
\sigma^{2} E \tau^{m} & =E R_{\tau^{m}}^{2}+E \sum_{1}^{\tau^{m}} \phi\left(R_{i-1}\right) \\
& \leq \sigma^{2} r^{2} E \tau^{m}+E \Delta_{\tau^{m}}^{2}+2 \sigma r E\left(\sqrt{\tau^{m}} \Delta_{\tau^{m}}\right)+\varepsilon E \tau^{m} \\
& \leq \frac{\sigma^{2}}{2}\left(1+r^{2}\right) E \tau^{m}+E \Delta_{\tau^{m}}^{2}+2 \sigma r \sqrt{E \tau^{m}} \sqrt{E \Delta_{\tau^{m}}^{2}} \\
& =\left(\sqrt{E \Delta_{\tau^{m}}^{2}}+\sigma r \sqrt{E \tau^{m}}\right)^{2}+\frac{\sigma^{2}}{2}\left(1-r^{2}\right) E \tau^{m} .
\end{aligned}
$$

From this we see that the ratio $E \Delta_{\tau^{m}}^{2} / E \tau^{m}$ is bounded below by the constant $\sigma^{2}\left\{\sqrt{\frac{1}{2}\left(1+r^{2}\right)}-r\right\}^{2}>0$ for $m \geq m_{\varepsilon}$. The contradiction will follow by showing that $E \Delta_{\tau^{m}}^{2} / E \tau^{m} \rightarrow 0$ as $m \rightarrow \infty$.

To see this: take any $\delta>0$. We can choose $M=M(\varepsilon, \delta)$ so large that, whenever $m \geq M$,

$$
\begin{equation*}
\max _{i \geq 1} E\left(\Delta_{i}^{2} I\left(\Delta_{i}^{2}>\varepsilon E \tau^{m}\right) \mid \mathcal{F}_{i-1}\right) \leq \delta, \text { a.s. } \tag{4.30}
\end{equation*}
$$

This is done as follows. Since $E X^{2}<\infty$, given $\delta>0$ we can choose $y_{0}(\delta)$ so large that

$$
\begin{equation*}
\int_{|z|>y} z^{2} d F(z)+\sup _{z>y} z^{2} \bar{F}(z) \leq \delta, \forall y \geq y_{0} . \tag{4.31}
\end{equation*}
$$

Since we assumed $E \tau=\infty$, we have $\lim _{m \rightarrow \infty} E \tau^{m}=\infty$. So we can choose $M(\varepsilon, \delta)$ so large that $\sqrt{\varepsilon E \tau^{m}} \geq y_{0}$ when $m \geq M$. Now for any $a>0$, using the representation (1.4),

$$
I\left(\Delta_{i}^{2}>a\right)=I\left(X_{i}^{2}>a\right) I\left(X_{i} \leq R_{i-1}\right)+I\left(R_{i-1}^{2}>a\right) I\left(X_{i}>R_{i-1}\right)
$$

so

$$
\begin{aligned}
& \Delta_{i} I\left(\Delta_{i}^{2}>a\right) \\
& \quad=-X_{i} I\left(X_{i}^{2}>a\right) I\left(X_{i} \leq R_{i-1}\right)-R_{i-1} I\left(R_{i-1}^{2}>a\right) I\left(X_{i}>R_{i-1}\right)
\end{aligned}
$$

hence

$$
\begin{align*}
& E\left(\Delta_{i}^{2} I\left(\Delta_{i}^{2}>a\right) \mid \mathcal{F}_{i-1}\right) \\
& =\int_{|y|>\sqrt{a}} y^{2} I\left(y \leq R_{i-1}\right) d F(y)+R_{i-1}^{2} I\left(R_{i-1}>\sqrt{a}\right) \bar{F}\left(R_{i-1}\right) \\
& \leq \int_{|y|>\sqrt{a}} y^{2} d F(y)+\sup _{y>\sqrt{a}} y^{2} \bar{F}(y) . \tag{4.32}
\end{align*}
$$

