Unsolved problems in group theory. The Kourovka notebook. No. 18

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UNSOLVED PROBLEMS IN GROUP THEORY

THE KOUROVKA NOTEBOOK

No. 18

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Editors: V. D. Mazurov and E. I. Khukhro

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Preface

The idea of publishing a collection of unsolved problems in Group Theory was proposed by M. I. Kargapolov (1928–1976) at the Problem Day of the First All–Union (All–USSR) Symposium on Group Theory which took place in Kourovka, a small village near Sverdlovsk, on February, 16, 1965. This is why this collection acquired the name "Kourovka Notebook". Since then every 2–4 years a new issue has appeared containing new problems and incorporating the problems from the previous issues with brief comments on the solved problems.

For more than 40 years the "Kourovka Notebook" has served as a unique means of communication for researchers in Group Theory and nearby fields of mathematics. Maybe the most striking illustration of its success is the fact that more than 3/4 of the problems from the first issue have now been solved. Having acquired international popularity the "Kourovka Notebook" includes problems by more than 300 authors from all over the world. Starting from the 12th issue it is published simultaneously in Russian and in English.

This is the 18th issue of the "Kourovka Notebook". It contains 120 new problems. Comments have been added to those problems from the previous issues that have been recently solved. Some problems and comments from the previous issues had to be altered or corrected. The Editors thank all those who sent their remarks on the previous issues and helped in the preparation of the new issue. We thank A. N. Ryaskin for assistance in dealing with the electronic publication.

The section "Archive of Solved Problems" contains *all* solved problems that have already been commented on in one of the previous issues with a reference to a detailed publication containing a complete answer. However, those problems that are commented on with a complete reference for the first time in this issue remain in the main part of the "Kourovka Notebook", among the unsolved problems of the corresponding section. (An inquisitive reader may notice that some numbers of the problems appear neither in the main part nor in the Archive; these are the few problems that were removed at the request of the authors either as ill-conceived, or as no longer topical, for example, due to CFSG.)

The abbreviation CFSG stands for The Classification of the Finite Simple Groups, which means that every finite simple non-abelian group is isomorphic either to an alternating group, or to a group of Lie type over a finite field, or to one of the twenty-six sporadic groups (see D. Gorenstein, *Finite Simple Groups: The Introduction to Their Classification*, Plenum Press, New York, 1982). A note "mod CFSG" in a comment means that the solution uses the CFSG.

Wherever possible, references to papers published in Russian are given to their English translations. The numbering of the problems is the same as in the Russian original: sections correspond to issues, and within sections problems are ordered lexicographically by the Russian names of the authors (or Russian transliterations). The index of names may help the reader to find a particular problem.

V. D. Mazurov, E. I. Khukhro

Novosibirsk, January 2014

Problems from the 1st Issue (1965)

1.3. (Well-known problem). Can the group ring of a torsion-free group contain zero divisors?

L. A. Bokut'

1.5. (Well-known problem). Does there exist a group whose group ring does not contain zero divisors and is not embeddable into a skew field?

L. A. Bokut'

1.6. (A. I. Mal'cev). Is the group ring of a right-ordered group embeddable into a skew field?

L. A. Bokut'

1.12. (W. Magnus). The problem of the isomorphism to the trivial group for all groups with n generators and n defining relations, where $n > 2$.

M. D. Greendlinger

***1.19.** (A. I. Mal'cev). Which subgroups (subsets) are first order definable in a free group? Which subgroups are relatively elementarily definable in a free group? In particular, is the derived subgroup first order definable (relatively elementarily definable) in a free group?

Yu. L. Ershov

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*All definable (and relatively elementarily definable) sets are described; in particular, a subgroup is relatively elementarily definable if and only if it is cyclic (O. Kharlampovich, A. Myasnikov, *Int. J. Algebra Comput.*, **23**, no. 1 (2013), 91–110).

1.20. For which groups (classes of groups) is the lattice of normal subgroups first order definable in the lattice of all subgroups?

Yu. L. Ershov

1.27. Describe the universal theory of free groups.

M. I. Kargapolov

1.28. Describe the universal theory of a free nilpotent group.

M. I. Kargapolov

1.31. Is a residually finite group with the maximum condition for subgroups almost polycyclic?

M. I. Kargapolov

1.33. (A. I. Mal'cev). Describe the automorphism group of a free solvable group.

M. I. Kargapolov

1.35. c) (A. I. Mal'cev, L. Fuchs). Do there exist simple pro-orderable groups? A group is said to be *pro-orderable* if each of its partial orderings can be extended to a linear ordering.

M. I. Kargapolov

1.40. Is a group a nilgroup if it is the product of two normal nilsubgroups?

By definition, a nilgroup is a group consisting of nilelements, in other words, of (not necessarily boundedly) Engel elements.

Sh. S. Kemkhadze

1.46. What conditions ensure the normalizer of a relatively convex subgroup to be relatively convex?

A. I. Kokorin

1.51. What conditions ensure a matrix group over a field (of complex numbers) to be orderable?

A. I. Kokorin

1.54. Describe all linear orderings of a free metabelian group with a finite number of generators.

A. I. Kokorin

1.55. Give an elementary classification of linearly ordered free groups with a fixed number of generators.

A. I. Kokorin

1.65. Is the class of groups of abelian extensions of abelian groups closed under taking direct sums $(A, B) \mapsto A \oplus B$?

L. Ya. Kulikov

1.67. Suppose that G is a finitely presented group, F a free group whose rank is equal to the minimal number of generators of G, with a fixed homomorphism of F onto G with kernel N. Find a complete system of invariants of the factor-group of N by the commutator subgroup $[F, N]$.

L. Ya. Kulikov

1.74. Describe all minimal topological groups, that is, non-discrete groups all of whose closed subgroups are discrete. The minimal locally compact groups can be described without much effort. At the same time, the problem is probably complicated in the general case.

V. P. Platonov

1.86. Is it true that the identical relations of a polycyclic group have a finite basis?

A. L. Shmel'kin

1.87. The same question for matrix groups (at least over a field of characteristic 0).

A. L. Shmel'kin

Problems from the 2nd Issue (1967)

2.5. According to Plotkin, a group is called an NR-group if the set of its nil-elements coincides with the locally nilpotent radical, or, which is equivalent, if every inner automorphism of it is locally stable. Can an NR-group have a nil-automorphism that is not locally stable?

V. G. Vilyatser

2.6. In an NR-group, the set of generalized central elements coincides with the nil-kernel. Is the converse true, that is, must a group be an NR-group if the set of generalized central elements coincides with the nil-kernel?

V. G. Vilyatser

2.9. Do there exist regular associative operations on the class of groups satisfying the weakened Mal'cev condition (that is, monomorphisms of the factors of an arbitrary product can be glued together, generally speaking, into a homomorphism of the whole product), but not satisfying the analogous condition for epimorphisms of the factors?

O. N. Golovin

2.22. a) An abstract group-theoretic property Σ is said to be *radical* (*in our sense*) if, in any group G, the subgroup $\Sigma(G)$ generated by all normal Σ-subgroups is a Σ-subgroup itself (called the Σ-*radical* of G). A radical property Σ is said to be *strongly radical* if, for any group G, the factor-group $G/\Sigma(G)$ contains no non-trivial normal Σ-subgroups. Is the property $\overline{RN}$ radical? strongly radical?

Sh. S. Kemkhadze

2.24. Are Engel torsion-free groups orderable?

A. I. Kokorin

2.25. a) (L. Fuchs). Describe the groups which are linearly orderable in only finitely many ways.

A. I. Kokorin

2.26. (L. Fuchs). Characterize as abstract groups the multiplicative groups of orderable skew fields.

A. I. Kokorin

2.28. Can every orderable group be embedded in a pro-orderable group? (See 1.35 for the definition of pro-orderable groups.)

A. I. Kokorin

2.40. c) The *I-theory* (*Q-theory*) of a class $\mathfrak{K}$ of universal algebras is the totality of all identities (quasi-identities) that are valid on all algebras in $\mathfrak{K}$. Does there exist a finitely axiomatizable variety

*(2) of associative rings

(i) whose I-theory is non-decidable?

(ii) whose Q-theory is non-decidable?

(3) of Lie rings

(i) whose I-theory is non-decidable?

*(ii) whose Q-theory is non-decidable?

Remark: it is not difficult to find a recursively axiomatizable variety of semigroups with identity whose I-theory is non-recursive (see also A. I. Mal'cev, *Mat. Sbornik*, **69**, no. 1 (1966), 3–12 (Russian)).

A. I. Mal'cev

*(2i) No, it does not (A. Ya. Belov, *Izv. Math.* **74**, no. 1 (2010), 1–126).

*(2ii) and (3ii): Yes, it exists (A. I. Budkin, *Izv. Altai Univ.*, **65**, no. 1 (2010), 15–17 (Russian)).

2.42. What is the structure of the groupoid of quasivarieties

a) of all semigroups?

b) of all rings?

c) of all associative rings?

Compare with A. I. Mal'cev, *Siberian Math. J.*, **8**, no. 2 (1967), 254–267).

A. I. Mal'cev

2.45. b) (P. Hall). Prove or refute the following conjecture: If the marginal subgroup v^*G has finite index m in G, then the order of vG is finite and divides a power of m.

Editors' comment: There are examples when $|vG|$ does not divide a power of m (Yu. G. Kleiman, *Trans. Moscow Math. Soc.*, **1983**, no. 2, 63–110), but the question of finiteness remains open.

Yu. I. Merzlyakov

2.48. (N. Aronszajn). Let G be a connected topological group locally satisfying some identical relation $f|_U = 1$, where U is a neighborhood of the identity element of G. Is it then true that $f|_G = 1$?

V. P. Platonov

2.53. Describe the maximal subgroups of $SL_n(k)$ over an arbitrary infinite field k.

V. P. Platonov

2.56. Classify up to isomorphism the abelian connected algebraic unipotent linear groups over a field of positive characteristic. This is not difficult in the case of a field of characteristic zero. On the other hand, C. Chevalley has solved the classification problem for such groups up to isogeny.

V. P. Platonov

2.67. Find conditions which ensure that the nilpotent product of pure nilpotent groups (from certain classes) is determined by the lattice of its subgroups. This is known to be true if the product is torsion-free.

L. E. Sadovskiĭ

2.68. What can one say about lattice isomorphisms of a pure soluble group? Is such a group strictly determined by its lattice? It is well-known that the answer is affirmative for free soluble groups.

L. E. Sadovskiĭ

2.74. (Well-known problem). Describe the finite groups all of whose involutions have soluble centralizers.

A. I. Starostin

2.78. Any set of all subgroups of the same given order of a finite group G that contains at least one non-normal subgroup is called an $IE_{\bar{n}}$-system of G. A positive integer k is called a *soluble* (*non-soluble; simple; composite; absolutely simple*) *group-theoretic number* if every finite group having exactly k $IE_{\bar{n}}$-systems is soluble (respectively, if there is at least one non-soluble finite group having k $IE_{\bar{n}}$-systems; if there is at least one simple finite group having k $IE_{\bar{n}}$-systems; if there are no simple finite groups having k $IE_{\bar{n}}$-systems; if there is at least one simple finite group having k $IE_{\bar{n}}$-systems and there are no non-soluble non-simple finite groups having k $IE_{\bar{n}}$-systems).

Are the sets of all soluble and of all absolutely simple group-theoretic numbers finite or infinite? Do there exist composite, but not soluble group-theoretic numbers?

P. I. Trofimov

2.80. Does every non-trivial group satisfying the normalizer condition contain a non-trivial abelian normal subgroup?

S. N. Chernikov

2.81. a) Does there exist an axiomatizable class of lattices $\mathfrak{K}$ such that the lattice of all subsemigroups of a semigroup S is isomorphic to some lattice in $\mathfrak{K}$ if and only if S is a free group?

b) The same question for free abelian groups.

Analogous questions have affirmative answers for torsion-free groups, for non-periodic groups, for abelian torsion-free groups, for abelian non-periodic groups, for orderable groups (the corresponding classes of lattices are even finitely axiomatizable). Thus, in posed questions one may assume from the outset that the semigroup S is a torsion-free group (respectively, a torsion-free abelian group).

L. N. Shevrin

2.82. Can the class of groups with the nth Engel condition $[x, \underbrace{y, \ldots, y}_{n}] = 1$ be defined by identical relations of the form $u = v$, where u and v are words without negative powers of variables? This can be done for $n = 1, 2, 3$ (A. I. Shirshov, *Algebra i Logika*, **2**, no. 5 (1963), 5–18 (Russian)).

Editors' comment: This has also been done for $n = 4$ (G. Traustason, *J. Group Theory*, **2**, no. 1 (1999), 39–46). As remarked by O. Macedońska, this is also true for the class of locally graded n-Engel groups because by (Y. Kim, A. Rhemtulla, in *Groups–Korea'94*, de Gruyter, Berlin, 1995, 189–197) such a group is locally nilpotent and then by (R. G. Burns, Yu. Medvedev, *J. Austral. Math. Soc.*, **64** (1998), 92–100) such a group is an extension of a nilpotent group of n-bounded class by a group of n-bounded exponent; then a classical result of Mal'cev implies that such groups satisfy a positive law.

A. I. Shirshov

2.84. Suppose that a locally finite group G is a product of two locally nilpotent subgroups. Is G necessarily locally soluble?

V. P. Shunkov

Problems from the 3rd Issue (1969)

3.2. Classify the faithful irreducible (infinite-dimensional) representations of the nilpotent group defined by generators a, b, c and relations $[a, b] = c$, $ac = ca$, $bc = cb$. (A condition for a representation to be monomial is given in (A. E. Zalesskiĭ, *Math. Notes*, **9** (1971), 117–123).

S. D. Berman, A. E. Zalesskiĭ

3.3. (Well-known problem). Describe the automorphism group of the free associative algebra with n generators, $n \geqslant 2$.

L. A. Bokut'

3.5. Can a subgroup of a relatively free group be radicable? In particular, can a verbal subgroup of a relatively free group be radicable?

N. R. Brumberg

3.12. (Well-known problem). Is a locally finite group with a full Sylow basis locally soluble?

Yu. M. Gorchakov

3.16. The word problem for a group admitting a single defining relation in the variety of soluble groups of derived length n, $n \geqslant 3$.

M. I. Kargapolov

3.20. Does the class of orderable groups coincide with the smallest axiomatizable class containing pro-orderable groups? (See 1.35.)

A. I. Kokorin

3.34. (Well-known problem). The conjugacy problem for groups with a single defining relation.

D. I. Moldavanskiĭ

3.38. Describe the topological groups which have no proper closed subgroups.

Yu. N. Mukhin

3.43. Let μ be an infinite cardinal number. A group G is said to be μ-*overnilpotent* if every cyclic subgroup of G is a member of some ascending normal series of length less than μ reaching G. It is not difficult to show that the class of μ-overnilpotent groups is a radical class. Is it true that if $\mu_1 < \mu_2$ for two infinite cardinal numbers μ_1 and μ_2, then there exists a group G which is μ_2-overnilpotent and μ_1-semisimple?

Remark of 2001: In (S. Vovsi, *Sov. Math. Dokl.*, **13** (1972), 408–410) it was proved that for any two infinite cardinals $\mu_1 < \mu_2$ there exists a group that is μ_2-overnilpotent but not μ_1-overnilpotent.

B. I. Plotkin

3.44. Suppose that a group is generated by its subinvariant soluble subgroups. Is it necessarily locally soluble?

B. I. Plotkin

3.45. Let $\mathfrak{X}$ be a hereditary radical. Is it true that, in a locally nilpotent torsion-free group G, the subgroup $\mathfrak{X}(G)$ is isolated?

B. I. Plotkin

3.46. Does there exist a group having more than one, but finitely many maximal locally soluble normal subgroups?

B. I. Plotkin

3.47. (Well-known problem). Is it true that every locally nilpotent group is a homomorphic image of some torsion-free locally nilpotent group?

Editors' comment: An affirmative answer is known for periodic groups (E. M. Levich, A. I. Tokarenko, *Siberian Math. J.*, **11** (1970), 1033–1034) and for countable groups (N. S. Romanovskiĭ, Preprint, 1969).

B. I. Plotkin

3.48. It can be shown that hereditary radicals form a semigroup with respect to taking the products of classes. It is an interesting problem to find all indecomposable elements of this semigroup. In particular, we point out the problem of finding all indecomposable radicals contained in the class of locally finite p-groups.

B. I. Plotkin

3.49. Does the semigroup generated by all indecomposable radicals satisfy any identity?

B. I. Plotkin

3.50. Let G be a group of order $p^\alpha \cdot m$, where p is a prime, p and m are coprime, and let k be an algebraically closed field of characteristic p. Is it true that if the indecomposable projective module corresponding to the 1-representation of G has k-dimension p^α, then G has a Hall p'-subgroup? The converse is trivially true.

A. I. Saksonov

3.51. Is it true that every finite group with a group of automorphisms Φ which acts regularly on the set of conjugacy classes of G (that is, leaves only the identity class fixed) is soluble? The answer is known to be affirmative in the case where Φ is a cyclic group generated by a regular automorphism.

A. I. Saksonov

3.55. Is every binary soluble group all of whose abelian subgroups have finite rank, locally soluble?

S. P. Strunkov

3.57. Determine the laws of distribution of non-soluble and simple group-theoretic numbers in the sequence of natural numbers. See 2.78.

P. I. Trofimov

3.60. The notion of the *p-length* of an arbitrary finite group was introduced in (L. A. Shemetkov, *Math. USSR Sbornik*, **1** (1968), 83–92). Investigate the relations between the p-length of a finite group and the invariants c_p, d_p, e_p of its Sylow p-subgroup.

L. A. Shemetkov

Problems from the 4th Issue (1973)

4.2. a) Find an infinite finitely generated group of exponent < 100.
b) Do there exist such groups of exponent 5? *S. I. Adian*

4.5. b) Is it true that an arbitrary finitely presented group has either polynomial or exponential growth? *S. I. Adian*

4.6. (P. Hall). Are the projective groups in the variety of metabelian groups free? *V. A. Artamonov*

4.7. (Well-known problem). For which ring epimorphisms $R \to Q$ is the corresponding group homomorphism $SL_n(R) \to SL_n(Q)$ an epimorphism (for a fixed $n \geqslant 2$)? In particular, for what rings R does the equality $SL_n(R) = E_n(R)$ hold? *V. A. Artamonov*

4.8. Suppose G is a finitely generated free-by-cyclic group. Is G finitely presented? *G. Baumslag*

4.9. Let G be a finitely generated torsion-free nilpotent group. Are there only a finite number of non-isomorphic groups in the sequence αG, $\alpha^2 G, \ldots$? Here αG denotes the automorphism group of G and $\alpha^{n+1}G = \alpha(\alpha^n G)$ for $n = 1, 2, \ldots$ *G. Baumslag*

4.11. Let F be the free group of rank 2 in some variety of groups. If F is not finitely presented, is the multiplicator of F necessarily infinitely generated? *G. Baumslag*

4.13. Prove that every finite non-abelian p-group admits an automorphism of order p which is not an inner one. *Ya. G. Berkovich*

***4.14.** Let p be a prime number. What are necessary and sufficient conditions for a finite group G in order that the group algebra of G over a field of characteristic p be indecomposable as a two-sided ideal? There exist some nontrivial examples, for instance, the group algebra of the Mathieu group M_{24} is indecomposable when p is 2. *R. Brauer*

*A necessary and sufficient condition can be extracted from (G. R. Robinson, *J. Algebra*, **84** (1983), 493–502); another solution, which requires considering fewer subgroups, can be extracted from (B. Külshammer, *Arch. Math. (Basel)*, **56** (1991), 313–319).

4.17. If $\mathfrak{V}$ is a variety of groups, denote by $\widetilde{\mathfrak{V}}$ the class of all finite $\mathfrak{V}$-groups. How to characterize the classes of finite groups of the form $\widetilde{\mathfrak{V}}$ for $\mathfrak{V}$ a variety? *R. Baer*

4.18. Characterize the classes $\mathfrak{K}$ of groups, meeting the following requirements: subgroups, epimorphic images and groups of automorphisms of $\mathfrak{K}$-groups are $\mathfrak{K}$-groups, but not every countable group is a $\mathfrak{K}$-group. Note that the class of all finite groups and the class of all almost cyclic groups meet these requirements. *R. Baer*

4.19. Denote by $\mathfrak{C}$ the class of all groups G with the following property: if U and V are maximal locally soluble subgroups of G, then U and V are conjugate in G (or at least isomorphic). It is almost obvious that a finite group G belongs to $\mathfrak{C}$ if and only if G is soluble. What can be said about the locally finite groups in $\mathfrak{C}$? *R. Baer*

***4.20.** a) Let F be a free group, and N a normal subgroup of it. Is it true that the Cartesian square of N is m-reducible to N (that is, there is an algorithm that from a pair of words $w_1, w_2 \in N$ constructs a word $w \in N$ such that $w_1 \in N$ and $w_2 \in N$ if and only if $w \in N$)?

b) (Well-known problem). Do there exist finitely presented groups in which the word problem has an arbitrary pre-assigned recursively enumerable m-degree of insolubility? *M. K. Valiev*

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*a) No, it is not true (O. V. Belegradek, *Siberian Math. J.*, **19** (1978), 867–870).

*b) No, there exist m-degrees which do not contain the word problem of any recursively presented cancellation semigroup (C. G. Jockusch, jr., *Z. Math. Logik Grundlag. Math.*, **26**, no. 1 (1980), 93–95).

4.24. Suppose T is a non-abelian Sylow 2-subgroup of a finite simple group G.

a) Suppose T has nilpotency class n. The best possible bound for the exponent of the center of T is 2^{n-1}. This easily implies the bound $2^{n(n-1)}$ for the exponent of T, however, this is almost certainly too crude. What is the best possible bound?

d) Find a "small number" of subgroups $T_1, \ldots, T_n$ of T which depend only on the isomorphism class of T such that $\{N_G(T_1), \ldots, N_G(T_n)\}$ together control fusion in T with respect to G (in the sense of Alperin). *D. M. Goldschmidt*

4.30. Describe the groups (finite groups, abelian groups) that are the full automorphism groups of topological groups. *M. I. Kargapolov*

4.31. Describe the lattice of quasivarieties of nilpotent groups of nilpotency class 2. *M. I. Kargapolov*

4.33. Let $\mathfrak{K}_n$ be the class of all groups with a single defining relation in the variety of soluble groups of derived length n.

a) Under what conditions does a $\mathfrak{K}_n$-group have non-trivial center? Can a $\mathfrak{K}_n$-group, $n \geqslant 2$, that cannot be generated by two elements have non-trivial centre? *Editors' comment (1998):* These questions were answered for $n = 2$ (E. I. Timoshenko, *Siberian Math. J.*, **14**, no. 6 (1973), 954–957; *Math. Notes*, **64**, no. 6 (1998), 798–803).

b) Describe the abelian subgroups of $\mathfrak{K}_n$-groups.

c) Investigate the periodic subgroups of $\mathfrak{K}_n$-groups. *M. I. Kargapolov*

4.34. Let v be a group word, and let $\mathfrak{K}_v$ be the class of groups G such that there exists a positive integer $n = n(G)$ such that each element of the verbal subgroup vG can be represented as a product of n values of the word v on the group G.

a) For which v do all finitely generated soluble groups belong to the class $\mathfrak{K}_v$?

b) Does the word $v(x, y) = x^{-1}y^{-1}xy$ satisfy this condition? *M. I. Kargapolov*

4.40. Let C be a fixed non-trivial group (for instance, $C = \mathbb{Z}/2\mathbb{Z}$). As it is shown in (Yu. I. Merzlyakov, *Algebra and Logic*, **9**, no. 5 (1970), 326–337), for any two groups A and B, all split extensions of B by A can be imbedded in a certain unified way into the direct product $A \times \mathrm{Aut}\,(B \wr C)$. How are they situated in it?

Yu. I. Merzlyakov

4.42. For what natural numbers n does the following equality hold:

$$GL_n(\mathbb{R}) = D_n(\mathbb{R}) \cdot O_n(\mathbb{R}) \cdot UT_n(\mathbb{R}) \cdot GL_n(\mathbb{Z})?$$

For notation, see, for instance, (M. I. Kargapolov, Ju. I. Merzljakov, *Fundamentals of the Theory of Groups*, Springer, New York, 1979). For given n this equality implies the affirmative solution of Minkowski's problem on the product of n linear forms (A. M. Macbeath, *Proc. Glasgow Math. Assoc.*, **5**, no. 2 (1961), 86–89), which remains open for $n \geqslant 6$. It is known that the equality holds for $n \leqslant 3$ (Kh. N. Narzullayev, *Math. Notes*, **18** (1975), 713–719); on the other hand, it does not hold for all sufficiently large n (N. S. Akhmedov, *Zapiski Nauchn. Seminarov LOMI*, **67** (1977), 86–107 (Russian)).

Yu. I. Merzlyakov

4.44. (Well-known problem). Describe the groups whose automorphism groups are abelian.

V. T. Nagrebetskiĭ

4.46. a) We call a variety of groups a *limit variety* if it cannot be defined by finitely many laws, while each of its proper subvarieties has a finite basis of identities. It follows from Zorn's lemma that every variety that has no finite basis of identities contains a limit subvariety. Give explicitly (by means of identities or by a generating group) at least one limit variety.

A. Yu. Ol'shanskiĭ

4.48. A locally finite group is said to be an A-*group* if all of its Sylow subgroups are abelian. Does every variety of A-groups possess a finite basis of identities?

A. Yu. Ol'shanskiĭ

4.50. What are the soluble varieties of groups all of whose finitely generated subgroups are residually finite?

V. N. Remeslennikov

4.55. Let G be a finite group and $\mathbb{Z}_p$ the localization at p. Does the Krull–Schmidt theorem hold for projective $\mathbb{Z}_p G$-modules?

K. W. Roggenkamp

4.56. Let R be a commutative Noetherian ring with 1, and Λ an R-algebra, which is finitely generated as R-module. Put $T = \{U \in \mathrm{Mod}\,\Lambda \mid \exists$ an exact Λ-sequence $0 \to P \to \Lambda^{(n)} \to U \to 0$ for some n, with $P_m \cong \Lambda_m^{(n)}$ for every maximal ideal m of $R\}$. Denote by $\mathbb{G}(T)$ the Grothendieck group of T relative to short exact sequences.

a) Describe $\mathbb{G}(T)$, in particular, what does it mean: $[U] = [V]$ in $\mathbb{G}(T)$?

b) *Conjecture:* if $\dim(\max(R)) = d < \infty$, and there are two epimorphisms $\varphi: \Lambda^{(n)} \to U$, $\psi: \Lambda^{(n)} \to V$, $n > d$, and $[U] = [V]$ in $\mathbb{G}(T)$, then $\mathrm{Ker}\,\varphi = \mathrm{Ker}\,\psi$.

K. W. Roggenkamp

4.65. *Conjecture:* $\dfrac{p^q - 1}{p - 1}$ never divides $\dfrac{q^p - 1}{q - 1}$ if p, q are distinct primes. The validity of this conjecture would simplify the proof of solvability of groups of odd order (W. Feit, J. G. Thompson, *Pacific J. Math.*, **13**, no. 3 (1963), 775–1029), rendering unnecessary the detailed use of generators and relations.

J. G. Thompson

4.66. Let P be a presentation of a finite group G on m_p generators and r_p relations. The *deficiency* $\operatorname{def}(G)$ is the maximum of $m_p - r_p$ over all presentations P. Let G be a finite group such that $G = G' \neq 1$ and the multiplicator $M(G) = 1$. Prove that $\operatorname{def}(G^n) \to -\infty$ as $n \to \infty$, where G^n is the nth direct power of G. *J. Wiegold*

4.69. Let G be a finite p-group, and suppose that $|G'| > p^{n(n-1)/2}$ for some non-negative integer n. Prove that G is generated by the elements of breadths $\geqslant n$. The *breadth* of an element x of G is $b(x)$ where $|G : C_G(x)| = p^{b(x)}$. *J. Wiegold*

4.72. Is it true that every variety of groups whose free groups are residually nilpotent torsion-free, is either soluble or coincides with the variety of all groups? For an affirmative answer, it is sufficient to show that every variety of Lie algebras over the field of rational numbers which does not contain any finite-dimensional simple algebras is soluble. *A. L. Shmel'kin*

4.74. b) Is every binary-finite 2-group of order greater than 2 non-simple? *V. P. Shunkov*

4.75. Let G be a periodic group containing an involution i and suppose that the Sylow 2-subgroups of G are either locally cyclic or generalized quaternion. Does the element $iO_{2'}(G)$ of the factor-group $G/O_{2'}(G)$ always lie in its centre? *V. P. Shunkov*

Problems from the 5th Issue (1976)

5.1. b) Is every locally finite minimal non-FC-group distinct from its derived subgroup? The question has an affirmative answer for minimal non-BFC-groups.

V. V. Belyaev, N. F. Sesekin

5.5. If G is a finitely generated abelian-by-polycyclic-by-finite group, does there exist a finitely generated metabelian group M such that G is isomorphic to a subgroup of the automorphism group of M? If so, many of the tricky properties of G like its residual finiteness would become transparent.

B. A. F. Wehrfritz

5.14. An intersection of some Sylow 2-subgroups is called a *Sylow intersection* and an intersection of a pair of Sylow 2-subgroups is called a *paired Sylow intersection.*

a) Describe the finite groups all of whose 2-local subgroups have odd indices.

b) Describe the finite groups all of whose normalizers of Sylow intersections have odd indices.

c) Describe the finite groups all of whose normalizers of paired Sylow intersections have odd indices.

d) Describe the finite groups in which for any two Sylow 2-subgroups P and Q, the intersection $P \cap Q$ is normal in some Sylow 2-subgroup of $\langle P, Q\rangle$.

V. V. Kabanov, A. A. Makhnëv, A. I. Starostin

5.15. Do there exist finitely presented residually finite groups with recursive, but not primitive recursive, solution of the word problem?

F. B. Cannonito

5.16. Is every countable locally linear group embeddable in a finitely presented group? In (G. Baumslag, F. B. Cannonito, C. F. Miller III, *Math. Z.*, **153** (1977), 117–134) it is proved that every countable group which is locally linear of bounded degree can be embedded in a finitely presented group with solvable word problem.

F. B. Cannonito, C. Miller

5.21. Can every torsion-free group with solvable word problem be embedded in a group with solvable conjugacy problem? An example due to A. Macintyre shows that this question has a negative answer when the condition of torsion-freeness is omitted.

D. J. Collins

5.25. Prove that the factor-group of any soluble linearly ordered group by its derived subgroup is non-periodic.

V. M. Kopytov

5.26. Let G be a finite p-group with the minimal number of generators d, and let r_1 (respectively, r_2) be the minimal number of defining relations on d generators in the sense of representing G as a factor-group of a free discrete group (pro-p-group). It is well known that always $r_2 > d^2/4$. For each prime number p denote by $c(p)$ the exact upper bound for the numbers $b(p)$ with the property that $r_2 \geqslant b(p)d^2$ for all finite p-groups.

a) It is obvious that $r_1 \geqslant r_2$. Find a p-group with $r_1 > r_2$.

b) *Conjecture:* $\lim\limits_{p\to\infty} c(p) = 1/4$. It is proved (J. Wisliceny, *Math. Nachr.*, **102** (1981), 57–78) that $\lim\limits_{d\to\infty} r_2\,/d^2 = 1/4$.

H. Koch

5.27. Prove that if G is a torsion-free pro-p-group with a single defining relation, then $\operatorname{cd} G = 2$. This has been proved for a large class of groups in (J. P. Labute, *Inv. Math.*, **4**, no. 2 (1967), 142–158).

H. Koch

5.30. (Well-known problem). Suppose that G is a finite soluble group, $A \leqslant \operatorname{Aut} G$, $C_G(A) = 1$, the orders of G and A are coprime, and let $|A|$ be the product of n not necessarily distinct prime numbers. Is the nilpotent length of G bounded above by n? This is proved for large classes of groups (E. Shult, F. Gross, T. Berger, A. Turull), and there is a bound in terms of n if A is soluble (J. G. Thompson's $\leqslant 5^n$, H. Kurzweil's $\leqslant 4n$, A. Turull's $\leqslant 2n$), but the problem remains open.

V. D. Mazurov

5.33. (Y. Ihara). Consider the quaternion algebra Q with norm $f = x^2 - \tau y^2 - \rho z^2 + \rho\tau u^2$, $\rho, \tau \in \mathbb{Z}$. Assume that f is indefinite and of $\mathbb{Q}$-rank 0, i. e. $f = 0$ for $x, y, z, u \in \mathbb{Q}$ implies $x = y = z = u = 0$. Consider Q as the algebra of the matrices

$$X = \begin{pmatrix} x + \sqrt{\tau}y & \rho(z + \sqrt{\tau}u) \\ z - \sqrt{\tau}u & x - \sqrt{\tau}y \end{pmatrix}$$

with $x, y, z, u \in \mathbb{Q}$. Let p be a prime, $p \nmid \rho\tau$. Consider the group G of all X with $x, y, z, u \in \mathbb{Z}^{(p)}$, $\det X = 1$, where $\mathbb{Z}^{(p)} = \{m/p^t \mid m, t \in \mathbb{Z}\}$.

Conjecture: G has the congruence subgroup property, i. e. every non-central normal subgroup N of G contains a full congruence subgroup $N(\mathfrak{A}) = \{X \in G \mid X \equiv E \pmod{\mathfrak{A}}\}$ for some $\mathfrak{A}$. Notice that the congruence subgroup property is independent of the matrix representation of Q.

J. Mennicke

5.35. Let V be a vector space of dimension n over a field. A subgroup G of $GL_n(V)$ is said to be *rich in transvections* if $n \geqslant 2$ and for every hyperplane $H \subseteq V$ and every line $L \subseteq H$ there is at least one transvection in G with residual line L and fixed space H. Describe the automorphisms of the subgroups of $GL_2(V)$ which are rich in transvections.

Yu. I. Merzlyakov

5.36. What profinite groups satisfy the maximum condition for closed subgroups?

Yu. N. Mukhin

5.38. Is it the case that if A, B are finitely generated soluble Hopfian groups then $A \times B$ is Hopfian?

Peter M. Neumann

5.39. Prove that every countable group can act faithfully as a group of automorphisms of a finitely generated soluble group (of derived length at most 4). For background to this problem, in particular its relationship with problem 8.50, see (Peter M. Neumann, *in: Groups–Korea, Pusan, 1988* (*Lect. Notes Math.*, **1398**), Springer, Berlin, 1989, 124–139).

Peter M. Neumann

5.42. Does the free group of rank 2 have an infinite ascending chain of verbal subgroups each being generated as a verbal subgroup by a single element?

A. Yu. Ol'shanskiĭ

5.44. A union $\mathfrak{A} = \bigcup\limits_{\alpha} \mathfrak{V}_\alpha$ of varieties of groups (in the lattice of varieties) is called *irreducible* if $\bigcup\limits_{\beta\neq\alpha} \mathfrak{V}_\beta \neq \mathfrak{A}$ for each index α. Is every variety an irreducible union of (finitely or infinitely many) varieties each of which cannot be decomposed into a union of two proper subvarieties?

A. Yu. Ol'shanskiĭ

5.47. Is every countable abelian group embeddable in the center of some finitely presented group?

V. N. Remeslennikov

5.48. Suppose that G and H are finitely generated residually finite groups having the same set of finite homomorphic images. Are G and H isomorphic if one of them is a free (free soluble) group?

V. N. Remeslennikov

5.52. It is not hard to show that a finite perfect group is the normal closure of a single element. Is the same true for infinite finitely generated groups?

J. Wiegold

5.54. Let p, q, r be distinct primes and $u(x, y, z)$ a commutator word in three variables. Prove that there exist (infinitely many?) natural numbers n such that the alternating group $\mathbb{A}_n$ can be generated by three elements ξ, η, ζ satisfying $\xi^p = \eta^q = \zeta^r = \xi\eta\zeta \cdot u(\xi, \eta, \zeta) = 1$

J. Wiegold

5.55. Find a finite p-group that cannot be embedded in a finite p-group with trivial multiplicator. Notice that every finite group can be embedded in a group with trivial multiplicator.

J. Wiegold

5.56. a) Let p be a prime greater than 3. Is it true that every finite group of exponent p can be embedded in the commutator subgroup of a finite group of exponent p?

J. Wiegold

5.59. Let G be a locally finite group which is the product of a p-subgroup and a q-subgroup, where p and q are distinct primes. Is G a $\{p, q\}$-group?

B. Hartley

***5.65.** Is the class of all finite groups that have Hall π-subgroups closed under taking finite subdirect products?

L. A. Shemetkov

*Yes, it is (D. O. Revin, E. P. Vdovin, *J. Group Theory*, **14**, no. 1 (2011), 93–101).

5.67. Is a periodic residually finite group finite if it satisfies the weak minimum condition for subgroups?

V. P. Shunkov

Problems from the 6th Issue (1978)

6.1. A subgroup H of an arbitrary group G is said to be *C-closed* if $H = C^2(H) = C(C(H))$, and *weakly C-closed* if $C^2(x) \leqslant H$ for any element x of H. The structure of the finite groups all of whose proper subgroups are C-closed was studied by Gaschütz. Describe the structure of the locally finite groups all of whose proper subgroups are weakly C-closed.

V. A. Antonov, N. F. Sesekin

6.2. The totality of C-closed subgroups of an arbitrary group G is a complete lattice with respect to the operations $A \wedge B = A \cap B$, $A \vee B = C(C(A) \cap C(B))$. Describe the groups whose lattice of C-closed subgroups is a sublattice of the lattice of all subgroups.

V. A. Antonov, N. F. Sesekin

6.3. A group G is called *of type* $(FP)_\infty$ if the trivial G-module $\mathbb{Z}$ has a resolution by finitely generated projective G-modules. The class of all groups of type $(FP)_\infty$ has a couple of excellent closure properties with respect to extensions and amalgamated products (R. Bieri, *Homological dimension of discrete groups*, Queen Mary College Math. Notes, London, 1976). Is every periodic group of type $(FP)_\infty$ finite? This is related to the question whether there is an infinite periodic group with a finite presentation.

R. Bieri

6.5. Is it the case that every soluble group G of type $(FP)_\infty$ is *constructible* in the sense of (G. Baumslag, R. Bieri, *Math. Z.*, **151**, no. 3 (1976), 249–257)? In other words, can G be built up from the trivial group in terms of finite extensions and HNN-extensions?

R. Bieri

6.9. Is the derived subgroup of any locally normal group decomposable into a product of at most countable subgroups which commute elementwise?

Yu. M. Gorchakov

6.10. Let p be a prime and n be an integer with $p > 2n + 1$. Let x, y be p-elements in $GL_n(\mathbb{C})$. If the subgroup $\langle x, y\rangle$ is finite, then it is an abelian p-group (W. Feit, J. G. Thompson, *Pacif. J. Math.*, **11**, no. 4 (1961), 1257–1262). What can be said about $\langle x, y\rangle$ in the case it is an infinite group?

J. D. Dixon

6.11. Let $\mathfrak{L}$ be the class of locally compact groups with no small subgroups (see D. Montgomery, L. Zippin, *Topological transformation groups*, New York, 1955; V. M. Glushkov, *Uspekhi Matem. Nauk*, **12**, no. 2 (1957), 3–41 (Russian)). Study extensions of groups in this class with the objective of giving a direct proof of the following: For each $G \in \mathfrak{L}$ there exists an $H \in \mathfrak{L}$ and a (continuous) homomorphism $\vartheta : G \to H$ with a discrete kernel and an image $\operatorname{Im}\vartheta$ satisfying $\operatorname{Im}\vartheta \cap Z(H) = 1$ ($Z(H)$ is the center of H). This result follows from the Gleason–Montgomery–Zippin solution of Hilbert's 5th Problem (since $\mathfrak{L}$ is the class of finite-dimensional Lie groups and the latter are locally linear). On the other hand a direct proof of this result would give a substantially shorter proof of the 5th Problem since the adjoint representation of H is faithful on $\operatorname{Im}\vartheta$.

J. D. Dixon

6.21. G. Higman proved that, for any prime number p, there exists a natural number $\chi(p)$ such that the nilpotency class of any finite group G having an automorphism of order p without non-trivial fixed points does not exceed $\chi(p)$. At the same time he showed that $\chi(p) \geqslant (p^2-1)/4$ for any such *Higman's function* χ. Find the best possible Higman's function. Is it the function defined by equalities $\chi(p) = (p^2-1)/4$ for $p > 2$ and $\chi(2) = 1$? This is known to be true for $p \leqslant 7$. *V. D. Mazurov*

6.24. The occurrence problem for the braid group on four strings. It is known (T. A. Makanina, *Math. Notes*, **29**, no. 1 (1981), 16–17) that the occurrence problem is undecidable for braid groups with more than four strings. *G. S. Makanin*

6.26. Let D be a normal set of involutions in a finite group G and let $\Gamma(D)$ be the graph with vertex set D and edge set $\{(a,b) \mid a,b \in D,\ ab = ba \neq 1\}$. Describe the finite groups G with non-connected graph $\Gamma(D)$. *A. A. Makhnëv*

6.28. Let A be an elementary abelian 2-group which is a TI-subgroup of a finite group G. Investigate the structure of G under the hypothesis that the weak closure of A in a Sylow 2-subgroup of G is abelian. *A. A. Makhnëv*

6.29. Suppose that a finite group A is isomorphic to the group of all topological automorphisms of a locally compact group G. Does there always exist a discrete group whose automorphism group is isomorphic to A? This is true if A is cyclic; the condition of local compactness of G is essential (R. J. Wille, *Indag. Math.*, **25**, no. 2 (1963), 218–224). *O. V. Mel'nikov*

6.30. Let G be a residually finite Hopfian group, and let $\widehat{G}$ be its profinite completion. Is $\widehat{G}$ necessarily Hopfian (in topological sense)? *O. V. Mel'nikov*

6.31. b) Let G be a finitely generated residually finite group, $d(G)$ the minimal number of its generators, and $\delta(G)$ the minimal number of topological generators of the profinite completion of G. It is known that there exist groups G for which $d(G) > \delta(G)$ (G. A. Noskov, *Math. Notes*, **33**, no. 4 (1983), 249–254). Is the function d bounded on the set of groups G with the fixed value of $\delta(G) \geqslant 2$? *O. V. Mel'nikov*

6.32. Let F_n be the free profinite group of finite rank $n > 1$. Is it true that for each normal subgroup N of the free profinite group of countable rank, there exists a normal subgroup of F_n isomorphic to N? *O. V. Mel'nikov*

6.38. b) Is it true that an arbitrary subgroup of $GL_n(k)$ that intersects every conjugacy class is parabolic? How far is the same statement true for subgroups of other groups of Lie type? *Peter M. Neumann*

6.39. A class of groups $\mathfrak{K}$ is said to be *radical* if it is closed under taking homomorphic images and normal subgroups and if every group generated by its normal $\mathfrak{K}$-subgroups also belongs to $\mathfrak{K}$. The question proposed relates to the topic "Radical classes and formulae of Narrow (first order) Predicate Calculus (NPC)". One can show that the only non-trivial radical class definable by universal formulae of NPC is the class of all groups. Recently (in a letter to me), G. Bergman constructed a family of locally finite radical classes definable by formulae of NPC. Do there exist similar classes which are not locally finite and different from the class of all groups? In particular, does there exist a radical class of groups which is closed under taking Cartesian products, contains an infinite cyclic group, and is different from the class of all groups?

B. I. Plotkin

6.45. Construct a characteristic subgroup N of a finitely generated free group F such that F/N is infinite and simple. If no such exists, it would follow that $d(S^2) = d(S)$ for every infinite finitely generated simple group S. There is reason to believe that this is false.

J. Wiegold

6.47. (C. D. H. Cooper). Let G be a group, v a group word in two variables such that the operation $x \odot y = v(x, y)$ defines the structure of a new group $G_v = \langle G, \odot \rangle$ on the set G. Does G_v always lie in the variety generated by G?

E. I. Khukhro

6.48. Is every infinite binary finite p-group non-simple? Here p is a prime number.

N. S. Chernikov

6.51. Let $\mathfrak{F}$ be a local subformation of some formation $\mathfrak{X}$ of finite groups and let Ω be the set of all maximal homogeneous $\mathfrak{X}$-screens of $\mathfrak{F}$. Find a way of constructing the elements of Ω with the help of the maximal inner local screen of $\mathfrak{F}$. What can be said about the cardinality of Ω? For definitions see (L. A. Shemetkov, *Formations of finite groups*, Moscow, Nauka, 1978 (Russian)).

L. A. Shemetkov

6.55. A group G of the form $G = F \leftthreetimes H$ is said to be a *Frobenius group with kernel F and complement H* if $H \cap H^g = 1$ for any $g \in G \setminus H$ and $F \setminus \{1\} = G \setminus \bigcup\limits_{g\in G} H^g$. Do there exist Frobenius p-groups?

V. P. Shunkov

6.56. Let $G = F \cdot \langle a \rangle$ be a Frobenius group with the complement $\langle a \rangle$ of prime order.

a) Is G locally finite if it is binary finite?

b) Is the kernel F locally finite if the groups $\langle a, a^g \rangle$ are finite for all $g \in G$?

V. P. Shunkov

6.59. A group G is said to be (*conjugacy, p-conjugacy*) *biprimitively finite* if, for any finite subgroup H, any two elements of prime order (any two conjugate elements of prime order, of prime order p) in $N_G(H)/H$ generate a finite subgroup. Prove that an arbitrary periodic (conjugacy) biprimitively finite group (in particular, having no involutions) of finite rank is locally finite.

V. P. Shunkov

6.60. Do there exist infinite finitely generated simple periodic (conjugacy) biprimitively finite groups which contain both involutions and non-trivial elements of odd order?

V. P. Shunkov

6.61. Is every infinite periodic (conjugacy) biprimitively finite group without involutions non-simple?

V. P. Shunkov

6.62. Is a (conjugacy, p-conjugacy) biprimitively finite group finite if it has a finite maximal subgroup (p-subgroup)? There are affirmative answers for conjugacy biprimitively finite p-groups and for 2-conjugacy biprimitively finite groups (V. P. Shunkov, *Algebra and Logic*, **9**, no. 4 (1970), 291–297; **11**, no. 4 (1972), 260–272; **12**, no. 5 (1973), 347–353), while the statement does not hold for arbitrary periodic groups, see Archive, 3.9.

V. P. Shunkov

Problems from the 7th Issue (1980)

7.3. Prove that the periodic product of odd exponent $n \geqslant 665$ of non-trivial groups $F_1, \dots, F_k$ which do not contain involutions cannot be generated by less than k elements. This would imply, on the basis of (S. I. Adian, *Sov. Math. Doklady*, **19**, (1978), 910–913), the existence of k-generated simple groups which cannot be generated by less than k elements, for any $k > 0$.

S. I. Adian

7.5. We say that a group is *indecomposable* if any two of its proper subgroups generate a proper subgroup. Describe the indecomposable periodic metabelian groups.

V. V. Belyaev

7.15. Prove that if $F^*(G)$ is quasisimple and $\alpha \in \operatorname{Aut} G$, $|\alpha| = 2$, then $C_G(\alpha)$ contains an involution outside $Z(F^*(G))$, except when $F^*(G)$ has quaternion Sylow 2-subgroups.

R. Griess

***7.17.** Is the number of maximal subgroups of the finite group G at most $|G| - 1$?

Editors' remarks: This is proved for G soluble (G. E. Wall, *J. Austral. Math. Soc.*, **2** (1961–62), 35–59) and for symmetric groups $\mathbb{S}_n$ for sufficiently large n (M. Liebeck, A. Shalev, *J. Combin. Theory, Ser. A*, **75** (1996), 341–352); it is also proved (M. W. Liebeck, L. Pyber, A. Shalev, *J. Algebra*, **317** (2007), 184–197) that any finite group G has at most $2C|G|^{3/2}$ maximal subgroups, where C is an absolute constant.

R. Griess

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*No, not always (R. Guralnick, F. Lübeck, L. Scott, T. Sprowl; see `arxiv.org/pdf/1303.2752`).

7.19. Construct an explicit example of a finitely presented simple group with word problem not solvable by a primitive recursive function.

F. B. Cannonito

7.21. Is there an algorithm which decides if an arbitrary finitely presented solvable group is metabelian?

F. B. Cannonito

7.23. Does there exist an algorithm which decides, for a given list of group identities, whether the elementary theory of the variety given by this list is soluble, that is, by (A. P. Zamyatin, *Algebra and Logic*, **17**, no. 1 (1978), 13–17), whether this variety is abelian?

A. V. Kuznetsov

7.25. We associate with words in the alphabet x, x^{-1}, y, y^{-1}, z, $z^{-1}, \ldots$ operations which are understood as functions in variables $x, y, z, \ldots$. We say that a word A *is expressible in words* $B_1, \ldots, B_n$ *on the group* G if A can be constructed from the words $B_1, \ldots, B_n$ and variables by means of finitely many substitutions of words one into another and replacements of a word by another word which is identically equal to it on G. A list of words is said to be *functionally complete on* G if every word can be expressed on G in words from this list. A word B is called a *Schaeffer word on* G if every word can be expressed on G in B (compare with A. V. Kuznetsov, *Matem. Issledovaniya*, Kishinëv, **6**, no. 4 (1971), 75–122 (Russian)). Does there exist an algorithm which decides

a) by a word B, whether it is a Schaeffer word on every group? Compare with the problem of describing all such words in (A. G. Kurosh, *Theory of Groups*, Moscow, Nauka, 1967, p. 435 (Russian)); here are examples of such words: xy^{-1}, $x^{-1}y^2z$, $x^{-1}y^{-1}zx$.

b) by a list of words, whether it is functionally complete on every group?

c) by words $A, B_1, \ldots, B_n$, whether A is expressible in $B_1, \ldots, B_n$ on every group? (This is a problem from A. V. Kuznetsov, *ibid.*, p. 112.)

d) the same for every finite group? (For finite groups, for example, x^{-1} is expressible in xy.) For a fixed finite group an algorithm exists (compare with A. V. Kuznetsov, *ibid.*, § 8).

A. V. Kuznetsov

7.26. The *ordinal height* of a variety of groups is, by definition, the supremum of order types (ordinals) of all well-ordered by inclusion chains of its proper subvarieties. It is clear that its cardinality is either finite or countably infinite, or equal to ω_1. Is every countable ordinal number the ordinal height of some variety?

A. V. Kuznetsov

7.27. Is it true that the group $SL_n(q)$ contains, for sufficiently large q, a diagonal matrix which is not contained in any proper irreducible subgroup of $SL_n(q)$ with the exception of block-monomial ones? For $n = 2$, 3 the answer is known to be affirmative (V. M. Levchuk, *in: Some problems of the theory of groups and rings*, Inst. of Physics SO AN SSSR, Krasnoyarsk, 1973 (Russian)). Similar problems are interesting for other Chevalley groups.

V. M. Levchuk

7.28. Let $G(K)$ be the Chevalley group over a commutative ring K associated with the root system Φ as defined in (R. Steinberg, *Lectures on Chevalley groups*, Yale Univ., New Haven, Conn., 1967). This group is generated by the root subgroups $x_r(K)$, $r \in \Phi$. We define an *elementary carpet of type* Φ *over* K to be any collection of additive subgroups $\{\mathfrak{A}_r \mid r \in \Phi\}$ of K satisfying the condition

$$c_{ij,rs}\mathfrak{A}_r^i\mathfrak{A}_s^j \subseteq \mathfrak{A}_{ir+js} \qquad \text{for } r, s, ir+js \in \Phi,\ \ i > 0,\ \ j > 0,$$

where $c_{ij,rs}$ are constants defined by the Chevalley commutator formula and $\mathfrak{A}_r^i = \{a^i \mid a \in \mathfrak{A}_r\}$. What are necessary and sufficient conditions (in terms of the $\mathfrak{A}_r$) on the elementary carpet to ensure that the subgroup $\langle x_r(\mathfrak{A}_r) \mid r \in \Phi\rangle$ of $G(K)$ intersects with $x_r(K)$ in $x_r(\mathfrak{A}_r)$? See also 15.46.

V. M. Levchuk

7.31. Let A be a group of automorphisms of a finite non-abelian 2-group G acting transitively on the set of involutions of G. Is A necessarily soluble?

Such groups G were divided into several classes in (F. Gross, *J. Algebra*, **40**, no. 2 (1976), 348–353). For one of these classes a positive answer was given by E. G. Bryukhanova (*Algebra and Logic*, **20**, no. 1 (1981), 1–12).

V. D. Mazurov

7.33. An elementary TI-subgroup V of a finite group G is said to be a *subgroup of non-root type* if $1 \neq N_V(V^g) \neq V$ for some $g \in G$. Describe the finite groups G which contain a 2-subgroup V of non-root type such that $[V, V^g] = 1$ implies that all involutions in VV^g are conjugate to elements of V.

A. A. Makhnëv

7.34. In many of the sporadic finite simple groups the 2-ranks of the centralizers of 3-elements are at most 2. Describe the finite groups satisfying this condition.

A. A. Makhnëv

7.35. What varieties of groups $\mathfrak{V}$ have the following property: the group $G/\mathfrak{V}(G)$ is residually finite for any residually finite group G?

O. V. Mel'nikov

7.38. A *variety of profinite groups* is a non-empty class of profinite groups closed under taking subgroups, factor-groups, and Tikhonov products. A subvariety $\mathfrak{V}$ of the variety $\mathfrak{N}$ of all pronilpotent groups is said to be (*locally*) *nilpotent* if all (finitely generated) groups in $\mathfrak{V}$ are nilpotent.

a) Is it true that any non-nilpotent subvariety of $\mathfrak{N}$ contains a non-nilpotent locally nilpotent subvariety?

b) The same problem for the variety of all pro-p-groups (for a given prime p).

O. V. Mel'nikov

7.39. Let $G = \langle a, b \mid a^p = (ab)^3 = b^2 = (a^\sigma b a^{2/\sigma} b)^2 = 1 \rangle$, where p is a prime, σ is an integer not divisible by p. The group $PSL_2(p)$ is a factor-group of G so that there is a short exact sequence $1 \to N \to G \to PSL_2(p) \to 1$. For each $p > 2$ there is σ such that $N = 1$, for example, $\sigma = 4$. Let N^{ab} denote the factor-group of N by its commutator subgroup. It is known that for some p there is σ such that N^{ab} is infinite (for example, for $p = 41$ one can take $\sigma^2 \equiv 2 \pmod{41}$), whereas for some other p (for example, for $p = 43$) the group N^{ab} is finite for every σ.

a) Is the set of primes p for which N^{ab} is finite for every σ infinite?

b) Is there an arithmetic condition on σ which ensures that N^{ab} is finite?

J. Mennicke

7.40. Describe the (lattice of) subgroups of a given classical matrix group over a ring which contain the subgroup consisting of all matrices in that group with coefficients in some subring (see Ju. I. Merzljakov, *J. Soviet Math.*, **1** (1973), 571–593).

Yu. I. Merzlyakov

7.41. (J. S. Wilson). Is every linear $\overline{SI}$-group an $\overline{SN}$-group?

Yu. I. Merzlyakov

7.45. Does the Q-theory of the class of all finite groups (in the sense of 2.40) coincide with the Q-theory of a single finitely presented group?

D. M. Smirnov

7.49. Let G be a finitely generated group, and N a minimal normal subgroup of G which is an elementary abelian p-group. Is it true that either N is finite or the growth function for the elements of N with respect to a finite generating set of G is bounded below by an exponential function?

V. I. Trofimov

7.50. Study the structure of the primitive permutation groups (finite and infinite), in which the stabilizer of any three pairwise distinct points is trivial. This problem is closely connected with the problem of describing those group which have a Frobenius group as one of maximal subgroups.

A. N. Fomin

7.51. What are the primitive permutation groups (finite and infinite) which have a regular sub-orbit, that is, in which a point stabilizer acts faithfully and regularly on at least one of its orbits?

A. N. Fomin

7.52. Describe the locally finite primitive permutation groups in which the centre of any Sylow 2-subgroup contains involutions stabilizing precisely one symbol. The case of finite groups was completely determined by D. Holt in 1978.

A. N. Fomin

7.54. Does there exist a group of infinite special rank which can be represented as a product of two subgroups of finite special rank?

N. S. Chernikov

7.55. Is it true that a group which is a product of two almost abelian subgroups is almost soluble?

N. S. Chernikov

7.56. (B. Amberg). Does a group satisfy the minimum (respectively, maximum) condition on subgroups if it is a product of two subgroups satisfying the minimum (respectively, maximum) condition on subgroups?

N. S. Chernikov

7.57. a) A set of generators of a finitely generated group G consisting of the least possible number $d(G)$ of elements is called a *basis of* G. Let $r_M(G)$ be the least number of relations necessary to define G in the basis M and let $r(G)$ be the minimum of the numbers $r_M(G)$ over all bases M of G. It is known that $r_M(G) \leqslant d(G) + r(G)$ for any basis M. Can the equality hold for some basis M?

V. A. Churkin

7.58. Suppose that F is an absolutely free group, R a normal subgroup of F, and let $\mathfrak{V}$ be a variety of groups. It is well-known (H. Neumann, *Varieties of Groups*, Springer, Berlin, 1967) that the group $F/\mathfrak{V}(R)$ is isomorphically embeddable in the $\mathfrak{V}$-verbal wreath product of a $\mathfrak{V}$-free group of the same rank as F with F/R. Find a criterion indicating which elements of this wreath product belong to the image of this embedding. A criterion is known in the case where $\mathfrak{V}$ is the variety of all abelian groups (V. N. Remeslennikov, V. G. Sokolov, *Algebra and Logic*, **9**, no. 5 (1970), 342–349).

G. G. Yabanzhi

Problems from the 8th Issue (1982)

8.1. Characterize all groups (or at least all soluble groups) G and fields F such that every irreducible FG-module has finite dimension over the centre of its endomorphism ring. For some partial answers see (B. A. F. Wehrfritz, *Glasgow Math. J.*, **24**, no. 1 (1983), 169–176).

B. A. F. Wehrfritz

8.2. Let G be the amalgamated free product of two polycyclic groups, amalgamating a normal subgroup of each. Is G isomorphic to a linear group? G is residually finite (G. Baumslag, *Trans. Amer. Math. Soc.*, **106**, no. 2 (1963), 193–209) and the answer is "yes" if the amalgamated part is torsion-free, abelian (B. A. F. Wehrfritz, *Proc. London Math. Soc.*, **27**, no. 3 (1973), 402–424) and nilpotent (M. Shirvani, 1981, *unpublished*).

B. A. F. Wehrfritz

8.3. Let m and n be positive integers and p a prime. Let $P_n(\mathbb{Z}_{p^m})$ be the group of all $n \times n$ matrices (a_{ij}) over the integers modulo p^m such that $a_{ii} \equiv 1$ for all i and $a_{ij} \equiv 0 \pmod{p}$ for all $i > j$. The group $P_n(\mathbb{Z}_{p^m})$ is a finite p-group. For which m and n is it regular?

Editors' comment (2001): The group $P_n(\mathbb{Z}_{p^m})$ is known to be regular if $mn < p$ (Yu. I. Merzlyakov, *Algebra i Logika*, **3**, no. 4 (1964), 49–59 (Russian)). The case $m = 1$ is completely done by A. V. Yagzhev in (*Math. Notes*, **56**, no. 5–6 (1995), 1283–1290), although this paper is erroneous for $m > 1$. The case $m = 2$ is completed in (S. G. Kolesnikov, *Issledovaniya in analysis and algebra*, no. 3, Tomsk Univ., Tomsk, 2001, 117–125 (Russian)). *Editors' comment (2009):* The group $P_n(\mathbb{Z}_{p^m})$ was shown to be regular for any m if $n^2 < p$ (S. G. Kolesnikov, *Siberian Math. J.*, **47**, no. 6 (2006), 1054–1059).

B. A. F. Wehrfritz

8.4. Construct a finite nilpotent loop with no finite basis for its laws.

M. R. Vaughan-Lee

***8.5.** Prove that if X is a finite group, F is any field, and M is a non-trivial irreducible FX-module then $\frac{1}{|X|}\sum_{x\in X} \dim \operatorname{fix}(x) \leqslant \frac{1}{2} \dim M$.

M. R. Vaughan-Lee

*Proved, even with ... $\leqslant \frac{1}{p} \dim M$, where p is the least prime divisor of $|G|$ (R. M. Guralnick, A. Maróti, *Adv. Math.*, **226**, no. 1 (2011), 298–308).

8.8. b) (D. V. Anosov). Does there exist a non-cyclic finitely presented group G which contains an element a such that each element of G is conjugate to some power of a?

R. I. Grigorchuk

8.9. (C. Chou). We say that a group G has *property P* if for every finite subset F of G, there is a finite subset $S \supset F$ and a subset $X \subset G$ such that $x_1S \cap x_2S$ is empty for any $x_1, x_2 \in X$, $x_1 \neq x_2$, and $G = \bigcup_{x\in X} Sx$. Does every group have property P?

R. I. Grigorchuk

8.10. a) Is the group $G = \langle a, b \mid a^n = 1,\ ab = b^3a^3\rangle$ finite or infinite for $n = 7$? All other cases known. See Archive, 7.7 and 8.10 b.

D. L. Johnson

8.11. Consider the group

$$M = \langle x, y, z, t \mid [x, y] = [y, z] = [z, x] = (x, t) = (y, t) = (z, t) = 1\rangle$$

where $[x, y] = x^{-1}y^{-1}xy$ and $(x, t) = x^{-1}t^{-1}x^{-1}txt$. The subgroup $H = \langle x, t, y\rangle$ is isomorphic to the braid group $\mathfrak{B}_4$, and is normally complemented by $N = \big\langle (zx^{-1})^M \big\rangle$. Is N a free group (?) of countably infinite rank (?) on which H acts faithfully by conjugation?

D. L. Johnson

8.12. Let $D0$ denote the class of finite groups of deficiency zero, i. e. having a presentation $\langle X \mid R\rangle$ with $|X| = |R|$.

a) Does $D0$ contain any 3-generator p-group for $p \geqslant 5$?

c) Do soluble $D0$-groups have bounded derived length?

d) Which non-abelian simple groups can occur as composition factors of $D0$-groups?

D. L. Johnson, E. F. Robertson

8.14. b) Assume a group G is existentially closed in one of the classes $L\mathfrak{N}^+$, $L\mathfrak{S}_\pi$, $L\mathfrak{S}^+$, $L\mathfrak{S}$ of, respectively, all locally nilpotent torsion-free groups, all locally soluble π-groups, all locally soluble torsion-free groups, or all locally soluble groups. Is it true that G is characteristically simple?

The existential closedness of G is a local property, thus it seems difficult to obtain global properties of G from it. See also Archive, 8.14 a.

O. H. Kegel

8.15. (Well-known problem). Is every $\widetilde{N}$-group a $\overline{Z}$-group? For definitions, see (M. I. Kargapolov, Yu. I. Merzlyakov, *Fundamentals of the Theory of Groups*, 3rd Edition, Moscow, Nauka, 1982, p. 205 (Russian)).

Sh. S. Kemkhadze

8.16. Is the class of all $\overline{Z}$-groups closed under taking normal subgroups?

Sh. S. Kemkhadze

8.19. Which of the following properties of (soluble) varieties of groups are equivalent to one another:

1) to satisfy the minimum condition for subvarieties;

2) to have at most countably many subvarieties;

3) to have no infinite independent system of identities?

Yu. G. Kleiman

8.21. If the commutator subgroup G' of a relatively free group G is periodic, must G' have finite exponent?

L. G. Kovács

8.23. If the dihedral group D of order 18 is a section of a direct product $A \times B$, must at least one of A and B have a section isomorphic to D?

L. G. Kovács

8.24. Prove that every linearly ordered group of finite rank is soluble.

V. M. Kopytov

8.25. Does there exist an algorithm which recognizes by an identity whether it defines a variety of groups that has at most countably many subvarieties?

A. V. Kuznetsov

8.27. Does the lattice of all varieties of groups possess non-trivial automorphisms?

A. V. Kuznetsov

8.29. Do there exist locally nilpotent groups with trivial centre satisfying the weak maximal condition for normal subgroups?

L. A. Kurdachenko

8.30. Let $\mathfrak{X}$, $\mathfrak{Y}$ be Fitting classes of soluble groups which satisfy the Lockett condition, i. e. $\mathfrak{X} \cap \mathfrak{S}_* = \mathfrak{X}_*$, $\mathfrak{Y} \cap \mathfrak{S}_* = \mathfrak{Y}_*$ where $\mathfrak{S}$ denotes the Fitting class of all soluble groups and the lower star the bottom group of the Lockett section determined by the given Fitting class. Does $\mathfrak{X} \cap \mathfrak{Y}$ satisfy the Lockett condition?

Editors' comment (2001): The answer is affirmative if $\mathfrak{X}$ and $\mathfrak{Y}$ are local (A. Grytczuk, N. T. Vorob'ev, *Tsukuba J. Math.*, **18**, no. 1 (1994), 63–67).

H. Lausch

8.31. Is it true that $PSL_2(7)$ is the only finite simple group in which every proper subgroup has a complement in some larger subgroup?

V. M. Levchuk

8.33. Let a, b be two elements of a group, a having infinite order. Find a necessary and sufficient condition for $\bigcap_{n=1}^{\infty} \langle a^n, b\rangle = \langle b\rangle$.

F. N. Liman

***8.34.** Let G be a finite group. Is it true that indecomposable projective $\mathbb{Z}G$-modules are finitely generated (and hence locally free)?

P. A. Linnell

*Yes, it is; see Remark 2.13 and Corollary 4.2 in (P. Příhoda, *Rend. Semin. Mat. Univ. Padova*, **123** (2010), 141–167).

8.35. c) Determine the conjugacy classes of maximal subgroups in the sporadic simple group F_1. See the current status in *Atlas of Finite Group Representations* (`http://brauer.maths.qmul.ac.uk/Atlas/spor/M/`).

V. D. Mazurov

8.40. Describe the finite groups generated by a conjugacy class D of involutions which satisfies the following property: if $a, b \in D$ and $|ab| = 4$, then $[a, b] \in D$. This condition is satisfied, for example, in the case where D is a conjugacy class of involutions of a known finite simple group such that $\langle C_D(a)\rangle$ is a 2-group for every $a \in D$.

A. A. Makhnëv

8.41. What finite groups G contain a normal set of involutions D which contains a non-empty proper subset T satisfying the following properties:

1) $C_D(a) \subseteq T$ for any $a \in T$;
2) if $a, b \in T$ and $ab = ba \neq 1$, then $C_D(ab) \subseteq T$?

A. A. Makhnëv

8.42. Describe the finite groups all of whose soluble subgroups of odd indices have 2-length 1. For example, the groups $L_n(2^m)$ are known to satisfy this condition.

A. A. Makhnëv

8.43. (F. Timmesfeld). Let T be a Sylow 2-subgroup of a finite group G. Suppose that $\langle N(B) \mid B$ is a non-trivial characteristic subgroup of $T\rangle$ is a proper subgroup of G. Describe the group G if $F^*(M) = O_2(M)$ for any 2-local subgroup M containing T.

A. A. Makhnëv

8.44. Prove or disprove that for all, but finitely many, primes p the group

$$G_p = \langle a, b \mid a^2 = b^p = (ab)^3 = (b^r ab^{-2r} a)^2 = 1 \rangle ,$$

where $r^2+1 \equiv 0 \pmod p$, is infinite. A solution of this problem would have interesting topological applications. It was proved with the aid of computer that G_p is finite for $p \leqslant 17$.

J. Mennicke

8.45. When it follows that a group is residually a $(\mathfrak{X}\cap\mathfrak{Y})$-group, if it is both residually a $\mathfrak{X}$-group and residually a $\mathfrak{Y}$-group?

Yu. I. Merzlyakov

8.50. At a conference in Oberwolfach in 1979 I exhibited a finitely generated soluble group G (of derived length 3) and a non-zero cyclic $\mathbb{Z}G$-module V such that $V \cong V\oplus V$. Can this be done with G metabelian? I conjecture that it cannot. For background of this problem and for details of the construction see (Peter M. Neumann, *in: Groups–Korea, Pusan, 1988* (*Lect. Notes Math.*, **1398**), Springer, Berlin, 1989, 124–139).

Peter M. Neumann

8.51. (J. McKay). If G is a finite group and p a prime let $m_p(G)$ denote the number of ordinary irreducible characters of G whose degree is not divisible by p. Let P be a Sylow p-subgroup of G. Is it true that $m_p(G) = m_p(N_G(P))$?

J. B. Olsson

8.52. (Well-known problem). Do there exist infinite finitely presented periodic groups? Compare with 6.3.

A. Yu. Ol'shanskiĭ

8.53. a) Let n be a sufficiently large odd number. Describe the automorphisms of the free Burnside group $B(m,n)$ of exponent n on m generators.

A. Yu. Ol'shanskiĭ

8.54. a) (Well-known problem). Classify metabelian varieties of groups (or show that this is, in a certain sense, a "wild" problem).

b) Describe the identities of 2-generated metabelian groups, that is, classify varieties generated by such groups.

A. Yu. Ol'shanskiĭ

8.55. It is easy to see that the set of all quasivarieties of groups, in each of which quasivarieties a non-trivial identity holds, is a semigroup under multiplication of quasivarieties. Is this semigroup free?

A. Yu. Ol'shanskiĭ

8.58. Does a locally compact locally nilpotent nonabelian group without elements of finite order contain a proper closed isolated normal subgroup?

V. M. Poletskikh

8.59. Suppose that, in a locally compact locally nilpotent group G, all proper closed normal subgroups are compact. Does G contain an open compact normal subgroup?

V. M. Poletskikh, I. V. Protasov

8.60. Describe the locally compact primary locally soluble groups which are covered by compact subgroups and all of whose closed abelian subgroups have finite rank.

V. M. Poletskikh, V. S. Charin

8.62. Describe the locally compact locally pronilpotent groups for which the space of closed normal subgroups is compact in the E-topology. The corresponding problem has been solved for discrete groups.

I. V. Protasov

8.64. Does the class of all finite groups possess an independent basis of quasiidentities?

A. K. Rumyantsev, D. M. Smirnov

8.67. Do there exist Golod groups all of whose abelian subgroups have finite ranks? Here a *Golod group* means a finitely generated non-nilpotent subgroup of the adjoint group of a nil-ring. The answer is negative for the whole adjoint group of a nil-ring (Ya. P. Sysak, *Abstracts of the 17th All–Union Algebraic Conf., part 1*, Minsk, 1983, p. 180 (Russian)).

A. I. Sozutov

***8.68.** Let $G = \langle a, b \mid r = 1\rangle$ where r is a cyclically reduced word that is not a proper power of any word in a, b. If G is residually finite, is $G_t = \langle a, b \mid r^t = 1\rangle$, $t > 1$, residually finite?

Editors' comment: An analogous fact was proved for relative one-relator groups (S. J. Pride, *Proc. Amer. Math. Soc.*, **136**, no. 2 (2008), 377–386).

C. Y. Tang

*Yes, it is (D. T. Wise, *The structure of groups with a quasiconvex hierarchy*, Preprint, 2011, `http://www.math.mcgill.ca/wise/papers.html`).

***8.69.** Is every 1-relator group with non-trivial torsion conjugacy separable?

C. Y. Tang

*Yes, it is (A. Minasyan, P. Zalesskii, *J. Algebra*, **382** (2013), 39–45).

8.72. Does there exist a finitely presented group G which is not free or cyclic of prime order, having the property that every proper subgroup of G is free?

C. Y. Tang

8.74. A subnormal subgroup $H \lhd\lhd G$ of a group G is said to be *good* if and only if $\langle H, J\rangle \lhd\lhd G$ whenever $J \lhd\lhd G$. Is it true that when H and K are good subnormal subgroups of G, then $H \cap K$ is good?

J. G. Thompson

8.75. (A known problem). Suppose G is a finite primitive permutation group on Ω, and α, β are distinct points of Ω. Does there exist an element $g \in G$ such that $\alpha g = \beta$ and g fixes no point of Ω?

J. G. Thompson

8.77. Do there exist strongly regular graphs with parameters $\lambda = 0$, $\mu = 2$ of degree $k > 10$? Such graphs are known for $k = 5$ and $k = 10$, their automorphism groups are primitive permutation groups of rank 3.

D. G. Fon-Der-Flaass

8.78. It is known that there exists a countable locally finite group that contains a copy of every other countable locally finite group. For which other classes of countable groups does a similar "largest" group exist? In particular, what about periodic locally soluble groups? periodic locally nilpotent groups?

B. Hartley

8.79. Does there exist a countable infinite locally finite group G that is complete, in the sense that G has trivial centre and no outer automorphisms? An uncountable one exists (K. K. Hickin, *Trans. Amer. Math. Soc.*, **239** (1978), 213–227).

B. Hartley

8.82. Let $\mathfrak{H} = \mathbb{C} \times \mathbb{R} = \{(z, r) \mid z \in \mathbb{C},\ r > 0\}$ be the three-dimensional Poincaré space which admits the following action of the group $SL_2(\mathbb{C})$:

$$(z,r)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{(az+b)(\overline{cz+d})+a\bar{c}r^2}{|cz+d|^2+|c|^2r^2},\ \frac{r}{|cz+d|^2+|c|^2r^2}\right).$$

Let $\mathfrak{o}$ be the ring of integers of the field $K = \mathbb{Q}(\sqrt{D})$, where $D < 0$, π a prime such that $\pi\overline{\pi}$ is a prime in $\mathbb{Z}$ and let $\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathfrak{o}) \;\middle|\; b \equiv 0 \,(\mathrm{mod}\,\pi) \right\}$. Adding to the space $\Gamma \setminus \mathfrak{H}^3$ two vertices we get a three-dimensional compact space $\overline{\Gamma \setminus \mathfrak{H}^3}$.

Calculate $r(\pi) = \dim_{\mathbb{Q}} H_1(\overline{\Gamma \setminus \mathfrak{H}^3}, \mathbb{Q}) = \dim_{\mathbb{Q}}(\Gamma^{ab} \otimes \mathbb{Q})$.

For example, if $D = -3$ then $r(\pi)$ is distinct from zero for the first time for $\pi \mid 73$ (then $r(\pi) = 1$), and if $D = -4$ then $r(\pi) = 1$ for $\pi \mid 137$.

H. Helling

8.83. In the notation of 8.82, for $r(\pi) > 0$, one can define via Hecke algebras a formal Dirichlet series with Euler multiplication (see G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Princeton Univ. Press, 1971). Does there exist an algebraic Hasse–Weil variety whose ζ-function is this Dirichlet series? There are several conjectures.

H. Helling

8.85. Construct a finite p-group G whose Hughes subgroup $H_p(G) = \langle x \in G \mid |x| \neq p\rangle$ is non-trivial and has index p^3.

E. I. Khukhro

8.86. (Well-known problem). Suppose that all proper closed subgroups of a locally compact locally nilpotent group G are compact. Is G abelian if it is non-compact?

V. S. Charin

Problems from the 9th Issue (1984)

9.1. A group G is said to be *potent* if, to each $x \in G$ and each positive integer n dividing the order of x (we suppose ∞ is divisible by every positive integer), there exists a finite homomorphic image of G in which the image of x has order precisely n. Is the free product of two potent groups again potent?

R. B. J. T. Allenby

9.4. It is known that if $\mathfrak{M}$ is a variety (quasivariety, pseudovariety) of groups, then the class $I\mathfrak{M}$ of all quasigroups that are isotopic to groups in $\mathfrak{M}$ is also a variety (quasivariety, pseudovariety), and $I\mathfrak{M}$ is finitely based if and only if $\mathfrak{M}$ is finitely based. Is it true that if $\mathfrak{M}$ is generated by a single finite group then $I\mathfrak{M}$ is generated by a single finite quasigroup?

A. A. Gvaramiya

9.5. A variety of groups is called *primitive* if each of its subquasivarieties is a variety. Describe all primitive varieties of groups. Is every primitive variety of groups locally finite?

V. A. Gorbunov

9.6. Is it true that an independent basis of quasiidentities of any finite group is finite?

V. A. Gorbunov

9.7. (A. M. Stëpin). Does there exist an infinite finitely generated amenable group of bounded exponent?

R. I. Grigorchuk

9.8. Does there exist a finitely generated simple group of intermediate growth?

R. I. Grigorchuk

9.9. Does there exist a finitely generated group which is not nilpotent-by-finite and whose growth function has as a majorant a function of the form $c^{\sqrt{n}}$ where c is a constant greater than 1?

R. I. Grigorchuk

***9.10.** Do there exist finitely generated groups different from $\mathbb{Z}/2\mathbb{Z}$ which have precisely two conjugacy classes?

V. S. Guba

*Yes, they exist (D. Osin, *Ann. Math.*, **172**, no. 1 (2010), 1–39).

9.11. Is an abelian minimal normal subgroup A of a group G an elementary p-group if the factor-group G/A is a soluble group of finite rank? The answer is affirmative under the additional hypothesis that G/A is locally polycyclic (D. I. Zaitsev, *Algebra and Logic*, **19**, no. 2 (1980), 94–105).

D. I. Zaitsev

9.13. Is a soluble torsion-free group minimax if it satisfies the weak minimum condition for normal subgroups?

D. I. Zaitsev

9.14. Is a group locally finite if it decomposes into a product of periodic abelian subgroups which commute pairwise? The answer is affirmative in the case of two subgroups.

D. I. Zaitsev

9.15. Describe, without using CFSG, all subgroups L of a finite Chevalley group G such that $G = PL$ for some parabolic subgroup P of G.

A. S. Kondratiev

9.17. b) Let G be a locally normal residually finite group. Does there exist a normal subgroup H of G which is embeddable in a direct product of finite groups and is such that G/H is a divisible abelian group?

L. A. Kurdachenko

9.19. a) Let $n(X)$ denote the minimum of the indices of proper subgroups of a group X. A subgroup A of a finite group G is called *wide* if A is a maximal element by inclusion of the set $\{X \mid X \text{ is a proper subgroup of } G \text{ and } n(X) = n(G)\}$. Find all wide subgroups in finite projective special linear, symplectic, orthogonal, and unitary groups.

V. D. Mazurov

9.23. Let G be a finite group, B a block of characters of G, $D(B)$ its defect group and $k(B)$ (respectively, $k_0(B)$) the number of all irreducible complex characters (of height 0) lying in B. *Conjectures:*

a) (R. Brauer) $k(B) \leqslant |D(B)|$; this has been proved for p-soluble groups (D. Gluck, K. Magaard, U. Riese, P. Schmid, *J. Algebra*, **279** (2004), 694–719);

b) (J. B. Olsson) $k_0(B) \leqslant |D(B) : D(B)'|$, where $D(B)'$ is the derived subgroup of $D(B)$;

c) (R. Brauer) $D(B)$ is abelian if and only if $k_0(B) = k(B)$.

V. D. Mazurov

9.24. (J. G. Thompson). *Conjecture:* every finite simple non-abelian group G can be represented in the form $G = CC$, where C is some conjugacy class of G.

V. D. Mazurov

***9.25.** Find an algorithm which recognizes, by an equation $w(x_1, \ldots, x_n) = 1$ in a free group F and by a list of finitely generated subgroups $H_1, \ldots, H_n$ of F, whether there is a solution of this equation satisfying the condition $x_1 \in H_1, \ldots, x_n \in H_n$.

G. S. Makanin

*Such an algorithm is found in (V. Diekert, C. Gutiérrez, C. Hagenah, *Inf. Comput.*, **202**, no. 2 (2005), 105–140).

9.26. b) Describe the finite groups of 2-local 3-rank 1 which have non-cyclic Sylow 3-subgroups.

A. A. Makhnëv

9.28. Suppose that a finite group G is generated by a conjugacy class D of involutions and let $D_i = \{d_1 \cdots d_i \mid d_1, \ldots, d_i \text{ are different pairwise commuting elements of } D\}$. What is G, if $D_1, \ldots, D_n$ are all its different conjugacy classes of involutions? For example, the Fischer groups F_{22} and F_{23} satisfy this condition with $n = 3$.

A. A. Makhnëv

9.29. (Well-known problem). According to a classical theorem of Magnus, the word problem is soluble in 1-relator groups. Do there exist 2-relator groups with insoluble word problem?

Yu. I. Merzlyakov

9.31. Let k be a field of characteristic 0. According to (Yu. I. Merzlyakov, *Proc. Steklov Inst. Math.*, **167** (1986), 263–266), the family $\mathrm{Rep}_k(G)$ of all canonical matrix representations of a k-powered group G over k may be regarded as an affine k-variety. Find an explicit form of equations defining this variety.

Yu. I. Merzlyakov

9.32. What locally compact groups satisfy the following condition: the product of any two closed subgroups is also a closed subgroup? Abelian groups with this property were described in (Yu. N. Mukhin, *Math. Notes*, **8** (1970), 755–760).

Yu. N. Mukhin

9.35. A topological group is said to be *inductively compact* if any finite set of its elements is contained in a compact subgroup. Is this property preserved under lattice isomorphisms in the class of locally compact groups?

Yu. N. Mukhin

9.36. Characterize the lattices of closed subgroups in locally compact groups. In the discrete case this was done in (B. V. Yakovlev, *Algebra and Logic*, **13**, no. 6 (1974), 400–412).

Yu. N. Mukhin

9.37. Is a compactly generated inductively prosoluble locally compact group prosoluble?

Yu. N. Mukhin

9.38. A group is said to be *compactly covered* if it is the union of its compact subgroups. In a null-dimensional locally compact group, are maximal compactly covered subgroups closed?

Yu. N. Mukhin

9.39. Let Ω be a countable set and $\mathfrak{m}$ a cardinal number such that $\aleph_0 \leqslant \mathfrak{m} \leqslant 2^{\aleph_0}$ (we assume Axiom of Choice but not Continuum Hypothesis). Does there exist a permutation group G on Ω that has exactly $\mathfrak{m}$ orbits on the power set $\mathfrak{P}(\Omega)$?

Comment of 2001: It is proved (S. Shelah, S. Thomas, *Bull. London Math. Soc.*, **20**, no. 4 (1988), 313–318) that the answer is positive in set theory with Martin's Axiom. The question is still open in ZFC.

Peter M. Neumann

9.40. Let Ω be a countably infinite set. Define a *moiety* of Ω to be a subset Σ such that both Σ and $\Omega \setminus \Sigma$ are infinite. Which permutation groups on Ω are transitive on moieties?

Peter M. Neumann

9.41. Let Ω be a countably infinite set. For $k \geqslant 2$ define a *k-section* of Ω to be a partition of Ω as union of k infinite sets.

b) Does there exist a group that is transitive on k-sections but not on $(k+1)$-sections?

c) Does there exist a transitive permutation group on Ω that is transitive on $\aleph_0$-sections but which is a proper subgroup of $\mathrm{Sym}(\Omega)$?

Peter M. Neumann

9.42. Let Ω be a countable set and let D be the set of total order relations on Ω for which Ω is order-isomorphic with $\mathbb{Q}$. Does there exist a transitive proper subgroup G of $\mathrm{Sym}(\Omega)$ which is transitive on D?

Peter M. Neumann

9.43. b) The group G indicated in (N. D. Podufalov, *Abstracts of the 9th All–Union Symp. on Group Theory*, Moscow, 1984, 113–114 (Russian)) allows us to construct a projective plane of order 3: one can take as lines any class of subgroups of order $2 \cdot 5 \cdot 11 \cdot 17$ conjugate under S and add four more lines in a natural way. In a similar way, one can construct projective planes of order p^n for any prime p and any natural n. Could this method be adapted for constructing new planes?

N. D. Podufalov

9.44. A topological group is said to be *layer compact* if the full inverse images of all of its compacts under mappings $x \to x^n$, $n = 1, 2, \ldots$, are compacts. Describe the locally compact locally soluble layer compact groups.

V. M. Poletskikh

9.45. Let a be a vector in $\mathbb{R}^n$ with rational coordinates and set

$$S(a) = \{ka + b \mid k \in \mathbb{Z},\ \ b \in \mathbb{Z}^n\}.$$

It is obvious that $S(a)$ is a discrete subgroup in $\mathbb{R}^n$ of rank n. Find necessary and sufficient conditions in terms of the coordinates of a for $S(a)$ to have an orthogonal basis with respect to the standard scalar product in $\mathbb{R}^n$.

Yu. D. Popov, I. V. Protasov

9.47. Is it true that every scattered compact can be homeomorphically embedded into the space of all closed non-compact subgroups with E-topology of a suitable locally compact group?

I. V. Protasov

9.49. Let G be a compact group of weight $> \omega_2$. Is it true that the space of all closed subgroups of G with respect to E-topology is non-dyadic?

I. V. Protasov, Yu. V. Tsybenko

9.51. Does there exist a finitely presented soluble group satisfying the maximum condition on normal subgroups which has insoluble word problem?

D. J. S. Robinson

9.52. Does a finitely presented soluble group of finite rank have soluble conjugacy problem? Note: there is an algorithm to decide conjugacy to a given element of the group.

D. J. S. Robinson

9.53. Is the isomorphism problem soluble for finitely presented soluble groups of finite rank?

D. J. S. Robinson

9.55. Does there exist a finite p-group G and a central augmented automorphism φ of $\mathbb{Z}G$ such that φ, if extended to $\widehat{\mathbb{Z}}_pG$, the p-adic group ring, is *not* conjugation by unit in $\widehat{\mathbb{Z}}_pG$ followed by a homomorphism induced from a group automorphism?

K. W. Roggenkamp

***9.56.** Find all finite groups with the property that the tensor square of any ordinary irreducible character is multiplicity free.

J. Saxl

*Such groups are shown to be soluble (L. S. Kazarin, E. I. Chankov, *Sb. Math.*, **201**, no. 5 (2010), 655–668), and abundance of examples shows that a classification of such soluble groups is not feasible.

9.57. The set of subformations of a given formation is a lattice with respect to operations of intersection and generation. What formations of finite groups have distributive lattices of subformations?

A. N. Skiba

***9.59.** (W. Gaschütz). Prove that the formation generated by a finite group has finite lattice of subformations.

A. N. Skiba, L. A. Shemetkov

*Counterexamples are constructed (V. P. Burichenko, *J. Algebra*, **372** (2012), 428–458).

***9.60.** Let $\mathfrak{F}$ and $\mathfrak{H}$ be local formations of finite groups and suppose that $\mathfrak{F}$ is not contained in $\mathfrak{H}$. Does $\mathfrak{F}$ necessarily have at least one minimal local non-$\mathfrak{H}$-subformation?

A. N. Skiba, L. A. Shemetkov

*No, not necessarily (V. P. Burichenko, *Trudy Inst. Math. National Akad. Sci. Belarus'*, **21**, no. 1 (2013), 15–24 (Russian)).

9.61. Two varieties are said to be *S-equivalent* if they have the same Mal'cev theory (D. M. Smirnov, *Algebra and Logic*, **22**, no. 6 (1983), 492–501). What is the cardinality of the set of S-equivalent varieties of groups?

D. M. Smirnov

***9.64.** Is it true that, in a group of the form $G = AB$, every subgroup N of $A \cap B$ which is subnormal both in A and in B is subnormal in G? The answer is affirmative in the case of finite groups (H. Wielandt).

Ya. P. Sysak

*No, not always (C. Casolo, U. Dardano, *J. Group Theory*, **7** (2004), 507–520).

9.65. Is a locally soluble group periodic if it is a product of two periodic subgroups?

Ya. P. Sysak

9.66. a) *B. Jonsson's Conjecture:* elementary equivalence is preserved under taking free products in the class of all groups, that is, if $Th(G_1) = Th(G_2)$ and $Th(H_1) = Th(H_2)$ for groups G_1, G_2, H_1, H_2, then $Th(G_1 * H_1) = Th(G_2 * H_2)$.

b) It may be interesting to consider also the following *weakened conjecture:* if $Th(G_1) = Th(G_2)$ and $Th(H_1) = Th(H_2)$ for countable groups G_1, G_2, H_1, H_2, then for any numerations of the groups G_i, H_i there are m-reducibility $T_1 \underset{m}{\equiv} T_2$ and Turing reducibility $T_1 \underset{T}{\equiv} T_2$, where T_i is the set of numbers of all theorems in $Th(G_i * H_i)$. A proof of this weakened conjecture would be a good illustration of application of the reducibility theory to solving concrete mathematical problems.

A. D. Taimanov

9.68. Let $\mathfrak{V}$ be a variety of groups which is not the variety of all groups, and let p be a prime. Is there a bound on the p-lengths of the finite p-soluble groups whose Sylow p-subgroups are in $\mathfrak{V}$?

J. S. Wilson

9.69. (P. Cameron). Let G be a finite primitive permutation group and suppose that the stabilizer G_α of a point α induces on some of its orbits $\Delta \neq \{\alpha\}$ a regular permutation group. Is it true that $|G_\alpha| = |\Delta|$?

A. N. Fomin

9.70. Prove or disprove the conjecture of P. Cameron (*Bull. London Math. Soc.*, **13**, no. 1 (1981), 1–22): if G is a finite primitive permutation group of subrank m, then either the rank of G is bounded by a function of m or the order of a point stabilizer is at most m. The *subrank* of a transitive permutation group is defined to be the maximum rank of the transitive constituents of a point stabilizer.

A. N. Fomin

9.71. Is it true that every infinite 2-transitive permutation groups with locally soluble point stabilizer has a non-trivial irreducible finite-dimensional representation over some field?

A. N. Fomin

9.72. Let G_1 and G_2 be Lie groups with the following property: each G_i contains a nilpotent simply connected normal Lie subgroup B_i such that $G_i/B_i \cong SL_2(K)$, where $K = \mathbb{R}$ or $\mathbb{C}$. Assume that G_1 and G_2 are contained as closed subgroups in a topological group G, that $G_1 \cap G_2 \geqslant B_1B_2$, and that no non-identity Lie subgroup of $B_1 \cap B_2$ is normal in G. Can it then be shown (perhaps by using the method of "amalgams" from the theory of finite groups) that the nilpotency class and the dimension of B_1B_2 is bounded?

A. L. Chermak

9.75. Find all local formations $\mathfrak{F}$ of finite groups such that, in every finite group, the set of $\mathfrak{F}$-subnormal subgroups forms a lattice.

L. A. Shemetkov

9.76. We define a *Golod group* to be the r-generated, $r \geqslant 2$, subgroup $\langle 1 + x_1 + I, 1 + x_2 + I, \ldots, 1 + x_r + I\rangle$ of the adjoint group $1 + F/I$ of the factor-algebra F/I, where F is a free algebra of polynomials without constant terms in non-commuting variables $x_1, x_2, \ldots, x_r$ over a field of characteristic $p > 0$, and I is an ideal of F such that F/I is a non-nilpotent nil-algebra (see E. S. Golod, *Amer. Math. Soc. Transl. (2)*, **48** (1965), 103–106). Prove that in Golod groups the centralizer of every element is infinite.

Note that Golod groups with infinite centre were constructed by A. V. Timofeenko (*Math. Notes*, **39**, no. 5 (1986), 353–355); independently by other methods the same result was obtained in the 90s by L. Hammoudi.

V. P. Shunkov

9.77. Does there exist an infinite finitely generated residually finite binary finite group all of whose Sylow subgroups are finite? A. V. Rozhkov (Dr. of Sci. Disser., 1997) showed that such a group does exist if the condition of finiteness of the Sylow subgroups is weakened to local finiteness.

V. P. Shunkov

9.78. Does there exist a periodic residually finite F^*-group (see Archive, 7.42) all of whose Sylow subgroups are finite and which is not binary finite?

V. P. Shunkov

9.83. Suppose that G is a (periodic) p-conjugacy biprimitively finite group (see 6.59) which has a finite Sylow p-subgroup. Is it then true that all Sylow p-subgroups of G are conjugate?

V. P. Shunkov

9.84. a) Is every binary finite 2-group of finite exponent locally finite?

b) The same question for p-groups for $p > 2$.

V. P. Shunkov

Problems from the 10th Issue (1986)

10.2. A *mixed identity* of a group G is, by definition, an identity of an algebraic system obtained from G by supplementing its signature by some set of 0-ary operations. One can develop a theory of mixed varieties of groups on the basis of this notion (see, for example, V. S. Anashin, *Math. USSR Sbornik*, **57** (1987), 171–182). Construct an example of a class of groups which is not a mixed variety, but which is closed under taking factor-groups, Cartesian products, and those subgroups of Cartesian powers that contain the diagonal subgroup.

V. S. Anashin

10.3. Characterize (in terms of bases of mixed identities or in terms of generating groups) minimal mixed varieties of groups.

V. S. Anashin

10.4. Let p be a prime number. Is it true that a mixed variety of groups generated by an arbitrary finite p-group of sufficiently large nilpotency class is a variety of groups?

V. S. Anashin

10.5. Construct an example of a finite group whose mixed identities do not have a finite basis. Is the group constructed in (R. Bryant, *Bull. London Math. Soc.*, **14**, no. 2 (1982), 119–123) such an example?

V. S. Anashin

10.8. Does there exist a topological group which cannot be embedded in the multiplicative semigroup of a topological ring?

V. I. Arnautov, A. V. Mikhalëv

10.10. Is it true that a quasivariety generated by a finitely generated torsion-free soluble group and containing a non-abelian free metabelian group, can be defined in the class of torsion-free groups by an independent system of quasiidentities?

A. I. Budkin

10.11. Is it true that every finitely presented group contains either a free subsemigroup on two generators or a nilpotent subgroup of finite index?

R. I. Grigorchuk

10.12. Does there exist a finitely generated semigroup S with cancellation having non-exponential growth and such that its group of left quotients $G = S^{-1}S$ (which exists) is a group of exponential growth? An affirmative answer would give a positive solution to Problem 12 in (S. Wagon, *The Banach–Tarski Paradox*, Cambridge Univ. Press, 1985).

R. I. Grigorchuk

10.13. Does there exist a *massive set* of independent elements in a free group F_2 on free generators a, b, that is, a set E of irreducible words on the alphabet a, b, a^{-1}, b^{-1} such that

1) no element $w \in E$ belongs to the normal closure of $E \setminus \{w\}$, and

2) the massiveness condition is satisfied: $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|E_n|} = 3$, where E_n is the set of all words of length n in E?

R. I. Grigorchuk

10.15. For every (known) finite quasisimple group and every prime p, find the faithful p-modular absolutely irreducible linear representations of minimal degree.

A. S. Kondratiev

10.16. A class of groups is called a *direct variety* if it is closed under taking subgroups, factor-groups, and direct products (Yu. M. Gorchakov, *Groups with Finite Classes of Conjugate Elements*, Moscow, Nauka, 1978 (Russian)). It is obvious that the class of FC-groups is a direct variety. P. Hall (*J. London Math. Soc.*, **34**, no. 3 (1959), 289–304) showed that the class of finite groups and the class of abelian groups taken together do not generate the class of FC-groups as a direct variety, and it was shown in (L. A. Kurdachenko, *Ukrain. Math. J.*, **39**, no. 3 (1987), 255–259) that the direct variety of FC-groups is also not generated by the class of groups with finite derived subgroups. Is the direct variety of FC-groups generated by the class of groups with finite derived subgroups together with the class of FC-groups having quasicyclic derived subgroups?

L. A. Kurdachenko

10.17. (M. J. Tomkinson). Let G be an FC-group whose derived subgroup is embeddable in a direct product of finite groups. Must $G/Z(G)$ be embeddable in a direct product of finite groups?

L. A. Kurdachenko

10.18. (M. J. Tomkinson). Let G be an FC-group which is residually in the class of groups with finite derived subgroups. Must $G/Z(G)$ be embeddable in a direct product of finite groups?

L. A. Kurdachenko

10.19. Characterize the radical associative rings such that the set of all normal subgroups of the adjoint group coincides with the set of all ideals of the associated Lie ring.

V. M. Levchuk

10.20. In a Chevalley group of rank $\leqslant 6$ over a finite field of order $\leqslant 9$, describe all subgroups of the form $H = \langle H \cap U, H \cap V \rangle$ that are not contained in any proper parabolic subgroup, where U and V are opposite unipotent subgroups.

V. M. Levchuk

10.23. Is it true that extraction of roots in braid groups is unique up to conjugation?

G. S. Makanin

***10.25.** (Well-known problem). Does there exist an algorithm which decides for a given automorphism of a free group whether this automorphism has a non-trivial fixed point?

G. S. Makanin

*Yes, it does (O. Bogopolski, O. Maslakova, *A basis of the fixed point subgroup of an automorphism of a free group*, `arXiv:1204.6728`).

10.26. *a) Does there exist an algorithm which decides, for given elements a, b and an automorphism φ of a free group, whether the equation $ax^{\varphi} = xb$ is soluble in this group? This question seems to be useful for solving the problem of equivalence of two knots.

b) More generally: Does there exist an algorithm which decides whether the equation of the form $w(x_{i_1}^{\varphi_1}, \dots, x_{i_n}^{\varphi_n}) = 1$ is soluble in a free group where $\varphi_1, \dots, \varphi_n$ are automorphisms of this group?

G. S. Makanin

*a) Yes, it does (O. Bogopolskii, A. Martino, O. Maslakova, E. Ventura, *Bull. London Math. Soc.*, **38**, no. 5 (2006), 787–794).

10.27. a) Let t be an involution of a finite group G and suppose that the set $D = t^G \cup \{t^x t^y \mid x, y \in G,\ |t^x t^y| = 2\}$ does not intersect $O_2(G)$. Prove that if $t \in O_2(C(d))$ for any involution d from $C_D(t)$, then $D = t^G$ (and in this case the structure of the group $\langle D \rangle$ is known).

b) A significantly more general question. Let t be an involution of a finite group G and suppose that $t \in Z^*(N(X))$ for every non-trivial subgroup X of odd order which is normalized, but not centralized, by t. What is the group $\langle t^G \rangle$?

A. A. Makhnëv

***10.28.** Is it true that finite strongly regular graphs with $\lambda = 1$ have rank 3?

A. A. Makhnëv

*No, not always (A. Cossidente, T. Penttila, *J. London Math. Soc.*, **72** (2005), 731–741).

10.29. Describe the finite groups which contain a set of involutions D such that, for any subset D_0 of D generating a 2-subgroup, the normalizer $N_D(D_0)$ also generates a 2-subgroup.

A. A. Makhnëv

10.31. Suppose that G is an algebraic group, H is a closed normal subgroup of G and $f : G \to G/H$ is the canonical homomorphism. What conditions ensure that there exists a rational section $s\colon G/H \to G$ such that $sf = 1$? For partial results, see (Yu. I. Merzlyakov, *Rational Groups*, Nauka, Moscow, 1987, § 37 (Russian)).

Yu. I. Merzlyakov

10.32. A group word is said to be *universal on a group* G if its values on G run over the whole of G. For an arbitrary constant $c > 8/5$, J. L. Brenner, R. J. Evans, and D. M. Silberger (*Proc. Amer. Math. Soc.*, **96**, no. 1 (1986), 23–28) proved that there exists a number $N_0 = N_0(c)$ such that the word $x^r y^s$, $rs \neq 0$, is universal on the alternating group $\mathbb{A}_n$ for any $n \geqslant \max\{N_0,\ c \cdot \log m(r,s)\}$, where $m(r,s)$ is the product of all primes dividing rs if $r, s \notin \{-1, 1\}$, and $m(r,s) = 1$ if $r, s \in \{-1, 1\}$. For example, one can take $N_0(5/2) = 5$ and $N_0(2) = 29$. Find an analogous bound for the degree of the symmetric group $\mathbb{S}_n$ under hypothesis that at least one of the r, s is odd: up to now, here only a more crude bound $n \geqslant \max\{6,\ 4m(r,s) - 4\}$ is known (M. Droste, *Proc. Amer. Math. Soc.*, **96**, no. 1 (1986), 18–22).

Yu. I. Merzlyakov

10.34. Does there exist a non-soluble finite group which coincides with the product of any two of its non-conjugate maximal subgroups?

Editors' comment: the answer is negative for almost simple groups (T. V. Tikhonenko, V. N. Tyutyanov, *Siberian Math. J.*, **51**, no. 1 (2010), 174–177).

V. S. Monakhov

10.35. Is it true that every finitely generated torsion-free subgroup of $GL_n(\mathbb{C})$ is residually in the class of torsion-free subgroups of $GL_n(\overline{\mathbb{Q}})$?

G. A. Noskov

10.36. Is it true that $SL_n(\mathbb{Z}[x_1, \dots, x_r])$, for n sufficiently large, is a group of type $(FP)_m$ (which is defined in the same way as $(FP)_\infty$ was defined in 6.3, but with that weakening that the condition of being finitely generated is not imposed on the terms of the resolution with numbers $> m$). An affirmative answer is known for $m = 0$, $n \geqslant 3$ (A. A. Suslin, *Math. USSR Izvestiya*, **11** (1977), 221–238) and for $m = 1$, $n \geqslant 5$ (M. S. Tulenbayev, *Math. USSR Sbornik*, **45** (1983), 139–154; U. Rehmann, C. Soulé, in: *Algebraic K-theory, Proc. Conf., Northwestern Univ., Evanston, Ill., 1976,* (*Lect. Notes Math.*, **551**), Springer, Berlin, 1976, 164–169).

G. A. Noskov

10.38. (W. van der Kallen). Does $E_{n+3}(\mathbb{Z}[x_1, \dots, x_n])$, $n \geqslant 1$, have finite breadth with respect to the set of transvections?

G. A. Noskov

10.39. a) Is the occurrence problem soluble for the subgroup $E_n(R)$ of $SL_n(R)$ where R is a commutative ring?

b) Is the word problem soluble for the groups $K_i(R)$ where K_i are the Quillen K-functors and R is a commutative ring?

G. A. Noskov

10.40. (H. Bass). Let G be a group, e an idempotent matrix over $\mathbb{Z}G$ and let $\operatorname{tr} e = \sum\limits_{g \in G} e_g g$, $e_g \in \mathbb{Z}$. *Strong conjecture:* for any non-trivial $x \in G$, the equation $\sum\limits_{g \sim x} e_g = 0$ holds where $\sim$ denotes conjugacy in G. *Weak conjecture:* $\sum\limits_{g \in G} e_g = 0$. The strong conjecture has been proved for finite, abelian and linear groups and the weak conjecture has been proved for residually finite groups.

H. Bass' Comment of 2005: Significant progress has been made; a good up-to-date account is in the paper (A. J. Berrick, I. Chatterji, G. Mislin, *Math. Ann.*, **329** (2004), 597–621).

G. A. Noskov

10.42. (J. T. Stafford). Let G be a poly-$\mathbb{Z}$-group and let k be a field. Is it true that every finitely generated projective kG-module is either a free module or an ideal?

G. A. Noskov

10.43. Let R and S be associative rings with identity such that 2 is invertible in S. Let $\Lambda_I\colon GL_n(R) \to GL_n(R/I)$ be the homomorphism corresponding to an ideal I of R and let $E_n(R)$ be the subgroup of $GL_n(R)$ generated by elementary transvections $t_{ij}(x)$. Let $a_{ij} = t_{ij}(1)t_{ji}(-1)t_{ij}(1)$, let the bar denote images in the factor-group of $GL_n(R)$ by the centre and let $PG = \overline{G}$ for $G \leqslant GL_n(R)$. A homomorphism

$$\Lambda\colon E_n(R) \to GL(W) = GL_m(S)$$

is called *standard* if $S^m = P \oplus \cdots \oplus P \oplus Q$ (a direct sum of S-modules in which there are n summands P) and

$$\Lambda x = g^{-1}\tau(\delta^*(x)f + ({}^t\delta^*(x)^\nu)^{-1}(1-f))g, \qquad x \in E_n(R),$$

where $\delta^*: GL_n(R) \to GL_n(\operatorname{End} P)$ is the homomorphism induced by a ring homomorphism $\delta: R \to \operatorname{End} P$ taking identity to identity, g is an isomorphism of the module W onto S^m, $\tau: GL_n(\operatorname{End} P) \to GL(gW)$ is an embedding, f is a central idempotent of δR, t denotes transposition and ν is an antiisomorphism of δR. Let $n \geqslant 3$, $m \geqslant 2$. One can show that the homomorphism

$$\Lambda_0\colon PE_n(R) \to GL(W) = GL_m(S)$$

is induced by a standard homomorphism Λ if $\Lambda_0\bar{a}_{ij} = g^{-1}\tau(a_{ij}^*)g$ for some g and τ and for any $i \neq j$ where a_{ij}^* denotes the matrix obtained from a_{ij} by replacing 0 and 1 from R by 0 and 1 from $\operatorname{End} P$. Find (at least in particular cases) the form of a homomorphism Λ_0 that does not satisfy the last condition.

V. M. Petechuk

10.44. Prove that every standard homomorphism Λ of $E_n(R) \subset GL_n(R) = GL(V)$ into $E_m(S) \subset GL_m(S) = GL(W)$ originates from a collineation and a correlation, that is, has the form $\Lambda x = (g^{-1}xg)f + (h^{-1}xh)(1-f)$, where g is a semilinear isomorphism $V_R \to W_S$ (collineation) and h is a semilinear isomorphism $V_R \to_S W'_{S^0}$ (correlation).

V. M. Petechuk

10.45. Let $n \geqslant 3$ and suppose that N is a subgroup of $GL_n(R)$ which is normalized by $E_n(R)$. Prove that either N contains $E_n(R)$ or $\Lambda_I[N, E_n(R)] = 1$ for a suitable ideal $I \neq R$ of R.

V. M. Petechuk

10.46. Prove that for every element $\sigma \in GL_n(R)$, $n \geqslant 3$, there exist transvections τ_1, τ_2 such that $[[\sigma, \tau_1], \tau_2]$ is a unipotent element.

V. M. Petechuk

10.47. Describe the automorphisms of $PE_2(R)$ in the case where the ring R is commutative and 2 and 3 have inverses in it.

V. M. Petechuk

10.49. Does there exist a group G satisfying the following four conditions:

1) G is simple, moreover, there is an integer n such that $G = C^n$ for any conjugacy class C,

2) all maximal abelian subgroups of G are conjugate in G,

3) every maximal abelian subgroup of G is self-normalizing and it is the centralizer of any of its nontrivial elements,

4) there is an integer m such that if H is a maximal abelian subgroup of G and $a \in G \setminus H$ then every element in G is a product of m elements in aH?

Comment of 2005: There is a partial solution in (E. Jaligot, A. Ould Houcine, *J. Algebra*, **280** (2004), 772–796).

B. Poizat

10.50. Complex characters of a finite group G induced by linear characters of cyclic subgroups are called *induced cyclic characters* of G. Computer-aided computations (A. V. Rukolaine, *Abstracts of the 10th All–Union Sympos. on Group Theory*, Gomel', 1986, p. 199 (Russian)) show that there exist groups (for example, $\mathbb{S}_5$, $SL_2(13)$, M_{11}) all of whose irreducible complex characters are integral linear combinations of induced cyclic characters and the principal character of the group. Describe all finite groups with this property.

A. V. Rukolaine

10.51. Describe the structure of thin abelian p-groups. Such a description is known in the class of separable p-groups (C. Megibben, *Mich. Math. J.*, **13**, no. 2 (1966), 153–160).

S. V. Rychkov

10.52. (R. Mines). It is well known that the topology on the completion of an abelian group under the p-adic topology is the p-adic topology. R. Warfield has shown that this is also true in the category of nilpotent groups. Find a categorical setting for this theorem which includes the case of nilpotent groups.

S. V. Rychkov

10.53. (M. Dugas). Let $\mathfrak{R}$ be the *Reid class*, i. e. the smallest containing $\mathbb{Z}$ and closed under direct sums and direct products. Is $\mathfrak{R}$ closed under direct summands?

S. V. Rychkov

10.54. (R. Göbel). For a cardinal number μ, let

$$\mathbb{Z}^{<\mu} = \{f \in \mathbb{Z}^{\mu} \mid |\mathrm{supp}\,(f)| < \mu\} \quad \text{and} \quad G_{\mu} = \mathbb{Z}^{\mu}/\mathbb{Z}^{<\mu}.$$

a) Find a non-zero direct summand D of G_{ω_1} such that $D \ncong G_{\omega_1}$.
b) Investigate the structure of G_{μ} (the structure of G_{ω_0} is well known).

S. V. Rychkov

10.55. (A. Mader). "Standard B", that is, $B = \mathbb{Z}(p) \oplus \mathbb{Z}(p^2) \oplus \cdots$, is slender as a module over its endomorphism ring (A. Mader, in: *Abelian Groups and Modules, Proc., Udine, 1984*, Springer, 1984, 315–327). Which abelian p-groups are slender as modules over their endomorphism rings?

S. V. Rychkov

10.57. What are the minimal non-A-formations? An *A-formation* is, by definition, the formation of the finite groups all of whose Sylow subgroups are abelian.

A. N. Skiba

10.58. Is the subsemigroup generated by the undecomposable formations in the semigroup of formations of finite groups free?

A. N. Skiba

10.59. Is a p'-group G locally nilpotent if it admits a splitting automorphism φ of prime order p such that all subgroups of the form $\langle g, g^{\varphi}, \ldots, g^{\varphi^{p-1}}\rangle$ are nilpotent? An automorphism φ of order p is called *splitting* if $gg^{\varphi}g^{\varphi^2}\cdots g^{\varphi^{p-1}} = 1$ for all $g \in G$.

A. I. Sozutov

10.60. Does every periodic group (p-group) A of regular automorphisms of an abelian group have non-trivial centre?

Editors' comment (2001): The answer is affirmative if A contains an element of order 2 or 3 (A. Kh. Zhurtov, *Siberian Math. J.*, **41**, no. 2 (2000), 268–275).

A. I. Sozutov

10.61. Suppose that H is a proper subgroup of a group G, $a \in H$, $a^2 \neq 1$ and for every $g \in G \setminus H$ the subgroup $\langle a, a^g \rangle$ is a Frobenius group whose complement contains a. Does the set-theoretic union of the kernels of all Frobenius subgroups of G with complement $\langle a \rangle$ constitute a subgroup? For definitions see 6.55; see also (A. I. Sozutov, *Algebra and Logic*, **34**, no. 5 (1995), 295–305).

Editors' comment (2005): The answer is affirmative if the order of a is even (A. M. Popov, A. I. Sozutov, *Algebra and Logic*, **44**, no. 1 (2005), 40–45) or if the order of a is not 3 or 5 and the group $\langle a, a^g \rangle$ is finite for any $g \notin H$ (A. M. Popov, *Algebra and Logic*, **43**, no. 2 (2004), 123–127).

A. I. Sozutov

10.62. Construct an example of a (periodic) group without subgroups of index 2 which is generated by a conjugacy class of involutions X such that the order of the product of any two involutions from X is odd.

A. I. Sozutov

10.64. Does there exist a non-periodic doubly transitive permutation group with a periodic stabilizer of a point?

Ya. P. Sysak

10.65. Determine the structure of infinite 2-transitive permutation groups (G, Ω) in which the stabilizer of a point $\alpha \in \Omega$ has the form $G_\alpha = A \cdot G_{\alpha\beta}$ where $G_{\alpha\beta}$ is the stabilizer of two points α, β, $\alpha \neq \beta$, such that $G_{\alpha\beta}$ contains an element inverting the subgroup A. Suppose, in particular, that $A \setminus \{1\}$ contains at most two conjugacy classes of G_α; does G then possess a normal subgroup isomorphic to PSL_2 over a field?

A. N. Fomin

10.67. The class $LN\mathfrak{M}_p$ of locally nilpotent groups admitting a splitting automorphism of prime order p (for definition see 10.59) is a variety of groups with operators (E. I. Khukhro, *Math. USSR Sbornik*, **58** (1987), 119–126). Is it true that

$$LN\mathfrak{M}_p = (\mathfrak{N}_{c(p)} \cap LN\mathfrak{M}_p) \vee (\mathfrak{B}_p \cap LN\mathfrak{M}_p)$$

where $\mathfrak{N}_{c(p)}$ is the variety of nilpotent groups of some p-bounded class $c(p)$ and $\mathfrak{B}_p$ is the variety of groups of exponent p?

E. I. Khukhro

10.70. Find a geometrical justification for the Whitehead method for free products similar to the substantiation given for free groups by Whitehead himself and more visual than given in (D. J. Collins, H. Zieschang, *Math. Z.*, **185**, no. 4 (1984), 487–504; **186**, no. 3 (1984), 335–361).

D. J. Collins' comment: there is a partial solution in (D. McCullough, A. Miller, *Symmetric automorphisms of free products*, Mem. Amer. Math. Soc. **582** (1996)).

H. Zieschang

10.71. Is it true that the centralizer of any automorphism (any finite set of automorphisms) in the automorphism group of a free group of finite rank is a finitely presented group? This is true in the case of rank 2 and in the case of inner automorphisms for any finite rank.

V. A. Churkin

10.73. Enumerate all formations of finite groups all of whose subformations are S_n-closed.

L. A. Shemetkov

10.74. Suppose that a group G contains an element a of prime order such that its centralizer $C_G(a)$ is finite and all subgroups $\langle a, a^g\rangle$, $g \in G$, are finite and almost all of them are soluble. Is G locally finite? This problem is closely connected with 6.56. The question was solved in the positive for a number of very important partial cases by the author (*Abstracts on Group Theory of the Mal'cev Int. Conf. on Algebra*, Novosibirsk, 1989, p. 145 (Russian)).

V. P. Shunkov

10.75. Suppose that a group G contains an element a of prime order p such that the normalizer of every finite subgroup containing a has finite periodic part and all subgroups $\langle a, a^g\rangle$, $g \in G$, are finite and almost all of them are soluble. Does G possess a periodic part if $p > 2$? It was proved in (V. P. Shunkov, *Groups with involutions*, Preprints no. 4, 5, 12 of the Comput. centre of SO AN SSSR, Krasnoyarsk, 1986 (Russian)) that if a is a point, then the answer is affirmative; on the other hand, a group with a point a of order 2 satisfying the given hypothesis, which has no periodic part, was exhibited in the same works. For the definition of a *point* see (V. I. Senashov, V. P. Shunkov, *Algebra and Logic*, **22**, no. 1 (1983), 66–81).

V. P. Shunkov

10.77. Suppose that G is a periodic group containing an elementary abelian subgroup R of order 4. Must G be locally finite

a) if $C_G(R)$ is finite?

b) if the centralizer of every involution of R in G is a Chernikov group?

Remark of 1999: P. V. Shumyatsky (*Quart. J. Math. Oxford (2)*, **49**, no. 196 (1998), 491–499) gave a positive answer to the question a) in the case where G is residually finite.

V. P. Shunkov

10.78. Does there exist a non-Chernikov group which is a product of two Chernikov subgroups?

V. P. Shunkov

Problems from the 11th Issue (1990)

11.1. (Well-known problem). Describe the structure of the centralizers of unipotent elements in almost simple groups of Lie type.
R. Zh. Aleev

11.2. Classify the simple groups that are isomorphic to the multiplicative groups of finite rings, in particular, of the group rings of finite groups over finite fields and over $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{Z}$.
R. Zh. Aleev

11.3. (M. Aschbacher). A p-local subgroup H in a group G is said to be a *superlocal* if $H = N_G(O_p(H))$. Describe the superlocals in alternating groups and in groups of Lie type.
R. Zh. Aleev

11.5. Let k be a commutative ring and let G be a torsion-free almost polycyclic group. Suppose that P is a finitely generated projective module over the group ring kG and P contains two elements independent over kG. Is P a free module?
V. A. Artamonov

11.7. a) Find conditions for a group G, given by its presentation, under which the property *some term of the lower central series of G is a free group* $(*)$ implies the residual finiteness of G.

b) Find conditions on the structure of a residually finite group G which ensure property $(*)$ for G.
K. Bencsáth

11.8. For a finite group X, let $\chi_1(X)$ denote the totality of the degrees of all irreducible complex characters of X with allowance for their multiplicities. Suppose that $\chi_1(G) = \chi_1(H)$ for groups G and H. Clearly, then $|G| = |H|$. Is it true that

*a) H is simple if G is simple?

b) H is soluble if G is soluble?

It is known that H is a Frobenius group if G is a Frobenius group.
Ya. G. Berkovich

*a) Yes, it is (H. P. Tong-Viet, *J. Algebra*, **357** (2012), 61–68; *Monatsh. Math.*, **166**, no. 3–4 (2012), 559–577; *Algebr. Represent. Theory*, **15**, no. 2 (2012), 379–389).

11.9. (I. I. Pyatetskiĭ-Shapiro). Does there exist a finite non-soluble group G such that the set of characters induced by the trivial characters of representatives of all conjugacy classes of subgroups of G is linearly independent?
Ya. G. Berkovich

11.10. (R. C. Lyndon). a) Does there exist an algorithm that, given a group word $w(a, x)$, recognizes whether a is equal to the identity element in the group $\langle a, x \mid a^n = 1,\ \ w(a, x) = 1\rangle$?
V. V. Bludov

11.11. a) The well-known Baer–Suzuki theorem states that if every two conjugates of an element a of a finite group G generate a finite p-subgroup, then a is contained in a normal p-subgroup. Does such a theorem hold in the class of periodic groups?

The case $p = 2$ is of particular interest.
A. V. Borovik

11.12. a) Suppose that G is a simple locally finite group in which the centralizer of some element is a linear group, that is, a group admitting a faithful matrix representation over a field. Is G itself a linear group?

b) The same question with replacement of the word "linear" by "finitary linear". A group H is *finitary linear* if it admits a faithful representation on an infinite-dimensional vector space V such that the residue subspaces $V(1-h)$ have finite dimensions for all $h \in H$.

A. V. Borovik

***11.13.** Suppose that G is a periodic group with an involution i such that $i^g \cdot i$ has odd order for any $g \in G$. Is it true that the image of i in $G/O(G)$ belongs to the centre of $G/O(G)$?

A. V. Borovik

*No, not always (E. B. Durakov, A. I. Sozutov, *Abstracts of Int. Conf. "Algebra and Logic", 21–27 July 2013*, Siberian Federal Univ., Krasnoyarsk, 2013, 119–120 (Russian)).

11.14. Is every finite simple group characterized by its Cartan matrix over an algebraically closed field of characteristic 2? In (R. Brandl, *Arch. Math.*, **38** (1982), 322–323) it is shown that for any given finite group there exist only finitely many finite groups with the same Cartan matrix.

R. Brandl

11.15. It is known that for each prime number p there exists a series $a_1, a_2, \dots$ of words in two variables such that the finite group G has abelian Sylow p-subgroups if and only if $a_k(G) = 1$ for almost all k. For $p = 2$ such a series is known explicitly (R. Brandl, *J. Austral. Math. Soc.*, **31** (1981), 464–469). What about $p > 2$?

R. Brandl

11.16. Let V_r be the class of all finite groups G satisfying a law $[x,\, _ry] = [x,\, _sy]$ for some $s = s(G) > r$. Here $[x,\, _1y] = [x, y]$ and $[x,\, _{i+1}y] = [[x,\, _iy], y]$.

a) Is there a function f such that every soluble group in V_r has Fitting length $< f(r)$? For $r < 3$ see (R. Brandl, *Bull. Austral. Math. Soc.*, **28** (1983), 101–110).

b) Is it true that V_r contains only finitely many nonabelian simple groups? This is true for $r < 4$.

R. Brandl

11.17. Let G be a finite group and let $d = d(G)$ be the least positive integer such that G satisfies a law $[x,\, _ry] = [x,\, _{r+d}y]$ for some nonnegative integer $r = r(G)$.

a) Let $e = 1$ if $d(G)$ is even and $e = 2$ otherwise. Is it true that the exponent of $G/F(G)$ divides $e \cdot d(G)$?

b) If G is a nonabelian simple group, does the exponent of G divide $d(G)$?

Part a) is true for soluble groups (N. D. Gupta, H. Heineken, *Math. Z.*, **95** (1967), 276–287). I have checked part b) for $\mathbb{A}_n$, $PSL(2, q)$, and a number of sporadic groups.

R. Brandl

11.18. Let $G(a, b) = \langle x, y \mid x = [x,\, _ay],\ y = [y,\, _bx]\rangle$. Is $G(a, b)$ finite?

It is easy to show that $G(1, b) = 1$ and one can show that $G(2, 2) = 1$. Nothing is known about $G(2, 3)$. If one could show that every minimal simple group is a quotient of some $G(a, b)$, then this would yield a very nice sequence of words in two variables to characterize soluble groups, see (R. Brandl, J. S. Wilson, *J. Algebra*, **116** (1988), 334–341.)

R. Brandl

11.19. (C. Sims). Is the nth term of the lower central series of an absolutely free group the normal closure of the set of basic commutators (in some fixed free generators) of weight exactly n?

D. Jackson announced a positive answer for $n \leqslant 5$. *A. Gaglione, D. Spellman*

11.22. Characterize all p-groups, p a prime, that can be faithfully represented as $n \times n$ triangular matrices over a division ring of characteristic p.

If $p \geqslant n$ the solution is given in (B. A. F. Wehrfritz, *Bull. London Math. Soc.*, **19** (1987), 320–324). The corresponding question with $p = 0$ was solved by B. Hartley and P. Menal (*Bull. London Math. Soc.*, **15** (1983), 378–383), but see above reference for a second proof. *B. A. F. Wehrfritz*

11.23. An automorphism φ of a group G is called a *nil-automorphism* if, for every $a \in G$, there exists n such that $[a,\, {}_n\varphi] = 1$. Here $[x,\, {}_1y] = [x, y]$ and $[x,\, {}_{i+1}y] = [[x,\, {}_iy], y]$. An automorphism φ is called an *e-automorphism* if, for any two φ-invariant subgroups A and B such that $A \not\subseteq B$, there exists $a \in A \setminus B$ such that $[a, \varphi] \in B$. Is every e-automorphism of a group a nil-automorphism? *V. G. Vilyatser*

11.25. b) Do there exist local Fitting classes which are decomposable into a non-trivial product of Fitting classes and in every such a decomposition all factors are non-local? For the definition of the *product* of Fitting classes see (N. T. Vorob'ëv, *Math. Notes*, **43**, no. 2 (1988), 91–94). *N. T. Vorob'ëv*

11.28. Suppose the prime graph of the finite group G is disconnected. (This means that the set of prime divisors of the order of G is the disjoint union of non-empty subsets π and π' such that G contains no element of order pq where $p \in \pi$, $q \in \pi'$.) Then P. A. Linnell (*Proc. London Math. Soc.*, **47**, no. 1 (1983), 83–127) has proved mod CFSG that there is a decomposition of $\mathbb{Z}G$-modules $\mathbb{Z} \oplus \mathbb{Z}G = A \oplus B$ with A and B non-projective. Find a proof independent of CFSG. *K. Gruenberg*

11.29. Let F be a free group and $\mathfrak{f} = \mathbb{Z}F(F-1)$ the augmentation ideal of the integral group ring $\mathbb{Z}F$. For any normal subgroup R of F define the corresponding ideal $\mathfrak{r} = \mathbb{Z}F(R-1) = {}_{\mathrm{id}}(r - 1 \mid r \in R)$. One may identify, for instance, $F \cap (1 + \mathfrak{r}\mathfrak{f}) = R'$, where F is naturally imbedded into $\mathbb{Z}F$ and $1 + \mathfrak{r}\mathfrak{f} = \{1 + a \mid a \in \mathfrak{r}\mathfrak{f}\}$.

Identify in an analogous way in terms of corresponding subgroups of F

a) $F \cap (1 + \mathfrak{r}_1\mathfrak{r}_2 \cdots \mathfrak{r}_n)$, where R_i are normal subgroups of F, $i = 1, 2, \ldots, n$;

b) $F \cap (1 + \mathfrak{r}_1\mathfrak{r}_2\mathfrak{r}_3)$;

c) $F \cap (1 + \mathfrak{f}\mathfrak{s} + \mathfrak{f}^n)$, where F/S is finitely generated nilpotent;

d) $F \cap (1 + \mathfrak{f}\mathfrak{s}\mathfrak{f} + \mathfrak{f}^n)$;

e) $F \cap (1 + \mathfrak{r}(k) + \mathfrak{f}^n)$, $n > k \geqslant 2$, where $\mathfrak{r}(k) = \mathfrak{r}\mathfrak{f}^{k-1} + \mathfrak{f}\mathfrak{r}\mathfrak{f}^{k-2} + \cdots + \mathfrak{f}^{k-1}\mathfrak{r}$.

N. D. Gupta

11.30. Is it true that the rank of a torsion-free soluble group is equal to the rank of any of its subgroups of finite index? The answer is affirmative for groups having a rational series (D. I. Zaitsev, in: *Groups with restrictions on subgroups*, Naukova dumka, Kiev, 1971, 115–130 (Russian)). We note also that every torsion-free soluble group of finite rank contains a subgroup of finite index which has a rational series.

D. I. Zaitsev

11.31. Is a radical group polycyclic if it is a product of two polycyclic subgroups? The answer is affirmative for soluble and for hyperabelian groups (D. I. Zaitsev, *Math. Notes*, **29**, no. 4 (1981), 247–252; J. C. Lennox, J. E. Roseblade, *Math. Z.*, **170** (1980), 153–154).

D. I. Zaitsev

11.32. Describe the primitive finite linear groups that contain a matrix with *simple spectrum*, that is, a matrix all of whose eigenvalues are of multiplicity 1. Partial results see in (A. Zalesskii, I. D. Suprunenko, *Commun. Algebra*, **26**, no. 3 (1998), 863–888; **28**, no. 4 (2000), 1789–1833; Ch. Rudloff, A. Zalesski, *J. Group Theory*, **10** (2007), 585–612).

A. E. Zalesskiĭ

11.33. b) Let $G(q)$ be a simple Chevalley group over a field of order q. Prove that there exists m such that the restriction of every non-one-dimensional representation of $G(q^m)$ over a field of prime characteristic not dividing q to $G(q)$ contains all irreducible representations of $G(q)$ as composition factors.

A. E. Zalesskiĭ

11.34. Describe the complex representations of quasisimple finite groups which remain irreducible after reduction modulo any prime number q. An important example: representations of degree $(p^k-1)/2$ of the symplectic group $Sp(2k,p)$ where $k \in \mathbb{N}$ and p is an odd prime. Partial results see in (P. H. Tiep, A. E. Zalesski, *Proc. London Math. Soc.*, **84** (2002), 439–472).

A. E. Zalesskiĭ

11.36. Let $G = B(m,n)$ be the free Burnside group of rank m and of odd exponent $n \gg 1$. Are the following statements true?

a) Every 2-generated subgroup of G is isomorphic to the Burnside n-product of two cyclic groups.

b) Every automorphism φ of G such that $\varphi^n = 1$ and $b^{\varphi} \cdot b^{\varphi^2} \cdots b^{\varphi^n} = 1$ for all $b \in G$ is an inner automorphism (here $m > 1$).

Editors' comment: for large prime n this follows from (E. A. Cherepanov, *Int. J. Algebra Comput.*, **16** (2006), 839–847), for prime $n \geqslant 1009$ from (V. S. Atabekyan, *Izv. Math.*, **75**, no. 2 (2011), 223–237); this is also true for odd $n \geqslant 1003$ if in addition the order of φ is a prime power (V. S. Atabekyan, *to appear in Math. Notes*, **95**, no. 3 (2014)).

c) The group G is Hopfian if $m < \infty$.

d) All retracts of G are free.

*e) All zero divisors in the group ring $\mathbb{Z}G$ are trivial, which means that if $ab = 0$ then $a = a_1c$, $b = db_1$ where $a_1, c, b_1, d \in \mathbb{Z}G$, $cd = 0$, and the set $\operatorname{supp} c \cup \operatorname{supp} d$ is contained in a cyclic subgroup of G.

S. V. Ivanov

*e) No, there are nontrivial zero divisors (S. V. Ivanov, R. Mikhailov, `arXiv:1209.1443`).

11.37. a) Can the free Burnside group $B(m,n)$, for any m and n, be given by defining relations of the form $v^n = 1$ such that for any natural divisor d of n distinct from n the element v^d is not trivial in $B(m,n)$? This is true for odd $n \geqslant 665$, and for all $n \geqslant 2^{48}$ divisible by 2^9.

S. V. Ivanov

11.38. Does there exist a finitely presented Noetherian group which is not almost polycyclic?

S. V. Ivanov

11.39. (Well-known problem). Does there exist a group which is not almost polycyclic and whose integral group ring is Noetherian?

S. V. Ivanov

11.40. Prove or disprove that a torsion-free group G with the small cancellation condition $C'(\lambda)$ where $\lambda \ll 1$ necessarily has the $\mathscr{U}P$-property (and therefore KG has no zero divisors).

S. V. Ivanov

11.44. For a finite group X, we denote by $r(X)$ its sectional rank. Is it true that the sectional rank of a finite p-group, which is a product AB of its subgroups A and B, is bounded by some linear function of $r(A)$ and $r(B)$?

L. S. Kazarin

11.45. A t-(v,k,λ) *design* $\mathscr{D} = (X,\mathscr{B})$ contains a set X of v points and a set $\mathscr{B}$ of k-element subsets of X called *blocks* such that each t-element subset of X is contained in λ blocks. Prove that there are no nontrivial block-transitive 6-designs. (We have shown that there are no nontrivial block-transitive 8-designs and there are certainly some block-transitive, even flag-transitive, 5-designs.)

Comment of 2009: In (*Finite Geometry and Combinatorics (Deinze 1992)*, Cambridge Univ. Press, 1993, 103–119) we showed that a block-transitive group G on a nontrivial 6-design is either an affine group $AGL(d,2)$ or is between $PSL(2,q)$ and $P\Gamma L(2,q)$; in (M. Huber, *J. Combin. Theory Ser. A*, **117**, no. 2 (2010), 196–203) it is shown that for the case $\lambda = 1$ the group G may only be $P\Gamma L(2,p^e)$, where p is 2 or 3 and e is an odd prime power. *Comment of 2013:* In the case $\lambda = 1$ there are no block-transitive 7-designs (M. Huber, *Discrete Math. Theor. Comput. Sci.*, **12**, no. 1 (2010), 123–132).

P. J. Cameron, C. E. Praeger

11.46. a) Does there exist a finite 3-group G of nilpotency class 3 with the property $[a,\, a^{\varphi}] = 1$ for all $a \in G$ and all endomorphisms φ of G? (See A. Caranti, *J. Algebra*, **97**, no. 1 (1985), 1–13.)

*c) Does there exist a 2-Engel finite p-group G of nilpotency class greater than 2 such that $\mathrm{Aut}\, G = \mathrm{Aut}_c G \cdot \mathrm{Inn}\, G$, where $\mathrm{Aut}_c G$ is the group of central automorphisms of G?

A. Caranti

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*c) Yes, it does (A. Abdollahi, A. Faghihi, S. A. Linton, E. A. O'Brien, *Arch. Math. (Basel)*, **95**, no. 1 (2010), 1–7).

11.48. Is the commutator $[x,\, y,\, y,\, y,\, y,\, y,\, y]$ a product of fifth powers in the free group $\langle x, y\rangle$? If not, then the Burnside group $B(2,5)$ is infinite.

A. I. Kostrikin

11.49. B. Hartley (*Proc. London Math. Soc.*, **35**, no. 1 (1977), 55–75) constructed an example of a non-countable Artinian $\mathbb{Z}G$-module where G is a metabelian group with the minimum condition for normal subgroups. It follows that there exists a non-countable soluble group (of derived length 3) satisfying Min-n. The following question arises in connection with this result and with the study of some classes of soluble groups with the weak minimum condition for normal subgroups. Is an Artinian $\mathbb{Z}G$-module countable if G is a soluble group of finite rank (in particular, a minimax group)?

L. A. Kurdachenko

11.50. Let A, C be abelian groups. If $A[n] = 0$, i. e. for $a \in A$, $na = 0$ implies $a = 0$, then the sequence $\dfrac{\mathrm{Hom}\,(C,A)}{n\mathrm{Hom}\,(C,A)} \rightarrowtail \mathrm{Hom}\left(\dfrac{C}{nC}, \dfrac{A}{nA}\right) \twoheadrightarrow \mathrm{Ext}\,(C,A)[n]$ is exact. Given $f \in \mathrm{Hom}\left(C, \dfrac{A}{nA}\right) = \mathrm{Hom}\left(\dfrac{C}{nC}, \dfrac{A}{nA}\right)$ the corresponding extension X_f is obtained as a pull-back

$$\begin{array}{ccccc} A & \rightarrowtail & X_f & \twoheadrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ A & \rightarrowtail & A & \twoheadrightarrow & \dfrac{A}{nA} \end{array}.$$

Use this scheme to classify certain extensions of A by C. The case $nC = 0$, A being torsion-free is interesting. Here $\mathrm{Ext}\,(C,A)[n] = \mathrm{Ext}\,(C,A)$. (See E. L. Lady, A. Mader, *J. Algebra*, **140** (1991), 36–64.)

A. Mader

11.51. Are there (large, non-trivial) classes $\mathfrak{X}$ of torsion-free abelian groups such that for $A, C \in \mathfrak{X}$ the group $\mathrm{Ext}\,(C,A)$ is torsion-free? It is a fact (E. L. Lady, A. Mader, *J. Algebra*, **140** (1991), 36–64) that two groups of such a class are nearly isomorphic if and only if they have equal p-ranks for all p.

A. Mader

11.52. (Well-known problem). A permutation group on a set Ω is called *sharply doubly transitive* if for any two pairs (α, β) and (γ, δ) of elements of Ω such that $\alpha \neq \beta$ and $\gamma \neq \delta$, there is exactly one element of the group taking α to γ and β to δ. Does every sharply doubly transitive group possess a non-trivial abelian normal subgroup? A positive answer is well known for finite groups.

V. D. Mazurov

11.56. a) Does every infinite residually finite group contain an infinite abelian subgroup? This is equivalent to the following: does every infinite residually finite group contain a non-identity element with an infinite centralizer?

By a famous theorem of Shunkov a torsion group with an involution having a finite centralizer is a virtually soluble group. Therefore we may assume that in our group all elements have odd order. One should start, perhaps, with the following:

b) Does every infinite residually p-group contain an infinite abelian subgroup?

A. Mann

11.58. Describe the finite groups which contain a tightly embedded subgroup H such that a Sylow 2-subgroup of H is a direct product of a quaternion group of order 8 and a non-trivial elementary group.

A. A. Makhnëv

11.59. A TI-subgroup A of a group G is called a *subgroup of root type* if $[A, A^g] = 1$ whenever $N_A(A^g) \neq 1$. Describe the finite groups containing a cyclic subgroup of order 4 as a subgroup of root type.

A. A. Makhnëv

11.60. Is it true that the hypothetical Moore graph with 3250 vertices of valence 57 has no automorphisms of order 2? M. Aschbacher (*J. Algebra*, **19**, no. 4 (1971), 538–540) proved that this graph is not a graph of rank 3.

A. A. Makhnëv

11.61. Let F be a non-abelian free pro-p-group. Is it true that the subset $\{r \in F \mid \mathrm{cd}\,(F/(r)) \leqslant 2\}$ is dense in F? Here (r) denotes the closed normal subgroup of F generated by r.

O. V. Mel'nikov

11.62. Describe the groups over which any equation is soluble. In particular, is it true that this class of groups coincides with the class of torsion-free groups? It is easy to see that a group, over which every equation is soluble, is torsion-free. On the other hand, S. D. Brodskiĭ (*Siberian Math. J.*, **25**, no. 2 (1984), 235–251) showed that any equation is soluble over a locally indicable group.

D. I. Moldavanskiĭ

11.63. Suppose that G is a one-relator group containing non-trivial elements of finite order and N is a subgroup of G generated by all elements of finite order. Is it true that any subgroup of G that intersects N trivially is a free group? One can show that the answer is affirmative in the cases where G/N has non-trivial centre or satisfies a non-trivial identity.

D. I. Moldavanskiĭ

11.65. *Conjecture:* any finitely generated soluble torsion-free pro-p-group with decidable elementary theory is an analytic pro-p-group.

A. G. Myasnikov, V. N. Remeslennikov

11.66. (Yu. L. Ershov). Is the elementary theory of a free pro-p-group decidable?

A. G. Myasnikov, V. N. Remeslennikov

11.67. Does there exist a torsion-free group with exactly 3 classes of conjugate elements such that no non-trivial conjugate class contains a pair of inverse elements?

B. Neumann

11.69. A group G acting on a set Ω will be said to be 1-$\{2\}$-*transitive* if it acts transitively on the set $\Omega^{1,\{2\}} = \{(\alpha, \{\beta, \gamma\}) \mid \alpha, \beta, \gamma \text{ distinct}\}$. Thus G is 1-$\{2\}$-transitive if and only if it is transitive and a stabilizer G_α is 2-homogeneous on $\Omega \setminus \{\alpha\}$. The problem is to classify all (infinite) permutation groups that are 1-$\{2\}$-transitive but not 3-transitive.

Peter M. Neumann

11.70. Let F be an infinite field or a skew-field.

a) Find all transitive subgroups of $PGL(2, F)$ acting on the projective line $F \cup \{\infty\}$.

*b) What conditions on the subring R of F will ensure that $PGL(d + 1, R)$ is flag-transitive on the projective d-space $PG(d, F)$?

c) What are the flag-transitive subgroups of $PGL(d + 1, F)$?

d) What subgroups of $PGL(d + 1, F)$ are 2-transitive on the points of $PG(d, F)$?

Peter M. Neumann, C. E. Praeger

*b) Such necessary and sufficient conditions are given in (S. A. Zyubin, *submitted to Siberian Electron. Math. Rep.*, 2013 (Russian)).

11.71. Let A be a finite group with a normal subgroup H. A subgroup U of H is called an *A-covering subgroup* of H if $\bigcup_{a\in A} U^a = H$. Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that whenever $U < H < A$, where A is a finite group, H is a normal subgroup of A of index n, and U is an A-covering subgroup of H, the index $|H : U| \leqslant f(n)$? (We have shown that the answer is "yes" if U is a maximal subgroup of H.)

Peter M. Neumann, C. E. Praeger

11.72. Suppose that a variety of groups $\mathfrak{V}$ is *non-regular*, that is, the free group $F_{n+1}(\mathfrak{V})$ is embeddable in $F_n(\mathfrak{V})$ for some n. Is it true that then every countable group in $\mathfrak{V}$ is embeddable in an n-generated group in $\mathfrak{V}$?

A. Yu. Ol'shanskiĭ

11.73. If a relatively free group is finitely presented, is it almost nilpotent?

A. Yu. Ol'shanskiĭ

11.76. (Well-known problem). Is the group of collineations of a finite non-Desarguasian projective plane defined over a semi-field soluble? (The hypothesis on the plane means that the corresponding regular set, see Archive, 10.48, is closed under addition.)

N. D. Podufalov

11.77. (Well-known problem). Describe the finite translation planes whose collineation groups act doubly transitively on the set of points of the line at infinity.

N. D. Podufalov

11.78. An isomorphism of groups of points of algebraic groups is called *semialgebraic* if it can be represented as a composition of an isomorphism of translation of the field of definition and a rational morphism.

a) Is it true that the existence of an isomorphism of groups of points of two directly undecomposable algebraic groups with trivial centres over an algebraically closed field implies the existence of a semialgebraic isomorphism of the groups of points?

b) Is it true that every isomorphism of groups of points of directly undecomposable algebraic groups with trivial centres defined over algebraic fields is semialgebraic? A field is called *algebraic* if all its elements are algebraic over the prime subfield.

K. N. Ponomarëv

11.80. Let G be a primitive permutation group on a finite set Ω and suppose that, for $\alpha \in \Omega$, G_α acts 2-transitively on one of its orbits in $\Omega \setminus \{\alpha\}$. By (C. E. Praeger, *J. Austral. Math. Soc. (A)*, **45**, 1988, 66–77) either

(a) $T \leqslant G \leqslant \operatorname{Aut} T$ for some nonabelian simple group T, or

(b) G has a unique minimal normal subgroup which is regular on Ω.

For what classes of simple groups in (a) is a classification feasible? Describe the examples in as explicit a manner as possible. Classify all groups in (b).

Remark of 1999: Two papers by X. G. Fang and C. E. Praeger (*Commun. Algebra*, **27** (1999), 3727–3754 and 3755–3769) show that this is feasible for T a Suzuki and Ree simple group, and a paper of J. Wang and C. E. Praeger (*J. Algebra*, **180** (1996), 808–833) suggests that this may not be the case for T an alternating group. *Remark of 2001:* In (X. G. Fang, G. Havas, J. Wang, *European J. Combin.*, **20**, no. 6 (1999), 551–557) new examples are constructed with $G = PSU(3, q)$. *Remark of 2009:* It was shown in (D. Leemans, *J. Algebra*, **322**, no. 3 (2009), 882–892) that for part (a) classification is feasible for self-paired 2-transitive suborbits with T a sporadic simple group; in this case these suborbits correspond to 2-arc-transitive actions on undirected graphs.

C. E. Praeger

11.81. A topological group is said to be *F-balanced* if for any subset X and any neighborhood of the identity U there is a neighborhood of the identity V such that $VX \subseteq XU$. Is every F-balanced group *balanced*, that is, does it have a basis of neighborhoods of the identity consisting of invariant sets?

I. V. Protasov

11.83. Is the conjugacy problem soluble for finitely generated abelian-by-polycyclic groups?

V. N. Remeslennikov

11.84. Is the isomorphism problem soluble

a) for finitely generated metabelian groups?

b) for finitely generated soluble groups of finite rank? *V. N. Remeslennikov*

11.85. Let F be a free pro-p-group with a basis X and let $R = r^F$ be the closed normal subgroup generated by an element r. We say that an element s of F is *associated to* r if $s^F = R$.

a) Suppose that none of the elements associated to r is a pth power of an element in F. Is it true that F/R is torsion-free?

b) Among the elements associated to r there is one that depends on the minimal subset X' of the basis X. Let $x \in X'$. Is it true that the images of the elements of $X \setminus \{x\}$ in F/R freely generate a free pro-p-group? *N. S. Romanovskiĭ*

11.86. Does every group $G = \langle x_1, \ldots, x_n \mid r_1 = \cdots = r_m = 1\rangle$ possess, in a natural way, a homomorphic image $H = \langle x_1, \ldots, x_n \mid s_1 = \cdots = s_m = 1\rangle$

a) such that H is a torsion-free group?

b) such that the integral group ring of H is embeddable in a skew field?