Substituting $a=\varepsilon E \tau^{m}$, we have $\sqrt{a} \geq y_{0}$ when $m \geq M$, so (4.32) gives (4.30) via (4.31), when $m \geq M$. This proves (4.30). From this we deduce

$$
\begin{aligned}
E \Delta_{\tau^{m}}^{2} \mathbf{1}_{\left\{\Delta_{\left.\tau^{m}>\delta \tau^{m}\right\}}^{2}\right.} & \leq E \sum_{k=1}^{\tau^{m}} \Delta_{k}^{2} \mathbf{1}_{\left\{\Delta_{k}^{2}>\delta k\right\}} \\
& \leq E \sum_{k=1}^{\tau^{m}} E\left\{\Delta_{k}^{2} \mathbf{1}_{\left\{\Delta_{k}^{2}>\delta k\right\}} \mid \mathcal{F}_{k-1}\right\} \\
& \leq M \sigma^{2}+\delta E \tau^{m} \leq 2 \delta E \tau^{m}
\end{aligned}
$$

for sufficiently large $m$. We also have $E \Delta_{\tau^{m}}^{2} \mathbf{1}_{\left\{\Delta_{\tau^{m}}^{2} \leq \delta \tau^{m}\right\}} \leq \delta E \tau^{m}$. So to complete the proof it suffices to prove (4.29).

Note first that $\phi(x) / 2 \leq \sigma_{+}^{2}:=E\left(X^{+}\right)^{2}$ for all $x \geq 0$, and $\phi(x) \downarrow 0$ as $x \rightarrow \infty$. So we can choose $K_{\varepsilon}<\infty$ with $\phi\left(K_{\varepsilon}\right) \leq \varepsilon / 3$, and have the bound

$$
\sum_{1}^{n} \phi\left(R_{i-1}\right) \leq \frac{1}{3} n \varepsilon+2 \sigma_{+}^{2} \sum_{1}^{n} \mathbf{1}\left(R_{i-1} \leq K_{\varepsilon}\right) .
$$

Define

$$
N^{(\varepsilon)}=\max \left(n: \sum_{1}^{n} \mathbf{1}\left(R_{i-1} \leq K_{\varepsilon}\right) \geq \frac{n \varepsilon}{6 \sigma_{+}^{2}}\right)
$$

then it suffices to show that $E N^{(\varepsilon)}<\infty$, since this gives

$$
E \sum_{1}^{\tau^{m}} \phi\left(R_{i-1}\right) \leq \frac{2 \varepsilon}{3} E \tau^{m}+E N^{(\varepsilon)} \leq \varepsilon E \tau^{m} \text { for all large enough } m
$$

To show that $E N^{(\varepsilon)}<\infty$, introduce the r.v.'s $A_{n}, B_{n}, n \geq 1$, given recursively by:

$$
\begin{gathered}
A_{1}=\min \left\{n \geq 1: R_{n}>K_{\varepsilon}\right\}, B_{1}=\min \left\{n \geq 1: R_{A_{1}+n} \leq K_{\varepsilon}\right\}, \\
C_{1}=A_{1}+B_{1},
\end{gathered}
$$

and, for $i=2,3, \cdots$,

$$
\begin{gathered}
A_{i}=\min \left\{n \geq 1: R_{C_{i-1}+n}>K_{\varepsilon}\right\}, B_{i}=\min \left\{n \geq 1: R_{C_{i-1}+A_{i}+n} \leq K_{\varepsilon}\right\}, \\
C_{i}=C_{i-1}+A_{i}+B_{i}
\end{gathered}
$$

In view of (b) and (d) of Proposition 1.1, the $A_{i}$ and $B_{i}$ are finite, a.s. Then, by construction

$$
\begin{equation*}
\sum_{1}^{n} \mathbf{1}\left(R_{i-1} \leq K_{\varepsilon}\right)=M_{n}:=\sum_{1}^{D_{n}} A_{i}+\left(n-C_{D_{n}}\right) \tag{4.33}
\end{equation*}
$$

where $D_{n}=\max \left\{k: C_{k} \leq n\right\}$. Now write $\tilde{\varepsilon}=\varepsilon \sigma^{2} /\left(4 \sigma_{+}^{2}\right)$, assume without loss of generality that $\tilde{\varepsilon}<1$, and note that the maximum values of $n^{-1} M_{n}$ occur when $n=C_{k}+A_{k+1}$ for some $k \geq 0$, i.e. when $C_{D_{n}}=n-A_{k+1}$, at which times $M_{n}$ has value $\sum_{i=1}^{k+1} A_{i}$. So