If the stronger assertion "b" is true, then this will give an explicit method of finding elements $x_{i_1}, \ldots, x_{i_{n-m}}$ which generate a free group in G. Such elements exist by N. S. Romanovskiĭ's theorem (*Algebra and Logic*, **16**, no. 1 (1977), 62–67).

V. A. Roman'kov

11.87. (Well-known problem). Is the automorphism group of a free metabelian group of rank $n \geqslant 4$ finitely presented? *V. A. Roman'kov*

***11.88.** We define the *length* $l(g)$ of an Engel element g of a group G to be the smallest number l such that $[h, g;\, l] = 1$ for all $h \in G$. Here $[h, g;\, 1] = [h, g]$ and $[h, g;\, i + 1] = [[h, g;\, i], g]$. Does there exist a polynomial function $\varphi(x, y)$ such that $l(uv) \leqslant \varphi(l(u), l(v))$? Up to now, it is unknown whether a product of Engel elements is again an Engel element. *V. A. Roman'kov*

*No, it does not (L. V. Dolbak, *Siberian Math. J.*, **47**, no. 1 (2006), 55–57).

11.89. Let k be an infinite cardinal number. Describe the epimorphic images of the Cartesian power $\prod_k \mathbb{Z}$ of the group $\mathbb{Z}$ of integers. Such a description is known for $k = \omega_0$ (R. Nunke, *Acta Sci. Math. (Szeged)*, **23** (1963), 67–73). *S. V. Rychkov*

11.90. Let $\mathfrak{V}$ be a variety of groups and let k be an infinite cardinal number. A group $G \in \mathfrak{V}$ of rank k is called *almost free in* $\mathfrak{V}$ if each of its subgroups of rank less than k is contained in a subgroup of G which is free in $\mathfrak{V}$. For what k do there exist almost free but not free in $\mathfrak{B}$ groups of rank k? *S. V. Rychkov*

11.92. What are the soluble hereditary non-one-generator formations of finite groups all of whose proper hereditary subformations are one-generated? *A. N. Skiba*

***11.94.** Describe all *simply reducible* groups, that is, groups such that all their characters are real and the tensor product of any two irreducible representations contains no multiple components. This question is interesting for physicists. Is every finite simply reducible group soluble?

S. P. Strunkov

*Yes, it is (L. S. Kazarin, E. I. Chankov, *Sb. Math.*, **201**, no. 5 (2010), 655–668).

11.95. Suppose that G is a p-group G containing an element a of order p such that the subgroup $\langle a, a^g\rangle$ is finite for any g and the set $C_G(a) \cap a^G$ is finite. Is it true that G has non-trivial centre? This is true for 2-groups.

S. P. Strunkov

11.96. Is it true that, for a given number n, there exist only finitely many finite simple groups each of which contains an involution which commutes with at most n involutions of the group? Is it true that there are no infinite simple groups satisfying this condition?

S. P. Strunkov

11.98. a) (R. Brauer). Find the best-possible estimate of the form $|G| \leqslant f(r)$ where r is the number of conjugacy classes of elements in a finite (simple) group G.

S. P. Strunkov

11.99. Find (in group-theoretic terms) necessary and sufficient conditions for a finite group to have complex irreducible characters having defect 0 for more than one prime number dividing the order of the group. Express the number of such characters in the same terms.

S. P. Strunkov

11.100. Is it true that every periodic conjugacy biprimitively finite group (see 6.59) can be obtained from 2-groups and binary finite groups by taking extensions? This question is of independent interest for p-groups, p an odd prime.

S. P. Strunkov

11.102. Does there exist a residually soluble but insoluble group satisfying the maximum condition on subgroups?

J. Wiegold

11.105. b) Let $\mathfrak{V}$ be a variety of groups. Its relatively free group of given rank has a presentation F/N, where F is absolutely free of the same rank and N fully invariant in F. The associated Lie ring $\mathscr{L}(F/N)$ has a presentation L/J, where L is the free Lie ring of the same rank and J an ideal of L. Is J fully invariant in L if $\mathfrak{V}$ is the Burnside variety of all groups of given exponent q, where q is a prime-power, $q \geqslant 4$?

G. E. Wall

11.107. Is there a non-linear locally finite simple group each of whose proper subgroups is residually finite?

R. Phillips

11.109. Is it true that the union of an ascending series of groups of Lie type of equal ranks over some fields should be also a group of Lie type of the same rank over a field? (For locally finite fields the answer is "yes", the theorem in this case is due to V. Belyaev, A. Borovik, B. Hartley & G. Shute, and S. Thomas.)

R. Phillips

11.111. Does every infinite locally finite simple group G contain a nonabelian finite simple subgroup? Is there an infinite tower of such subgroups in G?

B. Hartley

11.112. Let $L = L(K(p))$ be the associated Lie ring of a free countably generated group $K(p)$ of the Kostrikin variety of locally finite groups of a given prime exponent p. Is it true that

a) L is a relatively free Lie ring?

c) all identities of L follow from a finite number of identities of L?

E. I. Khukhro

11.113. (B. Hartley). Is it true that the derived length of a nilpotent periodic group admitting a regular automorphism of prime-power order p^n is bounded by a function of p and n?

E. I. Khukhro

11.114. Is every locally graded group of finite special rank almost hyperabelian? This is true in the class of periodic locally graded groups (*Ukrain. Math. J.*, **42**, no. 7 (1990), 855–861).

N. S. Chernikov

11.115. Suppose that X is a free non-cyclic group, N is a non-trivial normal subgroup of X and T is a proper subgroup of X containing N. Is it true that $[T, N] < [X, N]$ if N is not maximal in X?

V. P. Shaptala

11.116. The *dimension* of a partially ordered set $\langle P, \leqslant\rangle$ is, by definition, the least cardinal number δ such that the relation $\leqslant$ is an intersection of δ relations of linear order on P. Is it true that, for any Chernikov group which does not contain a direct product of two quasicyclic groups over the same prime number, the subgroup lattice has finite dimension? *Expected answer:* yes.

L. N. Shevrin

11.117. Let $\mathfrak{X}$ be a soluble non-empty Fitting class. Is it true that every finite non-soluble group possesses an $\mathfrak{X}$-injector?

L. A. Shemetkov

11.118. What are the hereditary soluble local formations $\mathfrak{F}$ of finite groups such that every finite group has an $\mathfrak{F}$-covering subgroup?

L. A. Shemetkov

11.119. Is it true that, for any non-empty set of primes π, the π-length of any finite π-soluble group does not exceed the derived length of its Hall π-subgroup?

L. A. Shemetkov

11.120. Let $\mathfrak{F}$ be a soluble saturated Fitting formation. Is it true that $l_{\mathfrak{F}}(G) \leqslant f(c_{\mathfrak{F}}(G))$, where $c_{\mathfrak{F}}(G)$ is the length of a composition series of some $\mathfrak{F}$-covering subgroup of a finite soluble group G? This is Problem 17 in (L. A. Shemetkov, *Formations of Finite Groups*, Moscow, Nauka, 1978 (Russian)). For the definition of the $\mathfrak{F}$-*length* $l_{\mathfrak{F}}(G)$, see *ibid.*

L. A. Shemetkov

11.121. Does there exist a local formation of finite groups which has a non-trivial decomposition into a product of two formations and in any such a decomposition both factors are non-local? Local products of non-local formations do exist.

L. A. Shemetkov

11.122. Does every non-zero submodule of a free module over the group ring of a torsion-free group contain a free cyclic submodule?

A. L. Shmel'kin

11.123. (Well-known problem). For a given group G, define the following sequence of groups: $A_1(G) = G$, $A_{i+1}(G) = \mathrm{Aut}\,(A_i(G))$. Does there exist a finite group G for which this sequence contains infinitely many non-isomorphic groups?

M. Short

11.124. Let F be a non-cyclic free group and R a non-cyclic subgroup of F. Is it true that if $[R, R]$ is a normal subgroup of F then R is also a normal subgroup of F?

V. E. Shpilrain

11.125. Let G be a finite group admitting a regular elementary abelian group of automorphisms V of order p^n. Is it true that the subgroup $H = \bigcap_{v \in V \setminus \{1\}} [G, v]$ is nilpotent? In the case of an affirmative answer, does there exist a function depending only on p, n, and the derived length of G which bounds the nilpotency class of H?

P. V. Shumyatskiĭ

11.127. Is every group of exponent 12 locally finite?

V. P. Shunkov

Problems from the 12th Issue (1992)

12.1. a) H. Bass (*Topology*, **4**, no. 4 (1966), 391–400) has constructed explicitly a proper subgroup of finite index in the group of units of the integer group ring of a finite cyclic group. Calculate the index of Bass' subgroup.

R. Zh. Aleev

12.3. A p-group is called *thin* if every set of pairwise incomparable (by inclusion) normal subgroups contains $\leqslant p+1$ elements. Is the number of thin pro-p-groups finite?

R. Brandl

12.4. Let G be a group and assume that we have $[x,y]^3 = 1$ for all $x, y \in G$. Is G' of finite exponent? Is G soluble? By a result of N. D. Gupta and N. S. Mendelssohn, 1967, we know that G' is a 3-group.

R. Brandl

12.6. Let G be a finitely generated group and suppose that H is a p-subgroup of G such that H contains no non-trivial normal subgroups of G and $HX = XH$ for any subgroup X of G. Is then $G/C_G(H^G)$ a p-group where H^G is the normal closure of H?

G. Busetto

12.8. Let $\mathscr{V}$ be a non-trivial variety of groups and let $a_1, \ldots, a_r$ freely generate a free group $F_r(V)$ in $\mathscr{V}$. We say $F_r(\mathscr{V})$ *strongly discriminates* $\mathscr{V}$ just in case every finite system of inequalities $w_i(a_1, \ldots, a_r, x_1, \ldots, x_k) \neq 1$ for $1 \leqslant i \leqslant n$ having a solution in some $F_s(\mathscr{V})$ containing $F_r(\mathscr{V})$ as a varietally free factor in the sense of $\mathscr{V}$, already has a solution in $F_r(\mathscr{V})$. Does there exist $\mathscr{V}$ such that for some integer $r > 0$, $F_r(\mathscr{V})$ discriminates but does not strongly discriminate $\mathscr{V}$? What about the analogous question for general algebras in the context of universal algebra?

A. M. Gaglione, D. Spellman

12.9. Following Bass, call a group *tree-free* if there is an ordered Abelian group Λ and a Λ-tree X such that G acts freely without inversion on X.

a) Must every finitely generated tree-free group satisfy the maximal condition for Abelian subgroups?

b) The same question for finitely presented tree-free groups.

A. M. Gaglione, D. Spellman

12.11. Suppose that G, H are countable or finite groups and A is a proper subgroup of G and H containing no non-trivial subgroup normal in both G and H. Can $G *_A H$ be embedded in Sym$\,(\mathbb{N})$ so that the image is highly transitive?

Comment of 2013: for partial results, see (S. V. Gunhouse, *Highly transitive representations of free products on the natural numbers*, Ph. D. Thesis Bowling Green State Univ., 1993; K. K. Hickin, *J. London Math. Soc. (2)*, **46**, no. 1 (1992), 81–91).

A. M. W. Glass

12.12. Is the conjugacy problem for nilpotent finitely generated lattice-ordered groups soluble?

A. M. W. Glass

12.13. If $\langle\Omega,\leqslant\rangle$ is the countable universal poset, then $G = \mathrm{Aut}\,(\langle\Omega,\leqslant\rangle)$ is simple (A. M. W. Glass, S. H. McCleary, M. Rubin, *Math. Z.*, **214**, no. 1 (1993), 55–66). If H is a subgroup of G such that $|G : H| < 2^{\aleph_0}$ and H is transitive on Ω, does $H = G$?

A. M. W. Glass

12.15. Suppose that, in a finite 2-group G, any two elements are conjugate whenever their normal closures coincide. Is it true that the derived subgroup of G is abelian?

E. A. Golikova, A. I. Starostin

12.16. Is the class of groups of recursive automorphisms of arbitrary models closed with respect to taking free products?

S. S. Goncharov

12.17. Find a description of autostable periodic abelian groups. *S. S. Goncharov*

12.19. Is it true that, for every $n \geqslant 2$ and every two epimorphisms φ and ψ of a free group F_{2n} of rank $2n$ onto $F_n \times F_n$, there exists an automorphism α of F_{2n} such that $\alpha\varphi = \psi$?

R. I. Grigorchuk

12.20. (Well-known problem). Is R. Thompson's group

$$F = \langle x_0,\, x_1, \ldots \mid x_n^{x_i} = x_{n+1},\;\; i < n,\;\; n = 1,\, 2, \ldots\rangle =$$

$$= \langle x_0, \ldots, x_4 \mid x_1^{x_0} = x_2,\;\; x_2^{x_0} = x_3,\;\; x_2^{x_1} = x_3,\;\; x_3^{x_1} = x_4,\;\; x_3^{x_2} = x_4\rangle$$

amenable?

R. I. Grigorchuk

12.21. Let R be a commutative ring with identity and let G be a finite group. Prove that the ring $a(RG)$ of R-representations of G has no non-trivial idempotents.

P. M. Gudivok, V. P. Rud'ko

12.22. b) Let $\Delta(G)$ be the augmentation ideal of the integer group ring of an arbitrary group G. Then $D_n(G) = G \cap (1 + \Delta^n(G))$ contains the nth lower central subgroup $\gamma_n(G)$ of G. Is it true that $D_n(G)/\gamma_n(G)$ has exponent dividing 2?

N. D. Gupta, Yu. V. Kuz'min

12.23. The *index of permutability* of a group G is defined to be the minimal integer $k \geqslant 2$ such that for each k-tuple $x_1, \ldots, x_k$ of elements in G there is a non-identity permutation σ on k symbols such that $x_1 \cdots x_k = x_{\sigma(1)} \cdots x_{\sigma(k)}$. Determine the index of permutability of the symmetric group $\mathbb{S}_n$.

M. Gutsan

12.27. Let G be a simple locally finite group. We say that G is a *group of finite type* if there is a non-trivial permutational representation of G such that some finite subgroup of G has no regular orbits. Investigate and, perhaps, classify the simple locally finite groups of finite type. Partial results see in (B. Hartley, A. Zalesski, *Isr. J. Math.*, **82** (1993), 299–327, *J. London Math. Soc.*, **55** (1997), 210–230; F. Leinen, O. Puglisi, *Illinois J. Math.*, **47** (2003), 345–360).

A. E. Zalesskiĭ

12.28. Let G be a group. A function $f\colon G\to\mathbb{C}$ is called

1) *normed* if $f(1)=1$;
2) *central* if $f(gh)=f(hg)$ for all $g,h\in G$;
3) *positive-definite* if $\sum\limits_{k,\,l} f(g_k^{-1}g_l)\bar{c}_k c_l \geqslant 0$ for any $g_1,\dots,g_n \in G$ and any $c_1,\dots,c_n\in\mathbb{C}$.

Classify the infinite simple locally finite groups G which possess functions satisfying 1)–3). The simple Chevalley groups are known to have no such functions, while such functions exist on locally matrix (or stable) classical groups over finite fields. The question is motivated by the theory of C^*-algebras, see § 9 in (A. M. Vershik, S. V. Kerov, *J. Sov. Math.*, **38** (1987), 1701–1733). Partial results see in (F. Leinen, O. Puglisi, *J. Pure Appl. Algebra*, **208** (2007), 1003–1021, *J. London Math. Soc.*, **70** (2004), 678–690).

A. E. Zalesskiĭ

12.29. Classify the locally finite groups for which the augmentation ideal of the complex group algebra is a simple ring. The problem goes back to I. Kaplansky (1965). For partial results, see (K. Bonvallet, B. Hartley, D. S. Passman, M. K. Smith, *Proc. Amer. Math. Soc.*, **56**, no. 1 (1976), 79–82; A. E. Zalesskiĭ, *Algebra i Analiz*, **2**, no. 6 (1990), 132–149 (Russian); Ch. Praeger, A. E. Zalesskii, *Proc. London Math. Soc.*, **70**, no. 2 (1995), 313–335; B. Hartley, A. Zalesski, *J. London Math. Soc.*, **55** (1997), 210–230).

A. E. Zalesskiĭ

12.30. (O. N. Golovin). On the class of all groups, do there exist associative operations which satisfy the postulates of MacLane and Mal'cev (that is, which are free functorial and hereditary) and which are different from taking free and direct products?

S. V. Ivanov

12.32. Prove an analogue of Higman's theorem for the Burnside variety $\mathfrak{B}_n$ of groups of odd exponent $n \gg 1$, that is, prove that every recursively presented group of exponent n can be embedded in a finitely presented (in $\mathfrak{B}_n$) group of exponent n.

S. V. Ivanov

12.33. Suppose that G is a finite group and x is an element of G such that the subgroup $\langle x,y\rangle$ has odd order for any y conjugate to x in G. Prove, without using CFSG, that the normal closure of x in G is a group of odd order.

L. S. Kazarin

12.34. Describe the finite groups G such that the sum of the cubes of the degrees of all irreducible complex characters is at most $|G|\cdot\log_2|G|$. The question is interesting for applications in the theory of signal processing.

L. S. Kazarin

12.35. Suppose that $\mathfrak{F}$ is a radical composition formation of finite groups. Prove that $\langle H,K\rangle^{\mathfrak{F}} = \langle H^{\mathfrak{F}},\, K^{\mathfrak{F}}\rangle$ for every finite group G and any subnormal subgroups H and K of G.

S. F. Kamornikov

12.37. (J. G. Thompson). For a finite group G and natural number n, set $G(n) = \{x\in G \mid x^n=1\}$ and define the *type* of G to be the function whose value at n is the order of $G(n)$. Is it true that a group is soluble if its type is the same as that of a soluble one?

A. S. Kondratiev

12.38. (J. G. Thompson). For a finite group G, we denote by $N(G)$ the set of all orders of the conjugacy classes of G. Is it true that if G is a finite non-abelian simple group, H a finite group with trivial centre and $N(G) = N(H)$, then G and H are isomorphic?

A. S. Kondratiev

12.40. Let φ be an irreducible p-modular character of a finite group G. Find the best-possible estimate of the form $\varphi(1)_p \leqslant f(|G|_p)$. Here n_p is the p-part of a positive integer n.

A. S. Kondratiev

12.41. Let F be a free group on two generators x, y and let φ be the automorphism of F defined by $x \to y$, $y \to xy$. Let G be a semidirect product of $F/(F''(F')^2)$ and $\langle\varphi\rangle$. Then G is just-non-polycyclic. What is the cohomological dimension of G over $\mathbb{Q}$? (It is either 3 or 4.)

P. H. Kropholler

12.43. (Well-known problem). Does there exist an infinite finitely-generated residually finite p-group such that each subgroup is either finite or of finite index?

J. C. Lennox

12.48. Let G be a sharply doubly transitive permutation group on a set Ω (see 11.52 for a definition).

a) Does G possess a regular normal subgroup if a point stabilizer is locally finite?

b) Does G possess a regular normal subgroup if a point stabilizer has an abelian subgroup of finite index?

V. D. Mazurov

12.50. (Well-known problem). Find an algorithm which decides, by a given finite set of matrices in $SL_3(\mathbb{Z})$, whether the first matrix of this set is contained in the subgroup generated by the remaining matrices. An analogous problem for $SL_4(\mathbb{Z})$ is insoluble since a direct product of two free groups of rank 2 embeds into $SL_4(\mathbb{Z})$.

G. S. Makanin

12.51. Does the free group F_η $(1 < \eta < \infty)$ have a finite subset S for which there is a unique total order of F_η making all elements of S positive? Equivalently, does the free representable l-group of rank η have a basic element? (The analogs for right orders of F_η and free l-groups have negative answers.)

S. McCleary

12.52. Is the free l-group $\mathscr{F}_\eta$ of rank η $(1 < \eta < \infty)$ Hopfian? That is, are l-homomorphisms from $\mathscr{F}_\eta$ onto itself necessarily one-to-one?

S. McCleary

12.53. Is it decidable whether or not two elements of a free l-group are conjugate?

S. McCleary

12.54. Is there a normal valued l-group G for which there is no abelian l-group A with $C(A) \cong C(G)$? (Here $C(G)$ denotes the lattice of convex l-subgroups of G. If G is not required to be normal valued, this question has an affirmative answer.)

S. McCleary

12.55. Let $f(n, p)$ be the number of groups of order p^n. Is $f(n, p)$ an increasing function of p for any fixed $n \geqslant 5$?

A. Mann

12.56. Let $\langle X \mid R\rangle$ be a finite presentation (the words in R are assumed cyclically reduced). Define the *length* of the presentation to be the sum of the number of generators and the lengths of the relators. Let $F(n)$ be the number of (isomorphism types of) groups that have a presentation of length at most n. What can one say about the function $F(n)$? It can be shown that $F(n)$ is not recursive (D. Segal) and it is at least exponential (L. Pyber).

A. Mann

12.57. Suppose that a cyclic group A of order 4 is a TI-subgroup of a finite group G.

a) Does A centralize a component L of G if A intersects $L \cdot C(L)$ trivially?

b) What is the structure of the normal closure of A in the case where A centralizes each component of G?

A. A. Makhnëv

12.58. A *generalized quadrangle* $GQ(s,t)$ with parameters s,t is by definition an incidence system consisting of points and lines in which every line consists of $s+1$ points, every two different lines have at most one common point, each point belongs to $t+1$ lines, and for any point a not lying on a line L there is a unique line containing a and intersecting L.

a) Does $GQ(4,11)$ exist?

b) Can the automorphism group of a hypothetical $GQ(5,7)$ contain involutions?

A. A. Makhnëv

12.59. Does there exist a strongly regular graph with parameters (85, 14, 3, 2) and a non-connected neighborhood of some vertex?

A. A. Makhnëv

12.60. Describe the strongly regular graphs in which the neighborhoods of vertices are generalized quadrangles (see 12.58).

A. A. Makhnëv

12.62. A *Frobenius group* is a transitive permutation group in which the stabilizer of any two points is trivial. Does there exist a Frobenius group of infinite degree which is primitive as permutation group, and in which the stabilizer of a point is cyclic and has only finitely many orbits?

Conjecture: No. Note that such a group cannot be 2-transitive (Gy. Károlyi, S. J. Kovács, P. P. Pálfy, *Aequationes Math.*, **39** (1990), 161–166; V. D. Mazurov, *Siberian Math. J.*, **31** (1990), 615–617; Peter M. Neumann, P. J. Rowley, in: *Geometry and cohomology in group theory* (*London Math. Soc. Lect. Note Ser.*, **252**), Cambridge Univ. Press, 1998, 291–295).

Peter M. Neumann

12.63. Does there exist a soluble permutation group of infinite degree that has only finitely many orbits on triples? *Conjecture:* No.

Comment of 2001: It is proved (D. Macpherson, *in: Advances in algebra and model theory* (*Algebra Logic Appl.*, **9**), Gordon & Breach, Amsterdam, 1997, 87–92) that every infinite soluble permutation group has infinitely many orbits on quadruples.

Peter M. Neumann

12.64. Is it true that for a given number $k \geqslant 2$ and for any (prime) number n, there exists a number $N = N(k,n)$ such that every finite group with generators $A = \{a_1, \ldots, a_k\}$ has exponent $\leqslant n$ if $(x_1 \cdots x_N)^n = 1$ for any $x_1, \ldots, x_N \in A \cup \{1\}$?

For given k and n, a negative answer implies, for example, the infiniteness of the free Burnside group $B(k,n)$, and a positive answer, in the case of sufficiently large n, gives, for example, an opportunity to find a hyperbolic group which is not residually finite (and in this group a hyperbolic subgroup of finite index which has no proper subgroups of finite index).

A. Yu. Ol'shanskiĭ

12.66. Describe the finite translation planes whose collineation groups act doubly transitively on the affine points.

N. D. Podufalov

12.67. Describe the structure of the locally compact soluble groups with the maximum (minimum) condition for closed non-compact subgroups.

V. M. Poletskikh

12.68. Describe the locally compact abelian groups in which any two closed subgroups of finite rank generate a subgroup of finite rank.

V. M. Poletskikh

12.69. Let F be a countably infinite field and G a finite group of automorphisms of F. The group ring $\mathbb{Z}G$ acts on F in a natural way. Suppose that S is a subfield of F satisfying the following property: for any $x \in F$ there is a non-zero element $f \in \mathbb{Z}G$ such that $x^f \in S$. Is it true that either $F = S$ or the field extension F/S is purely inseparable? One can show that for an uncountable field an analogous question has an affirmative answer.

K. N. Ponomarëv

12.72. Let $\mathfrak{F}$ be a soluble local hereditary formation of finite groups. Prove that $\mathfrak{F}$ is radical if every finite soluble minimal non-$\mathfrak{F}$-group G is a minimal non-$\mathfrak{N}^{l(G)-1}$-group. Here $\mathfrak{N}$ is the formation of all finite nilpotent groups, and $l(G)$ is the nilpotent length of G.

V. N. Semenchuk

12.73. A formation $\mathfrak{H}$ of finite groups is said to have *length* t if there is a chain of formations $\varnothing = \mathfrak{H}_0 \subset \mathfrak{H}_1 \subset \cdots \subset \mathfrak{H}_t = \mathfrak{H}$ in which $\mathfrak{H}_{i-1}$ is a maximal subformation of $\mathfrak{H}_i$. Is the lattice of soluble formations of length $\leqslant 4$ distributive?

A. N. Skiba

12.75. (B. Jonsson). Is the class N of the lattices of normal subgroups of groups a variety? It is proved (C. Herrman, W. Poguntke, *Algebra Univers.*, **4**, no. 3 (1974), 280–286) that N is an infinitely based quasivariety.

D. M. Smirnov

12.77. (Well-known problem). Does the order (if it is greater than p^2) of a finite non-cyclic p-group divide the order of its automorphism group?

A. I. Starostin

12.79. Suppose that a and b are two elements of a finite group G such that the function

$$\varphi(g) = 1^G(g) - 1^G_{\langle a\rangle}(g) - 1^G_{\langle b\rangle}(g) - 1^G_{\langle ab\rangle}(g) + 2$$

is a character of G. Is it true that $G = \langle a, b\rangle$? The converse statement is true.

S. P. Strunkov

12.80. b) (K. W. Roggenkamp). Is it true that the number of p-blocks of defect 0 of a finite simple group G is equal to the number of the conjugacy classes of elements $g \in G$ such that the number of solutions of the equation $[x, y] = g$ in G is not divisible by p?

S. P. Strunkov

12.81. What is the cardinality of the set of subvarieties of the group variety $\mathfrak{A}_p^3$ (where $\mathfrak{A}_p$ is the variety of abelian groups of prime exponent p)?

V. I. Sushchanskiĭ

12.85. Does every variety which is generated by a (known) finite simple group containing a soluble subgroup of derived length d contain a 2-generator soluble group of derived length d?

S. A. Syskin

12.86. For each known finite simple group, find its maximal 2-generator direct power.

S. A. Syskin

12.87. Let Γ be a connected undirected graph without loops or multiple edges and suppose that the automorphism group $\mathrm{Aut}(\Gamma)$ acts transitively on the vertex set of Γ. Is it true that at least one of the following assertions holds?

1. The stabilizer of a vertex of Γ in $\mathrm{Aut}(\Gamma)$ is finite.
2. The group $\mathrm{Aut}(\Gamma)$ as a permutation group on the vertex set of Γ admits an imprimitivity system σ with finite blocks for which the stabilizer of a vertex of the factor-graph Γ/σ in $\mathrm{Aut}(\Gamma/\sigma)$ is finite.
3. There exists a natural number n such that the graph obtained from Γ by adding edges joining distinct vertices the distance between which in Γ is at most n contains a regular tree of valency 3.

V. I. Trofimov

12.88. An undirected graph is called a *locally finite Cayley graph of a group G* if its vertex set can be identified with the set of elements of G in such a way that, for some finite generating set $X = X^{-1}$ of G not containing 1, two vertices g and h are adjacent if and only if $g^{-1}h \in X$. Do there exist two finitely generated groups with the same locally finite Cayley graph, one of which is periodic and the other is not periodic?

V. I. Trofimov

12.89. Describe the infinite connected graphs to which the sequences of finite connected graphs with primitive automorphism groups converge. For definitions, see (V. I. Trofimov, *Algebra and Logic*, **28**, no. 3 (1989), 220–237).

V. I. Trofimov

12.92. For a field K of characteristic 2, each finite group $G = \{g_1, \dots, g_n\}$ of odd order n is determined up to isomorphism by its group determinant (Formanek–Sibley, 1990) and even by its *reduced norm* which is defined as the last coefficient $s_m(x)$ of the minimal polynomial $\varphi(\lambda; x) = \lambda^m - s_1(x)\lambda^{m-1} + \cdots + (-1)^m s_m(x)$ for the generic element $x = x_1g_1 + \cdots + x_ng_n$ of the group ring KG (Hoehnke, 1991). Is it possible in this theorem to replace $s_m(x)$ by some other coefficients $s_i(x)$, $i < m$? For notation see (G. Frobenius, *Sitzungsber. Preuss. Akad. Wiss. Berlin*, 1896, 1343–1382) and (K. W. Johnson, *Math. Proc. Cambridge Phil. Soc.*, **109** (1991), 299–311).

H.-J. Hoehnke

12.94. Let G be a finitely generated pro-p-group not involving the wreath product $C_p \wr \mathbb{Z}_p$ as a closed section (where C_p is a cyclic group of order p and $\mathbb{Z}_p$ is the group of p-adic integers). Does it follow that G is p-adic analytic?

A. Shalev

12.95. Let G be a finitely generated pro-p-group and let $g_1, \ldots, g_n \in G$. Let H be an open subgroup of G, and suppose there exists a non-trivial word $w = w(X_1, \ldots, X_n)$ such that $w(a_1, \ldots, a_n) = 1$ whenever $a_1 \in g_1H, \ldots, a_n \in g_nH$ (that is, G satisfies a *coset identity*). Does it follow that G satisfies some non-trivial identity?

A positive answer to this question would imply that an analogue of the Tits Alternative holds for finitely generated pro-p-groups. Note that J. S. Wilson and E. I. Zelmanov (*J. Pure Appl. Algebra*, **81** (1992), 103–109) showed that the graded Lie algebra $L_p(G)$ with respect to the dimension subgroups of G in characteristic p satisfies a polynomial identity.

A. Shalev

12.100. Is every periodic group with a regular automorphism of order 4 locally finite?

P. V. Shumyatsky

12.101. We call a group G containing an involution i a T_0*-group* if

1) the order of the product of any two involutions conjugate to i is finite;

2) all 2-subgroups of G are either cyclic or generalized quaternion;

3) the centralizer C of the involution i in G is infinite, distinct from G, and has finite periodic part;

4) the normalizer of any non-trivial i-invariant finite subgroup in G either is contained in C or has periodic part which is a Frobenius group (see 6.55) with abelian kernel and finite complement of even order;

5) for every element c not contained in C for which ci is an involution there is an element s of C such that $\langle c, c^s\rangle$ is an infinite subgroup.

Does there exist a simple T_0-group?

V. P. Shunkov

Problems from the 13th Issue (1995)

13.1. Let $U(K)$ denote the group of units of a ring K. Let G be a finite group, $\mathbb{Z}G$ the integral group ring of G, and $\mathbb{Z}_pG$ the group ring of G over the residues modulo a prime number p. Describe the homomorphism from $U(\mathbb{Z}G)$ into $U(\mathbb{Z}_pG)$ induced by reducing the coefficients modulo p. More precisely, find the kernel and the image of this homomorphism and an explicit transversal over the kernel.
R. Zh. Aleev

13.2. Does there exist a finitely based variety of groups $\mathfrak{V}$ such that the word problem is solvable in $F_n(\mathfrak{V})$ for all natural n, but is unsolvable in $F_\infty(\mathfrak{V})$ (with respect to a system of free generators)?
M. I. Anokhin

13.3. Let $\mathfrak{M}$ be an arbitrary variety of groups. Is it true that every infinitely generated projective group in $\mathfrak{M}$ is an $\mathfrak{M}$-free product of countably generated projective groups in $\mathfrak{M}$?
V. A. Artamonov

13.4. Let G be a group with a normal (pro-) 2-subgroup N such that G/N is isomorphic to $GL_n(2)$, inducing its natural module on $N/\Phi(N)$, the Frattini factor-group of N. For $n = 3$ determine G such that N is as large as possible. (For $n > 3$ it can be proved that already the Frattini subgroup $\Phi(N)$ of N is trivial; for $n = 2$ there exists G such that N is the free pro-2-group generated by two elements).
B. Baumann

13.5. Groups A and B are said to be *locally equivalent* if for every finitely generated subgroup $X \leqslant A$ there is a subgroup $Y \leqslant B$ isomorphic to X, and conversely, for every finitely generated subgroup $Y \leqslant B$ there is a subgroup $X \leqslant A$, isomorphic to Y. We call a group G *categorical* if G is isomorphic to any group that is locally equivalent to G. Is it true that a periodic locally soluble group G is categorical if and only if G is hyperfinite with Chernikov Sylow subgroups? (A group is *hyperfinite* if it has a well ordered ascending normal series with finite factors.)
V. V. Belyaev

13.6. (B. Hartley). Is it true that a locally finite group containing an element with Chernikov centralizer is almost soluble?
V. V. Belyaev

13.7. (B. Hartley). Is it true that a simple locally finite group containing a finite subgroup with finite centralizer is linear?
V. V. Belyaev

13.8. (B. Hartley). Is it true that a locally soluble periodic group has a finite normal series with locally nilpotent factors if it contains an element
a) with finite centralizer?
b) with Chernikov centralizer?
V. V. Belyaev

13.9. Is it true that a locally soluble periodic group containing a finite nilpotent subgroup with Chernikov centralizer has a finite normal series with locally nilpotent factors?
V. V. Belyaev, B. Hartley

13.12. Is the group of all automorphisms of an arbitrary hyperbolic group finitely presented?

Comment of 2001: This is proved for the group of automorphisms of a torsion-free hyperbolic group that cannot be decomposed into a free product (Z. Sela, *Geom. Funct. Anal.*, **7**, no. 3 (1997) 561–593).

O. V. Bogopol'skiĭ

13.13. Let G and H be finite p-groups with isomorphic Burnside rings. Is the nilpotency class of H bounded by some function of the class of G? There is an example where G is of class 2 and H is of class 3.

R. Brandl

13.14. Is the lattice of quasivarieties of nilpotent torsion-free groups of nilpotency class $\leqslant 2$ distributive?

A. I. Budkin

13.15. Does a free non-abelian nilpotent group of class 3 possess an independent basis of quasiidentities in the class of torsion-free groups?

A. I. Budkin

13.17. A representation of a group G on a vector space V is called a *nil-representation* if for every v in V and g in G there exists $n = n(v, g)$ such that $v(g - 1)^n = 0$. Is it true that in zero characteristic every irreducible nil-representation is trivial? (In prime characteristic it is not true.)

S. Vovsi

13.19. Suppose that both a group Q and its normal subgroup H are subgroups of the direct product $G_1 \times \cdots \times G_n$ such that for each i the projections of both Q and H onto G_i coincide with G_i. If Q/H is a p-group, is Q/H a regular p-group?

Yu. M. Gorchakov

13.20. Is it true that the generating series of the growth function of every finitely generated group with one defining relation represents an algebraic (or even a rational) function?

R. I. Grigorchuk

13.21. b) What is the minimal possible rate of growth of the function $\pi(n) = \max\limits_{\delta(g) \leqslant n} |g|$ for the class of groups indicated in part "a" of this problem (see Archive)? It is known (R. I. Grigorchuk, *Math. USSR–Izv.*, **25** (1985), 259–300) that there exist p-groups with $\pi(n) \leqslant n^{\lambda}$ for some $\lambda > 0$. At the same time, it follows from a result of E. I. Zel'manov that $\pi(n)$ is not bounded if G is infinite.

R. I. Grigorchuk

13.22. Let the group $G = AB$ be the product of two polycyclic subgroups A and B, and assume that G has an ascending normal series with locally nilpotent factors (i. e. G is radical). Is it true that G is polycyclic?

F. de Giovanni

13.23. Let the group G have a finite normal series with infinite cyclic factors (containing of course G and $\{1\}$). Is it true that G has a non-trivial outer automorphism?

F. de Giovanni

13.24. Let G be a non-discrete topological group with only finitely many ultrafilters that converge to the identity. Is it true that G contains a countable open subgroup of exponent 2?

Comment of 2005: This is proved for G countable (E. G. Zelenyuk, *Mat. Studii*, **7** (1997), 139–144).

E. G. Zelenyuk

13.25. Let (G,τ) be a topological group with finite semigroup $\tau(G)$ of ultrafilters converging to the identity (I. V. Protasov, *Siberian Math. J.*, **34**, no. 5 (1993), 938–951). Is it true that $\tau(G)$ is a semigroup of idempotents?

Comment of 2005: This is proved for G countable (E. G. Zelenyuk, *Mat. Studii*, **14** (2000), 121–140).

E. G. Zelenyuk

***13.26.** Is it true that a countable topological group of exponent 2 with unique free ultrafilter converging to the identity has a basis of neighborhoods of the identity consisting of subgroups?

E. G. Zelenyuk, I. V. Protasov

*Assuming Martin's Axiom, the answer is "No" (Ye. Zelenyuk, *Adv. Math.*, **229**, no. 4 (2012), 2415–2426).

13.27. (B. Amberg). Suppose that $G = AB = AC = BC$ for a group G and its subgroups A, B, C.

a) Is G a Chernikov group if A, B, C are Chernikov groups?

b) Is G almost polycyclic if A, B, C are almost polycyclic?

L. S. Kazarin

13.29. Given an infinite set Ω, define an algebra A (the *reduced incidence algebra of finite subsets*) as follows. Let V_n be the set of functions from the set of n-element subsets of Ω to the rationals $\mathbb{Q}$. Now let $A = \bigoplus V_n$, with multiplication as follows: for $f \in V_n$, $g \in V_m$, and $|X| = m+n$, let $(fg)(X) = \sum f(Y)g(X \setminus Y)$, where the sum is over the n-element subsets Y of X. If G is a permutation group on Ω, let A^G be the algebra of G-invariants in A.

Conjecture: If G has no finite orbits on Ω, then A^G is an integral domain.

P. J. Cameron

13.30. A group G is called a *B-group* if every primitive permutation group which contains the regular representation of G is doubly transitive. Are there any countable B-groups?

P. J. Cameron

13.31. Let G be a permutation group on a set Ω. A sequence of points of Ω is a *base* for G if its pointwise stabilizer in G is the identity. *The greedy algorithm* for a base chooses each point in the sequence from an orbit of maximum size of the stabilizer of its predecessors. Is it true that there is a universal constant c with the property that, for any finite primitive permutation group, the greedy algorithm produces a base whose size is at most c times the minimal base size?

P. J. Cameron

13.32. (Well-known problem). On a group, a partial order that is directed upwards and has linearly ordered cone of positive elements is said to be *semilinear* if this order is invariant under right multiplication by the group elements. Can every semilinear order of a group be extended to a right order of the group?

V. M. Kopytov

13.35. Does every non-soluble pro-p-group of cohomological dimension 2 contain a free non-abelian pro-p-subgroup?

O. V. Mel'nikov

13.36. For a finitely generated pro-p-group G set $a_n(G) = \dim_{\mathbb{F}_p} I^n/I^{n+1}$, where I is the augmentation ideal of the group ring $\mathbb{F}_p[[G]]$. We define the *growth* of G to be the growth of the sequence $\{a_n(G)\}_{n\in\mathbb{N}}$.

a) If the growth of G is exponential, does it follow that G contains a free pro-p-subgroup of rank 2?

c) Do there exist pro-p-groups of finite cohomological dimension which are not p-adic analytic, and whose growth is slower than an exponential one?

O. V. Mel'nikov

13.37. Let G be a torsion-free pro-p-group, U an open subgroup of G. Suppose that U is a pro-p-group with a single defining relation. Is it true that then G is also a pro-p-group with a single defining relation?

O. V. Mel'nikov

13.39. Let A be an associative ring with unity and with torsion-free additive group, and let F^A be the tensor product of a free group F by A (A. G. Myasnikov, V. N. Remeslennikov, *Siberian Math. J.*, **35**, no. 5 (1994), 986–996); then F^A is a *free exponential group over A*; in (A. G. Myasnikov, V. N. Remeslennikov, *Int. J. Algebra Comput.*, **6** (1996), 687–711), it is shown how to construct F^A in terms of free products with amalgamation.

a) (G. Baumslag). Is F^A residually nilpotent torsion-free?

b) Is F^A a linear group?

c) (G. Baumslag). Is the Magnus homomorphism of $F^{\mathbb{Q}}$ into the group of power series over the rational number field $\mathbb{Q}$ faithful or not?

In the case where $\langle 1\rangle$ is a pure subgroup of the additive group of A, there are positive answers to questions "a" and "b" (A. M. Gaglione, A. G. Myasnikov, V. N. Remeslennikov, D. Spellman, *Commun. Algebra*, **25** (1997), 631–648). In the same paper it is shown that "a" is equivalent to "c". It is also known that the Magnus homomorphism is one-to-one on any subgroup of $F^{\mathbb{Q}}$ of the type $\langle F, t \mid u = t^n\rangle$ (G. Baumslag, *Commun. Pure Appl. Math.*, **21** (1968), 491–506).

d) Is the universal theory of F^A decidable?

e) (G. Baumslag). Can free A-groups be characterized by a length function?

f) (G. Baumslag). Does a free $\mathbb{Q}$-group admit a free action on some Λ-tree? See definition in (R. Alperin, H. Bass, *in: Combinatorial group theory and topology, Alta, Utah, 1984* (*Ann. Math. Stud.*, **111**), Princeton Univ. Press, 1987, 265–378).

A. G. Myasnikov, V. N. Remeslennikov

13.41. Is the elementary theory of the class of all groups acting freely on Λ-trees decidable?

A. G. Myasnikov, V. N. Remeslennikov

13.42. Prove that the tensor A-completion (see A. G. Myasnikov, V. N. Remeslennikov, *Siberian Math. J.*, **35**, no. 5 (1994), 986–996) of a free nilpotent group can be non-nilpotent.

A. G. Myasnikov, V. N. Remeslennikov

13.43. (G. R. Robinson). Let G be a finite group and B be a p-block of characters of G. *Conjecture:* If the defect group $D = D(B)$ of the block B is non-abelian, and if $|D : Z(D)| = p^a$, then each character in B has height strictly less than a.

J. Olsson

13.44. For any partition of an arbitrary group G into finitely many subsets $G = A_1 \cup \cdots \cup A_n$, there exists a subset of the partition A_i and a finite subset $F \subseteq G$, such that $G = A_i^{-1}A_iF$ (I. V. Protasov, *Siberian Math. J.*, **34**, no. 5 (1993), 938–952). Can the subset F always be chosen so that $|F| \leqslant n$? This is true for amenable groups.

I. V. Protasov

13.45. Every infinite group G of regular cardinality $\mathfrak{m}$ can be partitioned into two subsets $G = A_1 \cup A_2$ so that $A_1F \neq G$ and $A_2F \neq G$ for every subset $F \subset G$ of cardinality less than $\mathfrak{m}$. Is this statement true for groups of singular cardinality?

I. V. Protasov

13.48. (W. W. Comfort, J. van Mill). A topological group is said to be *irresolvable* if every two of its dense subsets intersect non-trivially. Does every non-discrete irresolvable group contain an infinite subgroup of exponent 2?

I. V. Protasov

13.49. (V. I. Malykhin). Can a topological group be partitioned into two dense subsets, if there are infinitely many free ultrafilters on the group converging to the identity?

I. V. Protasov

***13.50.** Let $\mathfrak{F}$ be a local Fitting class. Is it true that there are no maximal elements in the partially ordered by inclusion set of the Fitting classes contained in $\mathfrak{F}$ and distinct from $\mathfrak{F}$?

A. N. Skiba

*No, there may exist maximal elements (N. V. Savel'eva, N. T. Vorob'ev, *Siberian Math. J.*, **49**, no. 6 (2008), 1124–1130).

13.51. Is every finite modular lattice embeddable in the lattice of formations of finite groups?

A. N. Skiba

13.52. The *dimension* of a finitely based variety of algebras $\mathscr{V}$ is defined to be the maximal length of a *basis* (that is, an independent generating set) of the SC-theory $SC(\mathscr{V})$, which consists of the strong Mal'cev conditions satisfied on $\mathscr{V}$. The dimension is defined to be infinite if the lengths of bases in $SC(\mathscr{V})$ are not bounded. Does every finite abelian group generate a variety of finite dimension?

D. M. Smirnov

13.53. Let a, b be elements of finite order of the infinite group $G = \langle a, b\rangle$. Is it true that there are infinitely many elements $g \in G$ such that the subgroup $\langle a, b^g\rangle$ is infinite?

A. I. Sozutov

13.54. a) Is it true that, for p sufficiently large, every (finite) p-group can be a complement in some Frobenius group (see 6.55)?

A. I. Sozutov

13.55. Does there exist a Golod group (see 9.76), which is isomorphic to an AT-group? For a definition of an AT-*group* see (A. V. Rozhkov, *Math. Notes*, **40**, no. 5 (1986), 827–836).

A. V. Timofeenko

13.57. Let φ be an automorphism of prime order p of a finite group G such that $C_G(\varphi) \leqslant Z(G)$.

a) Is G soluble if $p = 3$? V. D. Mazurov and T. L. Nedorezov proved in (*Algebra and Logic*, **35**, no. 6 (1996), 392–397) that the group G is soluble for $p = 2$, and there are examples of unsoluble G for all $p > 3$.

b) If G is soluble, is the derived length of G bounded in terms of p?

c) (V. K. Kharchenko). If G is a p-group, is the derived length of G bounded in terms of p? (V. V. Bludov produced a simple example showing that the nilpotency class cannot be bounded.)

E. I. Khukhro

13.59. One can show that any extension of shape $\mathbb{Z}^d . \mathscr{S}L_d(\mathbb{Z})$ is residually finite, unless possibly $d = 3$ or $d = 5$. Are there in fact any extensions of this shape that fail to be residually finite when $d = 5$? As shown in (P. R. Hewitt, *Groups/St. Andrews'93 in Galway, Vol. 2* (*London Math. Soc. Lecture Note Ser.*, **212**), Cambridge Univ. Press, 1995, 305–313), there is an extension of $\mathbb{Z}^3$ by $\mathscr{S}L_3(\mathbb{Z})$ that is not residually finite. Usually it is true that an extension of an arithmetic subgroup of a Chevalley group over a rational module is residually finite. Is it ever false, apart from the examples of shape $\mathbb{Z}^3 . \mathscr{S}L_3(\mathbb{Z})$?

P. R. Hewitt

13.60. If a locally graded group G is a product of two subgroups of finite special rank, is G of finite special rank itself? The answer is affirmative if both factors are periodic groups. (N. S. Chernikov, *Ukrain. Math. J.*, **42**, no. 7 (1990), 855–861).

N. S. Chernikov

13.64. Let $\pi_e(G)$ denote the set of orders of elements of a group G. A group G is said to be an *OC_n-group* if $\pi_e(G) = \{1, 2, \ldots, n\}$. Is every OC_n-group locally finite? Do there exist infinite OC_n-groups for $n \geqslant 7$?

W. J. Shi

13.65. A finite simple group is called a *K_n-group* if its order is divisible by exactly n different primes. The number of K_3-groups is known to be 8. The K_4-groups are classified mod CFSG (W. J. Shi, *in: Group Theory in China* (*Math. Appl.*, **365**), Kluwer, 1996, 163–181) and some significant further results are obtained in (Yann Bugeaud, Zhenfu Cao, M. Mignotte, *J. Algebra*, **241** (2001), 658–668). But the question remains: is the number of K_4-groups finite or infinite?

W. J. Shi

13.67. Let G be a T_0-group (see 12.101), i an involution in G and $G = \langle i^g \mid g \in G \rangle$. Is the centralizer $C_G(i)$ residually periodic?

V. P. Shunkov

Problems from the 14th Issue (1999)

14.1. Suppose that G is a finite group with no non-trivial normal subgroups of odd order, and φ is its 2-automorphism centralizing a Sylow 2-subgroup of G. Is it true that φ^2 is an inner automorphism of G?

R. Zh. Aleev

14.2. (S. D. Berman). Prove that every automorphism of the centre of the integral group ring of a finite group induces a monomial permutation on the set of the class sums.

R. Zh. Aleev

14.3. Is it true that every central unit of the integral group ring of a finite group is a product of a central element of the group and a symmetric central unit? (A unit is *symmetric* if it is fixed by the canonical antiinvolution that transposes the coefficients at the mutually inverse elements.)

R. Zh. Aleev

14.4. a) Is it true that there exists a nilpotent group G for which the lattice $\mathscr{L}(G)$ of all group topologies is not modular? (It is known that for abelian groups the lattice $\mathscr{L}(G)$ is modular and that there are groups for which this lattice is not modular: V. I. Arnautov, A. G. Topale, *Izv. Akad. Nauk Moldova Mat.*, **1997**, no. 1, 84–92 (Russian).)

b) Is it true that for every countable nilpotent non-abelian group G the lattice $\mathscr{L}(G)$ of all group topologies is not modular?

V. I. Arnautov

14.5. Let G be an infinite group admitting non-discrete Hausdorff group topologies, and $\mathscr{L}(G)$ the lattice of all group topologies on G.

a) Is it true that for any natural number k there exists a non-refinable chain $\tau_0 < \tau_1 < \cdots < \tau_k$ of length k of Hausdorff topologies in $\mathscr{L}(G)$? (For countable nilpotent groups this is true (A. G. Topale, *Deposited in VINITI*, 25.12.98, no. 3849–V 98 (Russian)).)

b) Let k, m, n be natural numbers and let G be a nilpotent group of class k. Suppose that $\tau_0 < \tau_1 < \cdots < \tau_m$ and $\tau'_0 < \tau'_1 < \cdots < \tau'_n$ are non-refinable chains of Hausdorff topologies in $\mathscr{L}(G)$ such that $\tau_0 = \tau'_0$ and $\tau_m = \tau'_n$. Is it true that $m \leqslant n\cdot k$ and this inequality is best-possible? (This is true if $k = 1$, since for G abelian the lattice $\mathscr{L}(G)$ is modular.)

c) Is it true that there exists a countable G such that in the lattice $\mathscr{L}(G)$ there are a finite non-refinable chain $\tau_0 < \tau_1 < \cdots < \tau_k$ of Hausdorff topologies and an infinite chain $\{\tau'_\gamma \mid \gamma \in \Gamma\}$ of topologies such that $\tau_0 < \tau'_\gamma < \tau_k$ for any $\gamma \in \Gamma$?

d) Let G be an abelian group, k a natural number. Let A_k be the set of all those Hausdorff group topologies on G that, for every topology $\tau \in A_k$, any non-refinable chain of topologies starting from τ and terminating at the discrete topology has length k. Is it true that $A_k \bigcap \{\tau'_\gamma \mid \gamma \in \Gamma\} \neq \varnothing$ for any infinite non-refinable chain $\{\tau'_\gamma \mid \gamma \in \Gamma\}$ of Hausdorff topologies containing the discrete topology? (This is true for $k = 1$.)

V. I. Arnautov

***14.6.** A group Γ is said to have *Property* P_{nai} if, for any finite subset F of $\Gamma \setminus \{1\}$, there exists an element $y_0 \in \Gamma$ of infinite order such that, for each $x \in F$, the canonical epimorphism from the free product $\langle x \rangle * \langle y_0 \rangle$ onto the subgroup $\langle x, y_0 \rangle$ of Γ generated by x and y_0 is an isomorphism. For $n \in \{2, 3, \ldots\}$, does $PSL_n(\mathbb{Z})$ have Property P_{nai}? More generally, if Γ is a lattice in a connected real Lie group G which is simple and with centre reduced to $\{1\}$, does Γ have Property P_{nai}?

Answers are known to be "yes" if $n = 2$, and more generally if G has real rank 1 (M. Bekka, M. Cowling, P. de la Harpe, *Publ. Math. IHES*, **80** (1994), 117–134).

P. de la Harpe

Yes, it does (T. Poznansky, Characterization of linear groups whose reduced C-algebras are simple, `http://arxiv.org/abs/0812.2486`).

14.7. Let Γ_g be the fundamental group of a closed surface of genus $g \geqslant 2$. For each finite system S of generators of Γ_g, let $\beta_S(n)$ denote the number of elements of Γ_g which can be written as products of at most n elements of $S \cup S^{-1}$, and let $\omega(\Gamma_g, S) = \limsup_{n\to\infty} \sqrt[n]{\beta_S(n)}$ be the growth rate of the sequence $(\beta_S(n))_{n\geqslant 0}$. Compute the infimum $\omega(\Gamma_g)$ of the $\omega(\Gamma_g, S)$ over all finite sets of generators of Γ_g.

It is easy to see that, for a free group F_k of rank $k \geqslant 2$, the corresponding infimum is $\omega(F_k) = 2k - 1$ (M. Gromov, *Structures métriques pour les variétés riemanniennes,* Cedic/F. Nathan, Paris, 1981, Ex. 5.13). As any generating set of Γ_g contains a subset of $2g - 1$ elements generating a subgroup of infinite index in Γ_g with abelianization $\mathbb{Z}^{2g-1}$, hence a subgroup which is free of rank $2g - 1$, it follows that $\omega(\Gamma_g) \geqslant 4g - 3$.

P. de la Harpe

14.8. Let G denote the group of germs at $+\infty$ of orientation-preserving homeomorphisms of the real line $\mathbb{R}$. Let $\alpha \in G$ be the germ of $x \mapsto x + 1$. What are the germs $\beta \in G$ for which the subgroup $\langle \alpha, \beta \rangle$ of G generated by α and β is free of rank 2?

If β is the germ of $x \mapsto x^k$ for an odd integer $k \geqslant 3$, it is known that $\langle \alpha, \beta \rangle$ is free of rank 2. The proofs of this rely on Galois theory (for k an odd prime: S. White, *J. Algebra*, **118** (1988), 408–422; for any odd $k \geqslant 3$: S. A. Adeleke, A. M. W. Glass, L. Morley, *J. London Math. Soc.*, **43** (1991), 255–268, and for any odd $k \neq \pm 1$ and any even $k > 0$ in several papers by S. D. Cohen and A. M. W. Glass). *P. de la Harpe*

14.9. (Well-known problem). Let $W^*(F_k)$ denote the von Neumann algebra of the free group of rank $k \in \{2, 3, \ldots, \aleph_0\}$. Is $W^*(F_k)$ isomorphic to $W^*(F_l)$ for $k \neq l$?

For a group G, recall that $W^*(G)$ is an appropriate completion of the group algebra $\mathbb{C}G$ (see e. g. S. Sakai, *C^*-algebras and W^*-algebras,* Springer, 1971, in particular Problem 4.4.44). It is known that either $W^*(F_k) \cong W^*(F_l)$ for all $k, l \in \{2, 3, \ldots, \aleph_0\}$, or that the $W^*(F_k)$ are pairwise non-isomorphic (F. Radulescu, *Invent. Math.*, **115** (1994), 347–389, Corollary 4.7). *P. de la Harpe*

14.10. a) (Well-known problem). It is known that any recursively presented group embeds in a finitely presented group (G. Higman, *Proc. Royal Soc. London Ser. A*, **262** (1961), 455–475). Find an explicit and *"natural"* finitely presented group Γ and an embedding of the additive group of the rationals $\mathbb{Q}$ in Γ. There is an analogous question for a group Γ_n and an embedding of $GL_n(\mathbb{Q})$ in Γ_n. Another phrasing of the same problem is: find a simplicial complex X which covers a finite complex such that the fundamental group of X is $\mathbb{Q}$ or, respectively, $GL_n(\mathbb{Q})$. *P. de la Harpe*

14.11. (Yu. I. Merzlyakov). It is a well-known fact that for the ring $R = \mathbb{Q}[x, y]$ the elementary group $E_2(R)$ is distinct from $SL_2(R)$. Find a minimal subset $A \subseteq SL_2(R)$ such that $\langle E_2(R), A\rangle = SL_2(R)$.

V. G. Bardakov

14.12. (Yu. I. Merzlyakov, J. S. Birman). Is it true that all braid groups B_n, $n \geqslant 3$, are conjugacy separable?

V. G. Bardakov

***14.13.** b) By definition the *commutator length* of an element z of the derived subgroup of a group G is the least possible number of commutators from G whose product is equal to z. Does there exist a finitely presented simple group on which the commutator length is not bounded?

V. G. Bardakov

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*Yes, it exists (P.-E. Caprace, K. Fujiwara, *Geom. Funct. Anal.*, **19** (2010), 1296–1319).

14.14. (C. C. Edmunds, G. Rosenberger). We call a pair of natural numbers (k, m) *admissible* if in the derived subgroup F_2' of a free group F_2 there is an element w such that the commutator length of the element w^m is equal to k. Find all admissible pairs.

It is known that every pair $(k, 2)$, $k \geqslant 2$, is admissible (V. G. Bardakov, *Algebra and Logic*, **39** (2000), 224–251).

V. G. Bardakov

14.15. For the automorphism group $A_n = \operatorname{Aut} F_n$ of a free group F_n of rank $n \geqslant 3$ find the supremum k_n of the commutator lengths of the elements of A_n'.

It is easy to show that $k_2 = \infty$. On the other hand, the commutator length of any element of the derived subgroup of $\varinjlim A_n$ is at most 2 (R. K. Dennis, L. N. Vaserstein, *K-Theory*, **2**, N 6 (1989), 761–767).

V. G. Bardakov

14.16. Following Yu. I. Merzlyakov we define the *width* of a verbal subgroup $V(G)$ of a group G with respect to the set of words V as the smallest $m \in \mathbb{N} \cup \{\infty\}$ such that every element of $V(G)$ can be written as a product of $\leqslant m$ values of words from $V \cup V^{-1}$. It is known that the width of any verbal subgroup of a finitely generated group of polynomial growth is finite. Is this statement true for finitely generated groups of intermediate growth?

V. G. Bardakov

14.17. Let $\mathrm{IMA}(G)$ denote the subgroup of the automorphism group $\operatorname{Aut} G$ consisting of all automorphisms that act trivially on the second derived quotient G/G''. Find generators and defining relations for $\mathrm{IMA}(F_n)$, where F_n is a free group of rank $n \geqslant 3$.

V. G. Bardakov

14.18. We say that a family of groups $\mathscr{D}$ *discriminates* a group G if for any finite subset $\{a_1, \ldots, a_n\} \subseteq G \setminus \{1\}$ there exists a group $D \in \mathscr{D}$ and a homomorphism $\varphi : G \to D$ such that $a_j\varphi \neq 1$ for all $j = 1, \ldots, n$. Is every finitely generated group acting freely on some Λ-tree discriminated by torsion-free hyperbolic groups?

G. Baumslag, A. G. Myasnikov, V. N. Remeslennikov

14.19. We say that a group G has the *Noetherian Equation Property* if every system of equations over G in finitely many variables is equivalent to some finite part of it. Does an arbitrary hyperbolic group have the Noetherian Equation Property?

Comment of 2013: the conjecture holds for torsion-free hyperbolic groups (Z. Sela, *Proc. London Math. Soc.*, **99**, no. 1 (2009), 217–273) and for the larger class of toral relatively hyperbolic groups (D. Groves, *J. Geom. Topol.*, **9** (2005), 2319–2358).

G. Baumslag, A. G. Myasnikov, V. N. Remeslennikov

14.20. Does a free product of groups have the Noetherian Equation Property if this property is enjoyed by the factors?

G. Baumslag, A. G. Myasnikov, V. N. Remeslennikov

14.21. Does a free pro-p-group have the Noetherian Equation Property?

G. Baumslag, A. G. Myasnikov, V. N. Remeslennikov

14.22. Prove that any irreducible system of equations $S(x_1,\dots,x_n) = 1$ with coefficients in a torsion-free linear group G is equivalent over G to a finite system $T(x_1,\dots,x_n) = 1$ satisfying an analogue of Hilbert's Nullstellensatz, i. e. $Rad_G(T) = \sqrt{T}$. This is true if G is a free group (O. Kharlampovich, A. Myasnikov, *J. Algebra*, **200** (1998), 472–570).

Here both S and T are regarded as subsets of $G[X] = G * F(X)$, a free product of G and a free group on $X = \{x_1,\dots,x_n\}$. By definition, $Rad_G(T) = \{w(x_1,\dots,x_n) \in G[X] \mid w(g_1,\dots,g_n) = 1$ for any solution $g_1,\dots,g_n \in G$ of the system $T(X) = 1\}$, and $\sqrt{T}$ is the minimal normal isolated subgroup of $G[X]$ containing T.

G. Baumslag, A. G. Myasnikov, V. N. Remeslennikov

14.23. Let F_n be a free group with basis $\{x_1,\dots,x_n\}$, and let $|\cdot|$ be the length function with respect to this basis. For $\alpha \in \operatorname{Aut} F_n$ we put $\|\alpha\| = \max\{|\alpha(x_1)|,\dots,|\alpha(x_n)|\}$. Is it true that there is a recursive function $f : \mathbb{N} \to \mathbb{N}$ with the following property: for any $\alpha \in \operatorname{Aut} F_n$ there is a basis $\{y_1,\dots,y_k\}$ of $\operatorname{Fix}(\alpha) = \{x \mid \alpha(x) = x\}$ such that $|y_i| \leqslant f(\|\alpha\|)$ for all $i = 1,\dots,k$?

O. V. Bogopol'skiĭ

14.24. Let $\operatorname{Aut} F_n$ be the automorphism group of a free group of rank n with norm $\|\cdot\|$ as in 14.23. Does there exist a recursive function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with the following property: for any two conjugate elements $\alpha, \beta \in \operatorname{Aut} F_n$ there is an element $\gamma \in \operatorname{Aut} F_n$ such that $\gamma^{-1}\alpha\gamma = \beta$ and $\|\gamma\| \leqslant f(\|\alpha\|, \|\beta\|)$?

O. V. Bogopol'skiĭ

14.26. A quasivariety $\mathscr{M}$ is *closed under direct $\mathbb{Z}$-wreath products* if the direct wreath product $G \wr \mathbb{Z}$ belongs to $\mathscr{M}$ for every $G \in \mathscr{M}$ (here $\mathbb{Z}$ is an infinite cyclic group). Is the quasivariety generated by the class of all nilpotent torsion-free groups closed under direct $\mathbb{Z}$-wreath products?

A. I. Budkin

14.28. Let $\mathfrak{F}$ be a soluble Fitting formation of finite groups with Kegel's property, that is, $\mathfrak{F}$ contains every finite group of the form $G = AB = BC = CA$ if A, B, C are in $\mathfrak{F}$. Is $\mathfrak{F}$ a saturated formation?

Comment of 2013: Some progress was made in (A. Ballester-Bolinches, L. M. Ezquerro, *J. Group Theory*, **8**, no. 5 (2005), 605–611).

A. F. Vasiliev

14.29. Is there a soluble Fitting class of finite groups $\mathfrak{F}$ such that $\mathfrak{F}$ is not a formation and $A_{\mathfrak{F}} \cap B_{\mathfrak{F}} \subseteq G_{\mathfrak{F}}$ for every finite soluble group of the form $G = AB$?

A. F. Vasiliev

14.30. Let $\mathrm{lFit}\mathfrak{X}$ be the local Fitting class generated by a set of groups $\mathfrak{X}$ and let $\Psi(G)$ be the smallest normal subgroup of a finite group G such that $\mathrm{lFit}(\Psi(G)\cap M) = \mathrm{lFit}M$ for every $M \lhd\lhd G$ (K. Doerk, P. Hauck, *Arch. Math.*, **35**, no. 3 (1980), 218–227). We say that a Fitting class $\mathfrak{F}$ is *saturated* if $\Psi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$. Is it true that every non-empty soluble saturated Fitting class is local?

N. T. Vorob'ëv

14.31. Is the lattice of Fitting subclasses of the Fitting class generated by a finite soluble group finite?

N. T. Vorob'ëv

***14.32.** Extending the classical definition of formations, let us define a *formation* of (not necessarily finite) groups as a nonempty class of groups closed under taking homomorphic images and subdirect products with finitely many factors. Must every first order axiomatizable formation of groups be a variety?

A. M. Gaglione, D. Spellman

*No, every variety of groups that contains a finite nonsolvable member contains an axiomatic subformation that is not a variety (K. A. Kearnes, *J. Group Theory*, **13**, No. 2 (2010), 233–241).

14.35. Is every finitely presented group of prime exponent finite?

N. D. Gupta

14.36. A group G is called a T*-group* if every subnormal subgroup of G is normal, while G is said to be a $\overline{T}$*-group* if all of its subgroups are T-groups. Is it true that every non-periodic locally graded $\overline{T}$-group must be abelian?

F. de Giovanni

14.37. Let $G(n)$ be one of the classical groups (special, orthogonal, or symplectic) of $(n\times n)$-matrices over an infinite field K of non-zero characteristic, and $M(n)$ the space of all $(n \times n)$-matrices over K. The group $G(n)$ acts diagonally by conjugation on the space $M(n)^m = \underbrace{M(n) \oplus \cdots \oplus M(n)}_{m}$. Find generators of the algebra of invariants $K[M(n)^m]^{G(n)}$.

In characteristic 0 they were found in (C. Procesi, *Adv. Math.*, **19** (1976), 306–381). *Comment of 2001:* in positive characteristic the problem is solved for all cases excepting the orthogonal groups in characteristic 2 and special orthogonal groups of even degree (A. N. Zubkov, *Algebra and Logic*, **38**, no. 5 (1999), 299–318). *Comment of 2009:* ...and for special orthogonal groups of even degree over (infinite) fields of odd characteristic (A. A. Lopatin, *J. Algebra*, **321** (2009), 1079–1106).

A. N. Zubkov

14.38. For every pro-p-group G of (2×2)-matrices for $p \neq 2$ an analogue of the Tits Alternative holds: either G is soluble, or the variety of pro-p-groups generated by G contains the group $\overline{\left\langle \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right\rangle} \leqslant SL_2(\mathbb{F}_p[[t]])$ (A. N. Zubkov, *Algebra and Logic*, **29**, no. 4 (1990), 287–301). Is the same result true for matrices of size $\geqslant 3$ for $p \neq 2$?

A. N. Zubkov

14.39. Let $F(V)$ be a free group of some variety of pro-p-groups V. Is there a uniform bound for the exponents of periodic elements in $F(V)$? This is true if the variety V is metabelian (A. N. Zubkov, *Dr. Sci. Diss.*, Omsk, 1997 (Russian)).

A. N. Zubkov

14.40. Is a free pro-p-group representable as an abstract group by matrices over a commutative-associative ring with 1?

A. N. Zubkov

14.41. A group G is said to be *para-free* if all factors $\gamma_i(G)/\gamma_{i+1}(G)$ of its lower central series are isomorphic to the corresponding lower central factors of some free group, and $\bigcap_{i=1}^{\infty} \gamma_i(G) = 1$. Is an arbitrary para-free group representable by matrices over a commutative-associative ring with 1?

A. N. Zubkov

14.42. Is a free pro-p-group representable by matrices over an associative-commutative profinite ring with 1?

A negative answer is equivalent to the fact that every linear pro-p-group satisfies a non-trivial pro-p-identity; this is known to be true in dimension 2 (for $p \neq 2$: A. N. Zubkov, *Siberian Math. J.*, **28**, no. 5 (1987), 742–747; for $p = 2$: E. I. Zelmanov, *unpublished*).

A. N. Zubkov, V. N. Remeslennikov

14.43. Suppose that a finite group G has the form $G = AB$, where A and B are nilpotent subgroups of classes α and β respectively; then G is soluble (O. H. Kegel, H. Wielandt, 1961). Although the derived length $\mathrm{dl}(G)$ of G need not be bounded by $\alpha + \beta$ (see Archive, 5.17), can one bound $\mathrm{dl}(G)$ by a (linear) function of α and β?

L. S. Kazarin

14.44. Let $k(X)$ denote the number of conjugacy classes of a finite group X. Suppose that a finite group $G = AB$ is a product of two subgroups A, B of coprime orders. Is it true that $k(AB) \leqslant k(A)k(B)$?

Note that one cannot drop the coprimeness condition and the answer is positive if one of the subgroups is normal, see Archive, 11.43.

L. S. Kazarin, J. Sangroniz

14.45. Does there exist a (non-abelian simple) linearly right-orderable group all of whose proper subgroups are cyclic?

U. E. Kaljulaid

14.46. A finite group G is said to be *almost simple* if $T \leqslant G \leqslant \mathrm{Aut}(T)$ for some nonabelian simple group T. By definition, a *finite linear space* consists of a set V of points, together with a collection of k-element subsets of V, called *lines* ($k \geqslant 3$), such that every pair of points is contained in exactly one line. Classify the finite linear spaces which admit a line-transitive almost simple subgroup G of automorphisms which acts transitively on points.

A. Camina, P. Neumann and C. E. Praeger have solved this problem in the case where T is an alternating group. In (A. Camina, C. E. Praeger, *Aequat. Math.*, **61** (2001), 221–232) it is shown that a line-transitive group of automorphisms of a finite linear space which is point-*quasiprimitive* (i. e. all of whose non-trivial normal subgroups are point-transitive) is almost simple or affine.

Remarks of 2005: The cases of the following groups, but not almost simple groups with this socle, were dealt with: $PSU(3, q)$ (W. Liu, *Linear Algebra Appl.*, **374** (2003), 291–305); $PSL(2, q)$ (W. Liu, *J. Combin. Theory (A)*, **103** (2003), 209–222); $Sz(q)$ (W. Liu, *Discrete Math.*, **269** (2003), 181–190); $Ree(q)$ (W. Liu, *Europ. J. Combin.*, **25** (2004), 311–325); sporadic simple groups were done in (A. R. Camina, F. Spiezia, *J. Combin. Des.*, **8** (2000), 353–362). *Remark of 2009:* It was shown (N. Gill, `arxiv0903.3302`) that for non-desarguesian projective planes a point-quasiprimitive group would be affine. *Remark of 2013:* The problem is solved for large-dimensional classical groups (A. Camina, N. Gill, A. E. Zalesski, *Bull. Belg. Math. Soc. Simon Stevin*, **15**, no. 4 (2008), 705–731).

A. Camina, C. E. Praeger

14.47. Is the lattice of all soluble Fitting classes of finite groups modular?

S. F. Kamornikov, A. N. Skiba

14.51. (Well-known problems). Is there a finite basis for the identities of any
a) abelian-by-nilpotent group?
b) abelian-by-finite group?
c) abelian-by-(finite nilpotent) group?

A. N. Krasil'nikov

14.53. *Conjecture:* Let G be a profinite group such that the set of solutions of the equation $x^n = 1$ has positive Haar measure. Then G has an open subgroup H and an element t such that all elements of the coset tH have order dividing n.

This is true in the case $n = 2$. It would be interesting to see whether similar results hold for profinite groups in which the set of solutions of some equation has positive measure.

L. Levai, L. Pyber

14.54. Let $k(G)$ denote the number of conjugacy classes of a finite group G. Is it true that $k(G) \leqslant |N|$ for some nilpotent subgroup N of G? This is true if G is simple; besides, always $k(G) \leqslant |S|$ for some soluble subgroup $S \leqslant G$.

M. W. Liebeck, L. Pyber

14.55. b) Prove that the Nottingham group $J = N(\mathbb{Z}/p\mathbb{Z})$ (as defined in Archive, 12.24) is finitely presented for $p = 2$.

C. R. Leedham-Green

14.56. Prove that if G is an infinite pro-p-group with $G/\gamma_{2p+1}(G)$ isomorphic to $J/\gamma_{2p+1}(J)$ then G is isomorphic to J, where J is the Nottingham group.

C. R. Leedham-Green

14.57. Describe the hereditarily just infinite pro-p-groups of finite width.

Definitions: A pro-p-group G has finite *width* p^d if $|\gamma_i(G)/\gamma_{i+1}(G)| \leqslant p^d$ for all i, and G is *hereditarily just infinite* if G is infinite and each of its open subgroups has no closed normal subgroups of infinite index.

C. R. Leedham-Green

14.58. a) Suppose that A is a periodic group of regular automorphisms of an abelian group. Is A cyclic if A has prime exponent?

V. D. Mazurov

14.59. Suppose that G is a triply transitive group in which a stabilizer of two points contains no involutions, and a stabilizer of three points is trivial. Is it true that G is similar to $PGL_2(P)$ in its natural action on the projective line $P \cup \{\infty\}$, for some field P of characteristic 2? This is true under the condition that the stabilizer of two points is periodic.

V. D. Mazurov

14.61. Determine all pairs $(\mathscr{S}, G)$, where $\mathscr{S}$ is a semipartial geometry and G is an almost simple flag-transitive group of automorphisms of $\mathscr{S}$. A system of points and lines (P, B) is a *semipartial geometry* with parameters (α, s, t, μ) if every point belongs to exactly $t+1$ lines (two different points belong to at most one line); every line contains exactly $s+1$ points; for any anti-flag $(a, l) \in (P, B)$ the number of lines containing a and intersecting l is either 0 or α; and for any non-collinear points a, b there are exactly μ points collinear with a and with b.

A. A. Makhnëv

14.64. (M. F. Newman). Classify the finite 5-groups of maximal class; computer calculations suggest some conjectures (M. F. Newman, in: *Groups–Canberra, 1989* (*Lecture Notes Math.*, **1456**), Springer, Berlin, 1990, 49–62).

A. Moretó

14.65. (Well-known problem). For a finite group G let $\rho(G)$ denote the set of prime numbers dividing the order of some conjugacy class, and $\sigma(G)$ the maximum number of primes dividing the order of some conjugacy class. Is it true that $|\rho(G)| \leqslant 3\sigma(G)$?

A possible linear bound for $|\rho(G)|$ in terms of $\sigma(G)$ cannot be better than $3\sigma(G)$, since there is a family of groups $\{G_n\}$ such that $\lim_{n\to\infty} |\rho(G_n)|/\sigma(G_n) = 3$ (C. Casolo, S. Dolfi, *Rend. Sem. Mat. Univ. Padova*, **96** (1996), 121–130).

Comment of 2009: A quadratic bound was obtained in (A. Moretó, *Int. Math. Res. Not.*, **2005**, no. 54, 3375–3383), and a linear bound in (C. Casolo, S. Dolfi, *J. Group Theory*, **10**, no. 5 (2007), 571–583).

A. Moretó

14.67. Suppose that a is a non-trivial element of a finite group G such that $|C_G(a)| \geqslant |C_G(x)|$ for every non-trivial element $x \in G$, and let H be a nilpotent subgroup of G which is normalized by $C_G(a)$. Is it true that $H \leqslant C_G(a)$? This is true if H is abelian, or if H is a p'-group for some prime number $p \in \pi(Z(C_G(a)))$.

I. T. Mukhamet'yanov, A. N. Fomin

14.68. (Well-known problem). Suppose that F is an automorphism of order 2 of the polynomial ring $R_n = \mathbb{C}[x_1, \ldots, x_n]$, $n > 2$. Does there exist an automorphism G of R_n such that $G^{-1}FG$ is a linear automorphism?

This is true for $n = 2$.

M. V. Neshchadim

14.69. For every finite simple group find the minimum of the number of generating involutions satisfying an additional condition, in each of the following cases.

a) The product of the generating involutions equals 1.

b) (Malle–Saxl–Weigel). All generating involutions are conjugate.

c) (Malle–Saxl–Weigel). The conditions a) and b) are simultaneously satisfied.

Editors' comment: partial progress was obtained in (J. Ward, PhD Thesis, QMW, 2009, `http://www.maths.qmul.ac.uk/~raw/JWardPhD.pdf`).

d) All generating involutions are conjugate and two of them commute.

Ya. N. Nuzhin

14.70. A group G is called *n-Engel* if it satisfies the identity $[x, y, \ldots, y] = 1$, where y is taken n times. Are there non-nilpotent finitely generated n-Engel groups?

B. I. Plotkin

14.72. Let X be a regular algebraic variety over a field of arbitrary characteristic, and G a finite cyclic group of automorphisms of X. Suppose that the fixed point variety X^G of G is a regular hypersurface of X (of codimension 1). Is the quotient variety X/G regular?

K. N. Ponomarëv

14.73. *Conjecture:* There is a function f on the natural numbers such that, if Γ is a finite, vertex-transitive, locally-quasiprimitive graph of valency v, then the number of automorphisms fixing a given vertex is at most $f(v)$. (By definition, a vertex-transitive graph Γ is *locally-quasiprimitive* if the stabilizer in $Aut(\Gamma)$ of a vertex α is quasiprimitive (see 14.46) in its action on the set of vertices adjacent to α.)

To prove the conjecture above one need only consider the case where $Aut(\Gamma)$ has the property that every non-trivial normal subgroup has at most two orbits on vertices (C. E. Praeger, *Ars Combin.*, **19 A** (1985), 149–163). The analogous conjecture for finite, vertex-transitive, locally-primitive graphs was made by R. Weiss in 1978 and is still open. For non-bipartite graphs, there is a "reduction" of Weiss' conjecture to the case where the automorphism group is almost simple (see 14.46 for definition) (M. Conder, C. H. Li, C. E. Praeger, *Proc. Edinburgh Math. Soc. (2)*, **43**, no. 1 (2000), 129–138).

Comment of 2013: These conjectures have been proved in the case where there is an upper bound on the degree of any alternating group occurring as a quotient of a subgroup (C. E. Praeger, L. Pyber, P. Spiga, E. Szabó, *Proc. Amer. Math. Soc.*, **140**, no. 7 (2012), 2307–2318).

C. E. Praeger

14.74. Let $k(G)$ denote the number of conjugacy classes of a finite group G. Is it true that $k(G) \leqslant k(P_1) \cdots k(P_s)$, where $P_1, \ldots, P_s$ are Sylow subgroups of G such that $|G| = |P_1| \cdots |P_s|$?

L. Pyber

14.75. Suppose that $\mathfrak{G} = \{G_1, G_2, \ldots\}$ is a family of finite 2-generated groups which generates the variety of all groups. Is it true that a free group of rank 2 is residually in $\mathfrak{G}$?

L. Pyber

14.76. Does there exist an absolute constant c such that any finite p-group P has an abelian section A satisfying $|A|^c > |P|$?

By a result of A. Yu. Ol'shanskiĭ (*Math. Notes*, **23** (1978), 183–185) we cannot require A to be a subgroup. By a result of J. G. Thompson (*J. Algebra*, **13** (1969), 149–151) the existence of such a section A would imply the existence of a class 2 subgroup H of P with $|H|^c > |P|$.

L. Pyber

14.78. Suppose that $\mathfrak{H} \subseteq \mathfrak{F}_1 \subseteq \mathfrak{M}$ and $H \subseteq \mathfrak{F}_2 \subseteq \mathfrak{M}$ where $\mathfrak{H}$ and $\mathfrak{M}$ are local formations of finite groups and $\mathfrak{F}_1$ is a complement for $\mathfrak{F}_2$ in the lattice of all formations between $\mathfrak{H}$ and $\mathfrak{M}$. Is it true that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are local formations? This is true in the case $\mathfrak{H} = (1)$.

A. N. Skiba

14.79. Suppose that $\mathfrak{F} = \mathfrak{M}H = \mathrm{lFit}(G)$ is a soluble one-generated local Fitting class of finite groups where $\mathfrak{H}$ and $\mathfrak{M}$ are Fitting classes and $\mathfrak{M} \neq \mathfrak{F}$. Is $\mathfrak{H}$ a local Fitting class?

A. N. Skiba

14.81. Prove that the formation generated by a finite group has only finitely many S_n-closed subformations.

A. N. Skiba, L. A. Shemetkov

***14.82.** (Well-known problem). Describe the finite simple groups in which every element is a product of two involutions.

A. I. Sozutov

*Described in (E. P. Vdovin, A. A. Gal't, *Siberian Math. J.*, **51** (2010), 610–615).

14.83. We say that an infinite simple group G is a *monster of the third kind* if for every non-trivial elements a, b, of which at least one is not an involution, there are infinitely many elements $g \in G$ such that $\langle a, b^g \rangle = G$. (Compare with V. P. Shunkov's definitions in Archive, 6.63, 6.64.) Is it true that every simple quasi-Chernikov group is a monster of the third kind? This is true for quasi-finite groups (A. I. Sozutov, *Algebra and Logic*, **36**, no. 5 (1997), 336–348). (We say that a non-σ group is *quasi-σ* if all of its proper subgroups have the property σ.)

A. I. Sozutov

14.84. An element g of a (relatively) free group $F_r(\mathfrak{M})$ of rank r of a variety $\mathfrak{M}$ is said to be *primitive* if it can be included in a basis of $F_r(\mathfrak{M})$.

a) Do there exist a group $F_r(\mathfrak{M})$ and a non-primitive element $h \in F_r(\mathfrak{M})$ such that for some monomorphism α of $F_r(\mathfrak{M})$ the element $\alpha(h)$ is primitive?

b) Do there exist a group $F_r(\mathfrak{M})$ and a non-primitive element $h \in F_r(\mathfrak{M})$ such that for some $n > r$ the element h is primitive in $F_n(\mathfrak{M})$?

E. I. Timoshenko

14.85. Suppose that an endomorphism φ of a free metabelian group of rank r takes every primitive element to a primitive one. Is φ necessarily an automorphism? This is true for $r \leqslant 2$.

E. I. Timoshenko, V. Shpilrain

14.89. (E. A. O'Brien, A. Shalev). Let P be a finite p-group of order p^m and let $m = 2n + e$ with $e = 0$ or 1. By a theorem of P. Hall the number of conjugacy classes of P has the form $n(p^2 - 1) + p^e + a(p^2 - 1)(p - 1)$ for some integer $a \geqslant 0$, which is called the *abundance* of P.

a) Is there a bound for the coclass of P which depends only on a? Note that $a = 0$ implies coclass 1 and that all known examples with $a = 1$ have coclass $\leqslant 3$. (The group P has *coclass* r if $|P| = p^{c+r}$ where c is the nilpotency class of P.)

b) Is there an element $s \in P$ such that $|C_P(s)| \leqslant p^{f(a)}$ for some $f(a)$ depending only on a? We already know that we can take $f(0) = 2$ and it seems that $f(1) = 3$.

Note that A. Jaikin-Zapirain (*J. Group Theory*, **3**, no. 3 (2000), 225–231) has proved that $|P| \leqslant p^{f(p,a)}$ for some function f of p and a only.

G. Fernández–Alcober

14.90. Let P be a finite p-group of abundance a and nilpotency class c. Does there exist an integer $t = t(a)$ such that $\gamma_i(G) = \zeta_{c-i+1}(G)$ for $i \geqslant t$? This holds for $a = 0$ with $t = 1$, since P has maximal class; it can be proved that for $a = 1$ one can take $t = 3$.

G. Fernández–Alcober

14.91. Let p be a fixed prime. Do there exist finite p-groups of abundance a for any $a \geqslant 0$?

G. Fernández–Alcober

14.93. Let $N(\mathbb{Z}/p\mathbb{Z})$ be the group defined in Archive, 12.24 (the so-called "Nottingham group", or the "Wild group"). Find relations of $N(\mathbb{Z}/p\mathbb{Z})$ as a pro-p-group (it has two generators, e. g., $x + x^2$ and $x/(1 - x)$). *Comment of 2009:* for $p > 2$ see progress in (M. V. Ershov, *J. London Math. Soc.*, **71** (2005), 362–378).

I. B. Fesenko

14.94. For each positive integer r find the p-cohomological dimension $\mathrm{cd}_p(H_r)$ where H_r is the closed subgroup of $N(\mathbb{Z}/p\mathbb{Z})$ consisting of the series $x\left(1 + \sum\limits_{i=1}^{\infty} a_i x^{p^r i}\right)$, $a_i \in \mathbb{Z}/p\mathbb{Z}$.

I. B. Fesenko

14.95. (C. R. Leedham-Green, Peter M. Neumann, J. Wiegold). For a finite p-group P, denote by $c = c(P)$ its nilpotency class and by $b = b(P)$ its *breadth*, that is, p^b is the maximum size of a conjugacy class in P. *Class-Breadth Problem:* Is it true that $c \leqslant b + 1$ if $p \neq 2$?

So far, the best known bound is $c < \frac{p}{p-1}b + 1$ (C. R. Leedham-Green, P. M. Neumann, J. Wiegold, *J. London Math. Soc. (2)*, **1** (1969), 409–420). For $p = 2$ for every $n \in \mathbb{N}$ there exists a 2-group T_n such that $c(T_n) \geqslant b(T_n) + n$ (W. Felsch, J. Neubüser, W. Plesken, *J. London Math. Soc. (2)*, **24** (1981), 113–122).

A. Jaikin-Zapirain

14.97. Is it true that for any two different prime numbers p and q there exists a non-primary periodic locally soluble $\{p, q\}$-group that can be represented as the product of two of its p-subgroups?

N. S. Chernikov

14.98. We say that a metric space is a 2-*end* one (a *narrow* one), if it is quasiisometric to the real line $\mathbb{R}$ (respectively, to a subset of $\mathbb{R}$). All other spaces are said to be *wide*. Suppose that the Caley graph $\Gamma = \Gamma(G, A)$ of a group G with a finite set of generators A in the natural metric contains a 2-end subset, and suppose that there is $\varepsilon > 0$ such that the complement in Γ to the ε-neighbourhood of any connected 2-end subset contains exactly two wide connected components. Is it true that the group G in the word metric is quasiisometric to the Euclidean or hyperbolic plane?

V. A. Churkin

14.99. A formation $\mathfrak{F}$ of finite groups is called *superradical* if it is S_n-closed and contains every finite group of the form $G = AB$ where A and B are $\mathfrak{F}$-subnormal F-subgroups.

a) Find all superradical local formations.

b) Prove that every S-closed superradical formation is a solubly saturated formation.

L. A. Shemetkov

14.100. Is it true that in a Shunkov group (i. e. conjugately biprimitively finite group, see 6.59) having infinitely many elements of finite order every element of prime order is contained in some infinite locally finite subgroup? This is true under the additional condition that any two conjugates of this element generate a soluble subgroup (V. P. Shunkov, M_p-*groups*, Moscow, Nauka, 1990 (Russian)).

A. K. Shlëpkin

14.101. A group G is *saturated* by groups from a class X if every finite subgroup $K \leqslant G$ is contained in a subgroup $L \leqslant G$ isomorphic to some group from X. Is it true that a periodic group saturated by finite simple groups of Lie type of uniformly bounded ranks is itself a simple group of Lie type of finite rank?

A. K. Shlëpkin

14.102. (V. Lin). Let B_n be the braid group on n strings, and let $n > 4$.

a) Does B_n have any non-trivial non-injective endomorphisms?

b) Is it true that every non-trivial endomorphism of the derived subgroup $[B_n,\, B_n]$ is an automorphism?

V. Shpilrain

Problems from the 15th Issue (2002)

15.1. (P. Longobardi, M. Maj, A. H. Rhemtulla). Let $w = w(x_1, \ldots, x_n)$ be a group word in n variables $x_1, \ldots, x_n$, and $V(w)$ the variety of groups defined by the law $w = 1$. Let $V(w^*)$ (respectively, $V(w^{\#})$) be the class of all groups G in which for every n infinite subsets $S_1, \ldots, S_n$ there exist $s_i \in S_i$ such that $w(s_1, \ldots, s_n) = 1$ (respectively, $\langle s_1, \ldots, s_n \rangle \in V(w)$).

a) Is there some word w and an infinite group G such that $G \in V(w^{\#})$ but $G \notin V(w)$?

b) Is there some word w and an infinite group G such that $G \in V(w^*)$ but $G \notin V(w^{\#})$?

The answer to both of these questions is likely to be "yes". It is known that in a) w cannot be any of several words such as x_1^n, $[x_1, \ldots, x_n]$, $[x_1, x_2]^2$, $(x_1x_2)^3x_2^{-3}x_1^{-3}$, and $x_1^{a_1} \cdots x_n^{a_n}$ for any non-zero integers $a_1, \ldots, a_n$.

A. Abdollahi

15.2. By a theorem of W. Burnside, if $\chi \in \mathrm{Irr}(G)$ and $\chi(1) > 1$, then there exists $x \in G$ such that $\chi(x) = 0$, that is, only the linear characters are "nonvanishing". It is interesting to consider the dual notion of *nonvanishing elements* of a finite group G, that is, the elements $x \in G$ such that $\chi(x) \neq 0$ for all $\chi \in \mathrm{Irr}(G)$.

a) It is proved (I. M. Isaacs, G. Navarro, T. R. Wolf, *J. Algebra*, **222**, no. 2 (1999), 413–423) that if G is solvable and $x \in G$ is a nonvanishing element of odd order, then x belongs to the Fitting subgroup $\mathbf{F}(G)$. Is this true for elements of even order too? (M. Miyamoto showed in 2008 that every nontrivial abelian normal subgroup of a finite group contains a nonvanishing element.)

b) Which nonabelian simple groups have nonidentity nonvanishing elements? (For example, $\mathbb{A}_7$ has.)

I. M. Isaacs

15.3. Let α and β be faithful non-linear irreducible characters of a finite group G. There are non-solvable groups G giving examples when the product $\alpha\beta$ is again an irreducible character (for some of such α, β). One example is $G = SL_2(5)$ with two irreducible characters of degree 2. In (I. Zisser, *Israel J. Math.*, **84**, no. 1–2 (1993), 147–151) it is proved that such an example exists in an alternating group $\mathbb{A}_n$ if and only if n is a square exceeding 4. But do solvable examples exist? Evidence (but no proof) that they do not is given in (I. M. Isaacs, *J. Algebra*, **223**, no. 2 (2000), 630–646).

I. M. Isaacs

***15.5.** (Well-known problem). Does there exist an infinite finitely generated group which is simple and amenable?

P. de la Harpe

*Yes, it does (K. Juschenko, N. Monod, *to appear in Ann. Math.*; `http://arxiv.org/abs/1204.2132`).

***15.6.** (Well-known problem). Is it true that Golod p-groups are non-amenable?

These infinite finitely generated torsion groups are defined in (E. S. Golod, *Amer. Math. Soc. Transl. (2)*, **48** (1965), 103–106).

P. de la Harpe

*Yes, moreover, such groups have infinite quotients with Kazhdan's property (T) and are uniformly non-amenable (M. Ershov, A. Jaikin-Zapirain, *Proc. London Math. Soc. (3)*, **102**, no. 4 (2011), 599–636).

15.7. (Well-known problem). Is it true that the reduced C^*-algebra of a countable group without amenable normal subgroups distinct from $\{1\}$ is always a simple C^*-algebra with unique trace? See (M. B. Bekka, P. de la Harpe, *Expos. Math.*, **18**, no. 3 (2000), 215–230).

Comment of 2009: This was proved for linear groups (T. Poznansky, *Characterization of linear groups whose reduced C*-algebras are simple* (2009), `arXiv:0812.2486v7`).

P. de la Harpe

15.8. b) Let G be any Lie group (indeed, any separable continuous group) made discrete: can G act faithfully on a countable set?

P. de la Harpe

15.9. An automorphism φ of the free group F_n on the free generators $x_1, x_2, \ldots, x_n$ is called *conjugating* if $x_i^{\varphi} = t_i^{-1} x_{\pi(i)} t_i$, $i = 1, 2, \ldots, n$, for some permutation $\pi \in \mathbb{S}_n$ and some elements $t_i \in F_n$. The set of conjugating automorphisms fixing the product $x_1 x_2 \cdots x_n$ forms the braid group B_n. The group B_n is linear for any $n \geqslant 2$, while the group of all automorphisms $\operatorname{Aut} F_n$ is not linear for $n \geqslant 3$. Is the group of all conjugating automorphisms linear for $n \geqslant 3$?

V. G. Bardakov

15.11. (M. Morigi). An automorphism of a group is called a *power automorphism* if it leaves every subgroup invariant. Is every finite abelian p-group the group of all power automorphisms of some group?

V. G. Bardakov

15.12. Let G be a group acting faithfully and level-transitively by automorphisms on a rooted tree $\mathscr{T}$. For a vertex v of $\mathscr{T}$, the *rigid vertex stabilizer* at v consists of those elements of G whose support in $\mathscr{T}$ lies entirely in the subtree $\mathscr{T}_v$ rooted at v. For a non-negative integer n, the *n-th rigid level stabilizer* is the subgroup of G generated by all rigid vertex stabilizers corresponding to the vertices at the level n of the tree $\mathscr{T}$. The group G is a *branch group* if all rigid level stabilizers have finite index in G. For motivation, examples and known results see (R. I. Grigorchuk, *in: New horizons in pro-p groups*, Birkhäuser, Boston, 2000, 121–179).

Do there exist branch groups with Kazhdan's T-property? (See 14.34.)

L. Bartholdi, R. I. Grigorchuk, Z. Šunić

15.13. Do there exist finitely presented branch groups?

L. Bartholdi, R. I. Grigorchuk, Z. Šunić

15.14. b) Do there exist finitely generated branch groups that are non-amenable and do not contain the free group F_2 on two generators?

L. Bartholdi, R. I. Grigorchuk, Z. Šunić

15.15. Is every maximal subgroup of a finitely generated branch group necessarily of finite index?

L. Bartholdi, R. I. Grigorchuk, Z. Šunić

15.16. Do there exist groups whose rate of growth is $e^{\sqrt{n}}$

a) in the class of finitely generated branch groups?

b) in the whole class of finitely generated groups? This question is related to 9.9.

L. Bartholdi, R. I. Grigorchuk, Z. Šunić

15.17. An infinite group is *just infinite* if all of its proper quotients are finite. Is every finitely generated just infinite group of intermediate growth necessarily a branch group?

L. Bartholdi, R. I. Grigorchuk, Z. Šunić

15.18. A group is *hereditarily just infinite* if it is residually finite and all of its non-trivial normal subgroups are just infinite.

a) Do there exist finitely generated hereditarily just infinite torsion groups? (It is believed there are none.)

b) Is every finitely generated hereditarily just infinite group necessarily linear?

A positive answer to the question b) would imply a negative answer to a).

L. Bartholdi, R. I. Grigorchuk, Z. Šunić

15.19. Let p be a prime, and $\mathscr{F}_p$ the class of finitely generated groups acting faithfully on a p-regular rooted tree by finite automata. Any group in $\mathscr{F}_p$ is residually-p (residually in the class of finite p-groups) and has word problem that is solvable in (at worst) exponential time. There exist therefore groups that are residually-p, have a solvable word problem, and do not belong to $\mathscr{F}_p$; though no concrete example is known. For instance:

a) Is it true that some (or even all) the groups given in (R. I. Grigorchuk, *Math. USSR–Sb.*, **54** (1986), 185–205) do not belong to $\mathscr{F}_p$ when the sequence ω is computable, but not periodic?

b) Does $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ belong to $\mathscr{F}_2$? (Here the wreath products are restricted.)

See (A. M. Brunner, S. Sidki, *J. Algebra*, **257** (2002), 51–64) and (R. I. Grigorchuk, V. V. Nekrashevich, V. I. Sushchanskiĭ, *in: Dynamical systems, automata, and infinite groups. Proc. Steklov Inst. Math.*, **231** (2000), 128–203).

L. Bartholdi, S. Sidki

15.20. (B. Hartley). An infinite transitive permutation group is said to be *barely transitive* if each of its proper subgroups has only finite orbits. Can a locally finite barely transitive group coincide with its derived subgroup?

Note that there are no simple locally finite barely transitive groups (B. Hartley, M. Kuzucuoğlu, *Proc. Edinburgh Math. Soc.*, **40** (1997), 483–490), any locally finite barely transitive group is a p-group for some prime p, and if the stabilizer of a point in a locally finite barely transitive group G is soluble of derived length d, then G is soluble of derived length bounded by a function of d (V. V. Belyaev, M. Kuzucuoğlu, *Algebra and Logic*, **42** (2003), 147–152).

V. V. Belyaev, M. Kuzucuoğlu

15.21. (B. Hartley). Do there exist torsion-free barely transitive groups?

V. V. Belyaev

15.22. A permutation group is said to be *finitary* if each of its elements moves only finitely many points. Do there exist finitary barely transitive groups?

V. V. Belyaev

15.23. A transitive permutation group is said to be *totally imprimitive* if every finite set of points is contained in some finite block of the group. Do there exist totally imprimitive barely transitive groups that are not locally finite?

V. V. Belyaev

15.26. A *partition* of a group is a representation of it as a set-theoretic union of some of its proper subgroups (*components*) that intersect pairwise trivially. Is it true that every nontrivial partition of a finite p-group has an abelian component?

Ya. G. Berkovich

15.28. Suppose that a finite p-group G is the product of two subgroups: $G = AB$.

a) Is the exponent of G bounded in terms of the exponents of A and B?

b) Is the exponent of G bounded if A and B are groups of exponent p?

Ya. G. Berkovich

15.29. (A. Mann). The dihedral group of order 8 is isomorphic to its own automorphism group. Are there other non-trivial finite p-groups with this property?

Ya. G. Berkovich

15.30. Is it true that every finite abelian p-group is isomorphic to the Schur multiplier of some nonabelian finite p-group?

Ya. G. Berkovich

15.31. M. R. Vaughan-Lee and J. Wiegold (*Proc. R. Soc. Edinburgh Sect. A*, **95** (1983), 215–221) proved that if a finite p-group G is generated by elements of breadth $\leqslant n$ (that is, having at most p^n conjugates), then G is nilpotent of class $\leqslant n^2+1$; the bound for the class was later improved by A. Mann (*J. Group Theory*, **4**, no. 3 (2001), 241–246) to $\leqslant n^2-n+1$. Is there a linear bound for the class of G in terms of n?

Ya. G. Berkovich

15.32. Suppose that a finite p-group G of order p^m has an automorphism of order p^{m-k}. Is it true that if m is sufficiently large relative to k, then G possesses a cyclic subgroup of index p^k?

Ya. G. Berkovich

***15.33.** Suppose that all 2-generator subgroups of a finite 2-group G are metacyclic. Is the derived length of G bounded? This is true for finite p-groups if $p \neq 2$, see 1.1.8 in (M. Suzuki, *Structure of a group and the structure of its lattice of subgroups*, Springer, Berlin, 1956).

Ya. G. Berkovich

*Yes, it is at most 2 (E. Crestani, F. Menegazzo, *J. Group Theory*, **15**, no. 3 (2012), 359–383).

15.35. Let F be the free group of finite rank r with basis $\{x_1, \dots, x_r\}$. Is it true that there exists a number $C = C(r)$ such that any reduced word of length $n > 1$ in the x_i lies outside some subgroup of F of index at most $C \log n$?

O. V. Bogopol'skiĭ

15.36. For a class M of groups let $L(M)$ be the class of groups G such that the normal subgroup $\langle a^G \rangle$ generated by an element a belongs to M for any $a \in G$. Is it true that the class $L(M)$ is finitely axiomatizable if M is the quasivariety generated by a finite group?

A. I. Budkin

15.37. Let G be a group satisfying the minimum condition for centralizers. Suppose that X is a normal subset of G such that $[x, y, \dots, y] = 1$ for any $x, y \in X$ with y repeated $f(x, y)$ times. Does X generate a locally nilpotent (whence hypercentral) subgroup?

If the numbers $f(x, y)$ can be bounded, the answer is affirmative (F. O. Wagner, *J. Algebra*, **217**, no. 2 (1999), 448–460).

F. O. Wagner

*__15.38.__ Does there exist a non-local hereditary composition formation $\mathfrak{F}$ of finite groups such that the set of all $\mathfrak{F}$-subnormal subgroups is a sublattice of the subgroup lattice in any finite group?

A. F. Vasil'ev, S. F. Kamornikov

*No, it does not (S. F. Kamornikov, *Dokl. Math.*, **81**, no. 1 (2010), 97–100).

15.40. Let N be a nilpotent subgroup of a finite simple group G. Is it true that there exists a subgroup N_1 conjugate to N such that $N \cap N_1 = 1$?

The answer is known to be affirmative if N is a p-group. *Comment of 2013:* an affirmative answer for alternating groups is obtained in (R. K. Kurmazov, *Siberian Math. J.*, **54**, no. 1 (2013), 73–77).

E. P. Vdovin

15.41. Let $R(m,p)$ denote the largest finite m-generator group of prime exponent p.

a) Can the nilpotency class of $R(m,p)$ be bounded by a polynomial in m? (This is true for $p = 2$, 3, 5, 7.)

b) Can the nilpotency class of $R(m,p)$ be bounded by a linear function in m? (This is true for $p = 2$, 3, 5.)

c) In particular, can the nilpotency class of $R(m,7)$ be bounded by a linear function in m?

My guess is "no" to the first two questions for general p, but "yes" to the third. By contrast, a beautiful and simple argument of Mike Newman shows that if $m \geqslant 2$ and $k \geqslant 2$ ($k \geqslant 3$ for $p = 2$), then the order of $R(m,p^k)$ is at least $p^{p^{\cdot^{\cdot^{\cdot^{p^m}}}}}$, with p appearing k times in the tower; see (M. Vaughan-Lee, E. I. Zelmanov, *J. Austral. Math. Soc. (A)*, **67**, no. 2 (1999), 261–271).

M. R. Vaughan-Lee

15.42. Is it true that the group algebra $k[F]$ of R. Thompson's group F (see 12.20) over a field k satisfies the Ore condition, that is, for any $a, b \in k[F]$ there exist $u, v \in k[F]$ such that $au = bv$ and either u or v is nonzero? If the answer is negative, then F is not amenable.

V. S. Guba

15.44. a) Let G be a reductive group over an algebraically closed field K of arbitrary characteristic. Let X be an affine G-variety such that, for a fixed Borel subgroup $B \leqslant G$, the coordinate algebra $K[X]$ as a G-module is the union of an ascending chain of submodules each of whose factors is an induced module $\mathrm{Ind}_B^G V$ of some one-dimensional B-module V. (See S. Donkin, *Rational representations of algebraic groups. Tensor products and filtration* (*Lect. Notes Math.*, **1140**), Springer, Berlin, 1985). Suppose in addition that $K[X]$ is a Cohen–Macaulay ring, that is, a free module over the subalgebra generated by any homogeneous system of parameters. Is then the ring of invariants $K[X]^G$ Cohen–Macaulay? *Comment of 2005:* This is proved in the case where X is a rational G-module (M. Hashimoto, *Math. Z.*, **236** (2001), 605–623).

A. N. Zubkov

15.45. We define the class of *hierarchically decomposable* groups in the following way. First, if $\mathfrak{X}$ is any class of groups, then let $\mathbf{H}_1\mathfrak{X}$ denote the class of groups which admit an admissible action on a finite-dimensional contractible complex in such a way that every cell stabilizer belongs to $\mathfrak{X}$. Then the "big" class $\mathbf{H}\mathfrak{X}$ is defined to be the smallest $\mathbf{H}_1$-closed class containing $\mathfrak{X}$.

a) Let $\mathfrak{F}$ be the class of all finite groups. Find an example of an $\mathbf{H}\mathfrak{F}$ group which is not in $\mathbf{H}_3\mathfrak{F}$ $(= \mathbf{H}_1\mathbf{H}_1\mathbf{H}_1 F)$.

b) Prove or disprove that there is an ordinal α such that $\mathbf{H}_\alpha\mathfrak{F} = \mathbf{H}\mathfrak{F}$, where $\mathbf{H}_\alpha$ is the operator on classes of groups defined by transfinite induction in the obvious way starting from $\mathbf{H}_1$.

P. Kropholler

15.46. Can the question 7.28 on conditions for admissibility of an elementary carpet $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ be reduced to Lie rank 1 if K is a field? A carpet $\mathfrak{A}$ of type Φ of additive subgroups of K is called *addmissible* if in the Chevalley group over K associated with the root system Φ the subgroup $\langle x_r(\mathfrak{A}_r) \mid r \in \Phi\rangle$ intersects $x_r(K)$ in $x_r(\mathfrak{A}_r)$. More precisely, is it true that the carpet $\mathfrak{A}$ is admissible if and only if the subcarpets $\{\mathfrak{A}_r, \mathfrak{A}_{-r}\}$, $r \in \Phi$, of rank 1 are admissible? The answer is known to be affirmative if the field K is locally finite (V. M. Levchuk, *Algebra and Logic*, **22**, no. 5 (1983), 362–371).

V. M. Levchuk

15.47. Let $M < G \leqslant \mathrm{Sym}(\Omega)$, where Ω is finite, be such that M is transitive on Ω and there is a G-invariant partition $\mathscr{P}$ of $\Omega \times \Omega \setminus \{(\alpha,\alpha) \mid \alpha \in \Omega\}$ such that G is transitive on the set of parts of $\mathscr{P}$ and M fixes each part of $\mathscr{P}$ setwise. (Here $\mathscr{P}$ can be identified with a decomposition of the complete directed graph with vertex set Ω into edge-disjoint isomorphic directed graphs.) If G induces a cyclic permutation group on $\mathscr{P}$, then we showed (*Trans. Amer. Math. Soc.*, **355**, no. 2 (2003), 637–653) that the numbers $n = |\Omega|$ and $k = |\mathscr{P}|$ are such that the r-part n_r of n satisfies $n_r \equiv 1 \pmod{k}$ for each prime r. Are there examples with G inducing a non-cyclic permutation group on $\mathscr{P}$ for any n, k not satisfying this congruence condition?

C. H. Li, C. E. Praeger

15.48. Let G be any non-trivial finite group, and let X be any generating set for G. Is it true that every element of G can be obtained from X using fewer than $2\log_2 |G|$ multiplications? (When counting the number of mutiplications on a path from the generators to a given element, at each step one can use the elements obtained at previous steps.)

C. R. Leedham-Green

15.49. A group G is a *unique product group* if, for any finite nonempty subsets X, Y of G, there is an element of G which can be written in exactly one way in the form xy with $x \in X$ and $y \in Y$. Does there exist a unique product group which is not left-orderable?

P. Linnell

15.50. Let G be a group of automorphisms of an abelian group of prime exponent. Suppose that there exists $x \in G$ such that x is regular of order 3 and the order of $[x, g]$ is finite for every $g \in G$. Is it true that $\langle x^G\rangle$ is locally finite?

V. D. Mazurov

15.51. Suppose that G is a periodic group satisfying the identity $[x, y]^5 = 1$. Is then the derived subgroup $[G, G]$ a 5-group?

V. D. Mazurov

15.52. (Well-known problem). By a famous theorem of Wielandt the sequence $G_0 = G, G_1, \ldots$, where $G_{i+1} = \mathrm{Aut}\, G_i$, stabilizes for any finite group G with trivial centre. Does there exist a function f of natural argument such that $|G_i| \leqslant f(|G|)$ for all $i = 0, 1, \ldots$ for an arbitrary finite group G and the same kind of sequence? *V. D. Mazurov*

15.53. Let S be the set of all prime numbers p for which there exists a finite simple group G and an absolutely irreducible G-module V over a field of characteristic p such that the order of any element in the natural semidirect product VG coincides with the order of some element in G. Is S finite or infinite? *V. D. Mazurov*

15.54. Suppose that G is a periodic group containing an involution i such that the centralizer $C_G(i)$ is a locally cyclic 2-group. Does the set of all elements of odd order in G that are inverted by i form a subgroup? *V. D. Mazurov*

15.55. a) The Monster, M, is a 6-transposition group. Pairs of Fischer transpositions generate 9 M-classes of dihedral groups. The order of the product of a pair is the coefficient of the highest root of affine type E_8. Similar properties hold for Baby B, and F_{24}' with respect to E_7 and E_6 when the product is read modulo centres ($2.B$, $3.F_{24}'$). Explain this. *Editors' Comment of 2005:* Some progress was made in (C. H. Lam, H. Yamada, H. Yamauchi, *Trans. Amer. Math. Soc.*, **359**, no. 9 (2007), 4107–4123).

b) Note that the Schur multiplier of the sporadic simple groups M, B, F_{24}' is the fundamental group of type E_8, E_7, E_6, respectively. Why?

See (J. McKay, in: *Finite groups, Santa Cruz Conf. 1979* (*Proc. Symp. Pure Math., 37*), Amer. Math. Soc., Providence, RI, 1980, 183–186) and (R. E. Borcherds, *Doc. Math., J. DMV Extra Vol. ICM Berlin 1998*, Vol. I (1998), 607–616). *J. McKay*

15.56. Is there a spin manifold such that the Monster acts on its loop space? Perhaps 24-dimensional? hyper-Kähler, non-compact? See (F. Hirzebruch, T. Berger, R. Jung, *Manifolds and modular forms*, Vieweg, Braunschweig, 1992). *J. McKay*

15.57. Suppose that H is a subgroup of $SL_2(\mathbb{Q})$ that is dense in the Zariski topology and has no finite quotients. Is then $H = SL_2(\mathbb{Q})$? *J. McKay, J.-P. Serre*

15.58. Suppose that a free profinite product $G * H$ is a free profinite group of finite rank. Must G and H be free profinite groups?

By (J. Neukirch, *Arch. Math.*, **22**, no. 4 (1971), 337–357) this may not be true if the rank of $G * H$ is infinite. *O. V. Mel'nikov*

15.59. Does there exist a profinite group G that is not free but can be represented as a projective limit $G = \varprojlim(G/N_\alpha)$, where all the G/N_α are free profinite groups of finite ranks?

The finiteness condition on the ranks of the G/N_α is essential. Such a group G cannot satisfy the first axiom of countability (O. V. Mel'nikov, *Dokl. AN BSSR*, **24**, no. 11 (1980), 968–970 (Russian)). *O. V. Mel'nikov*

15.61. Is it true that $l_\pi^n(G) \leqslant n(G_\pi) - 1 + \max_{p\in\pi} l_p(G)$ for any π-soluble group G? Here $n(G_\pi)$ is the nilpotent length of a Hall π-subgroup G_π of the group G and $l_\pi^n(G)$ is the nilpotent π-length of G, that is, the minimum number of π-factors in those normal series of G whose factors are either π'-groups, or nilpotent π-groups. The answer is known to be affirmative in the case when all proper subgroups of G_π are supersoluble.

V. S. Monakhov

15.62. Given an ordinary irreducible character χ of a finite group G write $p^{e_p(\chi)}$ to denote the p-part of $\chi(1)$ and put $e_p(G) = \max\{e_p(\chi) \mid \chi \in \mathrm{Irr}(G)\}$. Suppose that P is a Sylow p-subgroup of a group G. Is it true that $e_p(P)$ is bounded above by a function of $e_p(G)$?

Comment of 2005: An affirmative answer in the case of solvable groups has been given in (A. Moretó, T. R. Wolf, *Adv. Math.*, **184** (2004), 18–36).

A. Moretó

15.63. a) Let F_n be the free group of finite rank n on the free generators $x_1, \ldots, x_n$. An element $u \in F_n$ is called *positive* if u belongs to the semigroup generated by the x_i. An element $u \in F_n$ is called *potentially positive* if $\alpha(u)$ is positive for some automorphism α of F_n. Is the property of an element to be potentially positive algorithmically recognizable?

Comment of 2013: In (R. Goldstein, *Contemp. Math.*, Amer. Math. Soc., **421** (2006), 157–168) the problem was solved in the affirmative in the special case $n = 2$.

A. G. Myasnikov, V. E. Shpilrain

15.64. For finite groups G, X define $r(G; X)$ to be the number of inequivalent actions of G on X, that is, the number of equivalence classes of homomorphisms $G \to \mathrm{Aut}\, X$, where equivalence is defined by conjugation by an element of $\mathrm{Aut}\, X$. Now define $r_G(n) := \max\{r(G; X) : |X| = n\}$.

Is it true that $r_G(n)$ may be bounded as a function of $\lambda(n)$, the total number (counting multiplicities) of prime factors of n?

Peter M. Neumann

15.65. A square matrix is said to be *separable* if its minimal polynomial has no repeated roots, and *cyclic* if its minimal and characteristic polynomials are equal. For a matrix group G over a finite field define $s(G)$ and $c(G)$ to be the proportion of separable and of cyclic elements respectively in G. For a classical group $\mathrm{X}(d, q)$ of dimension d defined over the field with q elements let $S(\mathrm{X}; q) := \lim_{d\to\infty} s(\mathrm{X}(d, q))$ and $C(\mathrm{X}; q) := \lim_{d\to\infty} c(\mathrm{X}(d, q))$. Independently G. E. Wall (*Bull. Austral. Math. Soc.*, **60**, no. 2 (1999), 253–284) and J. Fulman (*J. Group Theory*, **2**, no. 3 (1999), 251–289) have evaluated $S(\mathrm{GL}; q)$ and $C(\mathrm{GL}; q)$, and have found them to be rational functions of q. Are $S(\mathrm{X}; q)$ and $C(\mathrm{X}; q)$ rational functions of q also for the unitary, symplectic, and orthogonal groups?

Peter M. Neumann

15.66. For a class $\mathfrak{X}$ of groups let $g_{\mathfrak{X}}(n)$ be the number of groups of order n in the class $\mathfrak{X}$ (up to isomorphism). Many years ago I formulated the following problem: find good upper bounds for the quotient $g_{\mathfrak{V}}(n)/g_{\mathfrak{U}}(n)$, where $\mathfrak{V}$ is a variety that is defined by its finite groups and $\mathfrak{U}$ is a subvariety of $\mathfrak{V}$. (This quotient is not defined for all n but only for those for which there are groups of order n in $\mathfrak{U}$.) Some progress has been made by G. Venkataraman (*Quart. J. Math. Oxford (2)*, **48**, no. 189 (1997), 107–125) when $\mathfrak{V}$ is a variety generated by finite groups all of whose Sylow subgroups are abelian. *Conjecture:* if $\mathfrak{V}$ is a locally finite variety of p-groups and $\mathfrak{U}$ is a non-abelian subvariety of $\mathfrak{V}$, then $g_{\mathfrak{V}}(p^m)/g_{\mathfrak{U}}(p^m) < p^{O(m^2)}$.

Moreover, this seems a possible way to attack the Sims Conjecture that when we write the number of groups of order p^m as $p^{\frac{2}{27}m^3+\varepsilon(m)}$ the error term $\varepsilon(m)$ is $O(m^2)$.

Peter M. Neumann

15.67. Which adjoint Chevalley groups (of normal type) over the integers are generated by three involutions two of which commute?

The groups $SL_n(\mathbb{Z})$, $n > 13$, satisfy this condition (M. C. Tamburini, P. Zucca, *J. Algebra*, **195**, no. 2 (1997), 650–661). The groups $PSL_n(\mathbb{Z})$ satisfy it if and only if $n > 4$ (N. Ya. Nuzhin, *Vladikavkaz. Mat. Zh.*, **10**, no. 1 (2008), 68–74 (Russian)). The group $PSp_4(\mathbb{Z})$ does not satisfy it, which follows from the corresponding fact for $PSp_4(3)$, see Archive, 7.30.

Ya. N. Nuzhin

15.68. Does there exist an infinite finitely generated 2-group (of finite exponent) all of whose proper subgroups are locally finite?

A. Yu. Ol'shanskiĭ

15.69. Is it true that every hyperbolic group has a free normal subgroup with the factor-group of finite exponent?

A. Yu. Ol'shanskiĭ

15.70. Do there exist groups of arbitrarily large cardinality that satisfy the minimum condition for subgroups?

A. Yu. Ol'shanskiĭ

15.71. (B. Huppert). Let G be a finite solvable group, and let $\rho(G)$ denote the set of prime divisors of the degrees of irreducible characters of G. Is it true that there always exists an irreducible character of G whose degree is divisible by at least $|\rho(G)|/2$ different primes?

P. P. Pálfy

15.72. For a fixed prime p does there exist a sequence of groups P_n of order p^n such that the number of conjugacy classes $k(P_n)$ satisfies $\lim_{n\to\infty} \log k(P_n)/\sqrt{n} = 0$?

Note that J. M. Riedl (*J. Algebra*, **218** (1999), 190–215) constructed p-groups for which the above limit is $2\log p$.

P. P. Pálfy

15.73. Is it true that for every finite lattice L there exists a finite group G and a subgroup $H \leqslant G$ such that the interval $Int(H;G)$ in the subgroup lattice of G is isomorphic to L? (Probably not.)

P. P. Pálfy

15.74. For every prime p find a finite p-group of nilpotence class p such that its lattice of normal subgroups cannot be embedded into the subgroup lattice of any abelian group. (Solved for $p = 2$, 3.)

P. P. Pálfy

15.75. b) Consider the sequence $u_1 = [x, y], \ldots, u_{n+1} = [[u_n, x], [u_n, y]]$. Is it true that an arbitrary finite group is soluble if and only if it satisfies one of these identities $u_n = 1$?

B. I. Plotkin

15.76. b) If Θ is a variety of groups, then let Θ^0 denote the category of all free groups of finite rank in Θ. It is proved (G. Mashevitzky, B. Plotkin, E. Plotkin *J. Algebra*, **282** (2004), 490–512) that if Θ is the variety of all groups, then every automorphism of the category Θ^0 is an inner one. The same is true if Θ is the variety of all abelian groups. Is this true for the variety of metabelian groups?

An automorphism φ of a category is called *inner* if it is isomorphic to the identity automorphism. Let $s : 1 \to \varphi$ be a function defining this isomorphism. Then for every object A we have an isomorphism $s_A : A \to \varphi(A)$ and for any morphism of objects $\mu : A \to B$ we have $\varphi(\mu) = s_B \mu s_A^{-1}$.

B. I. Plotkin

15.77. Elements a, b of a group G are said to be *symmetric with respect to an element* $g \in G$ if $a = gb^{-1}g$. Let G be an infinite abelian group, α a cardinal, $\alpha < |G|$. Is it true that for any n-colouring $\chi : G \to \{0, 1, \ldots, n-1\}$ there exists a monochrome subset of cardinality α that is symmetric with respect to some element of G? This is known to be true for $n \leqslant 3$.

I. V. Protasov

15.78. (R. I. Grigorchuk). Is it true that for any n-colouring of a free group of any rank there exists a monochrome subset that is symmetric with respect to some element of the group? This is true for $n = 2$.

I. V. Protasov

15.79. Does there exist a Hausdorff group topology on $\mathbb{Z}$ such that the sequence $\{2^n + 3^n\}$ converges to zero?

I. V. Protasov

15.80. A sequence $\{F_n\}$ of pairwise disjoint finite subsets of a topological group is called *expansive* if for every open subset U there is a number m such that $F_n \cap U \neq \varnothing$ for all $n > m$. Suppose that a group G can be partitioned into countably many dense subsets. Is it true that in G there exists an expansive sequence?

I. V. Protasov

15.82. Suppose that a periodic group G contains a strongly isolated 2-subgroup U. Is it true that either G is locally finite, or U is a normal subgroup of G?

A. I. Sozutov, N. M. Suchkov

15.83. (Yu. I. Merzlyakov). Does there exist a rational number α such that $|\alpha| < 2$ and the matrices $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$ generate a free group?

Yu. V. Sosnovskiĭ

15.84. Yu. I. Merzlyakov (*Sov. Math. Dokl.*, **19** (1978), 64–68) proved that if the complex numbers α, β, γ are each at least 3 in absolute value, then the matrices $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}$, and $\begin{pmatrix} 1-\gamma & -\gamma \\ \gamma & 1+\gamma \end{pmatrix}$ generate a free group of rank three. Are there rational numbers α, β, γ, each less than 3 in absolute value, with the same property?

Yu. V. Sosnovskiĭ

15.85. A torsion-free group all of whose subgroups are subnormal is nilpotent (H. Smith, *Arch. Math.*, **76**, no. 1 (2001), 1–6). Is a torsion-free group with the normalizer condition

a) hyperabelian?

b) hypercentral?

Yu. V. Sosnovskiĭ

15.87. Suppose that a 2-group G admits a regular periodic locally cyclic group of automorphisms that transitively permutes the set of involutions of G. Is G locally finite?

N. M. Suchkov

15.88. Let $\mathfrak{A}$ and $\mathfrak{N}_c$ denote the varieties of abelian groups and nilpotent groups of class $\leqslant c$, respectively. Let $F_r = F_r(\mathfrak{A}\mathfrak{N}_c)$ be a free group of rank r in $\mathfrak{A}\mathfrak{N}_c$. An element of F_r is called *primitive* if it can be included in a basis of F_r. Does there exist an algorithm recognizing primitive elements in F_r?

E. I. Timoshenko

15.89. Let Γ be an infinite undirected connected vertex-symmetric graph of finite valency without loops or multiple edges. Is it true that every complex number is an eigenvalue of the adjacency matrix of Γ?

V. I. Trofimov

15.90. Let Γ be an infinite directed graph, and $\overline{\Gamma}$ the underlying undirected graph. Suppose that the graph Γ admits a vertex-transitive group of automorphisms, and the graph $\overline{\Gamma}$ is connected and of finite valency. Does there exist a positive integer k (possibly depending on Γ) such that for any positive integer n there is a directed path of length at most $k \cdot n$ in the graph Γ whose initial and terminal vertices are at distance at least n in the graph $\overline{\Gamma}$?

Comment of 2005: It is proved that there exists a positive integer k' (depending only on the valency of $\overline{\Gamma}$) such that for any positive integer n there is a directed path of length at most $k' \cdot n^2$ in the graph Γ whose initial and terminal vertices are at distance at least n in the graph $\overline{\Gamma}$ (V. I. Trofimov, *Europ. J. Combinatorics*, **27**, no. 5 (2006), 690–700).

V. I. Trofimov

15.91. Is it true that any irreducible faithful representation of a linear group G of finite rank over a finitely generated field of characteristic zero is induced from an irreducible representation of a finitely generated dense subgroup of the group G?

A. V. Tushev

15.92. A group $\langle x, y \mid x^l = y^m = (xy)^n \rangle$ is called the *triangle group* with parameters $(l,\, m,\, n)$. It is proved in (A. M. Brunner, R. G. Burns, J. Wiegold, *Math. Scientist*, **4** (1979), 93–98) that the triangle group (2, 3, 30) has uncountably many non-isomorphic homomorphic images that are residually finite alternating groups. Is the same true for the triangle group (2, 3, n) for all $n > 6$?

J. Wiegold

15.93. Let G be a pro-p group, $p > 2$, and φ an automorphism of G of order 2. Suppose that the centralizer $C_G(\varphi)$ is abelian. Is it true that G satisfies a pro-p identity?

An affirmative answer would generalize a result of A. N. Zubkov (*Siberian Math. J.*, **28**, no. 5 (1987), 742–747) saying that non-abelian free pro-p groups cannot be represented by 2×2 matrices.

A. Jaikin-Zapirain

15.94. Define the *weight* of a group G to be the minimum number of generators of G as a normal subgroup of itself. Let $G = G_1 * \cdots * G_n$ be a free product of n nontrivial groups.

a) Is it true that the weight of G is at least $n/2$?

b) Is it true if the G_i are cyclic?

c) Is it true for $n = 3$ (without assuming the G_i cyclic)?

Problem 5.53 in Archive (now answered in the affirmative) is the special case with $n = 3$ and all the G_i cyclic.

J. Howie

15.95. (A. Mann, Ch. Praeger). Suppose that all fixed-point-free elements of a transitive permutation group G have prime order p. If G is a finite p-group, must G have exponent p?

This is true for $p = 2, 3$; it is also known that the exponent of G is bounded in terms of p (A. Hassani, M. Khayaty, E. I. Khukhro, C. E. Praeger, *J. Algebra*, **214**, no. 1 (1999), 317–337).

E. I. Khukhro

15.96. An automorphism φ of a group G is called a *splitting automorphism of order* n if $\varphi^n = 1$ and $xx^{\varphi}x^{\varphi^2}\cdots x^{\varphi^{n-1}} = 1$ for any $x \in G$.

a) Is it true that the derived length of a d-generated nilpotent p-group admitting a splitting automorphism of order p^n is bounded by a function of d, p, and n? This is true for $n = 1$, see Archive, 7.53.

b) The same question for $p^n = 4$.

E. I. Khukhro

15.97. Let p be a prime. A group G satisfies the *p-minimal condition* if there are no infinite descending chains $G_1 > G_2 > \ldots$ of subgroups of G such that each difference $G_i \setminus G_{i+1}$ contains a p-element (S. N. Chernikov). Suppose that a locally finite group G satisfies the p-minimal condition and has a subnormal series each of whose factors is finite or a p'-group. Is it true that all p-elements of G generate a Chernikov subgroup?

N. S. Chernikov

15.98. Let $\mathfrak{F}$ be a saturated formation, and G a finite soluble minimal non-$\mathfrak{F}$-group such that $G^{\mathfrak{F}}$ is a Sylow p-subgroup of G. Is it true that $G^{\mathfrak{F}}$ is isomorphic to a factor group of the Sylow p-subgroup of $PSU(3, p^{2n})$? This is true for the formation of finite nilpotent groups (V. D. Mazurov, S. A. Syskin, *Math. Notes*, **14**, no. 2 (1973), 683–686; A. Kh. Zhurtov, S. A. Syskin, *Siberian Math. J.*, **26**, no. 2 (1987), 235–239).

L. A. Shemetkov

15.99. Let $f(n)$ be the number of isomorphism classes of finite groups of order n. Is it true that the equation $f(n) = k$ has a solution for any positive integer k? The answer is affirmative for all $k \leqslant 1000$ (G. M. Wei, *Southeast Asian Bull. Math.*, **22**, no. 1 (1998), 93–102).

W. J. Shi

15.100. Is a periodic group locally finite if it has a non-cyclic subgroup of order 4 that coincides with its centralizer?

A. K. Shlëpkin

15.101. Is a periodic group locally finite if it has an involution whose centralizer is a locally finite group with Sylow 2-subgroup of order 2?

A. K. Shlëpkin

15.102. (W. Magnus). An element r of a free group F_n is called a *normal root* of an element $u \in F_n$ if u belongs to the normal closure of r in F_n. Can an element u that lies outside the commutator subgroup $[F_n, F_n]$ have infinitely many non-conjugate normal roots?

V. E. Shpilrain

15.103. (Well-known problem). Is the group $\mathrm{Out}(F_3)$ of outer automorphisms of a free group of rank 3 linear?

V. E. Shpilrain

15.104. Let n be a positive integer and let w be a group word in the variables $x_1, x_2, \ldots$. Suppose that a residually finite group G satisfies the identity $w^n = 1$. Does it follow that the verbal subgroup $w(G)$ is locally finite?

This is the Restricted Burnside Problem if $w = x_1$. A positive answer was obtained also in a number of other particular cases (P. V. Shumyatsky, *Quart. J. Math.*, **51**, no. 4 (2000), 523–528).

P. V. Shumyatsky

Problems from the 16th Issue (2006)

16.1. Let G be a finite non-abelian group, and $Z(G)$ its centre. One can associate a graph Γ_G with G as follows: take $G\backslash Z(G)$ as vertices of Γ_G and join two vertices x and y if $xy \neq yx$. Let H be a finite non-abelian group such that $\Gamma_G \cong \Gamma_H$.

*a) If H is simple, is it true that $G \cong H$? *Comment of 2009:* This is true for groups with disconnected prime graphs (L. Wang, W. Shi, *Commun. Algebra*, **36** (2008), 523–528).

b) If H is nilpotent, is it true that G is nilpotent? *Comment of 2009:* This is true if $|H| = |G|$ (*J. Algebra*, **298** (2006), 468–492).

c) If H is solvable, is it true that G is solvable?

A. Abdollahi, S. Akbari, H. R. Maimani

*a) Yes, it is mod CFSG (Ch. Khan', G. Chèn', S. Go, *Siberian Math. J.*, **49**, no. 6 (2008), 1138–1146, for sporadic simple groups; A. Abdollahi, H. Shahverdi, *J. Algebra*, **357** (2012), 203–207, for alternating groups; R. M. Solomon, A. J. Woldar, *to appear in J. Group Theory*, `DOI: 10.1515/jgt-2013-0021`, for simple groups of Lie type).

***16.2.** A group G is *subgroup-separable* if for any subgroup $H \leqslant G$ and element $x \in G\backslash H$ there is a homomorphism to a finite group $f : G \to F$ such that $f(x) \notin f(H)$. Is it true that a finitely generated solvable group is locally subgroup-separable if and only if it does not contain a solvable Baumslag–Solitar group? Solvable Baumslag–Solitar groups are $BS(1,n) = \langle a, b \mid bab^{-1} = a^n \rangle$ for $n > 1$.

Background: It is known that a finitely generated solvable group is subgroup-separable if and only if it is polycyclic (R. C. Alperin, in: *Groups–Korea'98 (Pusan)*, de Gruyter, Berlin, 2000, 1–5).

R. C. Alperin

*No, it is not (J. O. Button, *Ricerche di Matematica*, **61**, no. 1 (2012), 139–145).

16.3. Is it true that if G is a finite group with all conjugacy classes of distinct sizes, then $G \cong \mathbb{S}_3$?

This is true if G is solvable (J. Zhang, *J. Algebra*, **170** (1994), 608–624); it is also known that $F(G)$ is nontrivial (Z. Arad, M. Muzychuk, A. Oliver, *J. Algebra*, **280** (2004), 537–576).

Z. Arad

16.4. Let G be a finite group with C, D two nontrivial conjugacy classes such that CD is also a conjugacy class. Can G be a non-abelian simple group?

Z. Arad

16.5. A group is said to be *perfect* if it coincides with its derived subgroup. Does there exist a perfect locally finite p-group

a) all of whose proper subgroups are hypercentral?

b) all of whose proper subgroups are solvable?

c) all of whose proper subgroups are hypercentral and solvable?

A. O. Asar

16.6. Can a perfect locally finite p-group be generated by a subset of bounded exponent

a) if all of its proper subgroups are hypercentral?

b) if all of its proper subgroups are solvable?

A. O. Asar

16.7. Is it true that the occurrence problem is undecidable for any semidirect product $F_n \leftthreetimes F_n$ of non-abelian free groups F_n?

An affirmative answer would imply an answer to 6.24. Recall that the occurrence problem is decidable for F_n (M. Hall, 1949) and undecidable for $F_n \times F_n$ (K. A. Mikhailova, 1958).

V. G. Bardakov

16.9. An element g of a free group F_n on the free generators $x_1, \ldots, x_n$ is called a *palindrome* with respect to these generators if the reduced word representing g is the same when read from left to right or from right to left. The *palindromic length* of an element $w \in F_n$ is the smallest number of palindromes in F_n whose product is w. Is there an algorithm for finding the palindromic length of a given element of F_n?

V. G. Bardakov, V. A. Tolstykh, V. E. Shpilrain

16.10. Is there an algorithm for finding the primitive length of a given element of F_n? The definition of a primitive element is given in 14.84; the primitive length is defined similarly to the palindromic length in 16.9.

V. G. Bardakov, V. A. Tolstykh, V. E. Shpilrain

16.11. Let G be a finite p-group. Does there always exist a finite p-group H such that $\Phi(H) \cong [G,G]$?

Ya. G. Berkovich

16.13. Does there exist a finite p-group G all of whose maximal subgroups H are special, that is, satisfy $Z(H) = [H,H] = \Phi(H)$?

Ya. G. Berkovich

16.14. Let G be a finite 2-group such that $\Omega_1(G) \leqslant Z(G)$. Is it true that the rank of G/G^2 is at most double the rank of $Z(G)$?

Ya. G. Berkovich

16.15. An element g of a group G is an *Engel element* if for every $h \in G$ there exists k such that $[h, g, \ldots, g] = 1$, where g occurs k times; if there is such k independent of h, then g is said to be *boundedly Engel*.

a) (B. I. Plotkin). Does the set of boundedly Engel elements of a group form a subgroup?

b) The same question for torsion-free groups.

c) The same question for right-ordered groups.

d) The same question for linearly ordered groups.

V. V. Bludov

16.16. a) Does the set of (not necessarily boundedly) Engel elements of a group without elements of order 2 form a subgroup?

b) The same question for torsion-free groups.

c) The same question for right-ordered groups.

d) The same question for linearly ordered groups.

There are examples of 2-groups where a product of two (unboundedly) Engel elements is not an Engel element.

V. V. Bludov

16.18. Does there exist a linearly orderable soluble group of derived length exactly n that has a single proper normal relatively convex subgroup

a) for $n = 3$?

b) for $n > 4$?

Such groups do exist for $n = 2$ and for $n = 4$.

V. V. Bludov

16.19. Is the variety of lattice-ordered groups generated by nilpotent groups finitely based? (Here the variety is considered in the signature of group and lattice operations.)

V. V. Bludov

16.20. Let M be a quasivariety of groups. The *dominion* $\mathrm{dom}_A^M(H)$ of a subgroup H of a group A (in M) is the set of all elements $a \in A$ such that for any two homomorphisms $f, g : A \to B \in M$, if f, g coincide on H, then $f(a) = g(a)$. Suppose that the set $\{\mathrm{dom}_A^N(H) \mid N \text{ is a quasivariety}, N \subseteq M\}$ forms a lattice with respect to set-theoretic inclusion. Can this lattice be modular and non-distributive?

A. I. Budkin

16.21. Given a non-central matrix $\alpha \in SL_n(F)$ over a field F for $n > 2$, is it true that every non-central matrix in $SL_n(F)$ is a product of n matrices, each similar to α?

L. Vaserstein

16.22. (Well-known problem.) Let $E_n(A)$ be the subgroup of $GL_n(A)$ generated by elementary matrices. Is $SL_2(A) = E_2(A)$ when $A = \mathbb{Z}[x, 1/x]$?

L. Vaserstein

16.23. Is there, for some $n > 2$ and a ring A with 1, a matrix in $E_n(A)$ that is nonscalar modulo any proper ideal and is not a commutator?

L. Vaserstein

16.24. The *spectrum* of a finite group is the set of orders of its elements. Does there exist a finite group G whose spectrum coincides with the spectrum of a finite simple exceptional group L of Lie type, but G is not isomorphic to L?

A. V. Vasil'ev

***16.25.** Do there exist three pairwise non-isomorphic finite non-abelian simple groups with the same spectrum?

A. V. Vasil'ev

*No (A. A. Buturlakin, *Sibirsk. Electron. Math. Reports*, **7** (2010), 111–114).

16.26. We say that the prime graphs of finite groups G and H coincide if the sets of primes dividing their orders are the same, $\pi(G) = \pi(H)$, and for any distinct $p, q \in \pi(G)$ there is an element of order pq in G if and only if there is such an element in H. Does there exist a positive integer k such that there are no k pairwise non-isomorphic finite non-abelian simple groups with the same graphs of primes? *Conjecture:* $k = 5$.

A. V. Vasil'ev

***16.27.** Suppose that a finite group G has the same spectrum as an alternating group. Is it true that G has at most one non-abelian composition factor?

For any finite simple group other than alternating the answer to the corresponding question is affirmative (mod CFSG).

A. V. Vasil'ev and V. D. Mazurov

*Yes, it is (I. B. Gorshkov, *Algebra and Logic*, **52**, no. 1 (2013), 41–45).

16.28. Let G be a connected linear reductive algebraic group over a field of positive characteristic, X a closed subset of G, and let $X^k = \{x_1 \cdots x_k \mid x_i \in X\}$.

a) Is it true that there always exists a positive integer $c = c(X) > 1$ such that X^c is closed?

b) If X is a conjugacy class of G such that X^2 contains an open subset of G, then is $X^2 = G$?

E. P. Vdovin

16.29. Which finite simple groups of Lie type G have the following property: for every semisimple abelian subgroup A and proper subgroup H of G there exists $x \in G$ such that $A^x \cap H = 1$?

This property holds in the case $G = GL_n(q)$ (J. Siemons, A. Zalesskii, *J. Algebra*, **226** (2000), 451–478, **256** (2002), 611–625). Note that if $G = L_2(5)$, $A = 2 \times 2$, and $H = 5 : 2$ (in Atlas notation), then $A^x \cap H > 1$ for every $x \in G$ (this example was communicated to the author by V. I. Zenkov).

E. P. Vdovin

16.30. Suppose that A and B are subgroups of a group G and $G = AB$. Will G have composition (principal) series if A and B have composition (respectively, principal) series?

V. A. Vedernikov

***16.31.** Suppose that a group G has a composition series and let $\mathfrak{F}(G)$ be the formation generated by G. Is the set of all subformations of $\mathfrak{F}(G)$ finite? *V. A. Vedernikov*

*No, not always, even for G finite (V. P. Burichenko, *J. Algebra*, **372** (2012), 428–458).

16.32. Suppose that a group G has a composition series and let $\mathrm{Fit}(G)$ be the Fitting class generated by G. Is the set of all Fitting subclasses of $\mathrm{Fit}(G)$ finite? Cf. 14.31.

V. A. Vedernikov

16.33. Suppose that a finite p-group G has an abelian subgroup A of order p^n. Does G contain an abelian subgroup B of order p^n that is normal in $\langle B^G \rangle$

a) if $p = 3$?

b) if $p = 2$?

c) If $p = 3$ and A is elementary abelian, does G contain an elementary abelian subgroup B of order 3^n that is normal in $\langle B^G \rangle$?

The corresponding results have been proved for greater primes p, based on extensions of Thompson's Replacement Theorem (and the dihedral group of order 32 shows that the third question has negative answer for $p = 2$). See (G. Glauberman, *J. Algebra*, **196** (1997), 301–338; J. Alperin, G. Glauberman, *J. Algebra*, **203** (1998), 533–566; G. Glauberman, *J. Algebra*, **272** (2004), 128–153).

G. Glauberman

16.34. Suppose that G is a finitely generated group acting faithfully on a regular rooted tree by finite-state automorphisms. Is the conjugacy problem decidable for G?

See the definitions in (R. I. Grigorchuk, V. V. Nekrashevich, V. I. Sushchanskiĭ, *Proc. Steklov Inst. Math.*, **2000**, no. 4 (231), 128–203).

R. I. Grigorchuk, V. V. Nekrashevich, V. I. Sushchanskiĭ

***16.36.** (Well-known problem). We call a finite group *rational* if all of its ordinary characters are rational-valued. Is every Sylow 2-subgroup of a rational group also a rational group?

M. R. Darafsheh

*No, not always (I. M. Isaacs, G. Navarro, *Math. Z.*, **272** (2012), 937–945).

16.38. Let G be a soluble group, and let A and B be periodic subgroups of G. Is it true that any subgroup of G contained in the set $AB = \{ab \mid a \in A,\ b \in B\}$ is periodic? This is known to be true if AB is a subgroup of G.

F. de Giovanni

16.39. (J. E. Humphreys, D. N. Verma). Let G be a semisimple algebraic group over an algebraically closed field k of characteristic $p > 0$. Let $\mathfrak{g}$ be the Lie algebra of G and let $u = u(\mathfrak{g})$ be the restricted enveloping algebra of $\mathfrak{g}$. By a theorem of Curtis every irreducible restricted u-module (i. e. every irreducible restricted $\mathfrak{g}$-module) is the restriction to $\mathfrak{g}$ of a (rational) G-module. Is it also true that every projective indecomposable u-module is the restriction of a rational G-module? This is true if $p \geqslant 2h - 2$ (where h is the Coxeter number of G) by results of Jantzen.

S. Donkin

16.40. Let Δ be a subgroup of the automorphism group of a free pro-p group of finite rank F such that Δ is isomorphic (as a profinite group) to the group Z_p of p-adic integers. Is the subgroup of fixed points of Δ in F finitely generated (as a profinite group)?

P. A. Zalesskiĭ

16.41. Let F be a free pro-p group of finite rank $n > 1$. Does $\operatorname{Aut} F$ possess an open subgroup of finite cohomological dimension?

P. A. Zalesskiĭ

16.42. Is a topological Abelian group (G, τ) compact if every group topology $\tau' \subseteq \tau$ on G is complete? (The answer is yes if every continuous homomorphic image of (G, τ) is complete.)

E. G. Zelenyuk

16.44. Is there in ZFC a countable non-discrete topological group not containing discrete subsets with a single accumulation point? (Such a group is known to exist under Martin's Axiom.)

E. G. Zelenyuk

16.45. Let G be a permutation group on a set Ω. A sequence of points of Ω is a *base* for G if its pointwise stabilizer in G is the identity; it is *minimal* if no point may be removed. Let $b(G)$ be the maximum, over all permutation representations of the finite group G, of the maximum size of a minimal base for G. Let $\mu'(G)$ be the maximum size of an *independent set* in G, a set of elements with the property that no element belongs to the subgroup generated by the others. Is it true that $b(G) = \mu'(G)$? (It is known that $b(G) \leqslant \mu'(G)$, and that equality holds for the symmetric groups.)

Remark. An equivalent question is the following. Suppose that the Boolean lattice $B(n)$ of subsets of an n-element set is embeddable as a meet-semilattice of the subgroup lattice of G, and suppose that n is maximal with this property. Is it true that then there is such an embedding of $B(n)$ with the property that the least element of $B(n)$ is a normal subgroup of G?

P. J. Cameron

16.46. Among the finitely-presented groups that act arc-transitively on the (infinite) 3-valent tree with finite vertex-stabilizer are the two groups $G_3 = \big\langle h,\, a,\, P,\, Q \mid h^3,\, a^2,\, P^2,\, Q^2,\, [P,Q],\, [h,P],\, (hQ)^2,\, a^{-1}PaQ \big\rangle$ and $G_4 = \big\langle h,\, a,\, p,\, q,\, r \mid h^3,\, a^2,\, p^2,\, q^2,\, r^2,\, [p,q],\, [p,r],\, p(qr)^2,\, h^{-1}phq,\, h^{-1}qhpq,\, (hr)^2,\, [a,p],\, a^{-1}qar \big\rangle$, each of which contains the modular group $G_1 = \big\langle h, a \mid h^3,\, a^2 \big\rangle \cong PSL_2(\mathbb{Z})$ as a subgroup of finite index. The free product of G_3 and G_4 with subgroup $G_1 = \langle h, a\rangle$ amalgamated has a normal subgroup K of index 8 generated by $A = h$, $B = aha$, $C = p$, $D = PpP$, $E = QpQ$, and $F = PQpPQ$, with dihedral complement $\langle a, P, Q\rangle$. The group K has presentation $\big\langle A,\, B,\, C,\, D,\, E,\, F \mid A^3,\, B^3,\, C^2,\, D^2,\, E^2,\, F^2,\, (AC)^3,\, (AD)^3,\, (AE)^3,\, (AF)^3,\, (BC)^3,\, (BD)^3,\, (BE)^3,\, (BF)^3,\, (ABA^{-1}C)^2,\, (ABA^{-1}D)^2,\, (A^{-1}BAE)^2,\, (A^{-1}BAF)^2,\, (BAB^{-1}C)^2,\, (B^{-1}ABD)^2,\, (BAB^{-1}E)^2,\, (B^{-1}ABF)^2 \big\rangle$. Does this group have a non-trivial finite quotient?