$$
\begin{aligned}
N^{(\varepsilon)} & =\max \left\{n: M_{n} \geq n \tilde{\varepsilon}\right\} \\
& \leq \max \left\{C_{k}+A_{k+1}: \sum_{i=1}^{k+1} A_{i} \geq \tilde{\varepsilon}\left(C_{k}+A_{k+1}\right)\right\} \\
& \leq \frac{1}{\tilde{\varepsilon}} \max \left\{\sum_{i=1}^{k+1} A_{i}: \sum_{i=1}^{k+1} A_{i} \geq \tilde{\varepsilon}\left(\sum_{i=1}^{k+1} A_{i}+\sum_{i=1}^{k} B_{i}\right)\right\} \\
& =\frac{1}{\tilde{\varepsilon}} \max \left\{\sum_{i=1}^{k+1} A_{i}: \sum_{i=1}^{k}(1-\tilde{\varepsilon}) A_{i}-\tilde{\varepsilon} B_{i} \geq(\tilde{\varepsilon}-1) A_{k+1}\right\} .
\end{aligned}
$$

Thus, writing $Y_{i}=\tilde{\varepsilon} B_{i}-(1-\tilde{\varepsilon}) A_{i}$ and $k^{*}=\max \left\{k: \sum_{1}^{k} Y_{i} \leq(1-\tilde{\varepsilon}) A_{k+1}\right\}$, we have for any $c>0$

$$
\begin{align*}
\tilde{\varepsilon} P\left\{N^{(\varepsilon)} \geq m \tilde{\varepsilon}\right\} & \leq P\left\{\sum_{1}^{k^{*}+1} A_{i} \geq m\right\} \\
& \leq P\left\{k^{*} \geq m c-1\right\}+P\left\{\sum_{1}^{m c} A_{i} \geq m\right\} \tag{4.34}
\end{align*}
$$

Now $\sum_{1}^{m c} A_{i} \leq \sum_{1}^{m c} \tilde{A}_{i}$, where $\tilde{A}_{1}, \tilde{A}_{2}, \cdots$ are i.i.d. with the distribution of the time that $R$, starting from $K_{\varepsilon}$, exits $\left[0,2 K_{\varepsilon}\right]$. Part (a) of Theorem 2.1 shows that $E e^{\lambda \tilde{A}_{1}}<\infty$ for some $\lambda>0$, so using a standard exponential bound and choosing $c<1 / E \tilde{A}_{1}$, we see that the second term in (4.34) is summable. On the other hand we have

$$
\begin{equation*}
B_{k} \geq \tilde{B}_{k}:=\min \left\{n: R_{C_{k}+A_{1}+n} \leq R_{C_{k}+A_{1}}\right\}=\min \left\{n: \hat{S}_{n} \leq 0\right\} \tag{4.35}
\end{equation*}
$$

where $\hat{S}_{n}=S_{C_{k}+A_{1}+n}-S_{C_{k}+A_{1}}, n \geq 0$, and the $\tilde{B}_{n}$ are an iid sequence with infinite mean (since $E X=0$ ). Thus $\tilde{Y}_{i}:=\tilde{\varepsilon} \tilde{B}_{i}-(1-\tilde{\varepsilon}) A_{i}$ are the i.i.d. steps of a random walk that drifts to $+\infty$ a.s. Then with $A(y):=E\left(\left(Y_{1} \wedge y\right) \vee(-y)\right)$,
write

$$
\begin{aligned}
\sum_{j \geq 1} P\left(k^{*} \geq j\right) & =\sum_{j \geq 1} P\left(\text { for some } k \geq j, \sum_{i=1}^{k} \tilde{Y}_{i} \leq(1-\tilde{\varepsilon}) A_{k+1}\right) \\
& \leq \sum_{j \geq 1} \sum_{k \geq j} \sum_{a \geq 1} P\left(\sum_{i=1}^{k} \tilde{Y}_{i} \leq(1-\tilde{\varepsilon}) a\right) P\left(A_{k+1}=a\right) \\
& =\sum_{a \geq 1} \sum_{k \geq 1} k P\left(\sum_{i=1}^{k} \tilde{Y}_{i} \leq(1-\tilde{\varepsilon}) a\right) P\left(A_{k+1}=a\right) \\
& \leq c_{1}+c_{2} \sum_{a \geq a_{0}}\left(\frac{1-\tilde{\varepsilon}) a}{A((1-\tilde{\varepsilon}) a)}\right)^{2} P\left(A_{k+1}=a\right) \\
& \leq c_{1}+c_{3} E A_{1}^{2}<\infty .
\end{aligned}
$$