M. Conder

16.47. (P. Conrad). Is it true that every torsion-free abelian group admits an Archimedean lattice ordering?

V. M. Kopytov, N. Ya. Medvedev

16.48. A group H is said to have *generalized torsion* if there exists an element $h \neq 1$ such that $h^{x_1}h^{x_2}\cdots h^{x_n} = 1$ for some n and some $x_i \in H$. Is every group without generalized torsion right-orderable?

V. M. Kopytov, N. Ya. Medvedev

16.49. Is it true that a free product of groups without generalized torsion is a group without generalized torsion?

V. M. Kopytov, N. Ya. Medvedev

16.50. Do there exist simple finitely generated right-orderable groups?

There exist finitely generated right-orderable groups coinciding with the derived subgroup (G. Bergman).

V. M. Kopytov, N. Ya. Medvedev

16.51. Do there exist groups that can be right-ordered in infinitely countably many ways?

There exist groups that can be fully ordered in infinitely countably many ways (R. Buttsworth).

V. M. Kopytov, N. Ya. Medvedev

***16.52.** Is every finitely presented elementary amenable group solvable-by-finite?

P. Linnell, T. Schick

*No, not every. I. Belegradek and Y. Cornulier have pointed out that the groups in (C. H. Houghton, *Arch. Math. (Basel)*, **31**, no. 3 (1978/79), 254–258) are finitely presented as shown in (K. S. Brown, *J. Pure Appl. Algebra*, **44**, no. 1–3 (1987), 45–75) and elementary amenable, but not virtually solvable; `http://mathoverflow.net/questions/107996`

16.53. Let $d(G)$ denote the smallest cardinality of a generating set of the group G. Suppose that $G = \langle A, B\rangle$, where A and B are two d-generated finite groups of coprime orders. Is it true that $d(G) \leqslant d + 1$?

See Archive 12.71 and (A. Lucchini, *J. Algebra*, **245** (2001), 552–561).

A. Lucchini

16.54. We say that a group G acts *freely* on a group V if $vg \neq v$ for any nontrivial elements $g \in G$, $v \in V$. Is it true that a group G that can act freely on a non-trivial abelian group is embeddable in the multiplicative group of some skew-field?

V. D. Mazurov

16.56. The *spectrum* $\omega(G)$ of a group G is the set of orders of elements of G. Suppose that $\omega(G) = \{1, 2, 3, 4, 5, 6\}$. Is G locally finite?

V. D. Mazurov

***16.58.** Is $SU_2(\mathbb{C})$ the only group that has just one irreducible complex representation of dimension n for each $n = 1, 2, \ldots$?

(If $R[n]$ is the n-dimensional irreducible complex representation of $SU_2(\mathbb{C})$, then $R[2]$ is the natural two-dimensional representation, and $R[2]\otimes R[n] = R[n-1]+R[n+1]$ for $n > 1$.)

J. McKay

*No, it is not. It is known that $\mathbb{C}$ has infinitely many (discrete) automorphisms. For $\varphi \in \mathrm{Aut}(\mathbb{C})$ and a matrix x, let x^φ denote the matrix obtained from x by applying φ to each element. Then $T_\varphi : x \mapsto x^\varphi$ is an irreducible representation of $SU_2(\mathbb{C})$. It is easy to show that among these representations there are infinitely many pairwise non-equivalent ones. Therefore the group $SU_2(\mathbb{C})$ itself does not satisfy the condition of the problem. (This observation belongs to von Neumann.) One can show that any group satisfying the condition of the problem is isomorphic to a subgroup of $SL_2(\mathbb{Q})$ satisfying the condition of Problem 15.57. On the other hand, E. Cartan proved that an unitary representation of a simple compact Lie group is always continuous. Hence, if we restrict ourselves to the *unitary* representations, then $SU_2(\mathbb{C})$ does satisfy the condition of the problem. (V. P. Burichenko, *Letter of 16 July 2013.*)

16.59. Given a finite group K, does there exist a finite group G such that $K \cong \mathrm{Out}\, G = \mathrm{Aut}\, G/\mathrm{Inn}\, G$? (It is known that an infinite group G exists with this property.)

D. MacHale

16.60. If G is a finite group, let $T(G)$ be the sum of the degrees of the irreducible complex representations of G, $T(G) = \sum_{i=1}^{k(G)} d_i$, where G has $k(G)$ conjugacy classes. If $\alpha \in \mathrm{Aut}\, G$, let $S_\alpha = \{g \in G \mid \alpha(g) = g^{-1}\}$. Is it true that $T(G) \geqslant |S_\alpha|$ for all $\alpha \in \mathrm{Aut}\, G$?

D. MacHale

16.63. Is there a non-trivial finite p-group G of odd order such that $|\mathrm{Aut}\, G| = |G|$?

See also 12.77.

D. MacHale

16.64. A non-abelian variety in which all finite groups are abelian is called *pseudo-abelian.* A group variety is called a *t-variety* if for all groups in this variety the relation of being a normal subgroup is transitive. By (O. Macedońska, A. Storozhev, *Commun. Algebra*, **25**, no. 5 (1997), 1589–1593) each non-abelian t-variety is pseudo-abelian, and the pseudo-abelian varieties constructed in (A. Yu. Ol'shanskiĭ, *Math. USSR–Sb.*, **54** (1986), 57–80) are t-varieties. Is every pseudo-abelian variety a t-variety?

O. Macedońska

16.66. For a group G, let $D_n(G)$ denote the n-th dimension subgroup of G, and $\zeta_n(G)$ the n-th term of its upper central series. For a given integer $n \geqslant 1$, let $f(n) = \max\{m \mid \exists$ a nilpotent group G of class n with $D_m(G) \neq 1\}$ and $g(n) = \max\{m \mid \exists$ a nilpotent group G of class n such that $D_n(G) \not\subseteq \zeta_m(G)\}$.

a) What is $f(3)$?

b) (B. I. Plotkin). Is it true that $f(n)$ is finite for all n?

c) Is the growth of $f(n)$ and $g(n)$ polynomial, exponential, or intermediary?

It is known that both $f(n) - n$ and $g(n)$ tend to infinity as $n \to \infty$ (N. Gupta, Yu. V. Kuz'min, *J. Pure Appl. Algebra*, **104** (1995), 191–197).

R. Mikhailov, I. B. S. Passi

16.68. Let $W(x, y)$ be a non-trivial reduced group word, and G one of the groups $PSL(2,\mathbb{R})$, $PSL(2,\mathbb{C})$, or $SO(3,\mathbb{R})$. Are all the maps $W : G \times G \to G$ surjective?

For $SL(2,\mathbb{R})$, $SL(2,\mathbb{C})$, $GL(2,\mathbb{C})$, and for the group S^3 of quaternions of norm 1 there exist non-surjective words; see (J. Mycielski, *Amer. Math. Monthly*, **84** (1977), 723–726; **85** (1978), 263–264).

J. Mycielski

16.69. Let W be as above.

a) Must the range of the function $\mathrm{Tr}(W(x, y))$ for $x, y \in GL(2,\mathbb{R})$ include the interval $[-2, +\infty)$?

b) For x, y being non-zero quaternions, must the range of the function $\mathrm{Re}(W(x, y))$ include the interval $[-5/27, 1]$?

(For some comments see *ibid.*)

J. Mycielski

16.70. Suppose that a finitely generated group G acts freely on an A-tree, where A is an ordered abelian group. Is it true that G acts freely on a $\mathbb{Z}^n$-tree for some n?

Comment of 2013: It was proved that every finitely presented group acting freely on an A-tree acts freely on some $\mathbb{R}^n$-tree for a suitable n, where $\mathbb{R}$ has the lexicographical order (O. Kharlampovich, A. Myasnikov, D. Serbin, *Int. J. Algebra Comput.*, **23** (2013), 325–345).

A. G. Myasnikov, V. N. Remeslennikov, O. G. Kharlampovich

***16.71.** Is the elementary theory of a torsion-free hyperbolic group decidable?

A. G. Myasnikov, O. G. Kharlampovich

*Yes, it is (O. Kharlampovich, A. Myasnikov, `arXiv:1303.0760`).

16.72. Does there exist an exponential-time algorithm for obtaining a JSJ-decomposition of a finitely generated fully residually free group?

Some algorithm was found in (O. Kharlampovich, A. Myasnikov, in: *Contemp. Math. AMS* (*Algorithms, Languages, Logic*), **378** (2005), 87–212).

A. G. Myasnikov, O. G. Kharlampovich

16.73. Let G be a group generated by a finite set S, and let $l(g)$ denote the word length function of $g \in G$ with respect to S. The group G is said to be *contracting* if there exist a faithful action of G on the set X^* of finite words over a finite alphabet X and constants $0 < \lambda < 1$ and $C > 0$ such that for every $g \in G$ and $x \in X$ there exist $h \in G$ and $y \in X$ such that $l(h) < \lambda l(g) + C$ and $g(xw) = yh(w)$ for all $w \in X^*$.

*a) Can a contracting group have a non-abelian free subgroup?

b) Do there exist non-amenable contracting groups?

V. V. Nekrashevich

*a) No, it cannot (V. V. Nekrashevich, *Preprint*, 2008, `arXiv:0802.2554`).

16.74. *a) Let $G = \langle\alpha, \beta\rangle$ be the group generated by the following two permutations of $\mathbb{Z}$: $\alpha(n) = n + 1$; $\beta(0) = 0$, $\beta(2^k m) = 2^k(m + 2)$, where m is odd and n is a positive integer. Is G amenable?

b) More generally, is it true that all groups generated by automata of polynomial growth in the sense of S. Sidki (*Geom. Dedicata*, **108** (2004), 193–204) are amenable?

V. V. Nekrashevich

*a) Yes, it is (G. Amir, O. Angel, B. Virág, *J. Eur. Math. Soc.*, **15**, no. 3 (2013), 705–730).

16.76. We call a group G *strictly real* if each of its non-trivial elements is conjugate to its inverse by some involution in G. In which groups of Lie type over a field of characteristic 2 the maximal unipotent subgroups are strictly real?

Ya. N. Nuzhin

16.77. It is known that in every Noetherian group the nilpotent radical coincides with the collection of all Engel elements (R. Baer, *Math. Ann.*, **133** (1957), 256–270; B. I. Plotkin, *Izv. Vysš. Učebn. Zaved. Mat.*, **1958**, no. 1(2), 130–135 (Russian)). It would be nice to find a similar characterization of the solvable radical of a finite group. More precisely, let $u = u(x, y)$ be a sequence of words satisfying 15.75. We say that an element $g \in G$ is u-Engel if there exists $n = n(g)$ such that $u_n(x, g) = 1$ for every element $x \in G$. Does there exist a sequence $u = u(x, y)$ such that the solvable radical of a finite group coincides with the set of all u-Engel elements?

B. I. Plotkin

16.78. Do there exist linear non-abelian simple groups without involutions?

B. Poizat

16.79. Is it true that in any finitely generated AT-group over a sequence of cyclic groups of uniformly bounded orders all Sylow subgroups are locally finite? For the definition of an *AT-group* see (A. V. Rozhkov, *Math. Notes*, **40** (1986), 827–836)

A. V. Rozhkov

16.80. Suppose that a group G is obtained from the free product of torsion-free groups $A_1, \dots, A_n$ by imposing m additional relations, where $m < n$. Is it true that the free product of some $n - m$ of the A_i embeds into G?

N. S. Romanovskiĭ

***16.82.** Let $\mathscr{X}$ be a non-empty class of finite groups closed under taking homomorphic images, subgroups, and direct products. With every group $G \in \mathscr{X}$ we associate some set $\tau(G)$ of subgroups of G. We say that τ is a *subgroup functor on* $\mathscr{X}$ if:

1) $G \in \tau(G)$ for all $G \in \mathscr{X}$, and
2) for each epimorphism $\varphi : A \mapsto B$, where $A, B \in \mathscr{X}$, and for any $H \in \tau(A)$ and $T \in \tau(B)$ we have $H^{\varphi} \in \tau(B)$ and $T^{\varphi^{-1}} \in \tau(A)$.

A subgroup functor τ is *closed* if for each group $G \in \mathscr{X}$ and for every subgroup $H \in \mathscr{X} \cap \tau(G)$ we have $\tau(H) \subseteq \tau(G)$. The set $F(\mathscr{X})$ consisting of all closed subgroup functors on $\mathscr{X}$ is a lattice (in which $\tau_1 \leqslant \tau_2$ if and only if $\tau_1(G) \subseteq \tau_2(G)$ for every group $G \in \mathscr{X}$). It is known that $F(\mathscr{X})$ is a chain if and only if $\mathscr{X}$ a class of p-groups for some prime p (Theorem 1.5.17 in S. F. Kamornikov and M. V. Sel'kin, *Subgroups functors and classes of finite groups*, Belaruskaya Navuka, Minsk, 2001 (Russian)).

Is there a non-nilpotent class $\mathscr{X}$ such that the width of the lattice $F(\mathscr{X})$ is at most $|\pi(\mathscr{X})|$ where $\pi(\mathscr{X})$ is the set of all prime divisors of the orders of the groups in $\mathscr{X}$?

A. N. Skiba

*No such classes exist (S. F. Kamornikov, *Siberian Math. J.*, **51**, no. 5 (2010), 824–829).

16.83. Let E_n be a free locally nilpotent n-Engel group on countably many generators, and let $\pi(E_n)$ be the set of prime divisors of the orders of elements of the periodic part of E_n. It is known that 2, 3, $5 \in \pi(E_4)$.

*(a) Does there exist n for which $7 \in \pi(E_n)$?

(b) Is it true that $\pi(E_n) = \pi(E_{n+1})$ for all sufficiently large n? *Yu. V. Sosnovskiĭ*

*a) Yes, $n = 6$, since the 2-generator free nilpotent 6-Engel group has an element of order 7 (`http://www.mathematik.tu-darmstadt.de/~nickel/Engel/Engel.html`). (A. Abdollahi, *Letter of 19 August 2011.*)

16.84. Can the braid group $\mathfrak{B}_n$, $n \geqslant 4$, act faithfully on a regular rooted tree by finite-state automorphisms? Such action is known for $\mathfrak{B}_3$.

See the definitions in (R. I. Grigorchuk, V. V. Nekrashevich, V. I. Sushchanskiĭ, *Proc. Steklov Inst. Math.*, **2000**, no. 4 (231), 128–203). *V. I. Sushchanskiĭ*

16.85. Suppose that groups G, H act faithfully on a regular rooted tree by finite-state automorphisms. Can their free product $G * H$ act faithfully on a regular rooted tree by finite state automorphisms? *V. I. Sushchanskiĭ*

16.86. Does the group of all finite-state automorphisms of a regular rooted tree possess an irreducible system of generators? *V. I. Sushchanskiĭ*

16.87. Let $\mathfrak{M}$ be a variety of groups and let G_r be a free r-generator group in $\mathfrak{M}$. A subset $S \subseteq G_r$ is called a *test set* if every endomorphism of G_r identical on S is an automorphism. The minimum of the cardinalities of test sets is called the test rank of G_r. Suppose that the test rank of G_r is r for every $r \geqslant 1$.

a) Is it true that $\mathfrak{M}$ is an abelian variety?

b) Suppose that $\mathfrak{M}$ is not a periodic variety. Is it true that $\mathfrak{M}$ is the variety of all abelian groups? *E. I. Timoshenko*

16.88. (G. Bergman). The *width* of a group G with respect to a generating set X means the supremum over all $g \in G$ of the least length of a group word in elements of X expressing g. A group G has *finite width* if the width of G with respect to any generating set is finite. Does there exist a countably infinite group of finite width? All known infinite groups of finite width (infinite permutation groups, infinite-dimensional general linear groups, and some other groups) are uncountable. *V. Tolstykh*

16.89. (G. Bergman). Is it true that the automorphism group of an infinitely generated free group F has finite width? The answer is affirmative if F is countably generated. *V. Tolstykh*

16.90. Is it true that the automorphism group $\operatorname{Aut} F$ of an infinitely generated free group F is

a) the normal closure of a single element?

b) the normal closure of some involution in $\operatorname{Aut} F$?

For a free group F_n of finite rank n M. Bridson and K. Vogtmann have recently shown that $\operatorname{Aut} F_n$ is the normal closure of some involution, which permutes some basis of F_n. It is also known that the automorphism groups of infinitely generated free nilpotent groups have such involutions. *V. Tolstykh*

16.91. Let F be an infinitely generated free group. Is there an IA-automorphism of F whose normal closure in $\mathrm{Aut}\, F$ is the group of all IA-automorphisms of F?

V. Tolstykh

16.92. Let F be an infinitely generated free group. Is $\mathrm{Aut}\, F$ equal to its derived subgroup? This is true if F is countably generated (R. Bryant, V. A. Roman'kov, *J. Algebra*, **209** (1998), 713–723).

V. Tolstykh

16.93. Let F_n be a free group of finite rank $n \geqslant 2$. Is the group $\mathrm{Inn}\, F_n$ of inner automorphisms of F_n a first-order definable subgroup of $\mathrm{Aut}\, F_n$? It is known that the set of inner automorphisms induced by powers of primitive elements is definable in $\mathrm{Aut}\, F_n$.

V. Tolstykh

16.94. If $G = [G, G]$, then the *commutator width* of the group G is its width relative to the set of commutators. Let V be an infinite-dimensional vector space over a division ring. It is known that the commutator width of $GL(V)$ is finite. Is it true that the commutator width of $GL(V)$ is one?

V. Tolstykh

16.95. *Conjecture:* If F is a field and A is in $GL(n, F)$, then there is a permutation matrix P such that AP is cyclic, that is, the minimal polynomial of AP is also its characteristic polynomial.

J. G. Thompson

16.96. Let G be a locally finite n-Engel p-group where p a prime greater than n. Is G then a Fitting group? (Examples of N. Gupta and F. Levin show that the condition $p > n$ is necessary in general.)

G. Traustason

16.97. Let G be a torsion-free group with all subgroups subnormal of defect at most n. Must G then be nilpotent of class at most n? (This is known to be true for $n < 5$).

G. Traustason

16.98. Suppose that G is a solvable finite group and A is a group of automorphisms of G of relatively prime order. Is there a bound for the Fitting height $h(G)$ of G in terms of A and $h(C_G(A))$, or even in terms of the length $l(A)$ of the longest chain of nested subgroups of A and $h(C_G(A))$?

When A is solvable, it is proved in (A. Turull, *J. Algebra*, **86** (1984), 555–566) that $h(G) \leqslant h(C_G(A)) + 2l(A)$ and this bound is best possible for $h(C_G(A)) > 0$. For A non-solvable some results are in (H. Kurzweil, *Manuscripta Math.*, **41** (1983), 233–305).

A. Turull

16.99. Suppose that G is a finite solvable group, $A \leqslant \mathrm{Aut}\, G$, $C_G(A) = 1$, the orders of G and A are coprime, and let $l(A)$ be the length of the longest chain of nested subgroups of A. Is the Fitting height of G bounded above by $l(A)$?

For A solvable the question coincides with 5.30. It is proved that for any finite group A, first, there exist G with $h(G) = l(A)$ and, second, there is a finite set of primes π (depending on A) such that if $|G|$ is coprime to each prime in π, then $h(G) \leqslant l(A)$. See (A. Turull, *Math. Z.*, **187** (1984), 491–503).

A. Turull

16.100. Is there an (infinite) 2-generator simple group G such that $\mathrm{Aut}\,F_2$ is transitive on the set of normal subgroups N of the free group F_2 such that $F_2/N \cong G$? Cf. 6.45.

J. Wiegold

***16.101.** Do there exist uncountably many infinite 2-groups that are quotients of the group $\langle x, y \mid x^2 = y^4 = (xy)^8 = 1\rangle$? There certainly exists one, namely the subgroup of finite index in Grigorchuk's first group generated by b and ad; see (R. I. Grigorchuk, *Functional Anal. Appl.*, **14** (1980), 41–43).

J. Wiegold

*Yes, there exist (R. I. Grigorchuk, *Algebra Discrete Math.*, **2009**, no. 4 (2009), 78–96 (subm. 25 November 2009); A. Minasyan, A. Yu. Olshanskii, D. Sonkin, *Groups Geom. Dynam.*, **3**, no. 3 (2009) 423–452 (subm. 21 April 2008)).

16.102. We say that a group G is *rational* if any two elements $x, y \in G$ satisfying $\langle x\rangle = \langle y\rangle$ are conjugate. Is it true that for any d there exist only finitely many finite rational d-generated 2-groups?

A. Jaikin-Zapirain

***16.103.** Is there a rank analogue of the Leedham-Green–McKay–Shepherd theorem on p-groups of maximal class? More precisely, suppose that P is a 2-generator finite p-group whose lower central quotients $\gamma_i(P)/\gamma_{i+1}(P)$ are cyclic for all $i \geqslant 2$. Is it true that P contains a normal subgroup N of nilpotency class $\leqslant 2$ such that the rank of P/N is bounded in terms of p only?

E. I. Khukhro

*No, moreover, there are no functions $d(p)$ and $r(p)$ such that a group with these properties would necessarily have a normal subgroup of derived length $\leqslant d(p)$ with quotient of rank $\leqslant r(p)$ (E. I. Khukhro, *Siber. Math. J.*, **54**, no. 1 (2013), 174–184).

16.104. If G is a finite group, then every element a of the rational group algebra $\mathbb{Q}[G]$ has a unique Jordan decomposition $a = a_s + a_n$, where $a_n \in \mathbb{Q}[G]$ is nilpotent, $a_s \in \mathbb{Q}[G]$ is semisimple over $\mathbb{Q}$, and $a_s a_n = a_n a_s$. The integral group ring $\mathbb{Z}[G]$ is said to have the *additive Jordan decomposition* property (AJD) if a_s, $a_n \in \mathbb{Z}[G]$ for every $a \in \mathbb{Z}[G]$. If $a \in \mathbb{Q}[G]$ is invertible, then a_s is also invertible and so $a = a_s a_u$ with $a_u = 1 + a_s^{-1} a_n$ unipotent and $a_s a_u = a_u a_s$. Such a decomposition is again unique. We say that $\mathbb{Z}[G]$ has *multiplicative Jordan decomposition* property (MJD) if a_s, $a_u \in \mathbb{Z}[G]$ for every invertible $a \in \mathbb{Z}[G]$. See the survey (A. W. Hales, I. B. S. Passi, in: *Algebra, Some Recent Advances*, Birkhäuser, Basel, 1999, 75–87).

Is it true that there are only finitely many isomorphism classes of finite 2-groups G such that $\mathbb{Z}[G]$ has MJD but not AJD?

A. W. Hales, I. B. S. Passi

16.105. Is it true that a locally graded group which is a product of two almost polycyclic subgroups (equivalently, of two almost soluble subgroups with the maximal condition) is almost polycyclic?

By (N. S. Chernikov, *Ukrain. Math. J.*, **32** (1980) 476–479) a locally graded group which is a product of two Chernikov subgroups (equivalently, of two almost soluble subgroups with the minimal condition) is Chernikov.

N. S. Chernikov

***16.107.** Is it true that almost every alternating group A_n is uniquely determined in the class of finite groups by its set of element orders, i. e. that $h(\pi_e(A_n)) = 1$ for all large enough n?

W. J. Shi

*Yes, it is (I. B. Gorshkov, *Algebra and Logic*, **52**, no. 1 (2013), 41–45).

16.108. Do braid groups B_n, $n > 4$, have non-elementary hyperbolic factor groups?

V. E. Shpilrain

16.110. (I. Kapovich, P. Schupp). Is there an algorithm which, when given two elements u, v of a free group F_n, decides whether or not the cyclic length of $\phi(u)$ equals the cyclic length of $\phi(v)$ for every automorphism ϕ of F_n?

Comment of 2009: The answer was shown to be positive for $n = 2$ in (D. Lee, *J. Group Theory*, **10** (2007), 561–569).

V. E. Shpilrain

16.111. Must an infinite simple periodic group with a dihedral Sylow 2-subgroup be isomorphic to $L_2(P)$ for a locally finite field P of odd characteristic?

V. P. Shunkov

Problems from the 17th Issue (2010)

17.1. (I. M. Isaacs). Does there exist a finite group partitioned by subgroups of equal order not all of which are abelian? (Cf. 15.26.) It is known that such a group must be of prime exponent (I. M. Isaacs, *Pacific J. Math.*, **49** (1973), 109–116). *A. Abdollahi*

***17.2.** (P. Schmid). Does there exist a finite non-abelian p-group G such that $H^1(G/\Phi(G),\, Z(\Phi(G))) = 0$? *A. Abdollahi*

*Yes, it does (A. Abdollahi, *J. Algebra*, **342** (2011), 154–160).

17.3. Let G be a group in which every 4-element subset contains two elements generating a nilpotent subgroup. Is it true that every 2-generated subgroup of G is nilpotent? *A. Abdollahi*

17.4. Let x be a right 4-Engel element of a group G.

a) Is it true that the normal closure $\langle x\rangle^G$ of x in G is nilpotent if G is locally nilpotent?

b) If the answer to a) is affirmative, is there a bound on the nilpotency class of $\langle x\rangle^G$?

c) Is it true that $\langle x\rangle^G$ is always nilpotent? *A. Abdollahi*

17.5. Is the nilpotency class of a nilpotent group generated by d left 3-Engel elements bounded in terms of d? A group generated by two left 3-Engel elements is nilpotent of class at most 4 (*J. Pure Appl. Algebra*, **188** (2004), 1–6.) *A. Abdollahi*

17.6. a) Is there a function $f : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that every nilpotent group generated by d left n-Engel elements is nilpotent of class at most $f(n,d)$?

b) Is there a function $g : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that every nilpotent group generated by d right n-Engel elements is nilpotent of class at most $g(n,d)$? *A. Abdollahi*

17.7. a) Is there a group in which the set of right Engel elements does not form a subgroup?

b) Is there a group in which the set of bounded right Engel elements does not form a subgroup? *A. Abdollahi*

17.8. a) Is there a group containing a right Engel element which is not a left Engel element?

b) Is there a group containing a bounded right Engel element which is not a left Engel element? *A. Abdollahi*

17.9. Is there a group containing a left Engel element whose inverse is not a left Engel element? *A. Abdollahi*

17.10. Is there a group containing a right 4-Engel element which does not belong to the Hirsch–Plotkin radical? *A. Abdollahi*

17.11. Is there a group containing a left 3-Engel element which does not belong to the Hirsch–Plotkin radical?

A. Abdollahi

***17.12.** Are there functions $e, c : \mathbb{N} \to \mathbb{N}$ such that if in a nilpotent group G a normal subgroup H consists of right n-Engel elements of G, then $H^{e(n)} \leqslant \zeta_{c(n)}(G)$?

A. Abdollahi

*Yes, there are (P. G. Crosby, G. Traustason, *J. Algebra*, **324** (2010), 875–883).

17.13. Let G be a totally imprimitive p-group of finitary permutations on an infinite set. Suppose that the support of any cycle in the cyclic decomposition of every element of G is a block for G. Does G necessarily contain a minimal non-FC-subgroup?

A. O. Asar

17.14. Can the braid group B_n for $n \geqslant 4$ be embedded into the automorphism group $Aut(F_{n-2})$ of a free group F_{n-2} of rank $n-2$?

Artin's theorem implies that B_n can be embedded into $Aut(F_n)$, and an embedding of B_n, $n \geqslant 3$, into $Aut(F_{n-1})$ was constructed in (B. Perron, J. P. Vannier, *Math. Ann.*, **306** (1996), 231–245).

V. G. Bardakov

17.15. Construct an algorithm which, for a given polynomial $f \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$, finds (explicitly, in terms of generators) the maximal subgroup G_f of the group $Aut(\mathbb{Z}[x_1, x_2, \ldots, x_n])$ that leaves f fixed. This question is related to description of the solution set of the Diophantine equation $f = 0$.

V. G. Bardakov

17.16. Let A be an Artin group of finite type, that is, the corresponding Coxeter group is finite. It is known that the group A is linear (S. Bigelow, *J. Amer. Math. Soc.*, **14** (2001), 471–486; D. Krammer, *Ann. Math.*, **155** (2002), 131–156; F. Digne, *J. Algebra*, **268** (2003), 39–57; A. M. Cohen, D. B. Wales, *Israel J. Math.*, **131** (2002), 101–123). Is it true that the automorphism group $Aut(A)$ is also linear?

An affirmative answer for A being a braid group B_n, $n \geqslant 2$, was obtained in (V. G. Bardakov, O. V. Bryukhanov, *Vestnik Novosibirsk. Univ. Ser. Mat. Mekh. Inform.*, **7**, no. 3 (2007), 45–58 (Russian)).

V. G. Bardakov

17.17. If a finitely generated group G has $n < \infty$ maximal subgroups, must G be finite? In particular, what if $n = 3$?

G. M. Bergman

17.18. Let $\mathbf{A}$ be the class of compact groups A with the property that whenever two compact groups B and C contain A, they can be embedded in a common compact group D by embeddings agreeing on A. I showed (*Manuscr. Math.*, **58** (1987) 253–281) that all members of $\mathbf{A}$ are (not necessarily connected) finite-dimensional compact Lie groups satisfying a strong "local simplicity" property, and that all finite groups do belong to $\mathbf{A}$.

a) Is it true that $\mathbb{R}/\mathbb{Z} \in \mathbf{A}$?

b) Do any *nonabelian connected* compact Lie groups belong to $\mathbf{A}$?

c) If A belongs to $\mathbf{A}$, must the connected component of the identity in A belong to $\mathbf{A}$?

G. M. Bergman

17.19. If F is a free group of finite rank, R a retract of F, and H a subgroup of F of finite rank, must $H \cap R$ be a retract of H?

G. M. Bergman

17.20. If M is a real manifold with nonempty boundary, and G the group of self-homeomorphisms of M which fix the boundary pointwise, is G right-orderable?

G. M. Bergman

***17.21.** a) If A, B, C are torsion-free abelian groups with $A \cong A \oplus B \oplus C$, must $A \cong A \oplus B$?

b) What if, furthermore, $B \cong C$? *G. M. Bergman*

*No, not necessarily, even in case b). See (A. L. S. Corner, in *Proc. Colloq. Abelian Groups (Tihany, 1963)*, Budapest, 1964, 43–48); I also have an example with B and C of rank 1 as torsion-free abelian groups. (G. Bergman, *Letter of 20 February 2011.*)

17.22. Suppose A is a group which belongs to a variety $\mathbf{V}$ of groups, and which is embeddable in the full symmetric group S on an infinite set. Must the coproduct in $\mathbf{V}$ of two copies of A also be embeddable in S? (N. G. de Bruijn proved that this is true if $\mathbf{V}$ is the variety of all groups.)

G. M. Bergman

17.23. Suppose the full symmetric group S on a countably infinite set is generated by the union of two subgroups G and H. Must S be finitely generated over one of these subgroups?

G. M. Bergman

***17.24.** (A. Blass, J. Irwin, G. Schlitt). Does $\mathbb{Z}^\omega$ have a subgroup whose dual is free of uncountable rank?

G. M. Bergman

*Yes, it does (G. M. Bergman, *Portugaliae Math.*, **69** (2012) 69–84).

17.25. (S. P. Farbman). Let X be the set of complex numbers α such that the group generated by the two 2×2 matrices $I+\alpha e_{12}$ and $I+e_{21}$ is not free on those generators.

a) Does X contain all rational numbers in the interval $(-4, 4)$?

b) Does X contain any rational number in the interval $[27/7, 4)$?

Cf. 4.41 (Archive), 15.83, and (S. P. Farbman, *Publ. Mat.*, **39** (1995) 379–391).

G. M. Bergman

17.26. Are the classes of right-orderable and right-ordered groups closed under taking solutions of equations $w(a_1, \dots, a_k, x_1, \dots, x_n) = 1$? (Here the closures are under group embeddings and order-preserving embeddings, respectively.) This is true for equations with a single constant, when $k = 1$ (*J. Group Theory*, **11** (2008), 623–633).

V. V. Bludov, A. M. W. Glass

17.27. Can the free product of two ordered groups with order-isomorphic amalgamated subgroups be lattice orderable but not orderable?

V. V. Bludov, A. M. W. Glass

17.28. Is there a soluble right-orderable group with insoluble word problem?

V. V. Bludov, A. M. W. Glass

***17.29.** Construct a finitely presented orderable group with insoluble word problem.

V. V. Bludov, A. M. W. Glass

*Such a group is constructed (V. V. Bludov, A. M. W. Glass, *Bull. London Math. Soc.*, **44** (2012), 85–98).

17.30. Is there a constant l such that every finitely presented soluble group has a subgroup of finite index of nilpotent length at most l? Cf. 16.35.

V. V. Bludov, J. R. J. Groves

17.31. Can a soluble right-orderable group have finite quotient by the derived subgroup?

V. V. Bludov, A. H. Rhemtulla

17.32. Is the following analogue of the Cayley–Hamilton theorem true for the free group F_n of rank n: If $w \in F_n$ and $\varphi \in \operatorname{Aut} F_n$ are such that $\langle w, \varphi(w), \ldots, \varphi^n(w)\rangle = F_n$, then $\langle w, \varphi(w), \ldots, \varphi^{n-1}(w)\rangle = F_n$?

O. V. Bogopol'skiĭ

17.33. Can the quasivariety generated by the group $\langle a, b \mid a^{-1}ba = b^{-1}\rangle$ be defined by a set of quasi-identities in a finite set of variables? The answer is known for all other groups with one defining relation.

A. I. Budkin

17.34. Let N_c be the quasivariety of nilpotent torsion-free groups of class at most c. Is it true that the dominion in N_c (see the definition in 16.20) of a divisible subgroup in every group in N_c is equal to this subgroup?

A. I. Budkin

***17.35.** Suppose we have a finite two-colourable triangulation of the sphere, with triangles each coloured either black or white so that no pair of triangles with the same colour share an edge. On each vertex we write a generator, and we assume the generators commute. We use these generators to generate an abelian group G_W with relations stating that the generators around each white triangle add to zero. Doing the same thing with the black triangles, we generate a group G_B. *Conjecture:* G_W *is isomorphic to* G_B.

I. M. Wanless, N. J. Cavenagh

*Conjecture is proved (S. R. Blackburn, T. A. McCourt, *Triangulations of the sphere, bitrades and abelian groups*, `arXiv:1112.5423v2` (2011)).

17.36. Two groups are called *isospectral* if they have the same set of element orders. Find all finite non-abelian simple groups G for which there is a finite group H isospectral to G and containing a proper normal subgroup isomorphic to G. For every simple group G determine all groups H satisfying this condition.

It is easy to show that a group H must satisfy the condition $G < H \leqslant \operatorname{Aut} G$.

A. V. Vasil'ev

17.37. Is there an integer n such that for all $m > n$ the alternating group A_m has no non-trivial A_m-permutable subgroups? (See the definition in 17.112.)

A. F. Vasil'ev, A. N. Skiba

17.38. A formation $\mathfrak{F}$ is called *radical in a class* $\mathfrak{H}$ if $\mathfrak{F} \subseteq \mathfrak{H}$ and in every $\mathfrak{H}$-group the product of any two normal $\mathfrak{F}$-subgroups belongs to $\mathfrak{F}$. Let $\mathfrak{M}$ be the class of all saturated hereditary formations of finite groups such that the formation of all finite supersoluble groups is radical in every element of $\mathfrak{M}$. Is it true that $\mathfrak{M}$ has the largest (by inclusion) element?

A. F. Vasil'ev, L. A. Shemetkov

17.39. Is there a positive integer n such that the hypercenter of any finite soluble group coincides with the intersection of n system normalizers of that group? What is the least number with this property?

A. F. Vasil'ev, L. A. Shemetkov

17.40. Let N be a nilpotent subgroup of a finite group G. Do there always exist elements $x, y \in G$ such that $N \cap N^x \cap N^y \leqslant F(G)$?

E. P. Vdovin

17.41. Let S be a solvable subgroup of a finite group G that has no nontrivial solvable normal subgroups.

a) (L. Babai, A. J. Goodman, L. Pyber). Do there always exist seven conjugates of S whose intersection is trivial?

b) Do there always exist five conjugates of S whose intersection is trivial?

Comment of 2013: reduction to almost simple group G is obtained in (E. P. Vdovin, *J. Algebra Appl.*, **11**, no. 1 (2012), 1250015 (14 pages)).

E. P. Vdovin

17.42. Let $\overline{G}$ be a simple algebraic group of adjoint type over the algebraic closure $\overline{F}_p$ of a finite field F_p of prime order p, and σ a Frobenius map (that is, a surjective homomorphism such that $\overline{G}_\sigma = C_{\overline{G}}(\sigma)$ is finite). Then $G = O^{p'}(\overline{G}_\sigma)$ is a finite group of Lie type. For a maximal σ-stable torus $\overline{T}$ of $\overline{G}$, let $N = N_{\overline{G}}(\overline{T}) \cap G$. Assume also that G is simple and $G \not\cong \mathrm{SL}_3(2)$. Does there always exist $x \in G$ such that $N \cap N^x$ is a p-group?

E. P. Vdovin

17.43. Let π be a set of primes. Find all finite simple D_π-groups (see Archive, 3.62) in which

a) every subgroup is a D_π-group (H. Wielandt);

b) every subgroup possessing a Hall π-subgroup is a D_π-group.

All finite simple D_π-groups are known (D. O. Revin, *Algebra and Logic*, **47**, no. 3 (2008), 210–227). *Comment of 2013:* alternating and sporadic groups for both parts are found in (N. Ch. Manzaeva, *Siberian Electr. Math. Rep.*, **9** (2012), 294–305 (Russian)).

E. P. Vdovin, D. O. Revin

17.44. Let π be a set of primes. A finite group is said to be a C_π-group if it possesses exactly one class of conjugate Hall π-subgroups.

*(a) In a C_π-group, is an overgroup of a Hall π-subgroup always a C_π-group?

(b) In a D_π-group, is an overgroup of a Hall π-subgroup always a D_π-group?

An affirmative answer to (a) in the case $2 \notin \pi$ follows mod CFSG from (F. Gross, *Bull. London Math. Soc.*, **19**, no. 4 (1987), 311–319). *Comment of 2013:* an affirmative answer to (b) in case $2 \in \pi$ is announced in (N. Ch. Manzaeva, *in: Abstracts of Int. Conf. on Group Theory in Honor of the 70th Birthday of Professor Victor Mazurov*, Novosibirsk, 2013 (Russian)).

E. P. Vdovin, D. O. Revin

*a) Yes, it is (E. P. Vdovin, D. O. Revin, *Siberian Math. J.*, **54**, no. 1 (2013), 22–28).

17.45. A subgroup H of a group G is called *pronormal* if H and H^g are conjugate in $\langle H, H^g\rangle$ for every $g \in G$. We say that H is *strongly pronormal* if L^g is conjugate to a subgroup of H in $\langle H, L^g\rangle$ for every $L \leqslant H$ and $g \in G$.

*a) In a finite simple group, are Hall subgroups always pronormal?

b) In a finite simple group, are Hall subgroups always strongly pronormal?

c) In a finite group, is a Hall subgroup with a Sylow tower always strongly pronormal?

An affirmative answer to (a) is known modulo CFSG for Hall subgroups of odd order (F. Gross, *Bull. London Math. Soc.*, **19**, no. 4 (1987), 311–319). Notice that there exist finite (non-simple) groups with a non-pronormal Hall subgroup. Hall subgroups with a Sylow tower are known to be pronormal.

E. P. Vdovin, D. O. Revin

*a) Yes, they are (E. P. Vdovin, D. O. Revin, *Siberian Math. J.*, **53**, no 3 (2012), 419–430).

***17.46.** Let G be a finite p-group in which every 2-generator subgroup has cyclic derived subgroup. Is the derived length of G bounded?

If $p \neq 2$, then $G^{(2)} = 1$ (J. Alperin), but for $p = 2$ there are examples with $G^{(2)}$ cyclic and elementary abelian of arbitrary order.

B. M. Veretennikov

*Yes, it is at most 4 (B. Wilkens, *to appear in J. Group Theory*).

17.47. Let G be a nilpotent group in which every coset $x[G,G]$ for $x \notin [G,G]$ coincides with the conjugacy class x^G. Is there a bound for the nilpotency class of G?

If G is finite, then its class is at most 3 (R. Dark, C. Scoppola, *J. Algebra*, **181**, no. 3 (1996), 787–802).

S. H. Ghate, A. S. Muktibodh

17.48. Does the free product of two groups with stable first-order theory also have stable first-order theory?

E. Jaligot

17.49. Is it consistent with ZFC that every non-discrete topological group contains a nonempty nowhere dense subset without isolated points? Under Martin's Axiom, there are countable non-discrete topological groups in which each nonempty nowhere dense subset has an isolated point.

E. G. Zelenyuk

***17.50.** Is it true that for every finite group G, there is a finite group F and a surjective homomorphism $f : F \to G$ such that for each nontrivial subgroup H of F, the restriction $f|_H$ is not injective?

It is known that for every finite group G there is a finite group F and a surjective homomorphism $f : F \to G$ such that for each subgroup H of F the restriction $f|_H$ is not bijective.

E. G. Zelenyuk

*Yes, it is (V. P. Burichenko, *Math. Notes*, **92**, no. 3 (2012), 327–332).

17.51. Is it true that every non-discrete topological group containing no countable open subgroup of exponent 2 can be partitioned into two (infinitely many) dense subsets? If the group is Abelian or countable, the answer is yes.

E. G. Zelenyuk, I. V. Protasov

17.52. (N. Eggert). Let R be a commutative associative finite-dimensional nilpotent algebra over a finite field F of characteristic p. Let $R^{(p)}$ be the subalgebra of all elements of the form r^p $(r \in R)$. Is it true that $\dim R \geqslant p \dim R^{(p)}$?

An affirmative answer gives a solution to 11.44 for the case where the factors A and B are abelian.

L. S. Kazarin

17.53. A finite group G is called *simply reducible* (SR-group) if every element of G is conjugate to its inverse, and the tensor product of any two irreducible representations of G decomposes into a sum of irreducible representations of G with coefficients 0 or 1.

a) Is it true that the nilpotent length of a soluble SR-group is at most 5?

b) Is it true that the derived length of a soluble SR-group is bounded by some constant c?

L. S. Kazarin

17.54. Does there exist a non-hereditary local formation $\mathfrak{F}$ of finite groups such that in every finite group the set of all $\mathfrak{F}$-subnormal subgroups is a sublattice of the subgroup lattice?

A negative answer would solve 9.75, since all the hereditary local formations with the same property are already known.

S. F. Kamornikov

17.55. Does there exist an absolute constant k such that for any prefrattini subgroup H in any finite soluble group G there exist k conjugates of H whose intersection is $\Phi(G)$?

S. F. Kamornikov

17.56. Suppose that a subgroup H of a finite group G is such that $HM = MH$ for every minimal non-nilpotent subgroup M of G. Must H/H_G be nilpotent, where H_G is the largest normal subgroup of G contained in H?

V. N. Knyagina, V. S. Monakhov

17.57. Let $r(m) = \{r+km \mid k \in \mathbb{Z}\}$ for integers $0 \leqslant r < m$. For $r_1(m_1) \cap r_2(m_2) = \varnothing$ define the *class transposition* $\tau_{r_1(m_1),r_2(m_2)}$ as the involution which interchanges $r_1 + km_1$ and $r_2 + km_2$ for each integer k and fixes everything else. The group $\mathrm{CT}(\mathbb{Z})$ generated by all class transpositions is simple (*Math. Z.*, **264**, no. 4 (2010), 927–938). Is $\mathrm{Out}(\mathrm{CT}(\mathbb{Z})) = \langle \sigma \mapsto \sigma^{n \mapsto -n-1} \rangle \cong \mathrm{C}_2$?

S. Kohl

17.58. Does $\mathrm{CT}(\mathbb{Z})$ have subgroups of intermediate (word-) growth?

S. Kohl

17.59. A permutation of $\mathbb{Z}$ is called *residue-class-wise affine* if there is a positive integer m such that its restrictions to the residue classes (mod m) are all affine. Is $\mathrm{CT}(\mathbb{Z})$ the group of all residue-class-wise affine permutations of $\mathbb{Z}$ which fix the nonnegative integers setwise?

S. Kohl

17.60. Given a set $\mathcal{P}$ of odd primes, let $\mathrm{CT}_{\mathcal{P}}(\mathbb{Z})$ denote the subgroup of $\mathrm{CT}(\mathbb{Z})$ which is generated by all class transpositions which interchange residue classes whose moduli have only prime factors in $\mathcal{P} \cup \{2\}$. The groups $\mathrm{CT}_{\mathcal{P}}(\mathbb{Z})$ are simple (*Math. Z.*, **264**, no. 4 (2010), 927–938). Are they pairwise nonisomorphic?

S. Kohl

17.61. The group $\mathrm{CT}_{\mathcal{P}}(\mathbb{Z})$ is finitely generated if and only if $\mathcal{P}$ is finite. If $\mathcal{P} = \varnothing$, then it is isomorphic to the finitely presented (first) Higman–Thompson group (J. P. McDermott, see `http://www.gap-system.org/DevelopersPages/StefanKohl/preprints/ctpz-v2.pdf`). Is it always finitely presented if $\mathcal{P}$ is finite?

S. Kohl

17.62. Given a free group F_n and a proper characteristic subgroup C, is it ever possible to generate the quotient F_n/C by fewer than n elements?

J. Conrad

17.63. Prove that if p is an odd prime and s is a positive integer, then there are only finitely many p-adic space groups of finite coclass with point group of coclass s.

C. R. Leedham-Green

17.64. Say that a group G is an *n-approximation* to the Nottingham group $J = N(p)$ (as defined in Archive, 12.24) if G is an infinite pro-p group, and $G/\gamma_n(G)$ is isomorphic to $J/\gamma_n(J)$. Does there exist a function $f(p)$ such that, if G is an $f(p)$-approximation to the Nottingham group, then $\gamma_i(G)/\gamma_{i+1}(G)$ is isomorphic to $\gamma_i(J)/\gamma_{i+1}(J)$ for all i?

Cf. 14.56. Note that an affirmative solution to this problem trivially implies the now known fact that J is finitely presented as a pro-p group (if $p > 2$), see 14.55.

C. R. Leedham-Green

17.65. If Problem 17.64 has an affirmative answer, does it follow that, for some function $g(p)$, the number of isomorphism classes of $g(p)$-approximations to J is

a) countable?

b) one?

Note that if, for some $g(p)$, there are only finitely many isomorphism classes, then for some $h(p)$ there is only one.

C. R. Leedham-Green

17.66. (Well-known problem). Does there exist a positive integer d such that $\dim H^1(G, V) \leqslant d$ for any faithful absolutely irreducible module V for any finite group G? Cf. 16.55 in Archive.

V. D. Mazurov

17.67. (H. Zassenhaus). *Conjecture*: Every invertible element of the integral group ring $\mathbb{Z}G$ of a finite group G is conjugate in the rational group ring $\mathbb{Q}G$ to an element of $\pm G$.

V. D. Mazurov

17.68. Let C be a cyclic subgroup of a group G and $G = CFC$ where F is a finite cyclic subgroup. Is it true that $|G : C|$ is finite? Cf. 12.62.

V. D. Mazurov

17.69. Let G be a group of prime exponent acting freely on a non-trivial abelian group. Is G cyclic?

V. D. Mazurov

17.70. Let α be an automorphism of prime order q of an infinite free Burnside group $G = B(q, p)$ of prime exponent p such that α cyclically permutes the free generators of G. Is it true that α fixes some non-trivial element of G?

V. D. Mazurov

17.71. Let α be a fixed-point-free automorphism of prime order p of a periodic group G.

*a) Is it true that G does not contain a non-trivial p-element?

b) Suppose that G does not contain a non-trivial p-element. Is α a splitting automorphism?

V. D. Mazurov

*a) No, it may contain a non-trivial p-element (A. I. Sozutov, *Preprint*, 2013).

17.72. Let AB be a Frobenius group with kernel A and complement B. Suppose that AB acts on a finite group G so that GA is also a Frobenius group with kernel G and complement A.

*a) Is the nilpotency class of G bounded in terms of $|B|$ and the class of $C_G(B)$?

b) Is the exponent of G bounded in terms of $|B|$ and the exponent of $C_G(B)$?

V. D. Mazurov

*a) Yes, it is (N. Yu. Makarenko, P. Shumyatsky, *Proc. Amer. Math. Soc.*, **138** (2010), 3425–3436).

17.73. Let G be a finite simple group of Lie type defined over a field of characteristic p, and V an absolutely irreducible G-module over a field of the same characteristic. Is it true that in the cases

a) $G = U_4(q)$;
b) $G = S_{2n}(q), n \geqslant 3$;
c) $G = O_{2n+1}(q), n \geqslant 3$;
d) $G = O^{+}_{2n}(q), n \geqslant 4$;
e) $G = O^{-}_{2n}(q), n \geqslant 4$;
f) $G = {}^3D_4(q)$, $q \neq 2$;
g) $G = E_6(q)$;
h) $G = {}^2E_6(q)$;
i) $G = E_7(q)$;
*j) $G = G_2(2^m)$

the split extension of V by G must contain an element whose order is distinct from the order of any element of G?

Editors' comment: affirmative solution of parts (g), (h), (i) is announced in (M. A. Grechkoseeva, *Abstracts of Mal'cev Meeting-2011*, Novosibirsk, 2011), as well as of part (c) for $n > 3$ and parts (d), (e) for $n > 4$ (*ibid.*); `http://www.math.nsc.ru/conference/malmeet/11/plenary/2011MM_Grechkoseeva.pdf`.

V. D. Mazurov

*j) Yes, it is (A. V. Vasil'ev, A. M. Staroletov, *Algebra Logic*, **52**, no. 1 (2013), 1–14).

17.74. Let G be a finite simple group of Lie type defined over a field of characteristic p whose Lie rank is at least three, and V an absolutely irreducible G-module over a field of characteristic that does not divide p. It is true that the split extension of V by G must contain an element whose order is distinct from the order of any element of G? The case of $G = U_n(p^m)$ is of special interest.

Editors' comment: affirmative solution is announced in (M. A. Grechkoseeva, *Abstracts of Int. Conf. Algebra and Math. Logic ded. to V. V. Morozov*, Kazan', 2011, p. 206); for all classical groups published in (M. A. Grechkoseeva, *J. Algebra*, **339**, no. 1 (2011), 304–319).

V. D. Mazurov

17.75. Can the Monster M act on a nontrivial finite 3-group V so that all elements of M of orders 41, 47, 59, 71 have no nontrivial fixed points in V?

V. D. Mazurov

17.76. Does there exist a finite group G, with $|G| > 2$, such that there is exactly one element in G which is not a commutator?

D. MacHale

***17.77.** Let k be a positive integer such that there is an insoluble finite group with exactly k conjugacy classes. Is it true that a finite group of maximal order with exactly k conjugacy classes is insoluble?

R. Heffernan, D. MacHale

*No, it is not; see Table 1 in (A. Vera-Lopez, J. Sangroniz, *Math. Nachr.*, **280**, no. 5–6 (2007), 676–694).

17.78. Does there exist a finitely generated group without free subsemigroups generating a proper variety containing $\mathfrak{A}_p\mathfrak{A}$?

O. Macedońska

***17.79.** Does there exist a finitely generated torsion group of unbounded exponent generating a proper variety?

O. Macedońska

*Yes, moreover, there is a continuum of such groups (V. S. Atabekyan, *Infinite simple groups satisfying an identity*, Dep. VINITI no. 5381-V86, Moscow, 1986 (Russian)). Another example was suggested by D. Osin in a letter of 31 August 2013: a free group G in the variety $\mathfrak{M}$ defined by the law $x^n y = yx^n$ is a central extension of a free Burnside group of exponent n such that the centre of G is a free abelian group of countable rank (I. S. Ashmanov, A. Yu. Ol'shanskiĭ, *Izv. Vyssh. Uchebn. Zaved. Mat.*, **1985**, no. 11 (1985), 48–60 (Russian)). Let $x_1, x_2, \dots$ be a basis in $Z(G)$. By adding to G the relations $x_i^i = 1$ for all i we obtain a periodic group of unbounded exponent generating a proper variety (contained in $\mathfrak{M}$).

17.80. Let $[u,\, _nv] := [u, \overbrace{v, \dots, v}^{n}]$. Is the group $M_n = \langle x, y \mid x = [x,\, _ny],\;\; y = [y,\, _nx]\,\rangle$ infinite for every $n > 2$? (See also 11.18.)

O. Macedońska

17.81. Given normal subgroups $R_1, \dots, R_n$ of a group G, let $[[R_1, \dots, R_n]] := \prod \Big[\bigcap\limits_{i\in I} R_i, \bigcap\limits_{j\in J} R_j\Big]$, where the product is over all $I \cup J = \{1, \dots, n\}$, $I \cap J = \varnothing$. Let G be a free group, and let $R_i = \langle r_i \rangle^G$ be the normal closures of elements $r_i \in G$. It is known (B. Hartley, Yu. Kuzmin, *J. Pure Appl. Algebra*, **74** (1991), 247–256) that the quotient $(R_1 \cap R_2)/[R_1, R_2]$ is a free abelian group. On the other hand, the quotient $(R_1 \cap \dots \cap R_n)/[[R_1, \dots, R_n]]$ has, in general, non-trivial torsion for $n \geqslant 4$. Is this quotient always torsion-free for $n = 3$? This is known to be true if r_1, r_2, r_3 are not proper powers in G.

R. Mikhailov

***17.82.** Is it true that in every finitely presented group the intersection of the derived series has trivial abelianization?

R. Mikhailov

*No, not always (V. A. Roman'kov, *Siberian Math. J.*, **52**, no. 2 (2011), 348–351).

17.83. Does there exist a group such that every central extension of it is residually nilpotent, but there exists a central extension of a central extension of it which is not residually nilpotent?

R. Mikhailov

17.84. An associative algebra A is said to be *Calabi–Yau of dimension* d (for short, CYd) if there is a natural isomorphism of A-bimodules $\mathrm{Ext}^d_{A\text{-bimod}}(A,\, A\otimes A) \cong A$ and $\mathrm{Ext}^n_{A\text{-bimod}}(A,\, A \otimes A) = 0$ for $n \neq d$. By Kontsevich's theorem, the complex group algebra $\mathbb{C}G$ of the fundamental group G of a 3-dimensional aspherical manifold is CY3. Is every group with CY3 complex group algebra residually finite?

R. Mikhailov

17.85. Let $\mathcal{V}$ be a variety of groups such that any free group in $\mathcal{V}$ has torsion-free integral homology groups in all dimensions. Is it true that $\mathcal{V}$ is abelian?

It is known that free metabelian groups may have 2-torsion in the second homology groups, and free nilpotent groups of class 2 may have 3-torsion in the fourth homology groups.

R. Mikhailov

17.86. (Simplest questions related to the Whitehead asphericity conjecture). Let $\langle x_1, x_2, x_3 \mid r_1, r_2, r_3\rangle$ be a presentation of the trivial group.

a) Prove that the group $\langle x_1, x_2, x_3 \mid r_1, r_2\rangle$ is torsion-free.

*b) Let $F = F(x_1, x_2, x_3)$ and $R_i = \langle r_i\rangle^F$. Is it true that the group $F/[R_1, R_2]$ is residually soluble? If yes, then any subcomplex of a contractible two-dimensional complex with three cells is aspherical.

R. Mikhailov

*b) No, not always (V. A. Roman'kov, *to appear in Siberian Math. J.*).

17.87. Construct a group of intermediate growth with finitely generated Schur multiplier.

R. Mikhailov

17.88. Compute $K_0(\mathbb{F}_2G)$ and $K_1(\mathbb{F}_2G)$, where G is the first Grigorchuk group, $\mathbb{F}_2G$ its group algebra over the field of two elements, and K_0, K_1 the zeroth and first K-functors.

R. Mikhailov

17.89. By Bousfield's theorem, the free pronilpotent completion of a non-cyclic free group has uncountable Schur multiplier. Is it true that the free prosolvable completion (that is, the inverse limit of quotients by the derived subgroups) of a non-cyclic free group has uncountable Schur multiplier?

R. Mikhailov

17.90. (G. Baumslag). A group is *parafree* if it is residually nilpotent and has the same lower central quotients as a free group. Is it true that $H_2(G) = 0$ for any finitely generated parafree group G?

R. Mikhailov

17.91. Let $d(X)$ denote the derived length of a group X.

a) Does there exist an absolute constant k such that $d(G) - d(M) \leqslant k$ for every finite soluble group G and any maximal subgroup M?

b) Find the minimum k with this property.

V. S. Monakhov

***17.92.** What are the non-abelian composition factors of a finite non-soluble group all of whose maximal subgroups are Hall subgroups?

V. S. Monakhov

*Answer: $L_2(7)$, $L_2(11)$, $L_5(2)$ (N. V. Maslova, *Siberian Math. J.*, **53**, no. 5 (2012), 853–861).

17.93. (Well-known problem). Let G be a compact topological group which has elements of arbitrarily high orders. Must G contain an element of infinite order?

J. Mycielski

17.94. (Well-known problem). Can the free product $Z * G$ of the infinite cyclic group Z and a nontrivial group G be the normal closure of a single element?

J. Mycielski

17.95. Let G be a permutation group on the finite set Ω. A partition ρ of Ω is said to be *G-regular* if there exists a subset S of Ω such that S^g is a transversal of ρ for all $g \in G$. The group G is said to be *synchronizing* if $|\Omega| > 2$ and there are no non-trivial proper G-regular partitions on Ω.

a) Are the following affine-type primitive groups synchronizing:

$2^p.\mathrm{PSL}(2, 2p+1)$ where both p and $2p+1$ are prime, $p \equiv 3 \pmod 4$ and $p > 23$?

$2^{101}.He$?

b) For which finite simple groups S are the groups $S \times S$ acting on S by $(g,h) : x \mapsto g^{-1}xh$ non-synchronizing?

Peter M. Neumann

17.96. Does there exist a variety of groups which contains only countably many subvarieties but in which there is an infinite properly descending chain of subvarieties?

Peter M. Neumann

17.97. Is every variety of groups of exponent 4 finitely based? *Peter M. Neumann*

17.98. A variety of groups is said to be *small* if it contains only countably many non-isomorphic finitely generated groups.

a) Is it true that if G is a finitely generated group and the variety $\mathrm{Var}(G)$ it generates is small then G satisfies the maximal condition on normal subgroups?

b) Is it true that a variety is small if and only if all its finitely generated groups have the Hopf property?

Peter M. Neumann

17.99. Consider the group $B = \langle a, b \mid (bab^{-1})a(bab^{-1})^{-1} = a^2 \rangle$ introduced by Baumslag in 1969. The same relation is satisfied by the functions $f(x) = 2x$ and $g(x) = 2^x$ under the operation of composition in the group of germs of monotonically increasing to ∞ continuous functions on $(0, \infty)$, where two functions are identified if they coincide for all sufficiently large arguments. Is the representation $a \to f$, $b \to g$ of the group B faithful?

A. Yu. Ol'shanskiĭ

17.100. *Conjecture*: A finite group is not simple if it has an irreducible complex character of odd degree vanishing on a class of odd length.

If true, this implies the solvability of groups of odd order, so a proof independent of CFSG is of special interest.

V. Pannone

17.101. According to P. Hall, a group G is said to be *homogeneous* if every isomorphism of its finitely generated subgroups is induced by an automorphism of G. It is known (Higman–Neumann–Neumann) that every group is embeddable into a homogeneous one. Is the same true for the category of representations of groups? A representation (V, G) is finitely generated if G is a finitely generated group, and V a finitely generated module over the group algebra of G.

B. I. Plotkin

17.102. We say that two subsets A, B of an infinite group G are *separated* if there exists an infinite subset X of G such that $1 \in X$, $X = X^{-1}$, and $XAX \cap B = \varnothing$. Is it true that any two disjoint subsets A, B of an infinite group G satisfying $|A| < |G|$, $|B| < |G|$ are separated? This is so if A, B are finite, or G is Abelian.

I. V. Protasov

17.103. Does there exist a continuum of sets π of primes for which every finite group possessing a Hall π-subgroup is a D_π-group?

D. O. Revin

17.104. Let Γ be a finite non-oriented graph on the set of vertices $\{x_1, \ldots, x_n\}$ and let $S_\Gamma = \langle x_1, \ldots, x_n \mid x_i x_j = x_j x_i \Leftrightarrow (x_i, x_j) \in \Gamma;\ \mathfrak{A}^2 \rangle$ be a presentation of a partially commutative metabelian group S_Γ in the variety of all metabelian groups. Is the universal theory of the group S_Γ decidable? *V. N. Remeslennikov, E. I. Timoshenko*

17.105. An equation over a pro-p-group G is an expression $v(x) = 1$, where $v(x)$ is an element of the free pro-p-product of G and a free pro-p-group with basis $\{x_1, \ldots, x_n\}$; solutions are sought in the affine space G^n. Is it true that a free pro-p-group is equationally Noetherian, that is, for any n every system of equations in $x_1, \ldots, x_n$ over this group is equivalent to some finite subsystem of it? *N. S. Romanovskiĭ*

17.107. Does $G = \mathrm{SL}_2(\mathbb{C})$ contain a 2-generated free subgroup that is conjugate in G to a proper subgroup of itself?

A 6-generated free subgroup with this property has been found (D. Calegari, N. M. Dunfield, *Proc. Amer. Math. Soc.*, **134**, no. 11 (2006), 3131–3136). *M. Sapir*

17.108. Is the group $\langle a, b, t \mid a^t = ab,\ b^t = ba \rangle$ linear?

If not, this would be an easy example of a non-linear hyperbolic group. *M. Sapir*

17.109. A non-trivial group word w is *uniformly elliptic* in a class $\mathcal{C}$ if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that the width of w in every d-generator $\mathcal{C}$-group G is bounded by $f(d)$ (i. e. every element of the verbal subgroup $w(G)$ is equal to a product of $f(d)$ values of w or their inverses). If $\mathcal{C}$ is a class of finite groups, this is equivalent to saying that in every finitely generated pro-$\mathcal{C}$ group G the verbal subgroup $w(G)$ is closed. A. Jaikin-Zapirain (*Revista Mat. Iberoamericana*, **24** (2008), 617–630) proved that w is uniformly elliptic in finite p-groups if and only if $w \notin F''(F')^p$, where F is the free group on the variables of w. Is it true that w is uniformly elliptic in $\mathcal{C}$ if and only if $w \notin F''(F')^p$ for every prime p in the case where $\mathcal{C}$ is the class of all

a) finite soluble groups?

b) finite groups?

See also Ch. 4 of (D. Segal, *Words: notes on verbal width in groups*, LMS Lect. Note Series, **361**, Cambridge Univ. Press, 2009. *D. Segal*

17.110. a) Is it true that for each word w, there is a function $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that the width of w in every finite p-group of Prüfer rank r is bounded by $h(p, r)$?

b) If so, can $h(p, r)$ be made independent of p?

An affirmative answer to a) would imply A. Jaikin-Zapirain's result (*ibid.*) that every word has finite width in each p-adic analytic pro-p group, since a pro-p group is p-adic analytic if and only if the Prüfer ranks of all its finite quotients are uniformly bounded. *D. Segal*

***17.111.** Let G be a finite group, and p a prime divisor of $|G|$. Suppose that every maximal subgroup of a Sylow p-subgroup of G has a p-soluble supplement in G. Must G be p-soluble? *A. N. Skiba*

*Yes, it must (GuoHua Qian, *Science China Mathematics*, **56**, no. 5 (2013), 1015–1018).

17.112. A subgroup A of a group G is said to be *G-permutable in G* if for every subgroup B of G there exists an element $x \in G$ such that $AB^x = B^x A$. A subgroup A is said to be *hereditarily G-permutable in G* if A is E-permutable in every subgroup E of G containing A. Which finite non-abelian simple groups G possess

a) a non-trivial G-permutable subgroup?

b) a non-trivial hereditarily G-permutable subgroup?

A. N. Skiba, V. N. Tyutyanov

17.113. Do there exist for $p > 3$ 2-generator finite p-groups with deficiency zero (see 8.12) of arbitrarily high nilpotency class?

J. Wiegold

17.114. Does every generalized free product (with amalgamation) of two non-trivial groups have maximal subgroups?

J. Wiegold

17.115. Can a locally free non-abelian group have non-trivial Frattini subgroup?

J. Wiegold

***17.116.** Let $n(G)$ be the maximum of positive integers n such that the n-th direct power of a finite simple group G is 2-generated. Is it true that $n(G) \geqslant \sqrt{|G|}$?

A. Erfanian, J. Wiegold

*Yes, it is, even with $n(G) > 2\sqrt{|G|}$ (A. Maróti, M. C. Tamburini, *Commun. Algebra*, **41**, no. 1 (2013), 34–49).

17.117. (Well-known problem). If groups A and B have decidable elementary theories $Th(A)$ and $Th(B)$, must $Th(A * B)$ be decidable?

O. Kharlampovich

17.118. Suppose that a finite p-group G has a subgroup of exponent p and of index p. Must G also have a characteristic subgroup of exponent p and of index bounded in terms of p?

E. I. Khukhro

17.119. Suppose that a finite soluble group G admits a soluble group of automorphisms A of coprime order such that $C_G(A)$ has rank r. Let $|A|$ be the product of l not necessarily distinct primes. Is there a linear function f such that $G/F_{f(l)}(G)$ has $(|A|, r)$-bounded rank, where $F_{f(l)}(G)$ is the $f(l)$-th Fitting subgroup?

An exponential function f with this property was found in (E. Khukhro, V. Mazurov, *Groups St. Andrews 2005*, vol. II, Cambridge Univ. Press, 2007, 564–585). It is also known that $|G/F_{2l+1}(G)|$ is bounded in terms of $|C_G(A)|$ and $|A|$ (Hartley–Isaacs).

E. I. Khukhro

17.120. Is a residually finite group all of whose subgroups of infinite index are finite necessarily cyclic-by-finite?

N. S. Chernikov

17.121. Let G be a group whose set of elements is the real numbers and which is "nicely definable" (see below). Does G being $\aleph_\omega$-free imply it being free if "nicely definable" means

a) being F_σ?

b) being Borel?

c) being analytic?

d) being projective $\mathbb{L}[\mathbb{R}]$?

See `arXiv:math/0212250v2` for justification of the restrictions. *S. Shelah*

17.122. The same questions as 17.121 for an abelian group G. *S. Shelah*

17.123. Do there exist finite groups G_1, G_2 such that $\pi_e(G_1) = \pi_e(G_2)$, $h(\pi_e(G_1)) < \infty$, and each non-abelian composition factors of each of the groups G_1, G_2 is not isomorphic to a section of the other?

For definitions see Archive, 13.63. *W. J. Shi*

17.124. Is the set of finitely presented metabelian groups recursively enumerable? *V. Shpilrain*

17.125. Does every finite group G contain a pair of conjugate elements a, b such that $\pi(G) = \pi(\langle a, b\rangle)$? This is true for soluble groups.

Comment of 2013: it was proved in (A. Lucchini, M. Morigi, P. Shumyatsky, *Forum Math.*, **24** (2012), 875–887) that every finite group G contains a 2-generator subgroup H such that $\pi(G) = \pi(H)$. *P. Shumyatsky*

17.126. Suppose that G is a residually finite group satisfying the identity $[x, y]^n = 1$. Must $[G, G]$ be locally finite?

An equivalent question: let G be a finite soluble group satisfying the identity $[x, y]^n = 1$; is the Fitting height of G bounded in terms of n? Cf. Archive, 13.34 *P. Shumyatsky*

17.127. Suppose that a finite soluble group G of derived length d admits an elementary abelian p-group of automorphisms A of order p^n such that $C_G(A) = 1$. Must G have a normal series of n-bounded length with nilpotent factors of (p, n, d)-bounded nilpotency class?

This is true for $p = 2$. An affirmative answer would follow from an affirmative answer to 11.125. *P. Shumyatsky*

17.128. Let T be a finite p-group admitting an elementary abelian group of automorphisms A of order p^2 such that in the semidirect product $P = TA$ every element of $P \setminus T$ has order p. Does it follow that T is of exponent p? *E. Jabara*

New Problems

18.1. Given a group G of finite order n, does there necessarily exist a bijection f from G onto a cyclic group of order n such that for each element $x \in G$, the order of x divides the order of $f(x)$? An affirmative answer in the case where G is solvable was given by F. Ladisch.

I. M. Isaacs

18.2. Let the group $G = AB$ be the product of two Chernikov subgroups A and B each of which has an abelian subgroup of index at most 2. Is G soluble?

B. Amberg

18.3. Let the group $G = AB$ be the product of two central-by-finite subgroups A and B. By a theorem of N. Chernikov, G is soluble-by-finite. Is G metabelian-by-finite?

B. Amberg

18.4. (A. Rhemtulla, S. Sidki). (a) Is a group of the form $G = ABA$ with cyclic subgroups A and B always soluble? (This is known to be true if G is finite.)

(b) The same question when in addition A and B are conjugate in G.

B. Amberg

18.5. Let the (soluble) group $G = AB$ with finite torsion-free rank $r_0(G)$ be the product of two subgroups A and B. Is the equation $r_0(G) = r_0(A)+r_0(B)-r_0(A\cap B)$ valid in this case? It is known that the inequality $\leqslant$ always holds.

B. Amberg

18.6. Is every finitely generated one-relator group residually amenable?

G. N. Arzhantseva

18.7. Let $n \geqslant 665$ be an odd integer. Is it true that the group of outer automorphisms $\mathrm{Out}(B(m,n))$ of the free Burnside group $B(m,n)$ for $m > 2$ is complete, that is, has trivial center and all its automorphisms are inner?

V. S. Atabekyan

18.8. Let $\mathcal{X}$ be a class of finite simple groups such that $\pi(\mathcal{X}) = char(\mathcal{X})$. A formation of finite groups $\mathfrak{F}$ is said to be $\mathcal{X}$-saturated if a finite group G belongs to $\mathfrak{F}$ whenever the factor group $G/\Phi(O_{\mathcal{X}}(G))$ is in $\mathfrak{F}$, where $O_{\mathcal{X}}(G)$ is the largest normal subgroup of G whose composition factors are in $\mathcal{X}$. Is every $\mathcal{X}$-saturated formation $\mathcal{X}$-local in the sense of Förster? (See the definition in (P. Förster, *Publ. Sec. Mat. Univ. Autònoma Barcelona*, **29**, no. 2–3 (1985), 39–76).)

A. Ballester-Bolinches

18.9. Does there exist a subgroup-closed saturated formation $\mathfrak{F}$ of finite groups properly contained in $\mathfrak{E}_\pi$, where $\pi = char(\mathcal{F})$, satisfying the following property: if $G \in \mathfrak{F}$, then there exists a prime p (depending on the group G) such that the wreath product $C_p \wr G$ belongs to $\mathfrak{F}$, where C_p is the cyclic group of order p?

A. Ballester-Bolinches

18.10. A formation $\mathfrak{F}$ of finite groups satisfies the *Wielandt property for residuals* if whenever U and V are subnormal subgroups of $\langle U, V\rangle$ in a finite group G, then the $\mathfrak{F}$-residual $\langle U, V\rangle^{\mathfrak{F}}$ of $\langle U, V\rangle$ coincides with $\langle U^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle$. Does every Fitting formation $\mathfrak{F}$ satisfy the Wielandt property for residuals?

A. Ballester-Bolinches

18.11. Let G and H be subgroups of the automorphism group $Aut(F_n)$ of a free group F_n of rank $n \geqslant 2$. Is it true that the free product $G * H$ embeds in the automorphism group $Aut(F_m)$ for some m?

V. G. Bardakov

18.12. Let G be a finitely generated group of intermediate growth. Is it true that there is a positive integer m such that every element of the derived subgroup G' is a product of at most m commutators?

This assertion is valid for groups of polynomial growth, since they are almost nilpotent by Gromov's theorem. On the other hand, there are groups of exponential growth (for example, free groups) for which this assertion is not true.

V. G. Bardakov

18.13. (D. B. A. Epstein). Is it true that the group $H = (\mathbb{Z}_3 \times \mathbb{Z}) * (\mathbb{Z}_2 \times \mathbb{Z}) = \langle x, y, z, t \mid x^3 = z^2 = [x, y] = [z, t] = 1 \rangle$ cannot be defined by three relators in the generators x, y, z, t?

It is known that the relation module of the group H has rank 3 (K. W. Gruenberg, P. A. Linnell, *J. Group Theory*, **11**, no. 5 (2008), 587–608). An affirmative answer would give a solution of the relation gap problem.

V. G. Bardakov, M. V. Neshchadim

18.14. For an automorphism $\varphi \in Aut(G)$ of a group G, let $[e]_\varphi = \{g^{-1}g^\varphi \mid g \in G\}$.

Conjecture: If $[e]_\varphi$ is a subgroup for every $\varphi \in Aut(G)$, then the group G is nilpotent. If in addition G is finitely generated, then G is abelian.

V. G. Bardakov, M. V. Neshchadim, T. R. Nasybullov

18.15. Is it true that if a group G has trivial center, then there is an inner automorphism φ such that $[e]_\varphi$ is not a subgroup?

V. G. Bardakov, M. V. Neshchadim, T. R. Nasybullov

18.16. Is it true that any definable endomorphism of any ordered abelian group is of the form $x \mapsto rx$, for some rational number r?

O. V. Belegradek

18.17. Is there a torsion-free group which is finitely presented in the quasi-variety of torsion-free groups but not finitely presentable in the variety of all groups?

O. V. Belegradek

18.18. Mal'cev proved that the set of sentences that hold in all finite groups is not computably enumerable, although its complement is. Is it true that both the set of sentences that hold in almost all finite groups and its complement are not computably enumerable?

O. V. Belegradek

18.19. Is any torsion-free, relatively free group of infinite rank not $\aleph_1$-homogeneous? This is true for group varieties in which all free groups are residually finite (O. Belegradek, *Arch. Math. Logic*, **51** (2012), 781–787).

O. V. Belegradek

18.20. Characters φ and ψ of a finite group G are said to be *semiproportional* if they are not proportional and there is a normal subset M of G such that $\varphi|_M$ is proportional to $\psi|_M$ and $\varphi|_{G\setminus M}$ is proportional to $\psi|_{G\setminus M}$.

Conjecture: If φ and ψ are semiproportional irreducible characters of a finite group, then $\varphi(1) = \psi(1)$.

V. A. Belonogov

18.21. (B. H. Neumann and H. Neumann). Fix an integer $d \geqslant 2$. If $\mathfrak{V}$ is a variety of groups such that all d-generated groups in $\mathfrak{V}$ are finite, must $\mathfrak{V}$ be locally finite?

G. M. Bergman

18.22. (H. Neumann). Is the Kostrikin variety of all locally finite groups of given prime exponent p determined by finitely many identities?

G. M. Bergman

18.23. The normal covering number of the symmetric group S_n of degree n is the minimum number $\gamma(S_n)$ of proper subgroups $H_1, \ldots, H_{\gamma(S_n)}$ of S_n such that every element of S_n is conjugate to an element of H_i, for some $i = 1, \ldots, \gamma(S_n)$. Write $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ for primes $p_1 < \cdots < p_r$ and positive integers $\alpha_1, \ldots, \alpha_r$.

Conjecture:

$$\gamma(S_n) = \begin{cases} \frac{n}{2}\left(1-\frac{1}{p_1}\right) & \text{if } r=1 \text{ and } \alpha_1 = 1 \\ \frac{n}{2}\left(1-\frac{1}{p_1}\right)+1 & \text{if } r=1 \text{ and } \alpha_1 \geqslant 2 \\ \frac{n}{2}\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)+1 & \text{if } r=2 \text{ and } \alpha_1+\alpha_2 = 2 \\ \frac{n}{2}\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)+2 & \text{if } r \geqslant 2 \text{ and } \alpha_1+\cdots+\alpha_r \geqslant 3 \end{cases}$$

This is the strongest form of the conjecture. We would be also interested in a proof that this holds for n sufficiently large. The result for $r \leqslant 2$, which includes the first three cases above, is proved for n odd (D. Bubboloni, C. E. Praeger, *J. Combin. Theory (A)*, **118** (2011), 2000–2024). When $r \geqslant 3$ we know that $cn \leqslant \gamma(S_n) \leqslant \frac{2}{3}n$ for some positive constant c (D. Bubboloni, C. E. Praeger, P. Spiga, *J. Algebra*, **390** (2013) 199–215). We showed that the conjectured value for $\gamma(S_n)$ is an upper bound, by constructing a normal covering for S_n with this number of conjugacy classes of maximal subgroups, and gave further evidence for the truth of the conjecture in other cases (D. Bubboloni, C. E. Praeger, P. Spiga, *to appear in Int. J. Group Theory*).

D. Bubboloni, C. E. Praeger, P. Spiga

18.24. For a group G, a function $\phi : G \to \mathbb{R}$ is a *quasimorphism* if there is a least non-negative number $D(\phi)$ (called the *defect*) such that $|\phi(gh) - \phi(g) - \phi(h)| \leqslant D(\phi)$ for all $g, h \in G$. A quasimorphism is *homogeneous* if in addition $\phi(g^n) = n\phi(g)$ for all $g \in G$. Let ϕ be a homogeneous quasimorphism of a free group F. For any quasimorphism ψ of F (not required to be homogeneous) with $|\phi - \psi| <$ const, we must have $D(\psi) \geqslant D(\phi)/2$. Is it true that there is some ψ with $D(\psi) = D(\phi)/2$?

M. Burger, D. Calegari

18.25. Let form(G) be the formation generated by a finite group G. Suppose that G has a unique composition series $1 \lhd G_1 \lhd G_2 \lhd G$ and the consecutive factors of this series are Z_p, X, Z_q, where p and q are primes and X is a non-abelian simple group. Is it true that form(G) has infinitely many subformations if and only if $p = q$?

V. P. Burichenko

18.26. Suppose that a finite group G has a normal series $1 < G_1 < G_2 < G$ such that the groups G_1 and G_2/G_1 are elementary abelian p-groups, $G/G_2 \cong A_5 \times A_5$ (where A_5 is the alternating group of degree 5), G_1 and G_2/G_1 are minimal normal subgroups of G and G/G_1, respectively. Is it true that form(G) has finitely many subformations?

V. P. Burichenko

18.27. Does there exist an algorithm that determines whether there are finitely many subformations in form(G) for a given finite group G?

V. P. Burichenko

18.28. Is it true that any subformation of every one-generator formation form(G) is also one-generator?

V. P. Burichenko

18.29. Let $\mathfrak{F}$ be a Fitting class of finite soluble groups which contains every soluble group $G = AB$, where A and B are abnormal $\mathfrak{F}$-subgroups of G. Is $\mathfrak{F}$ a formation?

A. F. Vasil'ev

18.30. A subgroup H of a group G is called $\mathbb{P}$-*subnormal* in G if either $H = G$, or there is a chain of subgroups $H_0 \subset H_1 \subset \cdots \subset H_{n-1} \subset H_n = G$ such that $|H_i : H_{i-1}|$ is a prime for all $i = 1, \ldots, n$. Must a finite group be soluble if every Shmidt subgroup of it is $\mathbb{P}$-subnormal?

A. F. Vasil'ev, T. I. Vasil'eva, V. N. Tyutyanov

18.31. Let π be a set of primes. Is it true that in any D_π-group G (see Archive, 3.62) there are three Hall π-subgroups whose intersection coincides with $O_\pi(G)$?

E. P. Vdovin, D. O. Revin

18.32. Is every Hall subgroup of a finite group pronormal in its normal closure?

E. P. Vdovin, D. O. Revin

18.33. A group in which the derived subgroup of every 2-generated subgroup is cyclic is called an *Alperin group*. Is there a bound for the derived length of finite Alperin groups?

G. Higman proved that finite Alperin groups are soluble, and finite Alperin p-groups have bounded derived length, see 17.46.

B. M. Veretennikov

18.34. Let G be a topological group and $S = G_0 \leqslant \ldots \leqslant G_n = G$ a subnormal series of closed subgroups. The *infinite-length* of S is the cardinality of the set of infinite factors G_i/G_{i-1}. The *virtual length* of G is the supremum of the infinite lengths taken over all such series of G. It is known that if G is a pronilpotent group with finite virtual length then every closed subnormal subgroup is topologically finitely generated (N. Gavioli, V. Monti, C. M. Scoppola, *to appear in J. Austral. Math. Soc.*, `doi:10.1017/S144678871300027X`). Let G be a pro-p group (or more generally a pronilpotent group). If every closed subnormal subgroup of G is tolopologically finitely generated, is it true that G has finite virtual length?

N. Gavioli, V. Monti, C. M. Scoppola

18.35. A family $\mathscr{F}$ of group homomorphisms $A \to B$ is *separating* if for every nontrivial $a \in A$ there is $f \in \mathscr{F}$ such that $f(a) \neq 1$, and *discriminating* if for any finitely many nontrivial elements $a_1, \ldots, a_n \in A$ there is $f \in \mathscr{F}$ such that $f(a_i) \neq 1$ for all $i = 1, \ldots, n$. Let G be a relatively free group of rank 2 in the variety of metabelian groups. Let T be a metabelian group in which the centralizer of every nontrivial element is abelian. If T admits a separating family of surjective homomorphisms $T \to G$ must it also admit a discriminating family of surjective homomorphisms $T \to G$?

A positive answer would give a metabelian analogue of a classical theorem of B. Baumslag.

A. Gaglione, D. Spellman

18.36. There are groups of cardinality at most $2^{\aleph_0}$, even nilpotent of class 2, that cannot be embedded in $S := Sym(\mathbb{N})$ (V. A. Churkin, *Algebra and model theory* (Novosibirsk State Tech. Univ.), **5** (2005), 39–43 (Russian)). One condition which might characterize subgroups of S is the following. Consider the metric d on S where for all $x \neq y$ we define $d(x, y) := 2^{-k}$ when k is the least element of the set $\mathbb{N}$ such that $k^x \neq k^y$. Then (S, d) is a separable topological group, and so each subgroup of S (not necessarily a closed subgroup of (S, d)) is a separable topological group under the induced metric. Is it true that every group G for which there is a metric d' such that (G, d') is a separable topological group is (abstractly) embeddable in S?

Note that any such group G can be embedded as a *section* in S. *J. D. Dixon*

18.37. (Well-known problem). Is every locally graded group of finite rank almost locally soluble? *M. Dixon*

18.38. Let E_π denote the class of finite groups that contain a Hall π-subgroup. Does the inclusion $E_{\pi_1} \cap E_{\pi_2} \subseteq E_{\pi_1 \cap \pi_2}$ hold for arbitrary sets of primes π_1 and π_2?

A. V. Zavarnitsine

18.39. *Conjecture*: Let G be a hyperbolic group. Then every 2-dimensional rational homology class is virtually represented by a sum of closed surface subgroups, that is, for any $\alpha \in H_2(G; \mathbb{Q})$ there are finitely many closed oriented surfaces S_i and injective homomorphisms $\rho_i : \pi_1(S_i) \to G$ such that $\sum_i [S_i] = n\alpha$, where $[S_i]$ denotes the image of the fundamental class of S_i in $H_2(G)$. *D. Calegari*

18.40. The commutator length $cl(g)$ of $g \in [G, G]$ is the least number of commutators in G whose product is g, and the stable commutator length is $scl(g) := \lim_{n\to\infty} cl(g^n)/n$.

Conjecture: Let G be a hyperbolic group. Then the stable commutator length takes on rational values on $[G, G]$. *D. Calegari*

18.41. Let F be a free group of rank 2.

(a) It is known that $scl(g) = 0$ only for $g = 1$, and $scl(g) \geqslant 1/2$ for all $1 \neq g \in [F, F]$. Is 1/2 an isolated value?

(b) Are there any intervals J in $\mathbb{R}$ such that the set of values of scl on $[F, F]$ is dense in J?

(c) Is there some $T \in \mathbb{R}$ such that every rational number $\geqslant T$ is a value of $scl(g)$ for $g \in [F, F]$? *D. Calegari*

18.42. For an orientation-preserving homeomorphism $f : S^1 \to S^1$ of a unit circle S^1, let $\tilde{f}$ be its lifting to a homeomorphism of $\mathbb{R}$; then the *rotation number* of f is defined to be the limit $\lim_{n\to\infty}(\tilde{f}^n(x) - x)/n$ (which is independent of the point $x \in S^1$). Let F be a free group of rank 2 with generators a, b. For $w \in F$ and $r, s \in \mathbb{R}$, let $R(w, r, s)$ denote the maximum value of the (real-valued) rotation number of w, under all representations from F to the universal central extension of $Homeo^+(S^1)$ for which the rotation number of a is r, and the rotation number of b is s.

(a) If r, s are rational, must $R(w, r, s)$ be rational?

(b) Let $R(w, r-, s-)$ denote the supremum of the rotation number of w (as above) under all representations for which a and b are conjugate to rotations through r and s, respectively (in the universal central extension of $Homeo^+(S^1)$). Is $R(w, r-, s-)$ always rational if r and s are rational?

(c) *Weak Slippery Conjecture*: Let w be a word containing only positive powers of a and b. A pair of values (r, s) is *slippery* if there is a strict inequality $R(w, r', s') < R(w, r-, s-)$ for all $r' < r$, $s' < s$. Is it true that then $R(w, r-, s-) = h_a(w)r + h_b(w)s$, where $h_a(w)$ counts the number of a's in w, and $h_b(w)$ counts the number of b's in w?

(d) *Slippery Conjecture*: More precisely, is it always (without assuming (r, s) being slippery) true that if w has the form $w = a^{\alpha_1}b^{\beta_1}\cdots a^{\alpha_m}b^{\beta_m}$ (all positive) and $R(w, r, s) = p/q$ where p/q is reduced, then $|p/q - h_a(w)r - h_b(w)s| \leqslant m/q$?

D. Calegari, A. Walker

18.43. (a) Do there exist elements u, v of a 2-generator free group $F(a, b)$ such that u, v are not conjugate in $F(a, b)$ but for any matrices $A, B \in GL(3, \mathbb{C})$ we have $trace(u(A, B)) = trace(v(A, B))$?

(b) The same question if we replace $GL(3, \mathbb{C})$ by $SL(3, \mathbb{C})$. *I. Kapovich*

18.44. Let G be a finite non-abelian group and V a finite faithful irreducible G-module. Suppose that $M = |G/G'|$ is the largest orbit size of G on V, and among orbits of G on V there are exactly two orbits of size M. Does this imply that G is dihedral of order 8, and $|V| = 9$? *T. M. Keller*

18.45. Let G be a finite p-group of maximal class such that all its irreducible characters are induced from linear characters of normal subgroups. Let S be the set of the derived subgroups of the members of the central series of G. Is there a logarithmic bound for the derived length of G in terms of $|S|$? *T. M. Keller*

18.46. Every finite group G can be embedded in a group H in such a way that every element of G is a square of an element of H. The overgroup H can be chosen such that $|H| \leqslant 2|G|^2$. Is this estimate sharp?

It is known that the best possible estimate cannot be better than $|H| \leqslant |G|^2$ (D. V. Baranov, Ant. A. Klyachko, *Siber. Math. J.*, **53**, no. 2 (2012), 201–206).

Ant. A. Klyachko

18.47. Is it algorithmically decidable whether a group generated by three given class transpositions (for the definition, see 17.57)

(a) has only finite orbits on $\mathbb{Z}$?

(b) acts transitively on the set of nonnegative integers in its support?

A difficult case is the group $\langle\tau_{1(2),4(6)}, \tau_{1(3),2(6)}, \tau_{2(3),4(6)}\rangle$, which acts transitively on $\mathbb{N} \setminus 0(6)$ if and only if Collatz' $3n + 1$ conjecture is true. *S. Kohl*

18.48. Is it true that there are only finitely many integers which occur as orders of products of two class transpositions? (For the definition, see 17.57.)

S. Kohl

18.49. Let $n \in \mathbb{N}$. Is it true that for any $a, b, c \in \mathbb{N}$ satisfying $1 < a, b, c \leqslant n - 2$ the symmetric group S_n has elements of order a and b whose product has order c?

S. Kohl

18.50. Let $n \in \mathbb{N}$. Is it true that for every $k \in \{1, \ldots, n!\}$ there is some group G and pairwise distinct elements $g_1, \ldots, g_n \in G$ such that the set $\{g_{\sigma(1)} \cdots g_{\sigma(n)} \mid \sigma \in \mathrm{S}_n\}$ of all products of the g_i obtained by permuting the factors has cardinality k?

S. Kohl

18.51. Given a prime p and $n \in \mathbb{N}$, let $f_p(n)$ be the smallest number such that there is a group of order $p^{f_p(n)}$ into which every group of order p^n embeds. Is it true that $f_p(n)$ grows faster than polynomially but slower than exponentially when n tends to infinity?

S. Kohl

18.52. Is every finite simple group generated by two elements of prime-power orders m, n? (Here numbers m, n may depend on the group.) The work of many authors shows that it remains to verify this property for a small number of finite simple groups.

J. Krempa

18.53. Is there a non-linear simple locally finite group in which every centralizer of a non-trivial element is almost soluble, that is, has a soluble subgroup of finite index?

M. Kuzucuoğlu

18.54. Can a group be equal to the union of conjugates of a proper finite nonabelian simple subgroup?

G. Cutolo

18.55. (a) Can a locally finite p-group G be the union of conjugates of an abelian proper subgroup?

(b) Can this happen when G is of exponent p?

G. Cutolo

18.56. Let G be a finite 2-group, of order greater than 2, such that $|H/H_G| \leqslant 2$ for all $H \leqslant G$, where H_G denotes the largest normal subgroup of G contained in H. Must G have an abelian subgroup of index 4?

G. Cutolo

18.57. Let G be a finite 2-group generated by involutions in which $[x, u, u] = 1$ for every $x \in G$ and every involution $u \in G$. Is the derived length of G bounded?

D. V. Lytkina

18.58. Let G be a group generated by finite number n of involutions in which $(uv)^4 = 1$ for all involutions $u, v \in G$. Is it true that G is finite? is a 2-group? This is true for $n \leqslant 3$. What about $n = 4$?

D. V. Lytkina

18.59. Does there exist a periodic group G such that G contains an involution, all involutions in G are conjugate, and the centralizer of every involution i is isomorphic to $\langle i \rangle \times L_2(P)$, where P is some infinite locally finite field of characteristic 2?

D. V. Lytkina

18.60. Let V be an infinite countable elementary abelian additive 2-group. Does $\operatorname{Aut} V$ contain a subgroup G such that

(a) G is transitive on the set of non-zero elements of V, and

(b) if H is the stabilizer in G of a non-zero element $v \in V$, then $V = \langle v \rangle \oplus V_v$, where V_v is H-invariant, H is isomorphic to the multiplicative group P^* of a locally finite field P of characteristic 2, and the action H on V_v is similar to the action of P^* on P by multiplication?

Conjecture: such a group G does not exist. If so, then the group G in the previous problem does not exist too.

D. V. Lytkina

18.61. Is a 2-group nilpotent if all its finite subgroups are nilpotent of class at most 3? This is true if 3 is replaced by 2.

D. V. Lytkina.

18.62. The *spectrum* of a finite group is the set of orders of its elements. Let ω be a finite set of positive integers. A group G is said to be ω-*critical* if the spectrum of G coincides with ω, but the spectrum of every proper section of G is not equal to ω.

(a) Does there exist a number n such that for every finite simple group G the number of $\omega(G)$-critical groups is less than n?

(b) For every finite simple group G, find all $\omega(G)$-critical groups.

V. D. Mazurov

18.63. Let G be a periodic group generated by two fixed-point-free automorphisms of order 5 of an abelian group. Is G finite?

V. D. Mazurov

18.64. (K. Harada). *Conjecture*: Let G be a finite group, p a prime, and B a p-block of G. If J is a non-empty subset of $Irr(B)$ such that $\sum\limits_{\chi \in J} \chi(1)\chi(g) = 0$ for every p-singular element $g \in G$, then $J = Irr(B)$.

V. D. Mazurov

18.65. (R. Guralnick, G. Malle). *Conjecture*: Let p be a prime different from 5, and C a class of conjugate p-elements in a finite group G. If CC^{-1} consists of p-elements, then $C \subseteq O_p(G)$.

V. D. Mazurov

18.66. Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that GF is also a Frobenius group with kernel G and complement F. Is the derived length of G bounded in terms of $|H|$ and the derived length of $C_G(H)$?

N. Yu. Makarenko, E. I. Khukhro, P. Shumyatsky

18.67. Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. Is the exponent of G bounded in terms of $|F|$ and the exponent of $C_G(H)$?

This was proved when F is cyclic (E. I. Khukhro, N. Y. Makarenko, P. Shumyatsky, *Forum Math.*, 2011; `doi:10.1515/FORM.2011.152`; `arxiv.org/abs/1010.0343`) and for $FH \cong \mathbb{A}_4$ when $(|G|, 3) = 1$ (P. Shumyatsky, *J. Algebra*, **331** (2011), 482–489).

N. Yu. Makarenko, E. I. Khukhro, P. Shumyatsky

18.68. What are the nonabelian composition factors of a finite nonsoluble group all of whose maximal subgroups have complements?

Note that finite simple groups with all maximal subgroups having complements are, up to isomorphism, $L_2(7)$, $L_2(11)$, and $L_5(2)$ (V. M. Levchuk, A. G. Likharev, *Siberian Math. J.*, **47**, no. 4 (2006), 659–668). The same groups exhaust finite simple groups with Hall maximal subgroups and composition factors of groups with Hall maximal subgroups (N. V. Maslova, *Siberian Math. J.*, **53**, no. 5 (2012), 853–861). It is also proved that in a finite group with Hall maximal subgroups all maximal subgroups have complements (N. V. Maslova, D. O. Revin, *Siberian Adv. Math.*, **23**, no. 3 (2013), 196–209).

N. V. Maslova, D. O. Revin

18.69. Does there exist a relatively free group G containing a free subsemigroup and having $[G, G]$ finitely generated?

Note that G cannot be locally graded (*Publ. Math. Debrecen*, **81**, no. 3-4 (2012), 415–420.)

O. Macedońska

18.70. Is every finitely generated Coxeter group conjugacy separable?

Some special cases were considered in (P.-E. Caprace, A. Minasyan, `arxiv.org/abs/1210.4328`).

A. Minasyan

18.71. (N. Aronszajn). Suppose that $W(x, y) = 1$ in an open subset of $G \times G$, where G is a connected topological group. Must $W(x, y) = 1$ for all $(x, y) \in G \times G$?

For locally compact groups the answer is yes.

J. Mycielski

18.72. Is it true that an existentially closed subgroup of a nonabelian free group of finite rank is a nonabelian free factor of this group?

A. G. Myasnikov, V. A. Roman'kov

18.73. Does every finitely generated solvable group of derived length $l \geqslant 2$ embed into a 2-generated solvable group of length $l + 1$? Or at least, into some k-generated $(l + 1)$-solvable group, where $k = k(l)$?

A. Yu. Ol'shanskiĭ

18.74. Let G be a finitely generated elementary amenable group which is not virtually nilpotent. Is there a finitely generated metabelian non-virtually-nilpotent section in G?

A. Yu. Ol'shanskiĭ

18.75. Does every finite solvable group G have the following property: there is a number $d = d(G)$ such that G is a homomorphic image of every group with d generators and one relation?

This property holds for finite nilpotent groups and does not hold for every nonsolvable finite group; see (S. A. Zaĭtsev, *Moscow Univ. Math. Bull.*, **52**, no. 4 (1997), 42–44).

A. Yu. Ol'shanskiĭ

18.76. Let A be a division ring, G a subgroup of the multiplicative group of A, and E an extension of the additive group of A by G such that G acts by multiplication in A. Is it true that E splits? This is true if G is finite.

E. A. Palyutin.

18.77. Let G be a finite p-group and let p^e be the largest degree of an irreducible complex representation of G. If $p > e$, is it necessarily true that $\bigcap \ker \Theta = 1$, where the intersection runs over all irreducible complex representations Θ of G of degree p^e?

D. S. Passman

18.78. Let K^tG be a twisted group algebra of the finite group G over the field K. If K^tG is a simple K-algebra, is G necessarily solvable? This is known to be true if K^tG is central simple.

D. S. Passman

18.79. Let $K[G]$ be the group algebra of the finitely generated group G over the field K. Is the Jacobson radical $\mathscr{J}K[G]$ equal to the join of all nilpotent ideals of the ring? This is known to be true if G is solvable or linear.

D. S. Passman

18.80. (I. Kaplansky). For $G \neq 1$, show that the augmentation ideal of the group algebra $K[G]$ is equal to the Jacobson radical of the ring if and only if $\operatorname{char} K = p > 0$ and G is a locally finite p-group.

D. S. Passman

18.81. Let G be a finitely generated, residually finite p-group. Are all maximal subgroups of G necessarily normal?

D. S. Passman

18.82. Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that for any prime p, if $p^{f(n)}$ divides the order of a finite group G, then p^n divides the order of $\operatorname{Aut} G$?

R. M. Patne

18.83. A generating system X of a group G is *fast* if there is an integer n such that every element of G can be expressed as a product of at most n elements of X or their inverses. If not, we say that it is *slow*. For instance, in $(\mathbb{Z}, +)$, the squares are fast, but the powers of 2 are slow.

Do there exist countable infinite groups without an infinite slow generating set? Uncountable ones do exist.

B. Poizat

18.84. Let π be a set of primes. We say that a finite group is a BS_π-group if every conjugacy class in this group any two elements of which generate a π-subgroup itself generates a π-subgroup. Is every normal subgroup of a BS_π-group a BS_π-group?

In the case $2 \notin \pi$, an affirmative answer follows from (D. O. Revin, *Siberian Math. J.*, **52**, no. 2 (2011), 340–347).

D. O. Revin

18.85. A subset of a group is said to be *rational* if it can be obtained from finite subsets by finitely many rational operations, that is, taking union, product, and the submonoid generated by a set.

Conjecture: every finitely generated solvable group in which all rational subsets form a Boolean algebra is virtually abelian.

This is known to be true if the group is metabelian, or polycyclic, or of finite rank (G. A. Bazhenova).

V. A. Roman'kov

18.86. Is the group $G = \langle a, b \mid [[a, b], b] = 1\rangle$, which is isomorphic to the group of all unitriangular automorphisms of the free group of rank 3, linear?

V. A. Roman'kov

18.87. A system of equations with coefficients in a group G is said to be *independent* if the matrix composed of the sums of exponents of the unknowns has rank equal to the number of equations.

(a) The *Kervaire–Laudenbach Conjecture (KLC)*: every independent system of equations with coefficients in an arbitrary group G has a solution in some overgroup $\bar{G}$. This is true for every locally residually finite group G (M. Gerstenhaber and O. S. Rothaus).

(b) KLC — nilpotent version: every independent system of equations with coefficients in an arbitrary nilpotent group G has a solution in some nilpotent overgroup $\bar{G}$.

(c) KLC — solvable version: every independent system of equations with coefficients in an arbitrary solvable group G has a solution in some solvable overgroup $\bar{G}$.

V. A. Roman'kov

18.88. Can a finitely generated infinite group of finite exponent be the quotient of a residually finite group by a locally finite normal subgroup?

If not, then there exists a hyperbolic group that is not residually finite. *M. Sapir*

18.89. Consider the set of balanced presentations $\langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle$ with fixed n generators $x_1, \dots, x_n$. By definition, the group AC_n of Andrews–Curtis moves on this set of balanced presentations is generated by the Nielsen transformations together with conjugations of relators. Is AC_n finitely presented? *J. Swan, A. Lisitsa*

18.90. Prove that the group $Y(m,n) = \langle a_1, a_2, \dots, a_m \mid a_k^n = 1\ (1 \leqslant k \leqslant m),\ (a_k^i a_l^i)^2 = e\ (1 \leqslant k < l \leqslant m,\ 1 \leqslant i \leqslant \frac{n}{2})\rangle$ is finite for every pair (m,n).

Some known cases are (where C_n denotes a cyclic group of order n): $Y(m,2) = C_2^m$; $Y(m;3) = A_{m+2}$ (presentation by Carmichael); $Y(m;4)$ has order $2^{\frac{m(m+3)}{2}}$, nilpotency class 3 and exponent 4; $Y(2,n)$ is the natural extension of the augmentation ideal $\omega(C_n)$ of $GF(2)[C_n]$ by C_n (presentation by Coxeter); $Y(3,n) \cong SL(2, \omega(C_n))$.

These groups have connections with classical groups in characteristic 2. When n is odd, the group $Y(m,n)$ has the following presentation, where τ_{ij} are transpositions: $y(m,n) = \langle a,\ \mathbb{S}_m \mid a^n = e, [\tau_{12}^{a^i}, \tau_{12}] = e\ \ (1 \leqslant i \leqslant \frac{n}{2}), \tau_{12}^{1+a+\dots+a^{n-1}} = e,\ \tau_{i,i+1} a \tau_{i,i+1} = a^{-1}\ (2 \leqslant i \leqslant m-1)\rangle$, and it is known that, for example, $y(3,5) \cong SL(2,16) \cong \Omega^-(4,4)$, $y(4,5) \cong Sp(2,16) \cong \Omega(5,4)$, $y(5,5) \cong SU(4,16) \cong \Omega^-(6,4)$, $y(6,5) \cong 4^6\Omega^-(6,4)$, $y(3,7) \cong SL(2,8)^2 \cong \Omega^+(4,8)$, $y(4,7) \cong Sp(4,8) \cong \Omega(5,8)$. Extensive computations by Felsch, Neubüser and O'Brien confirm this trend. Different types reveal Bott periodicity and a connection with Clifford algebras. *S. Sidki*

18.91. A subgroup H of a group G is said to be *propermutable* in G if there is a subgroup $B \leqslant G$ such that $G = N_G(H)B$ and H permutes with every subgroup of B.

(a) Is there a finite group G with subgroups $A \leqslant B \leqslant G$ such that A is propermutable in G but A is not propermutable in B?

(b) Let P be a non-abelian Sylow 2-subgroup of a finite group G with $|P| = 2^n$. Suppose that there is an integer k such that $1 < k < n$ and every subgroup of P of order 2^k is propermutable in G, and also, in the case of $k = 1$, every cyclic subgroup of P of order 4 is propermutable in G. Is it true that then G is 2-nilpotent?

A. N. Skiba

18.92. A non-empty set θ of formations is called a *complete lattice* of formations if the intersection of any set of formations in θ belongs to θ and θ has the largest element (with respect to inclusion). If L is a complete lattice, then an element $a \in L$ is said to be compact if $a \leqslant \bigvee X$ for any $X \subseteq L$ implies that $a \leqslant \bigvee X_1$ for some finite $X_1 \subset X$. A complete lattice is called *algebraic* if every element is the join of a (possibly infinite) set of compact elements.

a) Is there a non-algebraic complete lattice of formations of finite groups?

b) Is there a non-modular complete lattice of formations of finite groups?

A. N. Skiba

18.93. Let $\mathscr{M}$ be a one-generated saturated formation, that is, the intersection of all saturated formations containing some fixed finite group. Let $\mathscr{F}$ be a subformation of $\mathscr{M}$ such that $\mathscr{F} \neq \mathscr{F}\mathscr{F}$.

(a) Is it true that then $\mathscr{F}$ can be written in the form $\mathscr{F} = \mathscr{F}_1 \cdots \mathscr{F}_t$, where $\mathscr{F}_i$ is a non-decomposable formation for every $i = 1, \dots, t$?

b) Suppose that $\mathscr{F} = \mathscr{F}_1 \cdots \mathscr{F}_t$, where $\mathscr{F}_i$ is a non-decomposable formation for every $i = 1, \dots, t$. Is it true that then all factors $\mathscr{F}_i$ are uniquely determined?

A. N. Skiba

18.94. Let G be a group without involutions, a an element of it that is not a square of any element of G, and n an odd positive integer. Is it true that the quotient $G/\langle (a^n)^G \rangle$ does not contain involutions?

A. I. Sozutov

18.95. Suppose that a group $G = AB$ is a product of an abelian subgroup A and a locally quaternion group B (that is, B is a union of an increasing chain of finite generalized quaternion groups). Is G soluble?

A. I. Sozutov

18.96. Suppose that a periodic group G contains involutions and the centralizer of each involution is locally finite. Is it true that G has a nontrivial locally finite normal subgroup?

N. M. Suchkov

18.97. Let G be a periodic Zassenhaus group, that is, a two-transitive permutation group with trivial stabilizer of every three points. Suppose that the stabilizer of a point is a Frobenius group with locally finite kernel U containing an involution. Is it true that U is a 2-group? This was proved for finite groups by Feit.

N. M. Suchkov

18.98. The work of many authors shows that most finite simple groups are generated by two elements of orders 2 and 3; for example, see the survey `http://math.nsc.ru/conference/groups2013/slides/MaximVsemirnov_slides.pdf`. Which finite simple groups cannot be generated by two elements of orders 2 and 3?

In particular, is it true that, among classical simple groups of Lie type, such exceptions, apart from $PSU(3, 5^2)$, arise only when the characteristic is 2 or 3?

M. C. Tamburini

18.99. Let $\mathrm{cd}(G)$ be the set of all complex irreducible character degrees of a finite group G. *Huppert's Conjecture:* if H is a nonabelian simple group and G is any finite group such that $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{H})$, then $G \cong H \times A$, where A is an abelian group.

H. P. Tong-Viet

18.100. For a given class of groups $\mathfrak{X}$, let $(\mathfrak{X},\infty)^*$ denote the class of groups in which every infinite subset contains two distinct elements x and y that satisfy $\langle x, x^y\rangle \in \mathfrak{X}$. Let G be a finitely generated soluble-by-finite group in the class $(\mathfrak{X},\infty)^*$, let m be a positive integer, and let $\mathfrak{F}$, $\mathfrak{E}_m$, $\mathfrak{E}$, $\mathfrak{N}$, and $\mathfrak{P}$ denote the classes of finite groups, groups of exponent dividing m, groups of finite exponent, nilpotent groups, and polycyclic groups, respectively.

a) If $\mathfrak{X} = \mathfrak{E}\mathfrak{N}$, then is G in $\mathfrak{E}\mathfrak{N}$?

b) If $\mathfrak{X} = \mathfrak{E}_m\mathfrak{N}$, then is G in $\mathfrak{E}_m(\mathfrak{F}\mathfrak{N})$?

c) If $\mathfrak{X} = \mathfrak{N}(\mathfrak{P}\mathfrak{F})$, then is G in $\mathfrak{N}(\mathfrak{P}\mathfrak{F})$?

N. Trabelsi

18.101. An element g of a group G is said to be left n-Engel if $[...[[x,\overbrace{g],g],\dots,g}^{n}] = 1$ for all $x \in G$. Is every left 3-Engel element contained in the locally nilpotent radical?

G. Traustason

18.102. (E. C. Dade). Let C be a Carter subgroup of a finite solvable group G, and let $\ell(C)$ be the number of primes dividing $|C|$ counting multiplicities. It was proved in (E. C. Dade, *Illinois J. Math.*, **13** (1969), 449–514) that there is an exponential function f such that the nilpotent length of G is at most $f(\ell(C))$. Is there a linear (or at least a polynomial) function f with this property?

A. Turull

18.103. A group is said to be *minimax* if it has a finite subnormal series each of whose factors satisfies either the minimum or the maximum condition on subgroups. Is it true that in the class of nilpotent minimax groups only finitely generated groups may have faithful irreducible primitive representations over a finitely generated field of characteristic zero?

A. V. Tushev

18.104. Construct an example of a pro-p group G and a proper abstract normal subgroup K such that G/K is perfect.

J. Wilson

18.105. Let R be the formal power series algebra in commuting indeterminates $x_1, \dots x_n, \dots$ over the field with p elements, with p a prime. (Thus its elements are (in general infinite) linear combinations of monomials $x_1^{a_1}\dots x_n^{a_n}$ with $n, a_1, \dots, a_n$ non-negative integers.) Let I be the maximal ideal of R and J the (abstract) ideal generated by all products of two elements of I. Is there an ideal U of R such that $U < I$ and $U + J = I$?

If so, then $\mathrm{SL}_3(R)$ and the kernel of the map to $\mathrm{SL}_3(R/U)$ provide an answer to the previous question.

J. Wilson

18.106. Let R be a group that acts coprimely on the finite group G. Let p be a prime, let P be the unique maximal $RC_G(R)$-invariant p-subgroup of G and assume that $C_G(O_p(G)) \leqslant O_p(G)$. If $p > 3$ then P contains a nontrivial characteristic subgroup that is normal in G (P. Flavell, *J. Algebra*, **257** (2002), 249–264). Does the same result hold for the primes 2 and 3?

P. Flavell

18.107. Is there a finitely generated infinite residually finite p-group such that every subgroup of infinite index is cyclic?

The answer is known to be negative for $p = 2$. Note that in (M. Ershov, A. Jaikin-Zapirain, Groups of positive weighted deficiency and their applications, *to appear in J. Reine Angew. Math.*) it is shown that for every prime p there is a finitely generated infinite residually finite p-group such that every finitely generated subgroup of infinite index is finite.

A. Jaikin-Zapirain

18.108. A group G is said to have *property* (τ) if its trivial representation is an isolated point in the subspace of irreducible representations with finite images (in the topological unitary dual space of G). Is it true that there exists a finitely generated group satisfying property (τ) that homomorphically maps onto every finite simple group?

The motivation is the theorem in (E. Breulliard, B. Green, M. Kassabov, A. Lubotzky, N. Nikolov, T. Tao) saying that there are $\epsilon > 0$ and k such that every finite simple group G contains a generating set S with at most k elements such that the Cayley graph of G with respect to S is an ϵ-expander. In (M. Ershov, A. Jaikin-Zapirain, M. Kassabov, `arxiv.org/pdf/1102.0031`) it is proved that there exists a group with Kazhdan's property (T) that maps onto every simple group of Lie type of rank $\geqslant 2$. (A group G is said to have *Kazhdan's property* (T) if its trivial representation is an isolated point in the unitary dual space of G.)

A. Jaikin-Zapirain

18.109. Is it true that a group satisfying Kazhdan's property (T) cannot homomorphically map onto infinitely many simple groups of Lie type of rank 1?

It is known that a group which maps onto $PSL_2(q)$ for infinitely many q does not have Kazhdan's property (T).

A. Jaikin-Zapirain

18.110. The *non-p-soluble length* of a finite group G is the number of non-p-soluble factors in a shortest normal series each of whose factors either is p-soluble or is a direct product of non-abelian simple groups of order divisible by p. For a given prime p and a given proper group variety $\mathfrak{V}$, is there a bound for the non-p-soluble length of finite groups whose Sylow p-subgroups belong to $\mathfrak{V}$?

E. I. Khukhro, P. Shumyatsky

18.111. Let G be a discrete countable group, given as a central extension $0 \to \mathbb{Z} \to G \to Q \to 0$. Assume that either G is quasi-isometric to $\mathbb{Z} \times Q$, or that $\mathbb{Z} \to G$ is a quasi-isometric embedding. Does that imply that G comes from a bounded cocycle on Q?

I. Chatterji, G. Mislin

18.112. Is it true that the orders of all elements of a finite group G are powers of primes if, for every divisor d ($d > 1$) of $|G|$ and for every subgroup H of G of order coprime to d, the order $|H|$ divides the number of elements of G of order d?

The converse is true (W. J. Shi, *Math. Forum (Vladikavkaz)*, **6** (2012), 152–154).

W. J. Shi

18.113. Let $\mathfrak{M}$ be a finite set of finite simple groups. Is it true that a periodic group saturated with groups from $\mathfrak{M}$ (see 14.101) is isomorphic to one of groups in $\mathfrak{M}$? The case where $\mathfrak{M}$ is one-element is of special interest.

A. K. Shlëpkin

18.114. Does there exist an irreducible $5'$-subgroup G of $GL(V)$ for some finite $\mathbb{F}_5$-space V such that the number of conjugacy classes of the semidirect product VG is equal to $|V|$ but G is not cyclic?

Such a subgroup exists if 5 is replaced by $p = 2, 3$ but does not exist for primes $p > 5$ (*J. Group Theory*, **14** (2011), 175–199).

P. Schmid

18.115. Let G be a finite simple group, and let X, Y be isomorphic simple maximal subgroups of G. Are X and Y conjugate in $\operatorname{Aut} G$?

P. Schmid

18.116. Given a finite 2-group G of order 2^n, does there exist a finite set S of rational primes such that $|S| \leqslant n$ and G is a quotient group of the absolute Galois group of the maximal 2-extension of $\mathbb{Q}$ unramified outside $S \cup \{\infty\}$?

This is true for p-groups, p odd, as noted by Serre.

P. Schmid

18.117. Is every element of a nonabelian finite simple group a commutator of two elements of coprime orders?

The answer is known to be "yes" for the alternating groups (P. Shumyatsky, *Forum Math.*, `doi:10.1515/forum-2012-0127`) and for $PSL(2, q)$ (M. A. Pellegrini, P. Shumyatsky, *Arch. Math.*, **99** (2012), 501–507). Without the coprimeness condition, this is Ore's conjecture proved in (M. W. Liebeck, E. A. O'Brien, A. Shalev, Pham Huu Tiep, *J. Eur. Math. Soc.*, **12**, no. 4 (2010), 939–1008).

P. Shumyatsky

18.118. Let G be a profinite group such that $G/Z(G)$ is periodic. Is $[G, G]$ necessarily periodic?

P. Shumyatsky

18.119. A multilinear commutator word is any commutator of weight n in n distinct variables. Let w be a multilinear commutator word and let G be a finite group. Is it true that every Sylow p-subgroup of the verbal subgroup $w(G)$ is generated by w-values?

P. Shumyatsky

18.120. Let $P = AB$ be a finite p-group factorized by an abelian subgroup A and a class-two subgroup B. Suppose, if necessary, $A \cap B = 1$. Then is it true that $\langle A, [B, B] \rangle = AB_0$, where B_0 is an abelian subgroup of B?

E. Jabara

Archive of solved problems

This section contains the solved problems that had already been commented on in one of the previous issues with a reference to a detailed publication containing a complete answer. Those problems that are commented on with a complete reference for the first time in this issue remain in the main part of the "Kourovka Notebook", among the unsolved problems of the corresponding section.

1.1. Do there exist non-trivial finitely-generated divisible groups? Equivalently, do there exist non-trivial finitely-generated divisible simple groups? *Yu. A. Bogan*

Yes, there exist (V. S. Guba, *Math. USSR–Izv.*, **29** (1987), 233–277).

1.2. Let G be a group, F a free group with free generators $x_1, \ldots, x_n$, and R the free product of G and F. An *equation* (*in the unknowns* $x_1, \ldots, x_n$) *over* G is an expression of the form $v(x_1, \ldots x_n) = 1$, where on the left is an element of R not conjugate in R to any element of G. We call G *algebraically closed* if every equation over G has a solution in G. Do there exist algebraically closed groups? *L. A. Bokut'*

Yes, there exist (S. D. Brodskiĭ, Dep. no. 2214-80, VINITI, Moscow, 1980 (Russian)).

1.4. (A. I. Mal'cev). Does there exist a ring without zero divisors which is not embeddable in a skew field, while the multiplicative semigroup of its non-zero elements is embeddable in a group? *L. A. Bokut'*

Yes, there exists (L. A. Bokut', *Soviet Math. Dokl.*, **8** (1967), 901–904; A. Bowtell, *J. Algebra*, **7** (1967), 126–139; A. Klein, *J. Algebra*, **7** (1967), 100–125).

1.9. Can the factor-group of a locally normal group by the second term of its upper central series be embedded (isomorphically) in a direct product of finite groups? *Yu. M. Gorchakov*

Yes, it can (Yu. M. Gorchakov, *Algebra and Logic*, **15** (1976), 386–390).

1.10. An automorphism φ of a group G is called *splitting* if $gg^{\varphi} \cdots g^{\varphi^{n-1}} = 1$ for any element $g \in G$, where n is the order of φ. Is a soluble group admitting a regular splitting automorphism of prime order necessarily nilpotent? *Yu. M. Gorchakov*

Yes, it is (E. I. Khukhro, *Algebra and Logic*, **17** (1978), 402–406) and even without the regularity condition on the automorphism (E. I. Khukhro, *Algebra and Logic*, **19** (1980), 77–84).

1.11. (E. Artin). The conjugacy problem for the braid groups $\mathscr{B}_n$, $n > 4$. *M. D. Greendlinger*

The solution is affirmative (G. S. Makanin, *Soviet Math. Dokl.*, **9** (1968), 1156–1157; F. A. Garside, *Quart. J. Math.*, **20** (1969), 235–254).

1.13. (J. Stallings). If a finitely presented group is trivial, is it always possible to replace one of the defining words by a primitive element without changing the group? *M. D. Greendlinger*

No, not always (S. V. Ivanov, *Invent. Math.*, **165**, no. 3 (2006), 525–549).

1.14. (B. H. Neumann). Does there exist an infinite simple finitely presented group?
M. D. Greendlinger

Yes, there exists (G. Higman, *Finitely presented infinite simple groups* (*Notes on Pure Mathematics, no.* 8) Dept. of Math., Inst. Adv. Studies Austral. National Univ., Canberra, 1974), where there is also a reference to R. Thompson.

1.17. Write an explicit set of generators and defining relations for a universal finitely presented group. *M. D. Greendlinger*

This was done (M. K. Valiev, *Soviet Math. Dokl.*, **14** (1973), 987–991).

1.18. (A. Tarski). a) Does there exist an algorithm for determining the solubility of equations in a free group?

b) Describe the structure of all solutions of an equation when it has at least one solution. *Yu. L. Ershov*

a) Yes, there exists (G. S. Makanin, *Math. USSR–Izv.*, **21** (1982), 546–582; **25** (1985), 75–88).
b) This was described (A. A. Razborov, *Math. USSR–Izv.*, **25** (1984), 115–162).

1.21. Are there only finitely many finite simple groups of a given exponent n?
M. I. Kargapolov

Yes, modulo CFSG, see, for example, (G. A. Jones, *J. Austral. Math. Soc.*, **17** (1974), 162–173).

1.23. Does there exist an infinite simple locally finite group of finite rank?
M. I. Kargapolov

No (V. P. Shunkov, *Algebra and Logic*, **10** (1971), 127–142).

1.24. Does every infinite group possess an infinite abelian subgroup?
M. I. Kargapolov

No (P. S. Novikov, S. I. Adian, *Math. USSR–Izv.*, **2** (1968), 1131–1144).

1.25. a) Is the universal theory of the class of finite groups decidable?

b) Is the universal theory of the class of finite nilpotent groups decidable?
M. I. Kargapolov

a) No (A. M. Slobodskoi, *Algebra and Logic*, **20** (1981), 139–156).
b) No (O. G. Kharlampovich, *Math. Notes*, **33** (1983), 254–263).

1.26. Does elementary equivalence of two finitely generated nilpotent groups imply that they are isomorphic? *M. I. Kargapolov*

No (B. I. Zil'ber, *Algebra and Logic*, **10** (1971), 192–197).

1.29. (A. Tarski). Is the elementary theory of a free group decidable?
M. I. Kargapolov

Yes, it is (O. Kharlampovich, A. Myasnikov, *J. Algebra*, **302**, no. 2 (2006), 451–552).

1.30. Is the universal theory of the class of soluble groups decidable?
M. I. Kargapolov

No (O. G. Kharlampovich, *Math. USSR–Izv.*, **19** (1982), 151–169).

1.32. Is the Frattini subgroup of a finitely generated matrix group over a field nilpotent? *M. I. Kargapolov*

Yes, it is (V. P. Platonov, *Soviet Math. Dokl.*, **7** (1966), 1557–1560).

1.34. Has every orderable polycyclic group a faithful representation by matrices over the integers? *M. I. Kargapolov*

Yes, it has (L. Auslander, *Ann. Math. (2)*, **86** (1967), 112–117; R. G. Swan, *Proc. Amer. Math. Soc.*, **18** (1967), 573–574).

1.35. A group is called *pro-orderable* if every partial ordering of the group extends to a linear ordering.

a) Is the wreath product of two arbitrary pro-orderable groups again pro-orderable?

b) (A. I. Mal'cev). Is every subgroup of a pro-orderable group again pro-orderable? *M. I. Kargapolov*

Not always, in both cases (V. M. Kopytov, *Algebra i Logika*, **5**, no. 6 (1966), 27–31 (Russian)).

1.36. If a group G is factorizable by p-subgroups, that is, $G = AB$, where A and B are p-subgroups, does it follows that G is itself a p-group? *Sh. S. Kemkhadze*

Not always (Ya. P. Sysak, *Products of infinite groups*, Math. Inst. Akad. Nauk Ukrain. SSR, Kiev, 1982 (Russian)).

1.37. Is it true that every subgroup of a locally nilpotent group is quasi-invariant? *Sh. S. Kemkhadze*

No, not always; for example, in the inductive limit of Sylow p-subgroups of symmetric groups of degrees p^i, $i = 1, 2, \ldots$. (V. I. Sushchanskiĭ, *Abstracts of 15th All-USSR Algebraic Conf.*, Krasnoyarsk, 1979, 154 (Russian)).

1.38. An N^0-*group* is a group in which every cyclic subgroup is a term of some normal system of the group. Is every N^0-group an $\widetilde{N}$-group? *Sh. S. Kemkhadze*

No. Every group G with a central system $\{H_i\}$ is an N^0-group, since for any $g \in G$ the system $\{\langle g, H_i\rangle\}$ refined by the trivial subgroup and the intersections of all the subsystems is a normal system of G containing $\langle g\rangle$. Free groups have central systems but are not $\widetilde{N}$-groups. (Yu. I. Merzlyakov, 1973.)

1.39. Is a group binary nilpotent if it is the product of two normal binary nilpotent subgroups? *Sh. S. Kemkhadze*

No, not always (A. I. Sozutov, *Algebra and Logic*, **30**, no. 1 (1991), 70–72).

1.41. A subgroup of a linearly orderable group is called *relatively convex* if it is convex with respect to some linear ordering of the group. Under what conditions is a subgroup of an orderable group relatively convex? *A. I. Kokorin*

Necessary and sufficient conditions are given in (A.I. Kokorin, V. M. Kopytov, *Fully ordered groups*, John Wiley, New York, 1974).

1.42. Is the centre of a relatively convex subgroup relatively convex? *A. I. Kokorin*

Not always (A. I. Kokorin, V. M. Kopytov, *Fully ordered groups*, John Wiley, New York, 1974).

1.43. Is the centralizer of a relatively convex subgroup relatively convex? *A. I. Kokorin*

Not always (A. I. Kokorin, V. M. Kopytov, *Siberian Math. J.*, **9** (1968), 622–626).

1.44. Is a maximal abelian normal subgroup relatively convex? *A. I. Kokorin*

Not always (D. M. Smirnov, *Algebra i Logika*, **5**, no. 6 (1966), 41–60 (Russian)).

1.45. Is the largest locally nilpotent normal subgroup relatively convex? *A. I. Kokorin*

Not always (D. M. Smirnov, *Algebra i Logika*, **5**, no. 6 (1966), 41–60 (Russian)).

1.47. A subgroup H of a group G is said to be *strictly isolated* if, whenever $xg_1^{-1}xg_1 \cdots g_n^{-1}xg_n$ belongs to H, so do x and each $g_i^{-1}xg_i$. A group in which the identity subgroup is strictly isolated is called an *S-group.* Do there exist S-groups that are not orderable groups? *A. I. Kokorin*

Yes, there exist (V. V. Bludov, *Algebra and Logic*, **13** (1974), 343–360).

1.48. Is a free product of two orderable groups with an amalgamated subgroup that is relatively convex in each of the factors again an orderable group? *A. I. Kokorin*

Not always (M. I. Kargapolov, A. I. Kokorin, V. M. Kopytov, *Algebra i Logika*, **4**, no. 6 (1965), 21–27 (Russian)).

1.49. An automorphism φ of a linearly ordered group is called *order preserving* if $x < y$ implies $x^\varphi < y^\varphi$. Is it possible to order an abelian strictly isolated normal subgroup of an S-group (see 1.47) in such a way that the order is preserved under the action of the inner automorphisms of the whole group? *A. I. Kokorin*

Not always (V. V. Bludov, *Algebra and Logic*, **11** (1972), 341–349).

1.50. Do the order-preserving automorphisms of a linearly ordered group form an orderable group? *A. I. Kokorin*

Not always (D. M. Smirnov, *Algebra i Logika*, **5**, no. 6 (1966), 41–60 (Russian)).

1.52. Describe the groups that can be ordered linearly in a unique way (reversed orderings are not regarded as different). *A. I. Kokorin*

This was done (V. V. Bludov, *Algebra and Logic*, **13** (1974), 343–360).

1.53. Describe all possible linear orderings of a free nilpotent group with finitely many generators. *A. I. Kokorin*

This was done (V. F. Kleimenov, in: *Algebra, Logic and Applications*, Irkutsk, 1994, 22–27 (Russian)).

1.56. Is a torsion-free group pro-orderable (see 1.35) if the factor-group by its centre is pro-orderable? *A. I. Kokorin*

Not always (M. I. Kargapolov, A. I. Kokorin, V. M. Kopytov, *Algebra i Logika*, **4**, no. 6 (1965), 21–27 (Russian)).

1.60. Can an orderable metabelian group be embedded in a radicable orderable group? *A. I. Kokorin*

Yes, it can (V. V. Bludov, N. Ya. Medvedev, *Algebra and Logic*, **13** (1974), 207–209).

1.61. Can any orderable group be embedded into an orderable group
a) with a radicable maximal locally nilpotent normal subgroup?
b) with a radicable maximal abelian normal subgroup? *A. I. Kokorin*

Yes, it can, in both cases (S. A. Gurchenkov, *Math. Notes*, **51**, no. 2 (1992), 129–132).

1.63. A group G is called *dense* if it has no proper isolated subgroups other than its trivial subgroup.

a) Do there exist dense torsion-free groups that are not locally cyclic?

b) Suppose that any two non-trivial elements x and y of a torsion-free group G satisfy the relation $x^k = y^l$, where k and l are non-zero integers depending on x and y. Does it follow that G is abelian? *P. G. Kontorovich*

a) Yes, there exist; b) Not necessarily (S. I. Adian, *Math. USSR–Izv.*, **5** (1971), 475–484).

1.64. A torsion-free group is said to be *separable* if it can be represented as the set-theoretic union of two of its proper subsemigroups. Is every R-group separable? *P. G. Kontorovich*

No (S. J. Pride, J. Wiegold, *Bull. London Math. Soc.*, **9** (1977), 36–37).

1.66. Suppose that T is a periodic abelian group, and $\mathfrak{m}$ an uncountable cardinal number. Does there always exist an abelian torsion-free group $U(T, \mathfrak{m})$ of cardinality $\mathfrak{m}$ with the following property: for any abelian torsion-free group A of cardinality $\leqslant \mathfrak{m}$, the equality $\operatorname{Ext}(A, T) = 0$ holds if and only if A is embeddable in $U(T, \mathfrak{m})$? *L. Ya. Kulikov*

No, not always. There is a model of ZFC in which for a certain class of cardinals $\mathfrak{m}$ the answer is negative (S. Shelah, L. Strüngmann, *J. London Math. Soc.*, **67**, no. 3 (2003), 626–642). On the other hand, under the assumption of Gödel's constructivity hypothesis ($V = L$) the answer is affirmative for any cardinal if T has only finitely many non-trivial bounded p-components (L. Strüngmann, *Ill. J. Math.*, **46**, no. 2 (2002), 477–490).

1.68. (A. Tarski). Let $\mathfrak{K}$ be a class of groups and $Q\mathfrak{K}$ the class of all homomorphic images of groups from $\mathfrak{K}$. If $\mathfrak{K}$ is axiomatizable, does it follow that $Q\mathfrak{K}$ is? *Yu. I. Merzlyakov*

Not always (F. Clare, *Algebra Universalis*, **5** (1975), 120–124).

1.69. (B. I. Plotkin). Do there exist locally nilpotent torsion-free groups without the property RN^*? *Yu. I. Merzlyakov*

Yes, there exist (E. M. Levich, A. I. Tokarenko, *Siberian Math. J.*, **11** (1970), 1033–1034; A. I. Tokarenko, *Trudy Rizhsk. Algebr. Sem.*, Riga Univ., Riga, **1969**, 280–281 (Russian)).

1.70. Let p be a prime number and let G be the group of all matrices of the form $\begin{pmatrix} 1+p\alpha & p\beta \\ p\gamma & 1+p\delta \end{pmatrix}$, where α, β, γ, δ are rational numbers with denominators coprime to p. Does G have the property $\overline{RN}$? *Yu. I. Merzlyakov*

Yes, it does (G. A. Noskov, *Siberian Math. J.*, **14** (1973), 475–477).

1.71. Let G be a connected algebraic group over an algebraically closed field. Is the number of conjugacy classes of maximal soluble subgroups of G finite? *V. P. Platonov*

Yes, it is (V. P. Platonov, *Siberian Math. J.*, **10** (1969), 800–804).

1.72. D. Hertzig has shown that a connected algebraic group over an algebraically closed field is soluble if it has a rational regular automorphism. Is this result true for an arbitrary field? *V. P. Platonov*

No (V. P. Platonov, *Soviet Math. Dokl.*, **7** (1966), 825–829).

1.73. Are there only finitely many conjugacy classes of maximal periodic subgroups in a finitely generated linear group over the integers? *V. P. Platonov*

Not always. The extension H of the free group on free generators a, b by the automorphism $\varphi : a \to a^{-1},\ \ b \to b$ is a linear group over the integers. For every $n \in \mathbb{Z}$ the element $c_n = \varphi b^{-n} a b^n$ has order 2 and $C_H(c_n) = \langle c_n \rangle$. Let the dash denote an isomorphism of H onto its copy H'. As a counterexample one can take the subgroup $G \leqslant H \times H'$ generated by the elements a, φ, $a'\varphi'$, bb'. Indeed, G contains the subgroups $T_n = \langle c_0, c_n' \rangle,\ \ n \in \mathbb{Z}$, which are maximal periodic (even in $H \times H'$). If T_n and T_m are conjugate by an element $xy' \in G$ (where $x \in H$ and $y' \in H'$), then $c_0^x = c_0$ and $c_n^y = c_m$, whence $x \in \langle c_0 \rangle$, $y \in b^{m-n} \langle c_m \rangle$. Since $xy' \in G$, the sums of the exponents at the occurrences of b in x and y (in any expression) must coincide; hence $m = n$. (Yu. I. Merzlyakov, 1973.)

1.75. Classify the infinite simple periodic linear groups over a field of characteristic $p > 0$. *V. P. Platonov*

They are classified for every p mod CFSG (V. V. Belyaev, in: *Investigations in Group Theory*, Sverdlovsk, UNC AN SSSR, 1984, 39–50 (Russian); A. V. Borovik, *Siberian Math. J.*, **24**, no. 6 (1983), 843–851; B. Hartley, G. Shute, *Quart. J. Math. Oxford (2)*, **35** (1984), 49–71; S. Thomas, *Arch. Math.*, **41** (1983), 103–116). For $p > 2$ there is a proof without using the CFSG (A. V. Borovik, *Siberian Math. J.*, **25**, no. 2 (1984), 221–235). For all p, without using CFSG, the classification also follows from the paper (M. J. Larsen, R. Pink, *J. Amer. Math. Soc.*, **24** (2011), 1105–1158) also known as a preprint of 1998.

1.76. Does there exist a simple, locally nilpotent, locally compact, topological group? *V. P. Platonov*

No (I. V. Protasov, *Soviet Math. Dokl.*, **19** (1978), 487–489).

1.78. Let a group G be the product of two divisible abelian p-groups of finite rank. Is then G itself a divisible abelian p-group of finite rank? *N. F. Sesekin*

Yet, it is (N. F. Sesekin, *Siberian Math. J.*, **9** (1968), 1070–1072).

1.80. Does there exist a finite simple group whose Sylow 2-subgroup is a direct product of quaternion groups? *A. I. Starostin*

No (G. Glauberman, *J. Algebra*, **4** (1966), 403–420).

1.81. The *width* of a group G is, by definition, the smallest cardinal $m = m(G)$ with the property that any subgroup of G generated by a finite set $S \subseteq G$ is generated by a subset of S of cardinality at most m.

a) Does a group of finite width satisfy the minimum condition for subgroups?

b) Does a group with the minimum condition for subgroups have finite width?

c) The same questions under the additional condition of local finiteness. In particular, is a locally finite group of finite width a Chernikov group? *L. N. Shevrin*

a) Not always (S. V. Ivanov, *Geometric methods in the study of groups with given subgroup properties*, Cand. Diss., Moscow Univ., Moscow, 1988 (Russian)).
b) Not always (G. S. Deryabina, *Math. USSR–Sb.*, **52** (1985), 481–490).
c) Yes, it is (V. P. Shunkov, *Algebra and Logic*, **9** (1970), 137–151; **10** (1971), 127–142).

1.82. Two sets of identities are said to be *equivalent* if they determine the same variety of groups. Construct an infinite set of identities which is not equivalent to any finite one. *A. L. Shmel'kin*

This was done (S. I. Adian, *Math. USSR–Izv.*, **4** (1970), 721–739).

1.83. Does there exist a simple group, the orders of whose elements are unbounded, in which a non-trivial identity relation holds? *A. L. Shmel'kin*

Yes, there exists (V. S. Atabekyan, Dep. no. 5381-V86, VINITI, Moscow, 1986 (Russian)).

1.84. Is it true that a polycyclic group G is residually a finite p-group if and only if G has a nilpotent normal torsion-free subgroup of p-power index? *A. L. Shmel'kin*

No, it is not (K. Seksenbaev, *Algebra i Logika*, **4**, no. 3 (1965), 79–83 (Russian)).

1.85. Is it true that the identity relations of a metabelian group have a finite basis? *A. L. Shmel'kin*

Yes, it is (D. E. Cohen, *J. Algebra*, **5** (1967), 267–273).

1.88. Is it true that if a matrix group over a field of characteristic 0 does not satisfy any non-trivial identity relation, then it contains a non-abelian free subgroup? *A. L. Shmel'kin*

Yes, it is (J. Tits, *J. Algebra*, **20** (1972), 250–270).

1.89. Is the following assertion true? Let G be a free soluble group, and a and b elements of G whose normal closures coincide. Then there is an element $x \in G$ such that $b^{\pm 1} = x^{-1}ax$. *A. L. Shmel'kin*

No (A. L. Shmel'kin, *Algebra i Logika*, **6**, no. 2 (1967), 95–109 (Russian)).

1.90. A subgroup H of a group G is called 2-*infinitely isolated* in G if, whenever the centralizer $C_G(h)$ in G of some element $h \neq 1$ of H contains at least one involution and intersects H in an infinite subgroup, it follows that $C_G(h) \leqslant H$. Let G be an infinite simple locally finite group whose Sylow 2-subgroups are Chernikov groups, and suppose G has a proper 2-infinitely isolated subgroup H containing some Sylow 2-subgroup of G. Does it follow that G is isomorphic to a group of the type $PSL_2(k)$, where k is a field of odd characteristic? *V. P. Shunkov*

Yes, it does (V. P. Shunkov, *Algebra and Logic*, **11** (1972), 260–272).

2.1. Classify the finite groups having a Sylow p-subgroup as a maximal subgroup.
V. A. Belonogov, A. I. Starostin

A description can be extracted from (B. Baumann, *J. Algebra*, **38** (1976), 119–135) and (L. A. Shemetkov, *Math. USSR–Izv.*, **2** (1968), 487–513).

2.2. A *quasigroup* is a groupoid $Q(\cdot)$ in which the equations $ax = b$ and $ya = b$ have a unique solution for any $a, b \in Q$. Two quasigroups $Q(\cdot)$ and $Q(\circ)$ are *isotopic* if there are bijections α, β, γ of the set Q onto itself such that $x \circ y = \gamma(\alpha x \cdot \beta y)$ for all $x, y \in Q$. It is well-known that all quasigroups that are isotopic to groups form a variety $\mathfrak{G}$. Let $\mathfrak{V}$ be a variety of quasigroups. Characterize the class of groups isotopic to quasigroups in $\mathfrak{G} \cap \mathfrak{V}$. For which identities characterizing $\mathfrak{V}$ is every group isotopic to a quasigroup in $\mathfrak{G} \cap \mathfrak{V}$? Under what conditions on $\mathfrak{V}$ does any group isotopic to a group in $\mathfrak{G} \cap \mathfrak{V}$ consist of a single element?
V. D. Belousov, A. A. Gvaramiya

Every part is answered (A. A. Gvaramiya, Dep. no. 6704-V84, VINITI, Moscow, 1984 (Russian)).

2.3. A finite group is called *quasi-nilpotent* (Γ-*quasi-nilpotent*) if any two of its (maximal) subgroups A and B satisfy one of the conditions 1) $A \leqslant B$, 2) $B \leqslant A$, 3) $N_A(A \cap B) \neq A \cap B \neq N_B(A \cap B)$. Do the classes of quasi-nilpotent and Γ-quasi-nilpotent groups coincide?
Ya. G. Berkovich, M. I. Kravchuk

No. The group $G = \langle x, y, z, t \mid x^4 = y^4 = z^2 = t^3 = 1,\ [x, y] = z,\ [x, z] = [y, z] = 1,\ x^t = y,\ y^t = x^{-1}y^{-1} \rangle$ is Γ-quasi-nilpotent, but not quasi-nilpotent. Since $G/\Phi(G) \cong \mathbb{A}_4$, the intersection of any two maximal subgroups A and B of G equals $\Phi(G)$, whence $N_A(A \cap B) \neq A \cap B \neq N_B(A \cap B)$; thus G is Γ-quasi-nilpotent. On the other hand, if $A_1 = \langle zx^2,\ zy^2,\ t \rangle$ and $B_1 = \langle z, t \rangle$, then $N_{A_1}(A_1 \cap B_1) = A_1 \cap B_1 = \langle t \rangle$; hence G is not quasi-nilpotent. (V. D. Mazurov, 1973.)

2.4. (S. Chase). Suppose that an abelian group A can be written as the union of pure subgroups A_α, $\alpha \in \Omega$, where Ω is the first non-denumerable ordinal, A_α is a free abelian group of denumerable rank, and, if $\beta < \alpha$, then A_β is a direct summand of A_α. Does it follow that A is a free abelian group?
Yu. A. Bogan

Not always (P. A. Griffith, *Pacific J. Math.*, **29** (1969), 279–284).

2.7. Find the cardinality of the set of all polyverbal operations acting on the class of all groups.
O. N. Golovin

It has the cardinality of the continuum (A. Yu. Ol'shanskiĭ, *Math. USSR–Izv.*, **4** (1970), 381–389).

2.8. (A. I. Mal'cev). Do there exist regular associative operations having the heredity property with respect to transition from the factors to their subgroups? *O. N. Golovin*

Yes, they exist (S. V. Ivanov, *Trans. Moscow Math. Soc.*, **1993**, 217–249).

2.10. Prove an analogue of the Remak–Shmidt theorem for decompositions of a group into nilpotent products.
O. N. Golovin

Such an analogue is proved (V. V. Limanskiĭ, *Trudy Moskov. Mat. Obshch.*, **39** (1979), 135–155 (Russian)).

2.13. (Well-known problem). Let G be a periodic group in which every π-element commutes with every π'-element. Does G decompose into the direct product of a maximal π-subgroup and a maximal π'-subgroup? *S. G. Ivanov*

Not always. S. I. Adian (*Math. USSR–Izv.*, **5** (1971), 475–484) has constructed a group $A = A(m,n)$ which is torsion-free and has a central element d such that $A(m,n)/\langle d\rangle \cong B(m,n)$, the free m-generator Burnside group of odd exponent $n \geqslant 4381$. Given a prime p coprime to n, a counterexample with $\pi = \{p\}$ can be found in the form $G = A/\langle d^{p^k}\rangle$ for some positive integer k. Indeed, $\langle d\rangle/\langle d^{p^k}\rangle$ is a maximal p-subgroup of G. Suppose that $A/\langle d^{p^k}\rangle = \langle d\rangle/\langle d^{p^k}\rangle \times H_k/\langle d^{p^k}\rangle$ for every k. Then $\langle d\rangle \cap H_k = \langle d^{p^k}\rangle$ for all k and therefore $\langle d\rangle \cap H = 1$, where $H = \bigcap_k H_k$. Since H is torsion-free and is isomorphic to a subgroup of $A/\langle d\rangle \cong B(m,n)$, we obtain $H = 1$. This implies that A is isomorphic to a subgroup of the Cartesian product of the abelian groups A/H_k, a contradiction. (Yu. I. Merzlyakov, 1973.)

2.15. Does there exist a torsion-free group such that the factor group by some term of its upper central series is nontrivial periodic, with a bound on the orders of the elements? *G. A. Karasëv*

Yes, there does (S. I. Adian, *Proc. Steklov Inst. Math.*, **112** (1971), 61–69).

2.16. A group G is called *conjugacy separable* if any two of its elements are conjugate in G if and only if their images are conjugate in every finite homomorphic image of G. Is G conjugacy separable in the following cases:

a) G is a polycyclic group,
b) G is a free soluble group,
c) G is a group of (all) integral matrices,
d) G is a finitely generated group of matrices,
e) G is a finitely generated metabelian group? *M. I. Kargapolov*

a) Yes (V. N. Remeslennikov, *Algebra and Logic*, **8** (1969), 404–411); E. Formanek, *J. Algebra*, **42** (1976), 1–10).
b) Yes (V. N. Remeslennikov, V. G. Sokolov, *Algebra and Logic*, **9** (1970), 342–349).
c), d) Not always (V. P. Platonov, G. V. Matveev, *Dokl. Akad. Nauk BSSR*, **14** (1970), 777–779 (Russian); V. N. Remeslennikov, V. G. Sokolov, *Algebra and Logic*, **9** (1970), 342–349; V. N. Remeslennikov, *Siberian Math. J.*, **12** (1971), 783–792).
e) Not always. Let p be a prime and let A_1, A_2 be two copies of the additive group $\{m/p^k \mid m, k \in \mathbb{Z}\}$. Let b_1, b_2 be the automorphisms of the direct sum $A = A_1 \oplus A_2$ defined by $a^{b_2} = pa$ for any $a \in A$ and $a_2^{b_1} = a_1 + a_2$ and $a_1^{b_1} = a_1$ for some fixed elements $a_1 \in A_1$, $a_2 \in A_2$. Let G be the semidirect product of A and the direct product $\langle b_1\rangle \times \langle b_2\rangle$ (of two infinite cyclics). It can be shown that the elements a_2 and $a_2 + a_1/p$ are not conjugate in G, but their images are conjugate in any finite quotient of G. (M. I. Kargapolov, E. I. Timoshenko, *Abstracts of the 4th All-Union Sympos. Group Theory, Akademgorodok, 1973*, Novosibirsk, 1973, 86–88 (Russian).)