Here the $c_{i}$ and $a_{0}$ are positive constants, $A(y)$ is bounded away from 0 for $a \geq a_{0}$, say (in fact $\lim _{y \rightarrow \infty} A(y)=\infty$ since $E \tilde{Y}_{1}=+\infty$ ), and the inequality in the fourth line follows from Theorem 2.2 of Kesten and Maller (1996). Thus $k^{*}$ has finite mean, and the result follows from (4.34).
(b) For the linear case, assume $E|X|<\infty$ and $E X<0$.
(i) We first show that $E \tau_{1}(r)<\infty$ for $r<-E X$. This follows easily from the strong law of large numbers for $S_{n}$ : we have $\lim _{n \rightarrow \infty} S_{n} / n=E X$ a.s. and $\lim _{n \rightarrow \infty} \max _{i \leq n} S_{i}:=S_{\infty}<\infty$ a.s., so $\lim _{n \rightarrow \infty} R_{n} / n=-E X$ a.s. Thus $P\left(R_{n}>n(|E X|-\varepsilon)\right.$ i.o. $)=1$ for every $\varepsilon>0$, giving $E \tau_{1}(|E X|-\varepsilon)<\infty$ for every $\varepsilon \in(0,|E X|)$.
(ii) It remains to show that $E \tau_{1}(r)=\infty$ for all $r \geq-E X$. This follows immediately from the next lemma, which proves a little more.

Lemma 4.4. Let $S$ be a random walk with steps $X$ having $E|X|<\infty$ and $E X=-1$, and for the corresponding reflected process $R_{n}=\max _{0 \leq i \leq n} S_{i}-S_{n}$, $R_{0}=0$, write

$$
\begin{equation*}
T_{a}=\min \left(n \geq 1: R_{n}>n+a\right) . \tag{4.36}
\end{equation*}
$$

Then $E T_{a}=\infty$ for $a \geq 0$.
Remark. Of course (4.36) for $a=0$ implies (4.36) but it doesn't seem possible to prove it without considering the case $a>0$.

Proof of Lemma 4.4. First we show the result holds for sufficiently large $a$. Note that $R_{n}=M_{n}+n-\tilde{S}_{n}$, where $\tilde{S}$ is a zero-mean random walk and $M_{n}=\max _{0 \leq i \leq n} S_{i} \leq M_{\infty}$, with $b:=E M_{\infty}<\infty$. So, assuming $E T_{a}<\infty$,

$$
\begin{aligned}
0 & =E \tilde{S}_{T_{a}}=E M_{T_{a}}+E T_{a}-E R_{T_{a}} \\
& \leq E M_{\infty}+E T_{a}-\left(a+E T_{a}\right)=b-a
\end{aligned}
$$

This is a contradiction when $a>b$, so $E T_{a}=\infty$ for $a>b$.
Next, observe that for $x \in[0, a], E\left\{T_{a} \mid R_{0}=x\right\} \geq E T_{a-x}$. So by considering the first step in $S$,

$$
\begin{aligned}
E T_{a} & =1+\int_{-(a+1)}^{0} P\{X \in d y\} E\left\{T_{a+1} \mid R_{0}=-y\right\}+P\{X \geq 0\} E T_{a+1} \\
& \geq \int_{-(a+1)}^{0} P\{X \in d y\} E T_{a+1+y}+P\{X \geq 0\} E T_{a+1}
\end{aligned}
$$

Now, excluding the degenerate case $X=-1$ a.s., there exists $\delta \in(0,1)$ with $P\{X \geq-1+\delta\}=c>0$, so since $E T_{a}$ is increasing in $a$,

$$
\begin{aligned}
E T_{a} & \geq \int_{-1+\delta}^{0} P\{X \in d y\} E T_{a+1+y}+P\{X \geq 0\} E T_{a+1} \\
& \geq P\{X \in[-1+\delta, 0)\} E T_{a+\delta}+P\{X \geq 0\} E T_{a+1} \\
& \geq c E T_{a+\delta}
\end{aligned}
$$

Thus if $E T_{0}$ were finite, $E T_{n \delta}$ would also be finite for $n=1,2, \cdots$. This proves the result.

With this, the proof of Theorem 2.2 is complete.

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