2.17. Is it true that the wreath product $A \wr B$ of two groups that are conjugacy separable is itself conjugacy separable if and only if either A is abelian or B is finite? *M. I. Kargapolov*

No (V. N. Remeslennikov, *Siberian Math. J.*, **12** (1971), 783–792).

2.18. Compute the ranks of the factors of the lower central series of a free soluble group. *M. I. Kargapolov*

They were computed (V. G. Sokolov, *Algebra and Logic*, **8** (1969), 212–215; Yu. M. Gorchakov, G. P. Egorychev, *Soviet Math. Dokl.*, **13** (1972), 565–568).

2.19. Are finitely generated subgroups of a free soluble group finitely separable? *M. I. Kargapolov*

Not always (S. A. Agalakov, *Algebra and Logic*, **22** (1983), 261–268).

2.20. Is it true that the wreath products $A \wr B$ and $A_1 \wr B_1$ are elementarily equivalent if and only if A, B are elementarily equivalent to A_1, B_1, respectively? *M. I. Kargapolov*

No, but if the word "elementarily" is replaced by the word "universally," then it is true (E. I. Timoshenko, *Algebra and Logic*, **7**, no. 4 (1968), 273–276).

2.21. Do the classes of Baer and Fitting groups coincide? *Sh. S. Kemkhadze*

No (R. S. Dark, *Math. Z.*, **105** (1968), 294–298).

2.22. b) An abstract group-theoretical property Σ is called *radical* (*in our sense*) if, in any group G, the join $\Sigma(G)$ of all normal Σ-subgroups is a Σ-subgroup. Is the property N^0 (see Archive, 1.38) radical? *Sh. S. Kemkhadze*

No. The group $SL_n(\mathbb{Z})$ for sufficiently large $n \geqslant 3$ contains a non-abelian finite simple group and therefore is not an N^0-group. On the other hand, as shown in (M. I. Kargapolov, Yu. I. Merzlyakov, *in: Itogi Nauki. Algebra. Topologiya. Geometriya, 1966*, VINITI, Moscow, 1968, 57–90 (Russian)), it is the product of its congruence-subgroups mod 2 and mod 3, which have central systems and hence are N^0-groups (see Archive, 1.38). (Yu. I. Merzlyakov, 1973.)

2.23. a) A subgroup H of a group G is called *quasisubinvariant* if there is a normal system of G passing through H. Let $\mathfrak{K}$ be a class of groups that is closed with respect to taking homomorphic images. A group G is called an *$R^0(\mathfrak{K})$-group* if each of its non-trivial homomorphic images has a non-trivial quasisubinvariant $\mathfrak{K}$-subgroup. Do $R^0(\mathfrak{K})$ and $\overline{RN}$ coincide when $\mathfrak{K}$ is the class of all abelian groups? *Sh. S. Kemkhadze*

No (J. S. Wilson, *Arch. Math.*, **25** (1974), 574–577).

2.25. b) Do there exist groups that are linearly orderable in countably many ways? *A. I. Kokorin*

Yes, there do (R. N. Buttsworth, *Bull. Austral. Math. Soc.*, **4** (1971), 97–104).

2.29. Does the class of finite groups in which every proper abelian subgroup is contained in a proper normal subgroup coincide with the class of finite groups in which every proper abelian subgroup is contained in a proper normal subgroup of prime index? *P. G. Kontorovich, V. T. Nagrebetskiĭ*

No. Let B be a finite group such that $B = [B, B] \neq 1$ and let r be the rank of B. Put $A = \bigoplus_{p \mid |B|} (\mathbb{Z}/p\mathbb{Z})^{r+1}$ and $G = A \wr B$. We define a homomorphism $\varphi : G \to A$ by setting $(bf)^\varphi = \sum_{x \in B} f(x)$, where $b \in B$ and $f \in F = \mathrm{Fun}(B, A)$. Suppose that $f^b = f$ for $b \in B$ and $f \in F$. It is clear that $f^\varphi \in nA$, where $n = |b|$. In particular, $C_F(b)^\varphi \leqslant pA \leqslant O_{p'}(A)$ if $p \mid n$. Every proper abelian subgroup H of G is contained in a proper normal subgroup. Indeed, we may assume that $H \not\leqslant F$. We fix an element $bf \in H$, where $f \in F$, $b \in B$, $b \neq 1$. Let p be a prime dividing $|b|$. For any $h \in H \cap F$ we have $bfh = hbf = bh^b f = bfh^b$, whence $h = h^b$. Hence $(H \cap F)^\varphi \leqslant C_F(b)^\varphi \leqslant O_{p'}(A)$. If T is the full preimage of $O_{p'}(A)$ in G, then $G/T \cong (\mathbb{Z}/p\mathbb{Z})^{r+1}$ and therefore $H/H \cap T$ is an elementary abelian p-group. The rank of it is $\leqslant r$, since $H/H \cap T$ embeds into B and $H \cap F \leqslant H \cap T$. Thus, HT is a proper normal subgroup containing H. On the other hand, F is a proper abelian subgroup that is not contained in any proper normal subgroup of prime index, since $G/F \cong B = [B, B]$. (G. Bergman, I. Isaacs, *Letter of June, 17, 1974.*)

2.30. Does there exist a finite group in which a Sylow p-subgroup is covered by other Sylow p-subgroups? *P. G. Kontorovich, A. L. Starostin*

Yes, there exists, for any prime p (V. D. Mazurov, *Ural Gos. Univ. Mat. Zap.*, **7**, no. 3 (1969/70), 129–132 (Russian)).

2.31. Can every group admitting an ordering with only finitely many convex subgroups be represented by matrices over a field? *V. M. Kopytov*

No. For example, the group $G = \big\langle a_n,\, b_n\ (n \in \mathbb{Z}),\ c,\ d \mid [a_n, b_n] = c,\ \ a_n^d = a_{n+1},\ b_n^d = b_{n+1},\ \ [a_n, a_m] = [b_n, b_m] = [a_n, c] = [b_n, c] = 1\ \ (n, m \in \mathbb{Z})\big\rangle$ is soluble and orderable with finitely many convex subgroups; but it is not residually finite and hence is not linear over a field. (V. A. Churkin, 1969.)

2.33. Is a direct summand of a direct sum of finitely generated modules over a Noetherian ring again a direct sum of finitely generated modules? *V. I. Kuz'minov*

Not always (P. A. Linnell, *Bull. London Math. Soc.*, **14** (1982), 124–126).

2.35. An inverse spectrum ξ of abelian groups is said to be *acyclic* if $\underleftarrow{\lim}^{(p)} \xi = 0$ for $p > 0$. Here $\underleftarrow{\lim}^{(p)}$ denotes the right derived functor of the projective limit functor. Let ξ be an acyclic spectrum of finitely generated groups. Is the spectrum $\bigoplus \xi_\alpha$ also acyclic, where each spectrum ξ_α coincides with ξ? *V. I. Kuz'minov*

The answer depends on the axioms of Set Theory (A. A. Khusainov, *Siberian Math. J.*, **37**, no. 2 (1996), 405–413).

2.36. (de Groot). Is the group of all continuous integer-valued functions on a compact space free abelian? *V. I. Kuz'minov*

Yes, it is. By (G. Nöbeling, *Invent. Math.*, **6** (1968) 41–55) the additive group of all bounded integer-valued functions on an arbitrary set is free. Hence the group of all continuous integer-valued functions on the Chech compactification of an arbitrary discrete space is also free. For any compact space X there exists a continuous mapping of the Chech compactification Y of a discrete space onto X. This induces an embedding of the group of continuous integer-valued functions on X into the free group of continuous integer-valued functions on Y. (V. I. Kuz'minov, 1969.)

2.37. Describe the finite simple groups whose Sylow p-subgroups are cyclic for all odd p. *V. D. Mazurov*

This was done (M. Aschbacher, *J. Algebra*, **54** (1978), 50–152).

2.38. (Old problem). The class of rings embeddable in associative division rings is universally axiomatizable. Is it finitely axiomatizable? *A. I. Mal'cev*

No (P. M. Cohn, *Bull. London Math. Soc.*, **6** (1974), 147–148).

2.39. Does there exist a non-finitely-axiomatizable variety of
a) (H. Neumann) groups?
b) of associative rings (the Specht problem)?
c) Lie rings? *A. I. Mal'cev*

a) Yes, there exists (S. I. Adian, *Math. USSR–Izv.*, **4** (1970), 721–739; A. Yu. Ol'shanskiĭ, *Math. USSR–Izv.*, **4** (1970), 381–389).
b) Yes, there exists (A. Ya. Belov, *Fundam. Prikl. Mat.*, **5**, no. 1 (1999), 47–66 (Russian), *Sb. Math.*, **191**, no. 3–4 (2000), 329–340; A. V. Grishin, *Fundam. Prikl. Mat.*, **5**, no. 1 (1999), 101–118 (Russian); V. V. Shchigolev, *Fundam. Prikl. Mat.*, **5**, no. 1 (1999), 307–312 (Russian)). But every variety of associative algebras over a field of characteristic 0 has a finite basis of identities (A. R. Kemer, *Algebra and Logic*, **26**, no. 5 (1987), 362–397).
c) Yes, there exists (M. R. Vaughan-Lee, *Quart. J. Math.*, **21** (1970), 297–308).

2.40. The *I-theory* (*Q-theory*) of a class $\mathfrak{K}$ of universal algebras is the totality of all identities (quasi-identities) that are true on all the algebras in $\mathfrak{K}$. Does there exist a finitely axiomatizable variety of
a) groups,
b) semigroups,
c) rings
whose I-theory (Q-theory) is non-decidable? *A. I. Mal'cev*

a) Yes (Yu. G. Kleiman, *Trans. Moscow Math. Soc.*, **1983**, no. 2, 63–110).
b) Yes (V. M. Murskiĭ, *Math. Notes*, **3** (1968), 423–427).
c) Yes (V. Yu. Popov, *Math. Notes*, **67** (2000), 495–504).

2.41. Is the variety generated by

a) a finite associative ring;

b) a finite Lie ring;

c) a finite quasigroup

finitely axiomatizable?

d) What is the cardinality n of the smallest semigroup generating a non-finitely-axiomatizable variety? *A. I. Mal'cev*

a) Yes (I. V. L'vov, *Algebra and Logic*, **12** (1973), 156–167; R. L. Kruse, *J. Algebra*, **26** (1973), 298–318).
b) Yes (Yu. A. Bakhturin, A. Yu. Ol'shanskiĭ, *Math. USSR–Sb.*, **25** (1975), 507–523).
c) Not always (M. R. Vaughan-Lee, *Algebra Universalis*, **9** (1979), 269–280).
d) It is proved that $n = 6$ (P. Perkins, *J. Algebra*, **11** (1969), 298–314; A. N. Trakhtman, *Semigroup Forum*, **27** (1983), 387–389).

2.43. A group G is called an *FN-group* if the groups $\gamma_i G/\gamma_{i+1}G$ are free abelian and $\bigcap_{i=1}^{\infty}\gamma_i G = 1$, where $\gamma_{i+1}G = [\gamma_i G, G]$. A variety of groups $\mathfrak{M}$ is called a *Σ-variety* (where Σ is an abstract property) if the $\mathfrak{M}$-free groups have the property Σ.

a) Which properties Σ are preserved under multiplication and intersection of varieties? Is the FN property preserved under these operations?

b) Are all varieties obtained by multiplication and intersection from the nilpotent varieties $\mathfrak{N}_1, \mathfrak{N}_2, \dots$ (where $\mathfrak{N}_1$ is the variety of abelian groups) FN-varieties? *A. I. Mal'cev*

a) The property FN is preserved by multiplication of varieties (A. L. Shmel'kin, *Trans. Moscow Math. Soc.*, **29** (1973), 239–252). The property FN is not preserved by the intersection of varieties. The following example is due to L. G. Kovács. Let $\mathfrak{U}$ and $\mathfrak{V}$ be the varieties of all nilpotent groups of class at most 4 satisfying the identities $[x, y, y, x] \equiv 1$ and $[[x, y], [z, t]] \equiv 1$, respectively. Then $\mathfrak{U}$ and $\mathfrak{V}$ are FN-varieties because 1) both of them are nilpotent of class 4 and contain all nilpotent groups of class $\leqslant 3$, and 2) relatively free groups in $\mathfrak{U}$ and $\mathfrak{V}$ are torsion-free (well-known for $\mathfrak{V}$ and follows for $\mathfrak{U}$ from (P. Fitzpatrick, L. G. Kovács, *J. Austral. Math. Soc. Ser. A*, **35**, no. 1 (1983), 59–73)). On the other hand, $\mathfrak{U}\cap\mathfrak{V}$ is not an FN-variety because the relatively free group in $\mathfrak{U}\cap\mathfrak{V}$ of rank 3 is not torsion-free. Indeed, every torsion-free group in $\mathfrak{U}\cap\mathfrak{V}$ is of class $\leqslant 3$ (follows from the paper by Fitzpatrick and Kovács cited above), and there is a 3-generated (exponent 2)-by-(exponent 2) group of class precisely 4 in $\mathfrak{U}\cap\mathfrak{V}$. (A. N. Krasil'nikov, *Letter of July, 17, 1998.*)
b) Yes (Yu. M. Gorchakov, *Algebra i Logika*, **6**, no. 3 (1967), 25–30 (Russian)).

2.44. Let $\mathfrak{A}$ and $\mathfrak{B}$ be subvarieties of a variety of groups $\mathfrak{M}$; then $(\mathfrak{A}\mathfrak{B})\cap\mathfrak{M}$ is called the *$\mathfrak{M}$-product of $\mathfrak{A}$ by $\mathfrak{B}$*, where $\mathfrak{A}\mathfrak{B}$ is the usual product. Does there exist a non-abelian variety $\mathfrak{M}$ with an infinite lattice of subvarieties and commutative $\mathfrak{M}$-multiplication? *A. I. Mal'cev*

Yes, there exists; for example, $\mathfrak{M} = \mathfrak{A}_p\mathfrak{A}_p$, where $\mathfrak{A}_p$ is the variety of all abelian groups of prime exponent p (Yu. M. Gorchakov, *Talk of June 21, 1967, Krasnoyarsk*).

2.45. (P. Hall). Prove or refute the following conjectures:

a) If a word v takes only finitely many values in a group G, then the verbal subgroup vG is finite.

c) If G satisfies the maximum condition and vG is finite, then v^*G has finite index in G. *Yu. I. Merzlyakov*

These conjectures are refuted in:
a) (S. V. Ivanov, *Soviet Math. (Izv. VUZ)*, **33**, no. 6 (1989), 59–70;
c) I. S. Ashmanov, A. Yu. Ol'shanskiĭ, *Soviet Math. (Izv. VUZ)*, **29**, no. 11 (1985), 65–82).

2.46. Find conditions under which a finitely-generated matrix group is almost residually a finite p-group for some prime p. *Yu. I. Merzlyakov*

A finitely generated matrix group over a field of characteristic zero has a subgroup of finite index which is residually a finite p-group for almost all primes p (M. I. Kargapolov, *Algebra i Logika*, **6**, no. 5 (1967), 17–20 (Russian); Yu. I. Merzlyakov, *Soviet Math. Dokl.*, **8** (1967), 1538–1541; V. P. Platonov, *Dokl. Akad. Nauk Belarus. SSR*, **12**, no. 6 (1968), 492–494 (Russian)). A finitely generated matrix group over a field of characteristic $p > 0$ has a subgroup of finite index which is residually a finite p-group (V. P. Platonov, *Dokl. Akad. Nauk Belarus. SSR*, **12**, no. 6 (1968), 492–494 (Russian); A. I. Tokarenko, *Proc. Riga Algebr. Sem.*, Latv. Univ., Riga, 1969, 280–281 (Russian)). See also some generalizations in (V. N. Remeslennikov, *Algebra i Logika*, **7**, no. 4 (1968), 106–113 (Russian); B. A. F. Wehrfritz, *Proc. London Math. Soc.*, **20**, no. 1 (1970), 101–122).

2.47. In which abelian groups is the lattice of all fully invariant subgroups a chain? *A. P. Mishina*

Such groups were characterized (M.-P. Brameret, *Séminaire P. Dubreil, M.-L. Dubreil-Jacotin, L. Lesieur, et C. Pisot*, **16** (1962/63), no. 13, Paris, 1967).

2.49. (A. Selberg). Let G be a connected semisimple linear Lie group whose corresponding symmetric space has rank greater than 1, and let Γ be an irreducible discrete subgroup of G such that G/Γ has finite volume. Does it follow that Γ is an arithmetic subgroup? *V. P. Platonov*

Yes, it does (G. A. Margulis, *Functional Anal. Appl.*, **8** (1974), 258–259).

2.50. (A. Selberg). Let Γ be an irreducible discrete subgroup of a connected Lie group G such that the factor space G/Γ is non-compact but has finite volume in the Haar measure. Prove that Γ contains a non-trivial unipotent element. *V. P. Platonov*

This is proved (D. A. Kazhdan, G. A. Margulis, *Math. USSR–Sb.*, **4** (1968), 147–152).

2.51. (A. Borel, R. Steinberg). Let G be a semisimple algebraic group and R_G the set of classes of conjugate unipotent elements of G. Is R_G finite? *V. P. Platonov*

Yes, it is (G. Lusztig, *Invent. Math.*, **34** (1976), 201–213).

2.52. (F. Bruhat, N. Iwahori, M. Matsumoto). Let G be a semisimple algebraic group over a locally compact, totally disconnected field. Do the maximal compact subgroups of G fall into finitely many conjugacy classes? If so, estimate this number. *V. P. Platonov*

Yes, they do, and a formula for this number is found (F. Bruhat, J. Tits, *Publ. Math. IHES*, **41** (1972), 5–251).

2.54. Can $SL(n,k)$ have maximal subgroups that are not closed in the Zariski topology? *V. P. Platonov*

Yes, it can if k is either $\mathbb{Q}$ or an algebraically closed field of characteristic zero (N. S. Romanovskiĭ, *Algebra i Logika*, **6**, no. 4 (1967), 75–82; **7**, no. 3 (1968), 123 (Russian)).

2.55. b) Does $SL_n(\mathbb{Z})$, $n \geqslant 2$, have maximal subgroups of infinite index? *V. P. Platonov*

Yes, it does (G. A. Margulis, G. A. Soifer, *Soviet Math. Dokl.*, **18** (1977), 847–851).

2.58. Let G be a vector space and Γ a group of automorphisms of G. Γ is called *locally finitely stable* if, for any finitely generated subgroup Δ of Γ, G has a finite series stable relative to Δ. If the characteristic of the field is zero and Γ is locally finitely stable, then Γ is locally nilpotent and torsion-free. Is it true that every locally nilpotent torsion-free group can be realized in this way? *B. I. Plotkin*

No (L. A. Simonyan, *Siberian Math. J.*, **12** (1971), 602–606).

2.59. Let Γ be any nilpotent group of class $n-1$. Does Γ always admit a faithful representation as a group of automorphisms of an abelian group with a series of length n stable relative to Γ? *B. I. Plotkin*

Not always (E. Rips, *Israel J. Math.*, **12** (1972), 342–346).

2.61. Let Γ be a Noetherian group of automorphisms of a vector space G such that every element of Γ is unipotent. Is Γ necessarily a stable group of automorphisms? *B. I. Plotkin*

Not if the field has non-zero characteristic (A. Yu. Ol'shanskiĭ, *The geometry of defining relations in groups*, Kluwer, Dordrecht, 1991). This is true for fields of characteristic zero if the indices of unipotence are uniformly bounded (B. I. Plotkin, S. M. Vovsi, *Varieties of group representations*, Zinatne, Riga, 1983 (Russian)).

2.62. If G is a finite-dimensional vector space over a field and Γ is a group of automorphisms of G in which every element is stable, then the whole of Γ is stable (E. Kolchin). Is Kolchin's theorem true for spaces over skew fields? *B. I. Plotkin*

Yes, it is, if the characteristic of the skew field is zero or sufficiently large compared to the dimension of the space (H. Y. Mochizuki, *Canad. Math. Bull.*, **21** (1978), 249–250).

2.63. Let G be a group of automorphisms of a vector space over a field of characteristic zero, and suppose that all elements of G are unipotent, with uniformly bounded unipotency indices. Must such a group be locally finitely stable? *B. I. Plotkin*

Yes, it must, which follows from (H. Heineken, *Arch. Math. (Basel)*, **13** (1962), 29–37). Moreover, such a group is even finitely stable, as follows from (E. I. Zel'manov, *Math. USSR-Sb.*, **66**, no. 1 (1990), 159–168).

2.64. Does the set of nil-elements of a finite-dimensional linear group coincide with its locally nilpotent radical? *B. I. Plotkin*

Not always. The non-nilpotent 3-generator nilgroup of E. S. Golod constructed by an algebra over $\mathbb{Q}$ is residually torsion-free nilpotent; hence it is orderable and therefore it is embeddable into GL_n over some skew field by Mal'cev's theorem (V. A. Roman'kov, 1978).

2.65. Does the adjoint group of a radical ring (in the sense of Jacobson) have a central series? *B. I. Plotkin*

Not always (O. M. Neroslavskiĭ, *Vesci Akad. Navuk BSSR, Ser. Fiz.-Mat. Navuk*, **1973**, No. 2, 5–10 (Russian)).

2.66. Is an R-group determined by its subgroup lattice? Is every lattice isomorphism of R-groups induced by a group isomorphism? *L. E. Sadovskiĭ*

Not always, in both cases (A. Yu. Ol'shanskiĭ, *C. R. Acad. Bulgare Sci.*, **32**, no. 9 (1979), 1165–1166).

2.69. Let a group G be the product of two subgroups A and B, each of which is nilpotent and satisfies the minimum condition. Prove or refute the following: a) G is soluble; b) the divisible parts of A and B commute elementwise. *N. F. Sesekin*

Both parts are proved (N. S. Chernikov, *Soviet Math. Dokl.*, **21** (1980), 701–703).

2.70. a) Let a group G be the product of two subgroups A and B, each of which is locally cyclic and torsion-free. Prove that either A or B has a non-trivial subgroup that is normal in G.

b) Characterize the groups that can be factorized in this way. *N. F. Sesekin*

a) This was proved (D. I. Zaitsev, *Algebra and Logic*, **19** (1980), 94–106).
b) They were characterized (Ya. P. Sysak, *Algebra and Logic*, **25** (1986), 425–433).

2.71. Does there exist a finitely generated right-orderable group which coincides with its derived subgroup and, therefore, does not have the property RN? *D. M. Smirnov*

Yes, there exists (G. M. Bergman, *Pacific J. Math.*, **147**, no. 2 (1991), 243–248).

2.72. (G. Baumslag). Suppose that F is a finitely generated free group, N its normal subgroup and V a fully invariant subgroup of N. Is F/V necessarily Hopfian if F/N is Hopfian? *D. M. Smirnov*

No, not necessarily (S. V. Ivanov, A. M. Storozhev, *Geom. Dedicata*, **114** (2005), 209–228).

2.73. (Well-known problem). Does there exist an infinite group all of whose proper subgroups have prime order? *A. I. Starostin*

Yes, there does (A. Yu. Ol'shanskiĭ, *Algebra and Logic*, **21** (1982), 369–418).

2.75. Let G be a periodic group containing an infinite family of finite subgroups whose intersection contains non-trivial elements. Does G contain a non-trivial element with infinite centralizer? *S. P. Strunkov*

Not always (K. I. Lossov, Dep. no. 5528-V88, VINITI, Moscow, 1988 (Russian)).

2.76. Let Γ be the holomorph of an abelian group A. Find conditions for A to be maximal among the locally nilpotent subgroups of Γ. *D. A. Suprunenko*

These are found (A. V. Yagzhev, *Mat. Zametki*, **46**, no. 6 (1989), 118 (Russian)).

2.77. Let A and B be abelian groups. Find conditions under which every extension of A by B is nilpotent. *D. A. Suprunenko*

These are found (A. V. Yagzhev, *Math. Notes*, **43** (1988), 244–245).

2.79. Do there exist divisible (simple) groups with maximal subgroups?
M. S. Tsalenko

Yes, there exist, for example, (V. G. Sokolov, *Algebra and Logic*, **7** (1968), 122–126).

2.83. Suppose that a periodic group G is the product of two locally finite subgroups. Is then G locally finite? *V. P. Shunkov*

Not always (V. I. Sushchanskiĭ, *Math. USSR–Sb.*, **67**, no. 2 (1990), 535–553).

2.88. Is every Hall π-subgroup of an arbitrary group a maximal π-subgroup?
M. I. Èidinov

Not always (R. Baer, *J. Reine Angew. Math.*, **239/240** (1969), 109–144).

3.4. (Well-known problems). a) Is there an algorithm that decides, for any set of group words $f_1, \ldots, f_m$ (in a fixed set of variables $x_1, x_2, \ldots$) and a separate word f, whether $f = 1$ is a consequence of $f_1 = 1, \ldots, f_m = 1$?

b) Given words $f_1, \ldots, f_m$, is there an algorithm that decides, for any word f, whether $f = 1$ is a consequence of $f_1 = 1, \ldots, f_m = 1$? *L. A. Bokut'*

No, in both cases (Yu. G. Kleiman, *Soviet Math. Dokl.*, **20** (1979), 115–119).

3.6. Describe the insoluble finite groups in which every soluble subgroup is either 2-closed, $2'$-closed, or isomorphic to $\mathbb{S}_4$. *V. A. Vedernikov*

This follows from (D. Gorenstein, J. H. Walter, *Illinois J. Math.*, **6** (1962) 553–593; V. D. Mazurov, *Soviet Math. Dokl.*, **7** (1966), 681–682; V. D. Mazurov, V. M. Sitnikov, S. A. Syskin, *Algebra and Logic*, **9** (1970), 187–204).

3.7. An automorphism σ of a group G is called *algebraic* if, for any $g \in G$, the minimal σ-invariant subgroup of G containing g is finitely generated. An automorphism σ of G is called an *e-automorphism* if, for any σ-invariant subgroups A and B, where A is a proper subgroup of B, $A \setminus B$ contains an element x such that $[x, \sigma] \in B$. Is every e-automorphism algebraic? *V. G. Vilyatser*

Yes, it is (A. V. Yagzhev, *Algebra and Logic*, **28** (1989), 83–85).

3.8. Let G be the free product of free groups A and B, and V the verbal subgroup of G corresponding to the equation $x^4 = 1$. Is it true that, if $a \in A \setminus V$ and $b \in B \setminus V$, then $(ab)^2 \notin V$? *V. G. Vilyatser*

No, it is not. If A and B are free groups with bases a_1, a_2 and b_1, b_2, respectively, then, for example, the commutators $a = [[a_1, a_2, a_2], [a_1, a_2]]$ and $b = [[b_1, b_2, b_2], [b_1, b_2]]$ do not belong to V, but $(ab)^2 \in V$. Indeed, it follows from (C. R. B. Wright, *Pacif. J. Math.*, **11**, no. 1 (1961), 387–394) that $a^2 \in V$, $b^2 \in V$, and $[a, b] \in V$. (V. A. Roman'kov, *Talk at the seminar Algebra and Logic*, March, 17, 1970.)

3.9. Does there exist an infinite periodic group with a finite maximal subgroup?
Yu. M. Gorchakov

Yes, there does (S. I. Adian, *Proc. Steklov Inst. Math.*, **112** (1971), 61–69).

3.10. Need the number of finite non-abelian simple groups contained in a proper variety of groups be finite? *Yu. M. Gorchakov*

Yes, mod CFSG; see, for example, (G. A. Jones, *J. Austral. Math. Soc.*, **17** (1974), 162–173).

3.11. An element g of a group G is said to be *generalized periodic* if there exist $x_1, \ldots, x_n \in G$ such that $x_1^{-1}gx_1 \cdots x_n^{-1}gx_n = 1$. Does there exist a finitely generated torsion-free group all of whose elements are generalized periodic? *Yu. M. Gorchakov*

Yes, there does (A. P. Goryushkin, *Siberian Math. J.*, **14** (1973), 146–148). Another example is $G = \langle a, b \mid (b^2)^a = b^{-2},\ (a^2)^b = a^{-2} \rangle$. Then $N = \langle a^2, b^2, (ab)^2 \rangle$ is an abelian normal subgroup of G and G/N is non-cyclic of order 4. If $a^{2l}b^{2m}(ab)^{2n} = 1$, then after conjugating by a we get $a^{2l}b^{-2m}(ab)^{-2n} = 1$, whence $a^{4l} = 1$. In view of the obvious homomorphism $G \to \mathbb{Z}(\operatorname{Aut}\mathbb{Z})$ that maps a to the number $1 \in \mathbb{Z}$ we have $l = 0$. Similarly, $m = n = 0$. Hence N is free abelian of rank 3. The squares of elements outside of N are non-trivial; for example, $\big(aa^{2l}b^{2m}(ab)^{2n}\big)^2 = a^2a^{-1}(a^{2l}b^{2m}(ab)^{2n})aa^{2l}b^{2m}(ab)^{2n} = a^{4l+2} \neq 1$. Hence G is torsion-free. Since the square of any element $x \in G$ belongs to N, we have $x^2(x^2)^a(x^2)^b(x^2)^{ab} = 1$. (V. A. Churkin, 1973.)

3.13. Is the elementary theory of subgroup lattices of finite abelian groups decidable? *Yu. L. Ershov, A. I. Kokorin*

No (G. T. Kozlov, *Algebra and Logic*, **9** (1970), 104–107).

3.14. Let G be a finite group of $n \times n$ matrices over a skew field T of characteristic zero. Prove that G has a soluble normal subgroup H whose index in G is not greater than some number $f(n)$ not depending on G or T. This problem is related to the representation theory of finite groups (the Schur index). *A. E. Zalesskiĭ*

This is proved mod CFSG (B. Hartley, M. A. Shahabi Shojaei, *Math. Proc. Cambridge Phil. Soc.*, **92** (1982), 55–64).

3.15. A group G is said to be *U-embeddable* in a class $\mathfrak{K}$ of groups if, for any finite submodel $M \subset G$, there is a group $A \in \mathfrak{K}$ such that M is isomorphic to some submodel of A. Are the following groups U-embeddable in the class of finite groups:

a) every group with one defining relation?

b) every group defined by one relation in the variety of soluble groups of a given derived length? *M. I. Kargapolov*

No, in both cases (A. I. Budkin, V. A. Gorbunov, *Algebra and Logic*, **14** (1975), 73–84). Another example. The group $G = \langle a, b \mid (b^2)^a = b^3 \rangle$ is non-Hopfian as proved in (G. Baumslag, D. Solitar, *Bull. Amer. Math. Soc.*, **68**, no. 3 (1962), 199–201) and therefore is not metabelian. We choose elements $a, b, x_1, x_2, x_3, x_4 \in G$ such that $w = [[x_1, x_2], [x_3, x_4]] \neq 1$, add to them 1, their inverses, and all the initial segments of the word w in the alphabet $\{x_i\}$, and all the initial segments of the word $a^{-1}b^2ab^{-3}$ and of each of the words x_i in the alphabet of $\{a, b\}$. Let M be the resulting model. If M was embeddable into a finite group G_0, then G_0 would have to be metacyclic, and also would have to have elements satisfying $[[x_1, x_2], [x_3, x_4]] \neq 1$, which is impossible. Hence G is not U-embeddable into finite groups. Since the group $G/G^{(k)}$ is also non-Hopfian for a suitable k (*ibid.*), it is not U-embeddable into finite groups for similar reasons. (Yu. I. Merzlyakov, 1969.)

3.17. a) Is a non-abelian group with a unique linear ordering necessarily simple?

b) Is a non-commutative orderable group simple if it has no non-trivial normal relatively convex subgroups? *A. I. Kokorin*

Not always, in both cases (V. V. Bludov, *Algebra and Logic*, **13** (1974), 343–360).

3.18. (B. H. Neumann). A group U is called *universal* for a class $\mathfrak{K}$ of groups if U contains an isomorphic image of every member of $\mathfrak{K}$. Does there exist a countable group that is universal for the class of countable orderable groups? *A. I. Kokorin*

No (M. I. Kargapolov, *Algebra and Logic*, **9** (1970), 257–261; D. B. Smith, *Pacific J. Math.*, **35** (1970), 499–502).

3.19. (A. I. Mal'cev). For an arbitrary linearly ordered group G, does there exist a linearly ordered abelian group with the same order-type as G? *A. I. Kokorin*

No (W. C. Holland, A. H. Mekler, S. Shelah, *Order*, **1** (1985), 383–397).

3.21. Let $\mathfrak{K}$ be the class of one-based models of signature σ, ϑ a property that makes sense for models in $\mathfrak{K}$, and $\mathfrak{K}_0$ the class of two-based models whose first base is a set M taken from $\mathfrak{K}$ and whose second base consists of all submodels of M with property ϑ and whose signature consists of the symbols in σ together with $\in$ and $\subseteq$ in the usual set-theoretic sense. The elementary theory of $\mathfrak{K}_0$ is called the *element-ϑ-submodel theory* of $\mathfrak{K}$. Are either of the following theories decidable:

a) the element-pure-subgroup theory of abelian groups?

b) the element-pure-subgroup theory of abelian torsion-free groups?

c) the element-ϑ-subgroup theory of abelian groups, when the set of ϑ-subgroups is linearly ordered by inclusion? *A. I. Kokorin*

No, in all cases (for a), c): G. T. Kozlov, *Algebra and Logic*, **9** (1970), 104–107, *Algebra*, no. 1, Irkutsk Univ., 1972, 21–23 (Russian); for b): É. I. Fridman, *Algebra*, no. 1, Irkutsk Univ., 1972 , 97–100 (Russian)).

3.22. Let $\xi = \{G_\alpha,\ \pi^\alpha_\beta \mid \alpha, \beta \in I\}$ be a projective system (over a directed set I) of finitely generated free abelian groups. If all the projections π^α_β are epimorphisms and all the G_α are non-zero, does it follow that $\varprojlim \xi \neq 0$? Equivalently, suppose every finite set of elements of an abelian group A is contained in a pure finitely-generated free subgroup of A. Then does it follow that A has a direct summand isomorphic to the infinite cyclic group? *V. I. Kuz'minov*

Under the assumption of the continuum hypothesis, not always (E. A. Palyutin, *Siberian Math. J.*, **19** (1978), 1415–1417).

3.26. (F. Gross). Is it true that finite groups of exponent $p^\alpha q^\beta$ have nilpotent length $\leqslant \alpha + \beta$? *V. D. Mazurov*

No (E. I. Khukhro, *Algebra and Logic*, **17** (1978), 473–482).

3.27. (J. G. Thompson). Is every finite simple group with a nilpotent maximal subgroup isomorphic to some $PSL_2(q)$? *V. D. Mazurov*

Yes, it is (B. Baumann, *J. Algebra*, **38** (1976), 119–135).

3.28. If G is a finite 2-group with cyclic centre and every abelian normal subgroup 2-generated, then is every abelian subgroup of G 3-generated? *V. D. Mazurov*

Not always (Ya. G. Berkovich, *Algebra and Logic*, **9** (1970), 75).

3.29. Under what conditions can a wreath product of matrix groups over a field be represented by matrices over a field? *Yu. I. Merzlyakov*

Conditions have been found (Yu. E. Vapne, *Soviet Math. Dokl.*, **11** (1970), 1396–1399).

3.30. A torsion-free abelian group is called *factor-decomposable* if, in all its factor groups, the periodic part is a direct summand. Characterize these groups.

A. P. Mishina

This is done (L. Bican, *Commentat. Math. Univ. Carolinae*, **19** (1978), 653–672).

3.31. Find necessary and sufficient conditions under which every pure subgroup of a completely decomposable torsion-free abelian group is itself completely decomposable.

A. P. Mishina

These have been found (L. Bican, *Czech. Math. J.*, **24** (1974), 176–191; A. A. Kravchenko, *Vestnik Moskov. Univ. Ser. 1 Mat. Mekh.*, **1980**, no. 3, p. 104 (Russian)).

3.33. Are two groups necessarily isomorphic if each of them can be defined by a single relation and is a homomorphic image of the other one? *D. I. Moldavanskiĭ*

No, not necessarily (A. V. Borshchev, D. I. Moldavanskiĭ, *Math. Notes*, **79**, no. 1 (2006), 31–40).

3.35. (K. Ross). Suppose that a group G admits two topologies σ and τ yielding locally compact topological groups G_σ and G_τ. If the sets of closed subgroups in G_σ and G_τ are the same, does it follow that G_σ and G_τ are topologically isomorphic?

Yu. N. Mukhin

Not always (A. I. Moskalenko, *Ukrain. Math. J.*, **30** (1978), 199–201).

3.37. Suppose that every finitely generated subgroup of a locally compact group G is pronilpotent. Then is it true that every maximal closed subgroup of G contains the derived subgroup G'? *Yu. N. Mukhin*

Yes, it is (I. V. Protasov, *Soviet Math. Dokl.*, **19** (1978), 1208–1210).

3.39. Describe the finite groups with a self-centralizing subgroup of prime order.

V. T. Nagrebetskiĭ

They are described. Self-centralizing subgroups of prime order are CC-subgroups. Finite groups with a CC-subgroup were fully classified in (Z. Arad, W. Herfort, *Commun. in Algebra*, **32** (2004), 2087–2098).

3.40. (I. R. Shafarevich). Let $SL_2(\mathbb{Z})\hat{}$ and $SL_2(\mathbb{Z})^-$ denote the completions of $SL_2(\mathbb{Z})$ determined by all subgroups of finite index and all congruence subgroups, respectively, and let $\psi : SL_2(\mathbb{Z})\hat{} \to SL_2(\mathbb{Z})^-$ be the natural homomorphism. Is $\operatorname{Ker}\psi$ a free profinite group? *V. P. Platonov*

Yes, it is (O. V. Mel'nikov, *Soviet Math. Dokl.*, **17** (1976), 867–870).

3.41. Is every compact periodic group locally finite? *V. P. Platonov*

Yes, it is (E. I. Zel'manov, *Israel J. Math.*, **77** (1992), 83–95).

3.42. (Kneser–Tits conjecture). Let G be a simply connected k-defined simple algebraic group, and $E_k(G)$ the subgroup generated by unipotent k-elements. If $E_k(G) \neq 1$, then $G_k = E_k(G)$. The proof is known for k-decomposable groups (C. Chevalley) and for local fields (V. P. Platonov). *V. P. Platonov*

The conjecture was refuted (V. P. Platonov, *Math. USSR–Izv.*, **10**, no. 2 (1976), 211–243).

3.52. Can the quasivariety generated by the free group of rank 2 be defined by a system of quasi-identities in finitely many variables? *D. M. Smirnov*

No (A. I. Budkin, *Algebra and Logic*, **15** (1976), 25–33).

3.53. Let $L(\mathfrak{N}_4)$ denote the lattice of subvarieties of the variety $\mathfrak{N}_4$ of nilpotent groups of class at most 4. Is $L(\mathfrak{N}_4)$ distributive? *D. M. Smirnov*

No (Yu. A. Belov, *Algebra and Logic*, **9** (1970), 371–374).

3.56. Is a 2-group with the minimum condition for abelian subgroups locally finite? *S. P. Strunkov*

Yes, it is (V. P. Shunkov, *Algebra and Logic*, **9** (1970), 291–297).

3.58. Let G be a compact 0-dimensional topological group all of whose Sylow p-subgroups are direct products of cyclic groups of order p. Then is every normal subgroup of G complementable? *V. S. Charin*

Yes, it is (M. I. Kabenyuk, *Siberian Math. J.*, **13** (1972), 654–657).

3.59. Let H be an insoluble minimal normal subgroup of a finite group G, and suppose that H has cyclic Sylow p-subgroups for every prime p dividing $|G : H|$. Prove that H has at least one complement in G. *L. A. Shemetkov*

This was proved (S. A. Syskin, *Siberian Math. J.*, **12** (1971), 342–344; L. A. Shemetkov, *Soviet Math. Dokl.*, **11** (1970), 1436–1438).

3.61. Let σ be an automorphism of prime order p of a finite group G, which has a Hall π-subgroup with cyclic Sylow subgroups. Suppose that $p \in \pi$. Does the centralizer $C_G(\sigma)$ have at least one Hall π-subgroup? *L. A. Shemetkov*

Yes, it does, mod CFSG (V. D. Mazurov, *Algebra and Logic*, **31**, no. 6 (1992), 360–366).

3.62. (Well-known problem). A finite group is said to be a *D_π-group* if any two of its maximal π-subgroups are conjugate. Is an extension of a D_π-group by a D_π-group always a D_π-group? *L. A. Shemetkov*

Yes, it is mod CFSG (E. P. Vdovin, D. O. Revin, *Contemp. Math.*, **402** (2006), 229–263).

3.64. Describe the finite simple groups with a Sylow 2-subgroup of the following type: $\langle a, t \mid a^{2^n} = t^2 = 1,\ \ tat = a^{2^{n-1}-1}\rangle,\ \ n > 2.$ *V. P. Shunkov*

This was done (J. L. Alperin, R. Brauer, D. Gorenstein, *Trans. Amer. Math. Soc.*, **151** (1970), 1–261).

4.1. Find an infinite finitely generated group with an identical relation of the form $x^{2^n} = 1$. *S. I. Adian*

It has been found (S. V. Ivanov, *Int. J. Algebra Comput.*, **4**, no. 1–2 (1994), 1–308; I. G. Lysënok, *Izv. Math.*, **60**, no. 3 (1996), 453–654).

4.3. Construct a finitely presented group with insoluble word problem and satisfying a non-trivial law. *S. I. Adian*

This has been done (O. G. Kharlampovich, *Math. USSR–Izv.*, **19** (1982), 151–169).

4.4. Construct a finitely presented group with undecidable word problem all of whose non-trivial defining relations have the form $A^2 = 1$. This problem is interesting for topologists. *S. I. Adian*

It is constructed (O. A. Sarkisyan, *The word problem for some classes of groups and semigroups*, Candidate disser., Moscow Univ., 1983 (Russian)).

4.5. a) (J. Milnor). Is it true that an arbitrary finitely generated group has either polynomial or exponential growth?

c) Is it true that every finitely generated group with undecidable word problem has exponential growth? *S. I. Adian*

a) No, c) No (R. I. Grigorchuk, *Soviet Math. Dokl.*, **28** (1983), 23–26).

4.10. A group G is called *locally indicable* if every non-trivial finitely generated subgroup of G has an infinite cyclic factor group. Is every torsion-free one-relator group locally indicable? *G. Baumslag*

Yes, it is (S. D. Brodskiĭ, Dep. no. 2214-80, VINITI, Moscow, 1980 (Russian)).

4.12. Let G be a finite group and A a group of automorphisms of G stabilizing a series of subgroups beginning with G and ending with its Frattini subgroup. Then is A nilpotent? *Ya. G. Berkovich*

Yes, it is (P. Schmid, *Math. Ann.*, **202** (1973), 57–69).

4.16. Suppose that $\mathfrak{K}$ is a class of groups meeting the following requirements: 1) subgroups and epimorphic images of $\mathfrak{K}$-groups are $\mathfrak{K}$-groups; 2) if the group $G = UV$ is the product of $\mathfrak{K}$-subgroups U and V (neither of which need be normal), then $G \in \mathfrak{K}$. If π is a set of primes, then the class of all finite π-groups meets these requirements. Are these the only classes $\mathfrak{K}$ with these properties? *R. Baer*

No (S. A. Syskin, *Siberian Math. J.*, **20** (1979), 475–476).

4.21. Let G be a finite group, p an odd prime number, and P a Sylow p-subgroup of G. Let the order of every non-identity normal subgroup of G be divisible by p. Suppose P has an element x that is conjugate to no other from P. Does x belong to the centre of G? For $p = 2$, the answer is positive (G. Glauberman, *J. Algebra*, **4** (1966), 403–420). *G. Glauberman*

Yes, it does, mod CFSG (O. D. Artemovich, *Ukrain. Math. J.*, **40** (1988), 343–345).

4.22. (J. G. Thompson). Let G be a finite group, A a group of automorphisms of G such that $|A|$ and $|G|$ are coprime. Does there exist an A-invariant soluble subgroup H of G such that $C_A(H) = 1$? *G. Glauberman*

Yes, it does (S. A. Syskin, *Siberian Math. J.*, **32**, no. 6 (1991), 1034–1037).

4.23. Let G be a finite simple group, τ some element of prime order, and α an automorphism of G whose order is coprime to $|G|$. Suppose α centralizes $C_G(\tau)$. Is $\alpha = 1$? *G. Glauberman*

Not always; for example, $G = Sz(8)$, $|\tau| = 5$, $|\alpha| = 3$ (N. D. Podufalov, *Letter of September, 3, 1975*).

4.24. Suppose that T is a non-abelian Sylow 2-subgroup of a finite simple group G.
b) Is it possible that T is the direct product of two proper subgroups?
c) Is $T' = \Phi(T)$? *D. Goldschmidt*

b) Yes, it is. For example, the Sylow 2-subgroups of the alternating groups $\mathbb{A}_{14}$ and $\mathbb{A}_{15}$ are isomorphic to the Sylow 2-subgroup of $\mathbb{S}_4 \times \mathbb{S}_8$. The Sylow 2-subgroups of $D_4(q)$ for q odd are also decomposable into direct products (A. S. Kondratiev, *Letter of October, 13, 1977*).
c) Not always; for example, for $G = PSL_3(q)$ with $q \equiv 1 \pmod 4$ (A. S. Kondratiev).

4.27. Describe all finite simple groups G which can be represented in the form $G = ABA$, where A and B are abelian subgroups. *I. P. Doktorov*

They are described mod CFSG (D. L. Zagorin, L. S. Kazarin, *Dokl. Math.*, **53**, no. 2 (1996), 237–239; D. L. Zagorin, *Some problems of ABA-factorizations of permutation groups and classical groups* (*Cand. Diss.*), Yaroslavl', 1994 (Russian); D. L. Zagorin, L. S. Kazarin, *Problems in algebra*, Gomel' Univ., Gomel', no. 11 (1997), 27–41 (Russian); E. P. Vdovin, *Algebra and Logic*, **38**, no. 2 (1999), 67–83).

4.28. For a given field k of characteristic $p > 0$, characterize the locally finite groups with semisimple group algebras over k. *A. E. Zalesskii*

They are characterized: simple groups in (D. S. Passman, A. E. Zalesskii, *Proc. London Math. Soc. (3)*, **67** (1993), 243–276); the general case in (D. S. Passman, *Proc. London Math. Soc. (3)*, **73** (1996), 323–357).

4.29. Classify the irreducible matrix groups over a finite field that are generated by reflections, that is, by matrices with Jordan form $\mathrm{diag}(-1, 1, \dots, 1)$. *A. E. Zalesskiĭ*

These are classified (A. E. Zalesskiĭ, V. N. Serëzhkin, *Math. USSR–Izv.*, **17** (1981), 477–503; A. Wagner, *Geom. Dedicata*, **9** (1980), 239–253; **10** (1981), 191–203, 475-523).

4.32. The conjugacy problem for metabelian groups. *M. I. Kargapolov*

This is algorithmically soluble (G. A. Noskov, *Math. Notes*, **31** (1982), 252–258).

4.35. (Well-known problem). Is there an infinite locally finite simple group satisfying the minimum condition for p-subgroups for every prime p? *O. H. Kegel*

No (V. V. Belyaev, *Algebra and Logic*, **20** (1981), 393–401).

4.36. Is there an infinite locally finite simple group G with an involution i such that the centralizer $C_G(i)$ is a Chernikov group? *O. H. Kegel*

No (A. O. Asar, *Proc. London Math. Soc.*, **45** (1982), 337–364).

4.37. Is there an infinite locally finite simple group G that cannot be represented by matrices over a field and is such that for some prime p the p-subgroups of G are either of bounded derived length or of finite exponent? *O. H. Kegel*

No (mod CFSG) by Theorem 4.8 of (O. H. Kegel, B. A. F. Wehrfritz, *Locally finite groups*, North Holland, Amsterdam, 1973).

4.38. Classify the composition factors of automorphism groups of finite (nilpotent of class 2) groups of prime exponent. *V. D. Mazurov*

Any finite simple group can be such a composition factor (P. M. Beletskiĭ, *Russian Math. Surveys*, **33**, no. 6 (1980), 85–86).

4.39. A countable group U is said to be *SQ-universal* if every countable group is isomorphic to a subgroup of a quotient group of U. Let G be a group that has a presentation with $r \geqslant 2$ generators and at most $r-2$ defining relations. Is G SQ-universal? *A. M. Macbeath, P. M. Neumann*

Yes, it is (B. Baumslag, S. J. Pride, *J. London Math. Soc.*, **17** (1978), 425–426).

4.41. A point z in the complex plane is called *free* if the matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ generate a free group. Are all points outside the rhombus with vertices ± 2, $\pm i$ free? *Yu. I. Merzlyakov*

No (Yu. A. Ignatov, *Math. USSR–Sb.*, **35** (1979), 49–56; A. I. Shkuratskiĭ, *Math. Notes*, **24** (1978), 720–721).

4.45. Let G be a free product amalgamating proper subgroups H and K of A and B, respectively.

a) Suppose that H, K are finite and $|A : H| > 2$, $|B : K| \geqslant 2$. Is G SQ-universal?

b) Suppose that A, B, H, K are free groups of finite ranks. Can G be simple? *P. M. Neumann*

a) Yes, it is (K. I. Lossov, *Siberian Math. J.*, **27**, no. 6 (1986), 890–899).
b) Yes, it can (M. Burger, S. Mozes, *Inst. Hautes Études Sci. Publ. Math.*, **92** (2000), 151–194). But if one of the indices $|A : H|$, $|B : K|$ is infinite, then no, it cannot (S. V. Ivanov, P. E. Schupp, in: *Algorithmic problems in groups and semigroups, Int. Conf., Lincoln, NE, 1998*, Boston MA, Birkhäuser, 2000, 139–142).

4.46. b) We call a variety of groups a *limit variety* if it cannot be defined by finitely many laws, while each of its proper subvarieties has a finite basis of identities. It follows from Zorn's lemma that every variety that has no finite basis of identities contains a limit subvariety. Is the set of limit varieties countable? *A. Yu. Ol'shanskiĭ*

No, there are continuum of such varieties (P. A. Kozhevnikov, *On varieties of groups of large odd exponent*, Dep. 1612-V00, VINITI, Moscow, 2000 (Russian); S. V. Ivanov, A. M. Storozhev, *Contemp. Math.*, **360** (2004), 55–62).

4.47. Does there exist a countable family of groups such that every variety is generated by a subfamily of it? *A. Yu. Ol'shanskiĭ*

No (Yu. G. Kleiman, *Math. USSR–Izv.*, **22** (1984), 33–65).

4.49. Let G and H be finitely generated torsion-free nilpotent groups such that $\operatorname{Aut} G \cong \operatorname{Aut} H$. Does it follow that $G \cong H$? *V. N. Remeslennikov*

No (G. A. Noskov, V. A. Roman'kov, *Algebra and Logic*, **13** (1974), 306–311).

4.51. (Well-known problem). Are knot groups residually finite? *V. N. Remeslennikov*

Yes, they are (W. P. Thurston, *Bull. Amer. Math. Soc.*, **6** (1982), 357–381).

4.52. Let G be a finitely generated torsion-free group that is an extension of an abelian group by a nilpotent group. Then is G almost a residually finite p-group for almost all primes p? *V. N. Remeslennikov*

Yes, it is (G. A. Noskov, *Algebra and Logic*, **13** (1974), 388–393; D. Segal, *J. Algebra*, **32** (1974), 389–399).

4.53. P. F. Pickel has proved that there are only finitely many non-isomorphic finitely-generated nilpotent groups having the same family of finite homomorphic images. Can Pickel's theorem be extended to polycyclic groups? *V. N. Remeslennikov*

Yes, it can (F. J. Grunewald, P. F. Pickel, D. Segal, *Ann. of Math. (2)*, **111** (1980), 155–195).

4.54. Are any two minimal relation-modules of a finite group isomorphic? *K. W. Roggenkamp*

Not always (P. A. Linnell, *Ph. D. Thesis*, Cambridge, 1979; A. J. Sieradski, M. N. Dyer, *J. Pure Appl. Algebra*, **15** (1979), 199–217).

4.57. Let a group G be the product of two of its abelian minimax subgroups A and B. Prove or refute the following statements:

a) $A_0B_0 \neq 1$, where $A_0 = \bigcap_{x\in G} A^x$ and similarly for B_0;

b) the derived subgroup of G is a minimax subgroup. *N. F. Sesekin*

Both parts were proved (D. I. Zaitsev, *Algebra and Logic*, **19** (1980), 94–106).

4.58. Let a finite group G be the product of two subgroups A and B, where A is abelian and B is nilpotent. Find the dependence of the derived length of G on the nilpotency class of B and the order of its derived subgroup. *N. F. Sesekin*

This was found (D. I. Zaitsev, *Math. Notes*, **33** (1983), 414–419).

4.59. (P. Hall). Find the smallest positive integer n such that every countable group can be embedded in a simple group with n generators. *D. M. Smirnov*

It is proved that $n = 2$ (A. P. Goryushkin, *Math. Notes*, **16** (1974), 725–727).

4.60. (P. Hall). What is the cardinality of the set of simple groups generated by two elements, one of order 2 and the other of order 3? *D. M. Smirnov*

The cardinality of the continuum. Every 2-generated group G is embeddable into a simple group H with two generators of orders 2 and 3 (P. E. Schupp, *J. London Math. Soc.*, **13**, no. 1 (1976), 90–94). Such a group H has at most countably many 2-generator subgroups, while there are continuum of groups G. (Yu. I. Merzlyakov, 1976.)

4.61. Does there exist a linear function f with the following property: if every abelian subgroup of a finite 2-group G is generated by n elements, then G is generated by $f(n)$ elements? *S. A. Syskin*

No (A. Yu. Ol'shanskiĭ, *Math. Notes*, **23** (1978), 183–185).

4.62. Does there exist a finitely based variety of groups whose universal theory is undecidable? *A. Tarski*

Yes, there does; for example, $\mathfrak{A}^5$. Indeed, it is shown in (V. N. Remeslennikov, *Algebra and Logic*, **12**, no. 5 (1975), 327–346) that there exists a finitely presented group $G = \langle x_1, \dots, x_n \mid w_1, \dots, w_m,\ \operatorname{mod} \mathfrak{A}^5\rangle$ in $\mathfrak{A}^5$ with undecidable word problem. We put $\Phi_w = (\forall x_1, \dots, x_n)((w_1 = 1) \wedge \dots \wedge (w_m = 1) \to (w = 1))$, where w runs over all the words in $x_1, \dots, x_n$. Clearly, there is no algorithm to decide whether a formula Φ_w is true in $\mathfrak{A}^5$. (V. N. Remeslennikov, 1976.)

4.63. Does there exist a non-abelian variety of groups (in particular, one that contains the variety of all abelian groups) whose elementary theory is decidable? *A. Tarski*

No (A. P. Zamyatin, *Algebra and Logic*, **27** (1978), 13–17).

4.64. Does there exist a variety of groups that does not admit an independent system of defining identities? *A. Tarski*

Yes, there does (Yu. G. Kleiman, *Math. USSR–Izv.*, **22** (1984), 33–65).

4.67. Let G be a finite p-group. Show that the rank of the multiplicator $M(G)$ of G is bounded in terms of the rank of G. *J. Wiegold*

This was done (A. Lubotzky, A. Mann, *J. Algebra*, **105** (1987), 484–505).

4.68. Construct a finitely generated (infinite) characteristically simple group that is not a direct power of a simple group. *J. Wiegold*

This was done (J. S. Wilson, *Math. Proc. Cambridge Phil. Soc.*, **80** (1976), 19–35).

4.70. Let k be a field of characteristic different from 2, and G_k the group of transformations $A = (a, \alpha) : x \to ax + \alpha \ \ (a, \alpha \in k, \ \ a \neq 0)$. Extend G_k to the projective plane by adjoining the symbols $(0, \alpha)$ and a line at infinity. Then the lines are just the centralizers $C_G(A)$ of elements $A \in G_k$ and their cosets. Do there exist other groups G complementable to the projective plane such that the lines are just the cosets of the centralizers of elements of G? *H. Schwerdtfeger*

No (E. A. Kuznetsov, Dep. no. 7028-V89, VINITI, Moscow, 1989 (Russian)).

4.71. Let A be a group of automorphisms of a finite group G which has a series of A-invariant subgroups $G = G_0 > \cdots > G_k = 1$ such that every $|G_i : G_{i+1}|$ is prime. Prove that A is supersoluble. *L. A. Shemetkov*

This was proved (L. A. Shemetkov, *Math. USSR–Sb.*, **23** (1974), 593–611).

4.73. (Well-known problem). Does there exist a non-abelian variety of groups

a) all of whose finite groups are abelian?

b) all of whose periodic groups are abelian? *A. L. Shmel'kin*

a) Yes, there exists (A. Yu. Ol'shanskiĭ, *Math. USSR–Sb.*, **54** (1986), 57–80).

b) Yes, there exists (P. A. Kozhevnikov, *On varieties of groups of large odd exponent*, Dep. 1612-V00, VINITI, Moscow, 2000 (Russian); S. V. Ivanov, A. M. Storozhev, *Contemp. Math.*, **360** (2004), 55–62).

4.74. a) Is every 2-group of order greater than 2 non-simple? *V. P. Shunkov*

No, the existence of simple infinite 2-groups follows from the existence of finitely generated infinite groups of exponent 2^n (S. V. Ivanov, *Int. J. Algebra Comput.*, **4**, no. 1–2 (1994), 1–308; I. G. Lysënok, *Izv. Math.*, **60**, no. 3 (1996), 453–654) and from the positive solution to the Restricted Burnside Problem for groups of exponent 2^n (E. I. Zel'manov, *Math. USSR–Sb.*, **72** (1992), 543–565).

4.76. Let G be a locally finite group containing an element a of prime order such that the centralizer $C_G(a)$ is finite. Is G almost soluble? *V. P. Shunkov*

Yes, it is. P. Fong (*Osaka J. Math.*, **13** (1976), 483–489) has shown (mod CFSG) that G is almost locally soluble. Given this, B. Hartley and T. Meixner (*Arch. Math.*, **36** (1981), 211–213) have proved that G is almost locally nilpotent. Therefore G is almost soluble by (J. L. Alperin, *Proc. Amer. Math. Soc.*, **13** (1962), 175–180) and (G. Higman, *J. London Math. Soc.*, **32** (1957), 321–334). Moreover, by (E. I. Khukhro, *Math. Notes*, **38**, no. 5-6 (1985), 867–870; E. I. Khukhro, *Math. USSR–Sb.*, **71**, no. 1 (1992), 51–63) then G is almost nilpotent.

4.77. In 1972, A. Rudvalis discovered a new simple group R of order $2^{14}{\cdot}3^3{\cdot}5^3{\cdot}7{\cdot}13{\cdot}29$. He has shown that R possesses an involution i such that $C_R(i) = V \times F$, where V is a 4-group (an elementary abelian group of order 4) and $F \cong Sz(8)$.

a) Show that R is the only finite simple group G that possesses an involution i such that $C_G(i) = V \times F$, where V is a 4-group and $F \cong Sz(8)$.

b) Let G be a non-abelian finite simple group that possesses an involution i such that $C_G(i) = V \times F$, where V is an elementary abelian 2-group of order 2^n, $n \geqslant 1$, and $F \cong Sz(2^m)$, $m \geqslant 3$. Show that $n = 2$ and $m = 3$. *Z. Janko*

a) This has been shown (V. D. Mazurov, *Math. Notes*, **31** (1982), 165–173).
b) This follows from the CFSG.

5.1. a) Is every locally finite minimal non-FC-group non-simple? *V. V. Belyaev, N. F. Sesekin*

Yes, it is (M. Kuzucuoḡlu, R. E. Phillips, *Proc. Cambridge Philos. Soc.*, **105**, no. 3 (1989), 417–420).

5.2. Is it true that the growth function f of any infinite finitely generated group satisfies the inequality $f(n) \leqslant (f(n-1) + f(n+1))/2$ for all sufficiently large n (for a fixed finite system of generators)? *V. V. Belyaev, N. F. Sesekin*

No (P. de la Harpe, R. I. Grigorchuk, *Algebra and Logic*, **37**, no. 6 (1998), 353–356).

5.3. (Well-known problem). Can every finite lattice L be embedded in the lattice of subgroups of a finite group? *G. M. Bergman*

Yet, it can (P. Pudlák, J. Tůma, *Algebra Universalis*, **10** (1980), 74–95).

5.4. Let g and h be positive elements of a linearly ordered group G. Can one always embed G in a linearly ordered group $\overline{G}$ in such a way that g and h are conjugate in $\overline{G}$? *V. V. Bludov*

No, not always (V. V. Bludov, *Algebra and Logic*, **44**, no. 6 (2005), 370–380).

5.6. If R is a finitely generated integral domain of characteristic $p > 0$, does the profinite (ideal) topology of R induce the profinite topology on the group of units of R? It does for $p = 0$. *B. A. F. Wehrfritz*

Yes, it does (D. Segal, *Bull. London Math. Soc.*, **11** (1979), 186–190).

5.7. An algebraic variety X over a field k is called *rational* if the field of functions $k(X)$ is purely transcendental over k, and it is called *stably rational* if $k(X)$ becomes purely transcendental after adjoining finitely many independent variables. Let T be a stably rational torus over a field k. Is T rational? Other formulations of this question and some related results see in (V. E. Voskresenskiĭ, *Russ. Math. Surveys*, **28**, no. 4 (1973), 79–105). *V. E. Voskresenskiĭ*

Yes, it is (V. E. Voskresenskii, *J. Math. Sci. (N. Y.)*, **161**, no. 1 (2009), 176–180).

5.8. Let G be a torsion-free soluble group of finite cohomological dimension $\operatorname{cd} G$. If hG denotes the Hirsch number of G, then it is known that $hG \leqslant \operatorname{cd} G \leqslant hG+1$. Find a purely group-theoretic criterion for $hG = \operatorname{cd} G$. *K. W. Gruenberg*

This has been found (P. H. Kropholler, *J. Pure Appl. Algebra*, **43** (1986), 281–287).

5.9. Let $1 \to R_i \xrightarrow{\pi_i} F \to G \to 1$, $i = 1, 2$, be two exact sequences of groups with G finite and F free of finite rank $d(F)$. If we assume that $d(F) = d(G) + 1$ (where $d(G)$ is the minimum number of generators of G), are the corresponding abelianized extensions isomorphic? *K. W. Gruenberg*

Yes, they are (P. A. Linnell, *J. Pure Appl. Algebra*, **22** (1981), 143–166).

5.10. Let $E = H * K$ and denote the augmentation ideals of the groups E, H, K by $\mathfrak{e}$, $\mathfrak{h}$, $\mathfrak{k}$, respectively. If I is a right ideal in $\mathbb{Z}E$, let $d_E(I)$ denote the minimum number of generators of I as a right ideal. Assuming H and K finitely generated, is it true that $d_E(\mathfrak{e}) = d_E(\mathfrak{h}E) + d_E(\mathfrak{k}E)$? Here $\mathfrak{h}E$ is, of course, the right ideal generated by $\mathfrak{h}$; similarly for $\mathfrak{k}E$. *K. W. Gruenberg*

Not always (P. A. Linnell, *unpublished*). A simple example: H is elementary abelian group of order 4, K is elementary abelian of order 9. Here $d_E(\mathfrak{e}) = 3$, although $d_E(\mathfrak{h}E) = d_E(\mathfrak{k}E) = 2$. (K. W. Gruenberg, *Letter of July, 27, 1995.*)

5.11. Let G be a finite group and suppose that there exists a non-empty proper subset π of the set of all primes dividing $|G|$ such that the centralizer of every non-trivial π-element is a π-subgroup. Does it follow that G contains a subgroup U such that $U^g \cap U = 1$ or U for every $g \in G$, and the centralizer of every non-trivial element of U is contained in U? *K. W. Gruenberg*

Yes, it does (mod CFSG) (J. S. Williams, *J. Algebra*, **69** (1981), 487–513).

5.12. Let G be a finite group with trivial soluble radical in which there are Sylow 2-subgroups having non-trivial intersection. Suppose that, for any two Sylow 2-subgroups P and Q of G with $P \cap Q \neq 1$, the index $|P : P \cap Q|$ does not exceed 2^n. Is it then true that $|P| \leqslant 2^{2n}$? *V. V. Kabanov*

Yes, it is, mod CFSG (V. I. Zenkov, *Algebra and Logic*, **36**, no. 2 (1997), 93–98).

5.13. Suppose that K and L are distinct conjugacy classes of involutions in a finite group G and $\langle x, y\rangle$ is a 2-group for all $x \in K$ and $y \in L$. Does it follow that $G \neq [K, L]$? *V. V. Kabanov*

Not always; for example, $G = Sp_4(2^n)$, $n \geqslant 2$, K, L being classes of involutions that have non-trivial intersections with the centres of $N_G(M)'$ and $N_G(N)'$, respectively, where M, N are distinct elementary abelian subgroups of order 8 in a Sylow 2-subgroup of G and the dash means taking the derived subgroup (A. A. Makhnëv, *Letter of October, 10, 1981*).

5.17. If the finite group G has the form $G = AB$ where A and B are nilpotent of classes α and β, respectively, then G is soluble. Is $G^{(\alpha+\beta)} = 1$? One can show that $G^{(\alpha+\beta)}$ is nilpotent (E. Pennington, 1973). One can ask the same question for infinite groups (or Lie algebras), but there is nothing known beyond Ito's theorem: $A' = B' = 1$ implies $G^{(2)} = 1$. *O. H. Kegel*

Not always (J. Cossey, S. Stonehewer, *Bull. London Math. Soc.*, **30**, no. 144 (1998), 247–250). See also new problem 14.43.

5.18. Let G be an infinite locally finite simple group. Is the centralizer of every element of G infinite? *O. H. Kegel*

Yes, it is (B. Hartley, M. Kuzucuoḡlu, *Proc. London Math. Soc. (3)*, **62**, no. 2 (1991), 301–324).

5.19. a) Let G be an infinite locally finite simple group satisfying the minimum condition for 2-subgroups. Is $G = PSL_2(F)$, F some locally finite field of odd characteristic, if the centralizer of every involution of G is almost locally soluble?

b) Can one characterize the simple locally finite groups with the min-2 condition containing a maximal radical non-trivial 2-subgroup of rank $\leqslant 2$ as linear groups of small rank? *O. H. Kegel*

a) Yes, it is (N. S. Chernikov, *Ukrain. Math. J.*, **35** (1983), 230–231).
b) Yes, one can (mod CFSG) by Theorem 4.8 of (O. H. Kegel, B. A. F. Wehrfritz, *Locally finite groups*, North Holland, Amsterdam, 1973).

5.20. Is the elementary theory of lattices of l-ideals of lattice-ordered abelian groups decidable? *A. I. Kokorin*

No, it is undecidable (N. Ya. Medvedev, *Algebra and Logic*, **44**, no. 5 (2005), 302–312).

5.22. Does there exist a version of the Higman embedding theorem in which the degree of unsolvability of the conjugacy problem is preserved? *D. J. Collins*

Yes, it exists (A. Yu. Ol'shanskii, M. V. Sapir, *The conjugacy problem and Higman embeddings* (*Memoirs AMS*, **804**), 2004, 133 p.).

5.23. Is it true that a free lattice-ordered group of the variety of lattice-ordered groups defined by the law $x^{-1}|y|x \ll |y|^2$ (or, equivalently, by the law $|[x, y]| \ll |x|$), is residually linearly ordered nilpotent? *V. M. Kopytov*

No, it is not (V. V. Bludov, A. M. W. Glass, *Trans. Amer. Math. Soc.*, **358** (2006), 5179–5192).

5.24. Is it true that a free lattice-ordered group of the variety of the lattice-ordered groups which are residually linearly ordered, is residually soluble linearly ordered? *V. M. Kopytov*

No, it is not (N. Ya. Medvedev, *Algebra and Logic*, **44**, no. 3 (2005), 197–204).

5.28. Let G be a group and H a torsion-free subgroup of G such that the augmentation ideal I_G of the integral group-ring $\mathbb{Z}G$ can be decomposed as $I_G = I_H\mathbb{Z}G \otimes M$ for some $\mathbb{Z}G$-submodule M. Prove that G is a free product of the form $G = H * K$. *D. E. Cohen*

This has been proved (W. Dicks, M. J. Dunwoody, *Groups acting on graphs*, Cambridge Univ. Press, Cambridge–New York, 1989).

5.29. Consider a group G given by a presentation with m generators and n defining relations, where $m \geqslant n$. Do some $m - n$ of the given generators generate a free subgroup of G? *R. C. Lyndon*

Yes, they do (N. S. Romanovskiĭ, *Algebra and Logic*, **16** (1977), 62–68).

5.32. Let p be a prime, C a conjugacy class of p-elements of a finite group G and suppose that for any two elements x and y of C the product xy^{-1} is a p-element. Is the subgroup generated by the class C a p-group? *V. D. Mazurov*

Yes, it is (for $p > 2$: W. Xiao, *Sci. in China*, **34**, no. 9 (1991), 1025–1031; for $p = 2$: V. I. Zenkov, in: *Algebra. Proc. IIIrd Int. Conf. on Algebra, Krasnoyarsk, 1993*, Berlin, De Gruyter, 1996, 297–302).

5.34. Let $\mathfrak{o}$ be a commutative ring with identity in which 2 is invertible and which is not generated by zero divisors. Do there exist non–standard automorphisms of $GL_n(\mathfrak{o})$ for $n \geqslant 3$? *Yu. I. Merzlyakov*

No (V. Ya. Bloshchitsyn, *Algebra and Logic*, **17** (1978), 415–417).

5.40. Let G be a countable group acting on a set Ω. Suppose that G is k-fold transitive for every finite k, and G contains no non-trivial permutations of finite support. Is it true that Ω can be identified with the rational line $\mathbb{Q}$ in such a way that G becomes a group of autohomeomorphisms? *P. M. Neumann*

Not always (A. H. Mekler, *J. London Math. Soc.*, **33** (1986), 49–58).

5.41. Does every non-trivial finite group, which is free in some variety, contain a non-trivial abelian normal subgroup? *A. Yu. Ol'shanskiĭ*

Yes, it does (mod CFSG). Otherwise, if G is a counterexample, the centralizer of the product E of all minimal normal subgroups of G is trivial and there is an element of G/E whose order is m, where m is the maximum of the orders of 2-elements of G. By Theorem 1 in (M. Aschbacher, P. B. Kleidman, M. W. Liebeck, *Math. Z.*, **208**, no. 3 (1991), 401–409), which is proved using CFSG, there is an element of order $2m$ in G, which contradicts the choice of m. (S. A. Syskin, *Letter of August, 20, 1992.*)

5.43. Does there exist a soluble variety of groups that is not generated by its finite groups? *A. Yu. Ol'shanskiĭ*

Yes, there exists (Yu. G. Kleiman, *Trans. Moscow Math. Soc.*, **1983**, No. 2, 63–110).

5.46. Is every recursively presented soluble group embeddable in a group finitely presented in the variety of all soluble groups of derived length n, for suitable n? *V. N. Remeslennikov*

Yes, it is (G. P. Kukin, *Soviet Math. Dokl.*, **21** (1980), 378–381).

5.49. Let $A_{m,n}$ be the group of all automorphisms of the free soluble group of derived length n and rank m. Is it true that

a) $A_{m,n}$ is finitely generated for any m and n?

b) every automorphism in $A_{m,n}$ is induced by some automorphism in $A_{m,n+1}$? *V. N. Remeslennikov*

a) No: $A_{3,2}$ is not finitely generated, although $A_{m,2}$ is whenever $m \neq 3$ (S. Bachmuth, H. Y. Mochizuki, *Trans. Amer. Math. Soc.*, **270** (1982), 693–700, **292** (1985), 81–101).

b) No (V. Shpilrain, *Int. J. Algebra and Comput.*, **1**, no. 2 (1991), 177–184).

5.50. Is there a finite group whose set of quasi-laws does not have an independent basis? *D. M. Smirnov*

Yes, there is (A. N. Fedorov, *Siberian Math. J.*, **21** (1980), 840–850).

5.51. Does a non-abelian free group have an independent basis for its quasi-laws? *D. M. Smirnov*

Yes, it does (A. I. Budkin, *Math. Notes*, **31** (1982), 413–417).

5.53. (P. Scott). Let p, q, r be distinct prime numbers. Prove that the free product $G = C_p * C_q * C_r$ of cyclics of orders p, q, r is not the normal closure of a single element. By Lemma 3.1 of (J. Wiegold, *J. Austral. Math. Soc.*, **17**, no. 2 (1974), 133–141), every soluble image, and every finite image, of G is the normal closure of a single element. *J. Wiegold*

This is proved in (J. Howie, *J. Pure Appl. Algebra*, **173**, no. 2 (2002), 167–176).

5.56. b) Does there exist a locally nilpotent group of prime exponent that coincides with its derived subgroup (and hence has no maximal subgroups)? *J. Wiegold*

Yes, there exists. Every non-soluble variety $\mathfrak{V}$ contains a non-trivial group coinciding with the derived subgroup, the direct limit of the spectrum $F \xrightarrow{\varphi} F \xrightarrow{\varphi} \ldots$, where F is the free group in $\mathfrak{V}$ on the free generators $x_1, x_2, \ldots$ and φ is the homomorphism given by $x_i \to [x_{2i-1}, x_{2i}]$. As shown in (Yu. P. Razmyslov, *Algebra and Logic*, **10** (1971), 21–29) the Kostrikin variety of locally nilpotent groups of prime exponent $p \geqslant 5$ is unsoluble. (E. I. Khukhro, I. V. L'vov, *Letter of June, 19, 1976.*) The same was also proved in (Yu. A. Kolmakov, *Math. Notes*, **35**, no. 5–6 (1984), 389–391). An example answering the question was also produced in (M. R. Vaughan-Lee, J. Wiegold, *Bull. London Math. Soc.*, **13**, no. 1 (1981), 45–46).

5.60. Is an arbitrary soluble group that satisfies the minimum condition for normal subgroups countable? *B. Hartley*

Not always (B. Hartley, *Proc. London Math. Soc.*, **33** (1977), 55–75).

5.61. (Well-known problem). Does an arbitrary uncountable locally finite group have only one end? *B. Hartley*

Yes, it does (D. Holt, *Bull. London Math. Soc.*, **13** (1981), 557–560).

5.62. Is a group locally finite if it contains infinite abelian subgroups and all of them are complementable? *S. N. Chernikov*

Not always (N. S. Chernikov, *Math. Notes*, **28** (1980), 788–792).

5.63. Prove that a finite group is not simple if it contains two non-identity elements whose centralizers have coprime indices. *S. A. Chunikhin*

This has been proved mod CFSG (L. S. Kazarin, in: *Studies in Group Theory*, Sverdlovsk, 1984, 81–99 (Russian)).

5.64. Suppose that a finite group G is the product of two subgroups A_1 and A_2. Prove that if A_i contains a nilpotent subgroup of index $\leqslant 2$ for $i = 1, 2$, then G is soluble. *L. A. Shemetkov*

This has been proved (L. S. Kazarin, *Math. USSR–Sb.*, **38** (1981), 47–59).

5.68. Let G be a finitely-presented group, and assume that G has polynomial growth in the sense of Milnor. Show that G has soluble word problem. *P. E. Schupp*

This has been shown (M. Gromov, *Publ. Math. IHES*, **53** (1981), 53–73).

5.69. Is every lattice isomorphism between torsion-free groups having no non-trivial cyclic normal subgroups induced by a group isomorphism? *B. V. Yakovlev*

No (A. Yu. Ol'shanskiĭ, *C. R. Acad. Bulgare Sci.*, **32** (1979), 1165–1166).

6.4. A group G is called *of type* $(FP)_\infty$ if the trivial G-module $\mathbb{Z}$ has a resolution by finitely generated projective G-modules. Is it true that every torsion-free group of type $(FP)_\infty$ has finite cohomological dimension? *R. Bieri*

No (K. S. Brown, R. Geoghegan, *Invent. Math.*, **77** (1984), 367–381).

6.6. Let P, Q be permutation representations of a finite group G with the same character. Suppose $P(G)$ is a primitive permutation group. Is $Q(G)$ necessarily primitive? The answer is known to be affirmative if G is soluble. *H. Wielandt*

No, not necessarily (mod CFSG) (R. M. Guralnick, J. Saxl, in: *Groups, Combinatorics, Geometry, Proc. Durham, 1990*, Cambridge Univ. Press, 1992, 364–367).

6.7. Suppose that P is a finite 2-group. Does there exist a characteristic subgroup $L(P)$ of P such that $L(P)$ is normal in H for every finite group H that satisfies the following conditions: 1) P is a Sylow 2-subgroup of H, 2) H is $\mathbb{S}_4$-free, and 3) $C_H(O_2(H)) \leqslant O_2(H)$? *G. Glauberman*

Yes, there exists (B. Stellmacher, *Israel J. Math.*, **94** (1996), 367–379).

6.8. Can a residually finite locally normal group be embedded in a Cartesian product of finite groups in such a way that each element of the group has at most finitely many central projections? *Yu. M. Gorchakov*

Yes, it can (M. J. Tomkinson, *Bull. London Math. Soc.*, **13** (1981), 133–137).

6.12. a) Is a metabelian group countable if it satisfies the weak minimum condition for normal subgroups?

b) Is such a group minimax if it is torsion-free? *D. I. Zaitsev*

Yes, it is, in both cases (D. I. Zaitsev, L. A. Kurdachenko, A. V. Tushev, *Algebra and Logic*, **24** (1985), 412–436).

6.13. Is it true that if a non-abelian Sylow 2-subgroup of a finite group G has a non-trivial abelian direct factor, then G is not simple? *A. S. Kondratiev*

Yes, it is, mod CFSG (V. V. Kabanov, A. S. Kondratiev, *Sylow 2-subgroups of finite groups (a survey)*, Inst. Math. Mech. UNC AN SSSR, Sverdlovsk, 1979 (Russian)).

6.14. Are the following lattices locally finite: the lattice of all locally finite varieties of groups? the lattice of all varieties of groups? *A. V. Kuznetsov*

No, they are not (M. I. Anokhin, *Izv. Math.*, **63**, no. 4 (1999), 649–665).

6.15. A variety is said to be *pro-locally-finite* if it is not locally finite while all of its proper subvarieties are locally finite. An example — the variety of abelian groups. How many pro-locally-finite varieties of groups are there? *A. V. Kuznetsov*

There are continuum of such varieties (P. A. Kozhevnikov, *On varieties of groups of large odd exponent*, Dep. 1612-V00, VINITI, Moscow, 2000 (Russian)).

6.16. A variety is called *sparse* if it has at most countably many subvarieties. How many sparse varieties of groups are there? *A. V. Kuznetsov*

There are continuum of such varieties (P. A. Kozhevnikov, *On varieties of groups of large odd exponent*, Dep. 1612-V00, VINITI, Moscow, 2000 (Russian); S. V. Ivanov, A. M. Storozhev, *Contemp. Math.*, **360** (2004), 55–62).

6.17. Is every variety of groups generated by its finitely generated groups that have soluble word problem? *A. V. Kuznetsov*

No (Yu. G. Kleiman, *Math. USSR–Izv.*, **22** (1984), 33–65).

6.18. (Well-known problem). Suppose that a class $\mathfrak{K}$ of 2-generator groups generates the variety of all groups. Is a non-cyclic free group residually in $\mathfrak{K}$? *V. M. Levchuk*

Not always (S. V. Ivanov, *Geometric methods in the study of groups with given subgroup properties*, Candidate Dissertation, Moscow University, 1988 (Russian)).

6.19. Let R be a nilpotent associative ring. Are the following two statements for a subgroup H of the adjoint group of R always equivalent: 1) H is a normal subgroup; 2) H is an ideal of the groupoid R with respect to Lie multiplication? *V. M. Levchuk*

Not always. Let R be the free nilpotent of index 3 associative algebra over $\mathbb{F}_2$ on the free generators x, y. Let S be the subalgebra generated by the elements $[x, y^2] = x{*}y^2$, $[x, xy] = x * (xy) = x(x * y)$, $[x, yx] = x * (yx) = (x * y)x$, where $[a, b]$ denotes the commutator in the adjoint group with multiplication $a \circ b = a + b + ab$, and $a * b = ab - ba$ is Lie multiplication. Let M, N be the subalgebras generated by S and the elements $x * y$ and $[x, y]$, respectively. The minimal subgroup of $(R, \circ)$ that contains x and is an ideal of the groupoid $(R, *)$ equals $\langle x\rangle \circ M = \langle x\rangle + M$, where $\langle x\rangle = \{0,\, x,\, x^2,\, x + x^2 + x^3\}$, while the minimal normal subgroup containing x equals $\langle x\rangle \circ N = \langle x\rangle + N$. Neither is contained in the other. (E. I. Khukhro, *Letter of July, 23, 1979.*)

6.20. Does there exist a supersoluble group of odd order, all of whose automorphisms are inner? *V. D. Mazurov*

Yes, there does (B. Hartley, D. J. S. Robinson, *Arch. Math.*, **35** (1980), 67–74).

6.22. Construct a braid that belongs to the derived subgroup of the braid group but is not a commutator. *G. S. Makanin*

Let $x = \sigma_1$, $y = \sigma_2$ be the standard generators of the braid group B_3; then the braid $(xyxyx)^{12}(xy)^{-12}(xyx)^{-12}$ belongs to the derived subgroup of B_3 but is not a commutator (Yu. S. Semënov, *Abstracts of the 10th All-Union Sympos. on Group Theory*, Minsk, 1986, p. 207 (Russian)).

6.23. A braid K of the braid group $\mathfrak{B}_{n+1}$ is said to be *smooth* if removing any of the threads in K transforms K into a braid that is equal to 1 in $\mathfrak{B}_n$. It is known that smooth braids form a free subgroup. Describe generators of this subgroup. *G. S. Makanin*

They are described in (D. L. Johnson, *Math. Proc. Camb. Philos. Soc.*, **92** (1982), 425–427).

6.25. (Well-known problem). Find an algorithm for calculating the rank of coefficient-free equations in a free group. The *rank* of an equation is the maximal rank of the free subgroup generated by a solution of this equation. *G. S. Makanin*

This was found (A. A. Razborov, *Math. USSR–Izv.*, **25** (1984), 115–162).

6.31. a) Suppose that G is a finitely-generated residually-finite group, $d(G)$ the minimal number of generators of G, and $\delta(G)$ the minimal number of topological generators of the profinite completion of G. Is $\delta(G) = d(G)$ always true? *O. V. Mel'nikov*

Not always (G. A. Noskov, *Math. Notes*, **33** (1983), 249–254).

6.34. Let $\mathfrak{o}$ be an associative ring with identity. A system of its ideals $\mathfrak{A} = \{\mathfrak{A}_{ij} \mid i, j \in \mathbb{Z}\}$ is called a *carpet of ideals* if $\mathfrak{A}_{ik}\mathfrak{A}_{kj} \subseteq \mathfrak{A}_{ij}$ for all $i, j, k \in \mathbb{Z}$. If $\mathfrak{o}$ is commutative, then the set $\Gamma_n(A) = \{x \in SL_n(\mathfrak{o}) \mid x_{ij} \equiv \delta_{ij} \pmod{\mathfrak{A}_{ij}}\}$ is a group, the (*special*) *congruenz-subgroup modulo the carpet* $\mathfrak{A}$ (the *"carpet subgroup"*). Under quite general conditions, it was proved in (Yu. I. Merzlyakov, *Algebra i Logika*, **3**, no. 4 (1964), 49–59 (Russian); see also M. I. Kargapolov, Yu. I. Merzlyakov, *Fundamentals of the Theory of Groups*, 3rd Ed., Moscow, Nauka, 1982, p. 145 (Russian)) that in the groups GL_n and SL_n the mutual commutator subgroup of the congruenz-subgroups modulo a carpet of ideals shifted by k and l steps is again the congruenz-subgroup modulo the same carpet shifted by $k + l$ steps. Prove analogous theorems a) for orthogonal groups; b) for unitary groups. *Yu. I. Merzlyakov*

These are proved (V. M. Levchuk, *Sov. Math. Dokl.*, **42**, no. 1 (1991), 82–86; *Ukrain. Math. J.*, **44**, no. 6 (1992), 710–718).

6.35. (R. Bieri, R. Strebel). Let $\mathfrak{o}$ be an associative ring with identity distinct from zero. A group G is said to be *almost finitely presented over* $\mathfrak{o}$ if it has a presentation $G = F/R$ where F is a finitely generated free group and the $\mathfrak{o}G$-module $R/[R, R] \otimes_{\mathbb{Z}} \mathfrak{o}$ is finitely generated. It is easy to see that every finitely presented group G is almost finitely presented over $\mathbb{Z}$ and therefore also over an arbitrary ring $\mathfrak{o}$. Is the converse true? *Yu. I. Merzlyakov*

No, it is not true (M. Bestvina, N. Brady, *Invent. Math.*, **129**, no. 3 (1997), 445–470).

6.36. (J. W. Grossman). The *nilpotent-completion diagram* of a group G is as follows: $G/\gamma_1 G \leftarrow G/\gamma_2 G \leftarrow \cdots$, where $\gamma_i G$ is the ith term of the lower central series and the arrows are natural homomorphisms. It is easy to see that every nilpotent-completion diagram $G_1 \leftarrow G_2 \leftarrow \cdots$ is a *γ-diagram*, that is, every sequence $1 \to \gamma_s G_{s+1} \to G_{s+1} \to G_s \to 1$, $s = 1, 2, \ldots$, is exact. Do γ-diagrams exist that are not nilpotent-completion diagrams? *Yu. I. Merzlyakov*

Yes, they exists (N. S. Romanovskiĭ, *Sibirsk. Mat. Zh.*, **26**, no. 4 (1985), 194–195 (Russian)).

6.37. (H. Wielandt, O. H. Kegel). Is a finite group G soluble if it has soluble subgroups A, B, C such that $G = AB = AC = BC$? *V. S. Monakhov*

Yes, it is (mod CFSG) (L. S. Kazarin, *Commun. Algebra*, **14** (1986), 1001–1066).

6.38. a) Let k be a (commutative) field. Find all irreducible subgroups G of $GL_n(k)$ having the property that $G \cap C \neq \varnothing$ for every conjugacy class C of $GL_n(k)$. I conjecture that $G = GL_n(k)$ except in case $n = \operatorname{char} k = 2$, the field k is quadratically closed, and G is conjugate to the group of all matrices of the form $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$ where $\alpha \neq 0$ and $\beta \neq 0$. *Peter M. Neumann*

a) The conjecture is refuted (S. A. Zyubin, *Algebra and Logic*, **45**, no. 5 (2006), 296–305).

6.42. Let H be a strongly 3-embedded subgroup of a finite group G. Suppose that $Z(H/O_{3'}(H))$ contains an element of order 3. Does $Z(G/O_{3'}(G))$ necessarily contain an element of order 3? *N. D. Podufalov*

Yes, it does (mod CFSG) (W. Xiao, *Sci. China (A)*, **33** (1990), 1172–1181).

6.43. Does the set of quasi-identities holding in the class of all finite groups possess a basis in finitely many variables? *D. M. Smirnov*

No (A. K. Rumyantsev, *Algebra and Logic*, **19** (1980), 297–311).

6.44. Construct a finitely generated infinite simple group requiring more than two generators. *J. Wiegold*

This has been done (V. S. Guba, *Siberian Math. J.*, **27** (1986), 670–684).

6.46. If G is d-generator group having no non-trivial finite homomorphic images (in particular, if G is an infinite simple d-generator group) for some integer $d \geqslant 2$, must $G \times G$ be a d-generator group? *J. S. Wilson*

No, it must not (V. N. Obraztsov, *Proc. Roy. Soc. Edinburgh*, **123** (1993), 839–855).

6.49. Is the minimal condition for abelian normal subgroups inherited by subgroups of finite index? This is true for the minimal condition for (all) abelian subgroups (J. S. Wilson, *Math. Z.*, **114** (1970), 19–21). *S. A. Chechin*

No, not always. Let $G = [(A \times B \times C \times D) \rtimes (\langle g \rangle \times \langle t \rangle)] \rtimes (Y \times \langle x \rangle)$, where A, B, C, D, Y are quasicyclic p-groups, $x^2 = 1$, while g and t are of infinite order. Let $A = \bigcup_{n=1}^{\infty} \langle a_n \rangle$, $B = \bigcup_{n=1}^{\infty} \langle b_n \rangle$, $C = \bigcup_{n=1}^{\infty} \langle c_n \rangle$, $D = \bigcup_{n=1}^{\infty} \langle d_n \rangle$, $Y = \bigcup_{n=1}^{\infty} \langle y_n \rangle$ with $a_{n+1}^p = a_n$, $b_{n+1}^p = b_n$, $c_{n+1}^p = c_n$, $d_{n+1}^p = d_n$, $y_{n+1}^p = y_n$. We impose the relations $[ABCD, Y] = [ABC, g] = [ABD, t] = 1$; $[x, g] = gt^{-1}$; $[x, a_n] = a_n b_n^{-1}$; $[x, c_n] = c_n d_n^{-1}$; $[g, d_n] = b_n$; $[t, c_n] = a_n$; $[y_n, g] = c_n$; $[y_n, t] = d_n$. Then all abelian normal subgroups of G satisfy the minimal condition for subgroups. The subgroup $H = [(A \times B \times C \times D) \rtimes (\langle g \rangle \times \langle t \rangle)] \rtimes Y$ does not satisfy the minimal condition for abelian normal subgroups, since the subgroups $E_n = A \times B \times C \times \langle g^{2^n} \rangle$ are normal in H and form a strictly decreasing chain. (S. A. Chechin, *Abstracts of 15th All-USSR Algebraic Conf.*, Krasnoyarsk, 1979).

6.52. Let f be a local screen of a formation which contains all finite nilpotent groups and let A be a group of automorphisms of a finite group G. Suppose that A acts f-stably on the socle of $G/\Phi(G)$. Is it true that A acts f-stably on $\Phi(G)$? *L. A. Shemetkov*

No, not always (A. N. Skiba, *Siber. Math. J.*, **34** (1993), 953–958).

6.53. A group G of the form $G = F \rtimes H$ is said to be a *Frobenius group with kernel F and complement H* if $H \cap H^g = 1$ for any $g \in G \setminus H$ and $F \setminus \{1\} = G \setminus \bigcup_{g \in G} H^g$. What can be said about the kernel and the complement of a Frobenius group? In particular, which groups can be kernels? complements? *V. P. Shunkov*

Every group can be embedded into the kernel of a Frobenius group, and every right-orderable group can be a complement in a Frobenius group (V. V. Bludov, *Siberian Math. J.*, **38**, no. 6 (1997), 1054–1056)

6.54. Are there infinite finitely generated Frobenius groups? *V. P. Shunkov*

Yes, such groups exist (A. I. Sozutov, *Siberian Math. J.*, **35**, no. 4 (1994), 795–801).

6.57. A group G is said to be (*conjugacy, p-conjugacy*) *biprimitively finite* if, for any finite subgroup H, any two elements of prime order (any two conjugate elements of prime order, of prime order p) in $N_G(H)/H$ generate a finite subgroup. Do the elements of finite order in a (conjugacy) biprimitively finite group G form a subgroup (the periodic part of G)? *V. P. Shunkov*

Not always (A. A. Cherep, *Algebra and Logic*, **29** (1987), 311–313).

6.58. Are
a) the Alёshin p-groups and
b) the 2-generator Golod p-groups
conjugacy biprimitively finite groups? *V. P. Shunkov*

a) Not always (A. V. Rozhkov, *Math. USSR–Sb.*, **57** (1987), 437–448).
b) Not always (A. V. Timofeenko, *Algebra and Logic*, **24** (1985), 129–139).

6.63. An infinite group G is called a *monster of the first kind* if it has elements of order > 2 and for any such an element a and for any proper subgroup H of G, there is an element g in $G \setminus H$, such that $\langle a, a^g \rangle = G$. Classify the monsters of the first kind all of whose proper subgroups are finite. *V. P. Shunkov*

The centre of such a group coincides with the set of elements of order $\leqslant 2$ (V. P. Shunkov, *Algebra and Logic*, **7** (1968), no. 1 (1970), 66–69). An infinite group all of whose proper subgroups are finite is a monster of the first kind if its centre coincides with the set of elements of order $\leqslant 2$ (A. I. Sozutov, *Algebra and Logic*, **36**, no. 5 (1997), 336–348). There are continuously many such groups (A. Yu. Ol'shanskii, *Geometry of defining relations in groups*, Kluwer, Dordrecht, 1991).

6.64. A group G is called a *monster of the second kind* if it has elements of order > 2 and if for any such element a and any proper subgroup H of G there exists an infinite subset $\mathfrak{M}_{a,H}$ consisting of conjugates of a by elements of $G \setminus H$ such that $\langle a, c \rangle = G$ for all $c \in \mathfrak{M}_{a,H}$. Do mixed monsters (that is, with elements of both finite and infinite orders) of the second kind exist? Do there exist torsion-free monsters of the second kind? *V. P. Shunkov*

Yes, they exist, in both cases (A. Yu. Ol'shanskiĭ, *The geometry of defining relations in groups*, Kluwer, Dordrecht, 1991).

7.1. The free periodic groups $B(m,p)$ of prime exponent $p > 665$ are known to possess many properties similar to those of absolutely free groups (see S. I. Adian, *The Burnside Problem and Identities in Groups*, Springer, Berlin, 1979). Is it true that all normal subgroups of $B(m,p)$ are not free periodic groups? *S. I. Adian*

Yes, it is true for all sufficiently large p (A. Yu. Ol'shanskii, in: *Groups, rings, Lie and Hopf algebras, Int. Workshop, Canada, 2001*, Dordrecht, Kluwer, 2003, 179–187); this is also proved for all primes $p \geqslant 1003$ (V. S. Atabekyan, *Fund. Prikl. Mat.*, **15**, no. 1 (2009), 3–21 (Russian)).

7.2. Prove that the free periodic groups $B(m,n)$ of odd exponent $n \geqslant 665$ with $m \geqslant 2$ generators are non-amenable and that random walks on these groups do not have the recurrence property. *S. I. Adian*

Both assertions are proved (S. I. Adian, *Math. USSR–Izv.*, **21** (1982), 425–434).

7.4. Is a finitely generated group with quadratic growth almost abelian? *V. V. Belyaev*

Yes, it is (M. Gromov, *Publ. Math. IHES*, **53** (1981), 53–73; J. A. Wolf, *Diff. Geometry*, **2** (1968), 421–446).

7.6. Describe the infinite simple locally finite groups with a Chernikov Sylow 2-subgroup. In particular, are such groups the Chevalley groups over locally finite fields of odd characteristic? *V. V. Belyaev, N. F. Sesekin*

No, as every countably infinite locally finite p-group can be embedded as a maximal p-subgroup of the simple Hall's universal group U, the unique countable existentially closed group in the class of all locally finite groups (K. Hickin, *Proc. London Math. Soc. (3)*, **52**, no. 1 (1986), 53–72). But if all Sylow 2-subgroups are Chernikov, then this is true mod CFSG (O. H. Kegel, *Math. Z.*, **95** (1967), 169–195; V. V. Belyaev, in: *Investigations in Group Theory,* Sverdlovsk, UNC AN SSSR, 1984, 39–50 (Russian); A. V. Borovik, *Siberian Math. J.*, **24**, no. 6 (1983), 843–851; B. Hartley, G. Shute, *Quart. J. Math. Oxford (2)*, **35** (1984), 49–71; S. Thomas, *Arch. Math.*, **41** (1983), 103–116). When all Sylow 2-subgroups are Chernikov, the same result, without using CFSG, follows from (M. J. Larsen, R. Pink, *J. Amer. Math. Soc.*, **24** (2011), 1105–1158 (also known as a preprint of 1998)).

7.7. (Well-known problem). Is the group $G = \langle a, b \mid a^9 = 1,\ \ ab = b^2a^2 \rangle$ finite? This group contains $F(2,9)$, the only Fibonacci group for which it is not yet known whether it is finite or infinite. *R. G. Burns*

No, it is infinite, since $F(2,9)$ is infinite (M. F. Newman, *Arch. Math.*, **54**, no. 3 (1990), 209–212).

7.8. Suppose that H is a normal subgroup of a group G, where H and G are subdirect products of the same n groups $G_1, \ldots, G_n$. Does the nilpotency class of G/H increase with n? *Yu. M. Gorchakov*

Yes, it does (E. I. Khukhro, *Sibirsk. Mat. Zh.*, **23**, no. 6 (1982), 178–180 (Russian)).

7.12. Find all groups with a Hall $2'$-subgroup. *R. L. Griess*

These are found mod CFSG (Z. Arad, M. B. Ward, *J. Algebra*, **77** (1982), 234–246).

7.16. If H is a proper subgroup of the finite group G, is there always an element of prime-power order not conjugate to an element of H? *R. L. Griess*

Yes, always (mod CFSG) (B. Fein, W. M. Kantor, M. Schacher, *J. Reine Angew. Math.*, **328** (1981), 39–57).

7.22. Suppose that a finite group G is realized as the automorphism group of some torsion-free abelian group. Is it true that for every infinite cardinal $\mathfrak{m}$ there exist $2^{\mathfrak{m}}$ non-isomorphic torsion-free abelian groups of cardinality $\mathfrak{m}$ whose automorphism groups are isomorphic to G? *S. F. Kozhukhov*

Yes, this is true in the Zermelo–Frenkel system with axioms of choice and 'weak diamond' (M. Dugas, R. Göbel, *Proc. London Math. Soc. (3)*, **45**, no. 2 (1982), 319–336), or if $\mathfrak{m}$ is smaller than the first measurable cardinal (V. A. Nikiforov, *Mat. Zametki*, **39**, no. 5 (1986), 641–646 (Russian)).

7.24. We say that a group is *sparse* if the variety generated by it has at most countably many subvarieties. Does there exist a finitely generated sparse group that has undecidable word problem? *A. V. Kuznetsov*

Yes, such groups do exist; for example, a free group of $\mathfrak{N}_3\mathfrak{A}$. By (A. N. Krasil'nikov, *Math. USSR–Izv.*, **37** (1991), 539–553) every subvariety of $\mathfrak{N}_3\mathfrak{A}$ has a finite basis for its laws; hence there are only countably many of them. It has undecidable word problem by (O. G. Kharlampovich, *Sov. Math.*, **32**, no. 11 (1988), 136–140).

7.30. Which finite simple groups can be generated by three involutions, two of which commute? *V. D. Mazurov*

The answer is known mod CFSG. For the alternating groups and groups of Lie type see (Ya. N. Nuzhin, *Algebra and Logic*, **36**, no. 4 (1997), 245–256). For sporadic groups B. L. Abasheev, A. V. Ershov, N. S. Nevmerzhitskaya, S. Norton, Ya. N. Nuzhin, A. V. Timofeenko have shown that the groups M_{11}, M_{22}, M_{23}, and McL cannot be generated as required, while the others can; see more details in (V. D. Mazurov, *Siberian Math. J.*, **44**, no. 1 (2003), 160–164).

7.36. Is it true that every residually finite group defined by a single defining relation (but neither trivial nor free) is isomorphic to the fundamental group of a compact surface? *O. V. Mel'nikov*

No; for example, let $H_n = \langle x, y \mid y^{-1}xy = x^n \rangle$, $n = 2, 3, \ldots$; then every subgroup of finite index in H_n is isomorphic to a group H_m for some m (V. A. Churkin, *Abstracts of 8th All-USSR Symp. on Group Theory*, Kiev, 1982, 139–140 (Russian)).

7.37. We say that a profinite group is *strictly complete* if each of its subgroups of finite index is open. It is known (B. Hartley, *Math. Z.*, **168**, no. 1 (1979), 71–76) that finitely generated profinite groups having a finite series with pronilpotent factors are strictly complete. Is a profinite group strictly complete if it is

a) finitely generated?

b) finitely generated and prosoluble? *O. V. Mel'nikov*

a) Yes, it is (N. Nikolov, D. Segal, *Ann. Math. (2)*, **165**, no. 1 (2007), 171–273).

b) Yes, it is (D. Segal, *Proc. London Math. Soc. (3)*, **81** (2000), 29–54).

7.42. A group U is called an F_q-*group* (where $q \in \pi(U)$) if, for each finite subgroup K of U and for any two elements a, b of order q in $T = N_U(K)/K$, there exists $c \in T$ such that the group $\langle a,\, b^c\rangle$ is finite. A group U is called an F^*-*group* if each subgroup H of U is an F_q-group for every $q \in \pi(H)$ (V. P. Shunkov, 1977).

a) Is every primary F^*-group satisfying the minimum condition for subgroups almost abelian?

b) Does every F^*-group satisfying the minimum condition for (abelian) subgroups possess the radicable part? *A. N. Ostylovskiĭ*

No. A counterexample to both questions is given by an infinite group all of whose subgroups are conjugate and have prime order (A. Yu. Ol'shanskiĭ, *Math. USSR–Izv.*, **16** (1981), 279–289).

7.44. Does a normal subgroup H in a finite group G possess a complement in G if each Sylow subgroups of H is a direct factor in some Sylow subgroup of G? *V. I. Sergiyenko*

Yes, it does, mod CFSG (W. Xiao, *J. Pure Appl. Algebra*, **87**, no. 1 (1993), 97–98).

7.48. (Well-known problem). Suppose that, in a finite group G, each two elements of the same order are conjugate. Is then $|G| \leqslant 6$? *S. A. Syskin*

Yes, it is, mod CFSG (P. Fitzpatrick, *Proc. Roy. Irish Acad. Sect. A*, **85**, no. 1 (1985), 53–58); see also (W. Feit, G. Seitz, *Illinois J. Math.*, **33**, no. 1 (1988), 103–131) and (R. W. van der Waall, A. Bensaïd, *Simon Stevin*, **65** (1991), 361–374).

7.53. Let p be a prime. The law $x \cdot x^{\varphi} \cdots x^{\varphi^{p-1}} = 1$ from the definition of a splitting automorphism (see Archive, 1.10) gives rise to a variety of groups with operators $\langle\varphi\rangle$ consisting of all groups that admit a splitting automorphism of order p. Does the analogue of Kostrikin's theorem hold for this variety, that is, do the locally nilpotent groups in this variety form a subvariety? *E. I. Khukhro*

Yes, it does (E. I. Khukhro, *Math. USSR–Sb.*, **58** (1987), 119–126).

7.57. A set of generators of a finitely presented group G that consists of the least possible number $d(G)$ of generators is called a *basis* for G. Let $r_M(G)$ be the least number of relations necessary to define G in the basis M, and $r(G)$ the minimum of $r_M(G)$ over all bases M for G. Let G_1, G_2 be any non-trivial groups.

b) Is it true that $r_{M_1 \cup M_2}(G_1 * G_2) = r_{M_1} G_1 + r_{M_2}(G_2)$ for any bases M_1, M_2 of G_1, G_2, respectively?

c) Is it true that $r(G_1 * G_2) = r(G_1) + r(G_2)$? *V. A. Churkin*

b) Not always; c) not always (C. Hog, M. Lusztig, W. Metzler, in: *Presentation classes, 3-manifolds and free products* (*Lecture Notes in Math.*, **1167**), Springer, Berlin, 1985, 154–167).

8.6. (M. M. Day). Do the classes of amenable groups and elementary groups coincide? The latter consists of groups that can be obtained from commutative and finite groups by forming subgroups, factor-groups, extensions, and direct limits. *R. I. Grigorchuk*

No (R. I. Grigorchuk, *Soviet Math. Dokl.*, **28** (1983), 23–26).

8.7. Does there exist a non-amenable finitely presented group which has no free subgroups of rank 2? *R. I. Grigorchuk*

Yes, it exists (A. Yu. Ol'shanskii, M. V. Sapir, *Publ. Math. Inst. Hautes Étud. Sci.*, **96** (2002), 43–169).

8.8. (D. V. Anosov). a) Does there exist a non-cyclic finitely-generated group G containing an element a such that every element of G is conjugate to a power of a? *R. I. Grigorchuk*

Yes, it exists (V. S. Guba, *Math. USSR–Izv.*, **29** (1986), 233–277).

8.10. b) Is the group $G = \langle a, b \mid a^n = 1,\ ab = b^3a^3 \rangle$ finite or infinite for $n = 9$ and $n = 15$? All other cases known, see Archive, 7.7. *D. L. Johnson*

Infinite (for $n = 15$: D. J. Seal, *Proc. Roy. Soc. Edinburgh (A)*, **92** (1982), 181–192; for $n = 9$ (and 15): M. I. Prishchepov, *Commun. Algebra*, **23** (1995), 5095–5117).

8.12. b) Let $D0$ denote the class of finite groups of deficiency zero, i. e. having a presentation $\langle X \mid R\rangle$ with $|X| = |R|$. Are the central factors of nilpotent $D0$-groups 3-generated? *D. L. Johnson, E. F. Robertson*

No, not always (G. Havas, E. F. Robertson, *Commun. Algebra*, **24** (1996), 3483–3487).

8.13. Let G be a simple algebraic group over an algebraically closed field of characteristic p and $\mathfrak{G}$ be the Lie algebra of G. Is the number of orbits of nilpotent elements of $\mathfrak{G}$ under the adjoint action of G finite? This is known to be true if p is not too small (i. e. if p is not a "bad prime" for G). *R. W. Carter*

Yes, it is (J. N. Spaltenstein; see Section 5.11 in R. W. Carter, *Finite Groups of Lie Type*, John Wiley, 1985.)

8.14. a) Assume a group G is existentially closed in the class $L\mathfrak{N}_p$ of all locally finite p-groups. Is it true that G is characteristically simple? This is true for G countable in $L\mathfrak{N}_p$ (Berthold Maier, Freiburg); in fact, up to isomorphism, there is only one such countable locally finite p-group. *O. H. Kegel*

Not always (S. R. Thomas, *Arch. Math.*, **44** (1985), 98–109).

8.17. An $\widetilde{RN}$*-group* is one whose every homomorphic image is an RN-group. Is the class of $\widetilde{RN}$-groups closed under taking normal subgroups? *Sh. S. Kemkhadze*

No (J. S. Wilson, *Arch. Math.*, **25** (1974), 574–577).

8.18. Is every countably infinite abelian group a verbal subgroup of some finitely generated (soluble) relatively free group? *Yu. G. Kleiman*

Yes, it is (A. Storozhev, *Commun. Algebra*, **22**, no. 7 (1994), 2677–2701).

8.20. What is the cardinality of the set of all varieties covering an abelian (nilpotent? Cross? hereditarily finitely based?) variety of groups? The question is related to 4.46 and 4.73. *Yu. G. Kleiman*

There are continually many varieties covering the variety A of abelian groups, as well as the variety A_n of abelian groups of sufficiently large odd exponent n (P. A. Kozhevnikov, *On varieties of groups of large odd exponent*, Dep. 1612-V00, VINITI, Moscow, 2000 (Russian); S. V. Ivanov, A. M. Storozhev, *Contemp. Math.*, **360** (2004), 55–62).

8.22. If G is a (non-abelian) finite group contained in a join $\mathfrak{A} \vee \mathfrak{B}$ of two varieties $\mathfrak{A}, \mathfrak{B}$ of groups, must there exist finite groups $A \in \mathfrak{A}$, $B \in \mathfrak{B}$ such that G is a section of the direct product $A \times B$? *L. G. Kovács*

Not always (A. Storozhev, *Bull. Austral. Math. Soc.*, **51**, no. 2 (1995), 287–290).

8.26. We call a variety *passable* if there exists an unrefinable chain of its subvarieties which is well-ordered by inclusion. For example, every variety generated by its finite groups is passable — this is an easy consequence of (H. Neumann, *Varieties of groups*, Berlin et al., Springer, 1967, Chapter 5). Do there exist non-passable varieties of groups? *A. V. Kuznetsov*

Yes, they exist (M. I. Anokhin, *Moscow Univ. Math. Bull.*, **51**, no. 1 (1996), 48–49).

8.28. Is the variety of groups finitely based if it is generated by a finitely based quasivariety of groups? *A. V. Kuznetsov*

No, not always (M. I. Anokhin, *Sb. Math.*, **189**, no. 7–8 (1998), 1115–1124).

8.32. Suppose G is a finitely generated group such that, for any set π of primes and any subgroup H of G, if $G/\langle H^G\rangle$ is a finite π-group then $|G:H|$ is a finite π-number. Is G nilpotent? This is true for finitely generated soluble groups. *J. C. Lennox*

Not always (V. N. Obraztsov, *J. Austral. Math. Soc. (A)*, **61**, no. 2 (1996), 267–288).

8.35. Determine the conjugacy classes of maximal subgroups in the sporadic simple groups

a) F_{24}';

b) F_2. *V. D. Mazurov*

a) They are determined mod CFSG (S. A. Linton, R. A. Wilson, *Proc. London Math. Soc.*, **63**, no. 1 (1991), 113–164).

b) They are determined mod CFSG (R. A. Wilson, *J. Algebra*, **211** (1999), 1–14).

8.37. (R. Griess). a) Is M_{11} a section in $O'N$?

b) The same question for M_{24} in F_2; J_1 in F_1 and in F_2; J_2 in F_{23}, F_{24}', and F_2. *V. D. Mazurov*

a) Yes (A. A. Ivanov, S. V. Shpektorov, *Abs. 18 All-Union Algebra Conf.*, Part 1, Kishinev, 1985, 209 (Russian); S. Yoshiara, *J. Fac. Sci. Univ. Tokyo (IA)*, **32** (1985), 105–141).

b) No (R. A. Wilson, *Bull. London Math. Soc.*, **18** (1986), 349–350).

8.39. b) Describe the irreducible subgroups of $SL_6(q)$. *V. D. Mazurov*

They are described mod CFSG (A. S. Kondratiev, *Algebra and Logic*, **28** (1989), 122–138; P. Kleidman, *Low-dimensional finite classical groups and their subgroups*, Harlow, Essex, 1989).

8.46. Describe the automorphisms of the symplectic group Sp_{2n} over an arbitrary commutative ring. *Conjecture:* they are all standard. *Yu. I. Merzlyakov*

The conjecture was proved (V. M. Petechuk, *Algebra and Logic*, **22** (1983), 397–405).

8.47. Do there exist finitely presented soluble groups in which the maximum condition for normal subgroups fails but all central sections are finitely generated? *Yu. I. Merzlyakov*

Yes (Yu. V. Sosnovskiĭ, *Math. Notes*, **36** (1984), 577–580).

8.48. If a finite group G can be written as the product of two soluble subgroups of odd index, then is G soluble? *V. S. Monakhov*

Yes, it is (mod CFSG) (L. S. Kazarin, *Commun. Algebra*, **14** (1986), 1001–1066).

8.49. Let G be a p-group acting transitively as a permutation group on a set Ω, let F be a field of characteristic p, and regard $F\Omega$ as an FG-module. Then do the descending and ascending Loewy series of $F\Omega$ coincide? *P. M. Neumann*

Yes, they do (J. L. Alperin, *Quart. J. Math. Oxford (2)*, **39**, no. 154 (1988), 129–133).

8.53. b) Let n be a sufficiently large odd number. Is it true that every non-cyclic subgroup of $B(m,n)$ has a subgroup isomorphic to $B(2,n)$? *A. Yu. Ol'shanskiĭ*

Yes, it is true for odd $n \geqslant 1003$ (V. S. Atabekyan, *Izv. Math.*, **73**, no. 5 (2009), 861–892).

8.56. Let X be a finite set and f a mapping from the set of subsets of X to the positive integers. Under the requirement that in a group generated by X, every subgroup $\langle Y\rangle$, $Y \subseteq X$, be nilpotent of class $\leqslant f(Y)$, is it true that the free group G_f relative to this condition is torsion-free? *A. Yu. Ol'shanskiĭ*

Not always (V. V. Bludov, V. F. Kleimenov, E. V. Khlamov, *Algebra and Logic*, **29** (1990), 95–96).

8.61. Suppose that a locally compact group G contains a subgroup that is topologically isomorphic to the additve group of the field of real numbers with natural topology. Is the space of all closed subgroups of G connected in the Chabauty topology? *I. V. Protasov*

No, not always: let H be the group of matrices of the form $\begin{pmatrix} 1 & z_1 & r \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}$, where $z_1, z_2 \in \mathbb{Z}$ and $r \in \mathbb{R}$. Then H is locally compact in the natural topology and contains a central subgroup topologically isomorphic to $\mathbb{R}$, but $L(H)$ is not connected (Yu. V. Tsybenko, *Abstracts of 17th All-USSR Algebraic Conf., Part 1*, Minsk, 1983, 213 (Russian)).

8.63. Suppose that the space of all closed subgroups of a locally compact group G is σ-compact in the E-topology. Is it true that the set of closed non-compact subgroups of G is at most countable? *I. V. Protasov*

Yes, it is (A. G. Piskunov, *Ukrain. J. Math.*, **40** (1988), 679–683).

8.66. Construct examples of residually finite groups which would separate Shunkov's classes of groups with (a,b)-finiteness condition, (weakly) conjugacy biprimitively finite groups and (weakly) biprimitively finite groups (see, in particular, 6.57). Can one derive such examples from Golod's construction? *A. I. Sozutov*

Such examples are constructed (L. Hammoudi, *Nil-algèbres non-nilpotentes et groupes périodiques infinis* (*Doctor Thesis*), Strasbourg, 1996; A. V. Rozhkov, *Finiteness conditions in groups of automorphisms of trees* (*Doctor of Sci. Thesis*), Krasnoyarsk, 1997; *Algebra and Logic*, **37**, no. 3 (1998), 192–203).

8.70. Let A, B be polycyclic-by-finite groups. Let $G = A *_H B$ where H is cyclic. Is G conjugacy separable? *C. Y. Tang*

Yes, it is (L. Ribes, D. Segal, P. A. Zalesskii, *J. London Math. Soc. (2)*, **57**, no. 3 (1998), 609–628).

8.71. Is every countable conjugacy-separable group embeddable in a 2-generator conjugacy-separable group? *C. Y. Tang*

Yes, it is (V. A. Roman'kov, *Embedding theorems for residually finite groups*, Preprint 84-515, Comp. Centre, Novosibirsk, 1984 (Russian)).

8.73. We say that a finite group G *separates cyclic subgroups* if, for any cyclic subgroups A and B of G, there is a $g \in G$ such that $A \cap B^g = 1$. Is it true that G separates cyclic subgroups if and only if G has no non-trivial cyclic normal subgroups? *J. G. Thompson*

Not always. For $p_1 = 2$, $p_2 = 5$, $p_3 = 11$, $p_4 = 17$ let R_i be an elementary abelian group of order p_i^2 and φ_i a regular automorphism of order 3 of R_i, $i = 1, 2, 3, 4$. In the direct product $R_1\langle\varphi_1\rangle \times R_2\langle\varphi_2\rangle \times R_3\langle\varphi_3\rangle \times R_4\langle\varphi_4\rangle$ let G be the subgroup generated by all the R_i and the elements $\varphi_2\varphi_3\varphi_4$ and $\varphi_1\varphi_3\varphi_4^{-1}$. Then G has no non-trivial cyclic normal subgroups and every (cyclic) subgroup of order $2 \cdot 5 \cdot 11 \cdot 17$ intersects any of its conjugates non-trivially. (N. D. Podufalov, *Abstracts of the 9th All-Union Group Theory Symp.*, Moscow, 1984, 113–114 (Russian).)

8.76. Give a realistic upper bound for the torsion-free rank of a finitely generated nilpotent group in terms of the ranks of its abelian subgroups. More precisely, for each integer n let $f(n)$ be the largest integer h such that there is a finitely generated nilpotent group of torsion-free rank h with the property that all abelian subgroups have torsion-free rank at most n. It is easy to see that $f(n)$ is bounded above by $n(n+1)/2$. Describe the behavior of $f(n)$ for large n. Is $f(n)$ bounded below by a quadratic in n? *J. S. Wilson*

Yes, it is (M. V. Milenteva, *J. Group Theory*, **7**, no. 3 (2004), 403–408).

8.80. Let G be a locally finite group containing a maximal subgroup which is Chernikov. Is G almost soluble? *B. Hartley*

Yes, it is (B. Hartley, *Algebra and Logic*, **37**, no. 1 (1998), 101–106).

8.81. Let G be a finite p-group admitting an automorphism α of prime order q with $|C_G(a)| \leqslant n$.

a) If $p = q$, then does G have a nilpotent subgroup of class at most 2 and index bounded by a function of n?

b) If $p \neq q$, then does G have a nilpotent subgroup of class bounded by a function of q and index bounded by a function of n (and, possibly, q)? *B. Hartley*

a) Not necessarily (E. I. Khukhro, *Math. Notes*, **38** (1985), 867–870).

b) Yes, it does (E. I. Khukhro, *Math. USSR–Sb.*, **71**, no. 1 (1992), 51–63).

8.84. We say that an automorphism φ of the group G is a *pseudo-identity* if, for all $x \in G$, there exists a finitely generated subgroup K_x of G such that $x \in K_x$ and $\varphi|_{K_x}$ is an automorphism of K_x. Let G be generated by subgroups H, K and let G be locally nilpotent. Let $\varphi : G \to G$ be an endomorphism such that $\varphi|_H$ is pseudo-identity of H and $\varphi|_K$ is an automorphism of K. Does it follow that φ is an automorphism of G? It is known that φ is a pseudo-identity of G if, additionally, $\varphi|_K$ is pseudo-identity of K; it is also known that φ is an automorphism if, additionally, K is normal in G. *P. Hilton*

No, not necessarily (A. V. Yagzhev, *Math. Notes*, **56**, no. 5 (1994), 1205–1207).

8.87. Find all hereditary local formations $\mathfrak{F}$ of finite groups satisfying the following condition: every finite minimal non-$\mathfrak{F}$-group is biprimary. A finite group is said to be *biprimary* if its order is divisible by precisely two distinct primes. *L. A. Shemetkov*

They are found (V. N. Semenchuk, *Problems of Algebra: Proc. of the Gomel' State Univ.*, no. 1 (15) (1999), 92–102 (Russian)).

9.2. Is $G = \langle a, b \mid a^l = b^m = (ab)^n = 1 \rangle$ conjugacy separable? *R. B. J. T. Allenby*

Yes, it is (B. Fine, G. Rosenberger, *Contemp. Math.*, **109** (1990), 11–18).

9.3. Suppose that a countable locally finite group G contains no proper subgroups isomorphic to G itself and suppose that all Sylow subgroups of G are finite. Does G possess a non-trivial finite normal subgroup? *V. V. Belyaev*

Yes, it does (S. D. Bell, *Locally finite groups with Černikov Sylow subgroups*, (*Ph. D. Thesis*), University of Manchester, 1994).

9.12. Is there a soluble group with the following properties: torsion-free of finite rank, not finitely generated, and having a faithful irreducible representation over a finite field? *D. I. Zaitsev*

Yes, there is (A. V. Tushev, *Ukrain. Math. J.*, **42** (1990), 1233–1238).

9.16. (Well-known problem). The *prime graph* of a finite group G is the graph with vertex set $\pi(G)$ and an edge joining p and q if and only if G has an element of order pq. Describe all finite Chevalley groups over a field of characteristic 2 whose prime graph is not connected and describe the connected components. *A. S. Kondratiev*

This was done in (A. S. Kondratiev, *Math. USSR–Sb.*, **67** (1990), 235–247).

9.17. a) Let G be a locally normal residually finite group. Can G be embedded in a direct product of finite groups when the factor group $G/[G, G]$ is a direct product of cyclic groups? *L. A. Kurdachenko*

Not always (L. A. Kurdachenko, *Math. Notes*, **39** (1986), 273–279).

9.18. Let $\mathfrak{S}_*$ be the smallest normal Fitting class. Are there Fitting classes which are maximal in $\mathfrak{S}_*$ (with respect to inclusion)? *H. Lausch*

No, there are no such classes (N. T. Vorob'yëv, *Dokl. Akad. Nauk Belorus. SSR*, **35**, no. 6 (1991), 485–487 (Russian)).

9.19. b) Let $n(X)$ denote the minimum of the indices of proper subgroups of a group X. A subgroup A of a finite group G is called *wide* if A is a maximal element by inclusion of the set $\{X \mid X \text{ is a proper subgroup of} G \text{ and } n(X) = n(G)\}$. Prove, without using CFSG, that $n(F_1) = |F_1 : 2F_2|$, where F_1 and F_2 are the Fischer simple groups and $2F_2$ is an extension of a group of order 2 by F_2. *V. D. Mazurov*

This was proved (S. V. Zharov, V. D. Mazurov, in: *Matematicheskoye programmirovaniye i prilozheniya*, Ekaterinburg, 1995, 96–97 (Russian)).

9.21. Let P be a maximal parabolic subgroup of the smallest index in a finite group G of Lie type E_6, E_7, E_8, or ${}^2E_6$ and let X be a subgroup such that $PX = G$. Is it true that $X = G$? *V. D. Mazurov*

Yes, it is true (mod CFSG) (C. Hering, M. W. Liebeck, J. Saxl, *J. Algebra*, **106**, no. 2 (1987), 517–527).

9.26. a) Describe the finite groups of 2-local 3-rank 1 which have 3-rank at least 3. *A. A. Makhnëv*

Described in (A. A. Makhnëv, *Siber. Math. J.*, **29** (1988), 951–959).

9.27. Let M be a subgroup of a finite group G, A an abelian 2-subgroup of M, and suppose that A^g is not contained in M for some g from G. Determine the structure of G under the hypothesis that $\langle A, A^x\rangle = G$ whenever the subgroup A^x, $x \in G$, is not contained in M. *A. A. Makhnëv*

It is described mod CFSG (V. I. Zenkov, *Algebra and Logic*, **35**, no. 3 (1996), 160–163).

9.30. (Well-known problem). A finite set of reductions $u_i \to v_i$ of words on a finite alphabet $\Sigma = \Sigma^{-1}$ is called a *group set of reductions* if *length* $(u_i) >$ *length* (v_i) or *length* $(u_i) =$ *length* (v_i) and $u_i > v_i$ in the lexicographical ordering, and every word in Σ can be reduced to the unique reduced form which does not depend on the sequence of reductions. Do there exist group sets of reductions satisfying the condition *length* $(v_i) \leqslant 1$ for all i, which are different from 1) sets of trivial reductions $x^{-\varepsilon}x^{\varepsilon} \to 1$, $\varepsilon = \pm 1$, 2) multiplication tables $xy \to z$ of finite groups, and 3) their finite unions? *Yu. I. Merzlyakov*

Yes, they exist (J. Avenhaus, K. Madlener, F. Otto, *Trans. Amer. Math. Soc.*, **297** (1986), 427–443).

9.33. (F. Kümmich, H. Scheerer). If H is a closed subgroup of a connected locally-compact group G such that $\overline{HX} = \overline{H}\,\overline{X}$ for every closed subgroup X of G, then is H normal? *Yu. N. Mukhin*

Yes, it is (C. Scheiderer, *Monatsh. Math.*, **98** (1984), 75–81).

9.34. (S. K. Grosser, W. N. Herfort). Does there exist an infinite compact p-group in which the centralizers of all elements are finite? *Yu. N. Mukhin*

No, it does not exist, in view of the connection of this problem with the Restricted Burnside Problem (S. K. Grosser, W. N. Herfort, *Trans. Amer. Math. Soc.*, **283**, no. 1 (1984), 211–224) and because of the positive solution to the latter (E. I. Zel'manov, *Math. USSR–Izv.*, **36** (1991), 41–60; *Math. USSR–Sb.*, **72** (1992), 543–565).

9.41. a) Let Ω be a countably infinite set. For $k \geqslant 2$, we define a *k-section* of Ω to be a partition of Ω into a union of k infinite subsets. Then does there exist a transitive permutation group on Ω that is transitive on k-sections but intransitive on ordered k-sections? *P. M. Neumann*

Yes, there does. Let U be a non-principal ultrafilter in $\mathfrak{P}(\Omega)$ and let $G = \{g \in \mathrm{Sym}(\Omega) \mid \mathrm{Fix}(g) \in U\}$. It is not hard to prove that G is transitive on k-sections but not on ordered k-sections for any k in the range $2 \leqslant k \leqslant \aleph_0$. (P. M. Neumann, *Letter of October, 5, 1989*.)

9.43. a) The group G described in the solution of 8.73 (Archive) enables us to construct a projective plane of order 3 in which the lines are the elements of any conjugacy class of subgroups of order $2 \cdot 5 \cdot 7 \cdot 11$ together with four lines added in a natural way. In a similar way, we can construct a projective plane of order p^n for any prime p and any positive integer n. Does the resulting projective plane have the Galois property? *N. D. Podufalov*

Not always. The answer is affirmative for $n = 1$. But for $n > 1$ the planes in question may not have the Galois property. For example, if there exists a near-field of p^n elements, then among the planes of order p^n indicated there will definitely be some non-Desarguesian planes. (N. D. Podufalov, *Letter of November, 24, 1986*.)

9.46. Let G be a locally compact group of countable weight and $L(G)$ the space of all its closed subgroups equipped with the E-topology. Then is $L(G)$ a k-space? *I. V. Protasov*

Yes (I. V. Protasov, *Dokl. Akad. Nauk Ukr. SSR Ser. A*, **10** (1986), 64–66 (Russian)).

9.48. In a null-dimensional locally compact group, is the set of all compact elements closed? *I. V. Protasov*

Yes, it is (G. A. Willis, *Math. Ann.*, **300** (1994), 341–363).

9.50. Is every 4-Engel group
a) without elements of order 2 and 5 necessarily soluble?
b) satisfying the identity $x^5 = 1$ necessarily locally finite?
c) (R. I. Grigorchuk) satisfying the identity $x^8 = 1$ necessarily locally finite?
Yu. P. Razmyslov

a) Yes; this follows from the local nilpotency of 4-Engel groups (G. Traustason, *Int. J. Algebra Comput.*, **15** (2005), 309–316; G. Havas, M. R. Vaughan-Lee, *Int. J. Algebra Comput.* **15** (2005), 649–682) and the affirmative answer for locally nilpotent groups in (A. Abdollahi, G. Traustason, *Proc. Amer. Math. Soc.*, **130** (2002), 2827–2836).
b) Yes, it is (M. R. Vaughan-Lee, *Proc. London Math. Soc. (3)*, **74** (1997), 306–334).
c) Yes; moreover, every 4-Engel 2-group is locally finite (G. Traustason, *J. Algebra*, **178**, no. 2 (1995), 414–429).

9.54. If $G = HK$ is a soluble group and H, K are minimax groups, is it true that G is a minimax group? *D. J. S. Robinson*

Yes, it is true (J. S. Wilson, *J. Pure Appl. Algebra*, **53**, no. 3 (1988), 297–318; Ya. P. Sysak, *Radical modules over groups of finite rank*, Inst. of Math. Acad. Sci. Ukrain. SSR, Kiev, 1989 (Russian)).

9.58. Can a product of non-local formations of finite groups be local?
A. N. Skiba, L. A. Shemetkov

Yes, it can (V. A. Vedernikov, *Math. Notes*, **46** (1989), 910–913; N. T. Vorob'ev, in: *Proc. 7th Regional Sci. Session Math.*, Zielona Goru, 1989, 79–86).

9.62. In any group G, the cosets of all of its normal subgroups together with the empty set form the *block lattice* $C(G)$ with respect to inclusion, which, for infinite $|G|$, is subdirectly irreducible and, for finite $|G| \geqslant 3$, is even simple (D. M. Smirnov, A. V. Reibol'd, *Algebra and Logic*, **23**, no. 6 (1984), 459–470). How large is the class of such lattices? Is every finite lattice embeddable in the lattice $C(G)$ for some finite group G? *D. M. Smirnov*

For each $g \in G$ the filter $\{x \in C(G) \mid g \in x\}$ is modular (and even arguesian); the variety generated by the block lattices of groups does not contain all finite lattices (D. M. Smirnov, *Siberian Math. J.*, **33**, no. 4 (1992), 663–668).

9.63. Is a finite group of the form $G = ABA$ soluble if A is an abelian subgroup and B is a cyclic subgroup? *Ya. P. Sysak*

Yes, it is soluble (mod CFSG) (D. L. Zagorin, L. S. Kazarin, *Dokl. Math.*, **53**, no. 2 (1996), 237–239).

9.67. (A. Tarski). Let F_n be a free group of rank n; is it true that $Th(F_2) = Th(F_3)$? *A. D. Taimanov*

Yes, it is (O. Kharlampovich, A. Myasnikov, *J. Algebra*, **302**, (2006), 451–552; Z. Sela, *Geom. Funct. Anal.*, **16** (2006), 707–730).

9.73. Let $\mathfrak{F}$ be any local formation of finite soluble groups containing all finite nilpotent groups. Prove that $H^{\mathfrak{F}}K = KH^{\mathfrak{F}}$ for any two subnormal subgroups H and K of an arbitrary finite group G. *L. A. Shemetkov*

This has been proved, even without the hypothesis of solubility of $\mathfrak{F}$ (S. F. Kamornikov, *Dokl. Akad. Nauk BSSR*, **33**, no. 5 (1989), 396–399 (Russian)).

9.74. Find all local formations $\mathfrak{F}$ of finite groups such that every finite minimal non-$\mathfrak{F}$-group is either a Shmidt group (that is, a non-nilpotent finite group all of whose proper subgroups are nilpotent) or a group of prime order. *L. A. Shemetkov*

They are found (S. F. Kamornikov, *Siberian Math. J.*, **35**, no. 4 (1994), 713–721).

9.79. (A. G. Kurosh). Is every group with the minimum condition countable? *V. P. Shunkov*

No (V. N. Obraztsov, *Math. USSR–Sb.*, **66** (1989), 541–553).

9.80. Are the 2-elements of a group with the minimum condition contained in its locally finite radical? *V. P. Shunkov*

Not necessarily (A. Yu. Ol'shanskiĭ, *The geometry of defining relations in groups*, Kluwer, Dordrecht, 1991).

9.81. Does there exist a simple group with the minimum condition possessing a non-trivial quasi-cyclic subgroup? *V. P. Shunkov*

Yes, there exists (V. N. Obraztsov, *Math. USSR–Sb.*, **66** (1989), 541–553).

9.82. An infinite group, all of whose proper subgroups are finite, is called *quasi-finite*. Is it true that an element of a quasi-finite group G is central if and only if it is contained in infinitely many subgroups of G? *V. P. Shunkov*

No (K. I. Lossov, Dep. no. 5529-V89, VINITI, Moscow, 1988 (Russian)).

10.1. Let p be a prime number. Describe the groups of order p^9 of nilpotency class 2 which contain subgroups X and Y such that $|X| = |Y| = p^3$ and any non-identity elements $x \in X$, $y \in Y$ do not commute. An answer to this question would yield a description of the semifields of order p^3. *S. N. Adamov, A. N. Fomin*

They are described (V. A. Antonov, *Preprint*, Chelyabinsk, 1999 (Russian)).

10.6. Is it true that in an abelian group every non-discrete group topology can be strengthened up to a non-discrete group topology such that the group becomes a complete topological group? *V. I. Arnautov*

This is true for group topologies satisfying the first axiom of countability (E. I. Marin, *in: Modules, Algebras, Topologies* (*Mat. Issledovaniya*, **105**), Kishinëv, 1988, 105–119 (Russian)), but this may not be true in general, see Archive, 12.2.

10.7. Is it true that in a countable group G, any non-discrete group topology satisfying the first axiom of countability can be strengthened up to a non-discrete group topology such that G becomes a complete topological group? *V. I. Arnautov*

Yes, it is (V. I. Arnautov, E. I. Kabanova, *Siberian Math. J.*, **31** (1990), 1–10).

10.9. Let p be a prime number and let L_p denote the set of all quasivarieties each of which is generated by a finite p-group. Is L_p a sublattice of the lattice of all quasivarieties of groups? *A. I. Budkin*

No, not always (S. A. Shakhova, *Math. Notes*, **53**, no. 3 (1993), 345–347).

10.14. a) Does every group satisfying the minimum condition on subgroups satisfy the weak maximum condition on subgroups?

b) Does every group satisfying the maximum condition on subgroups satisfy the weak minimum condition on subgroups? *D. I. Zaitsev*

a) No, not always (V. N. Obraztsov, *Math. USSR–Sb.*, **66**, no. 2 (1990), 541–553).
b) No, not always (V. N. Obraztsov, *Siberian Math. J.*, **32**, no. 1 (1991), 79–84).

10.24. A braid is said to be *coloured* if its strings represent the identity permutation. Is it true that links obtained by closing coloured braids are equivalent if and only if the original braids are conjugate in the braid group? *G. S. Makanin*

No. For non-oriented links: the braid words σ_1^2 and σ_1^{-2} both represent the Hopf link but are not conjugate in the braid group (easy to see using the Burau representation). For oriented links see Example 4 in (J. Birman, *Braids, links, and mapping class groups* (*Ann. Math. Stud. Princeton*, **82**), Princeton, NJ, 1975, p. 100). (V. Shpilrain, *Letters of May, 28, 1998 and January, 20, 2002*).

10.30. Does there exist a non-right-orderable group which is residually finite p-group for a finite set of prime numbers p containing at least two different primes? If a group is residually finite p-group for an infinite set of primes p, then it admits a linear order (A. H. Rhemtulla, *Proc. Amer. Math. Soc.*, **41**, no. 1 (1973), 31–33).

N. Ya. Medvedev

Yes, it exists (P. A. Linnell, *J. Algebra*, **248** (2002) 605–607).

10.33. For HNN-extensions of the form $G = \langle t, A \mid t^{-1}Bt = C,\ \varphi\rangle$, where A is a finitely generated abelian group and $\varphi : B \to C$ is an isomorphism of two of its subgroups, find

a) a criterion to be residually finite;

b) a criterion to be Hopfian. *Yu. I. Merzlyakov*

a) Such a criterion is found (S. Andreadakis, E. Raptis, D. Varsos, *Arch. Math.*, **50** (1988), 495–501.

b) Such a criterion is found (S. Andreadakis, E. Raptis, D. Varsos, *Commun. Algebra*, **20** (1992), 1511–1533).

10.37. Suppose that G is a finitely generated metabelian group all of whose integral homology groups are finitely generated. Is it true that G is a group of finite rank? The answer is affirmative if G splits over the derived subgroup (J. R. J. Groves, *Quart. J. Math.*, **33**, no. 132 (1982), 405–420). *G. A. Noskov*

Yes, it is (D. H. Kochloukova, *Groups St. Andrews 2001 in Oxford*, Vol. II, Cambridge Univ. Press, 2003, 332–343).

10.41. (Well-known problem). Let Γ be an almost polycyclic group with no non-trivial finite normal subgroups, and let k be a field. The complete ring of quotients $Q(k\Gamma)$ is a matrix ring $M_n(D)$ over a skew field. Conjecture: n is the least common multiple of the orders of the finite subgroups of Γ. An equivalent formulation (M. Lorenz) is as follows: $\rho(G_0(k\Gamma)) = \rho(G_0(k\Gamma)_{\mathscr{F}})$, where $G_0(k\Gamma)$ is the Grothendieck group of the category of finitely-generated $k\Gamma$-modules, $G_0(k\Gamma)_{\mathscr{F}}$ is the subgroup generated by classes of modules induced from finite subgroups of Γ, and ρ is the Goldie rank. There is a stronger conjecture: $G_0(k\Gamma) = G_0(k\Gamma)_{\mathscr{F}}$ *G. A. Noskov*

The strong conjecture $G_0(k\Gamma) = G_0(k\Gamma)_{\mathscr{F}}$ has been proved (J. A. Moody, *Bull. Amer. Math. Soc.*, **17** (1987), 113–116).

10.48. Let V be a vector space of finite dimension over a field of prime order. A subset R of $GL(V) \cup \{0\}$ is called *regular* if $|R| = |V|$, $0, 1 \in R$ and $vx \neq vy$ for any non-trivial vector $v \in V$ and any distinct elements $x, y \in R$. It is obvious that τ, ε, μ_g transform a regular set into a regular one, where $x^\tau = x^{-1}$ for $x \neq 0$ and $0^\tau = 0$, $x^\varepsilon = 1 - x$, $x^{\mu_g} = xg^{-1}$ and g is a non-zero element of the set being transformed. We say that two regular subsets are *equivalent* if one can be obtained from the other by a sequence of such transformations.

a) Study the equivalence classes of regular subsets.

b) Is every regular subset equivalent to a subgroup of $GL(V)$ together with 0?

N. D. Podufalov

a) They were studied (N. D. Podufalov, *Algebra and Logic*, **30**, no. 1 (1991), 62–69).

b) No. A regular set is closed with respect to multiplication if and only if the corresponding (γ, γ)-transitive plane is defined over a near-field. (N. D. Podufalov, *Letter of February, 13, 1989.*)

10.56. Is the lattice of formations of finite nilpotent groups of class $\leqslant 4$ distributive?
A. N. Skiba

No. The lattice L of all varieties of nilpotent 2-groups of class $\leqslant 4$ is non-distributive (Yu. A. Belov, *Algebra and Logic*, **9**, no. 6 (1970), 371-374; R. A. Bryce, *Philos. Trans. Roy. Soc. London (A)*, **266** (1970), 281–355, footnote on p. 335). The mapping $V \to V \cap F$, where F is the class of all finite groups, is an embedding of L into the lattice of formations. (L. Kovács, *Letter of May, 12, 1988.*) See also (L. A. Shemetkov, A. N. Skiba, *Formations of algebraic systems*, Nauka, Moscow, 1989 (Russian)).

10.63. Is there a doubly transitive permutation group in which the stabilizer of a point is infinite cyclic?
Ya. P. Sysak

No (V. D. Mazurov, *Siberian Math. J.*, **31** (1990), 615–617; Gy. Károlyi, S. J. Kovács, P. P. Pálfy, *Aequationes Math.*, **39**, no. 1 (1990), 161–166).

10.66. Is a group G non-simple if it contains two non-trivial subgroups A and B such that $AB \neq G$ and $AB^g = B^g A$ for any $g \in G$? This is true if G is finite (O. H. Kegel, *Arch. Math.*, **12**, no. 2 (1961), 90–93).
A. N. Fomin, V. P. Shunkov

No; for example, for G being any simple linearly ordered group and A and B arbitrary proper convex subgroups of G (V. V. Bludov, *Letter of February, 12, 1997*).

10.68. Suppose that a finite p-group G admits an automorphism of order p with exactly p^m fixed points. Is it true that G has a subgroup whose index is bounded by a function of p and m and whose nilpotency class is bounded by a function of m only? The results on p-groups of maximal class give an affirmative answer in the case $m = 1$.
E. I. Khukhro

Yes, it is (Yu. Medvedev, *J. London Math. Soc. (2)*, **59**, no. 3 (1999), 787–798).

10.69. Suppose that S is a closed oriented surface of genus $g > 1$ and $G = \pi_1 S = G_1 *_A G_2$, where $G_1 \neq A \neq G_2$ and the subgroup A is finitely generated (and hence free). Is this decomposition geometrical, that is, do there exist connected surfaces S_1, S_2, T such that $S = S_1 \cup S_2$, $S_1 \cap S_2 = T$, $G_i = \pi_1 S_i$, $A = \pi_1 T$ and embeddings $S_i \subset S$, $T \subset S$ induce, in a natural way, embeddings $G_i \subset G$, $A \subset G$? This is true for $A = \mathbb{Z}$.
H. Zieschang

No, not always (O. V. Bogopol'skiĭ, *Geom. Dedicata*, **94** (2002), 63–89).

10.72. Prove that the formation of all finite p-groups does not decompose into a product of two non-trivial subformations.
L. A. Shemetkov

This has been proved (A. N. Skiba, L. A. Shemetkov, *Dokl. Akad. Nauk BSSR*, **33** (1989), 581–582 (Russian)).

10.76. Suppose that G is a periodic group having an infinite Sylow 2-subgroup S which is either elementary abelian or a Suzuki 2-group, and suppose that the normalizer $N_G(S)$ is strongly embedded in G and is a Frobenius group (see 6.55) with locally cyclic complement. Must G be locally finite?
V. P. Shunkov

Yes, it must (A. I. Sozutov, *Algebra and Logic*, **39**, no. 5 (2000), 345–353; A. I. Sozutov, N. M. Suchkov, *Math. Notes*, **68**, no. 2 (2000), 237–247).

11.4. Is it true that the lattice of centralizers in a group is modular if it is a sublattice of the lattice of all subgroups? This is true for finite groups.
V. A. Antonov

No, not always (V. N. Obraztsov, *J. Algebra*, **199**, no. 1 (1998), 337–343).

11.6. Let p be an odd prime. Is it true that every finite p-group possesses a set of generators of equal orders? *C. Bagiński*

No, it is not true (E. A. O'Brien, C. M. Scoppola, M. R. Vaughan-Lee, *Proc. Amer. Math. Soc.*, **134**, no. 12 (2006), 3457–3464).

11.10. (R. C. Lyndon). b) Is it true that $a \neq 1$ in $G = \langle a, x \mid a^5 = 1,\ a^{x^2} = [a, a^x]\rangle$? *V. V. Bludov*

Yes, it is true, since the mapping $a \to (1\,3\,5\,2\,4)$, $x \to (1\,2\,4\,3\,5)$ can be extended to a homomorphism of the group G onto the alternating group $\mathbb{A}_5$ (D. N. Azarov, in: *Algebraicheskije Systemy*, Ivanovo, 1991, 4–5 (Russian)).

11.11. b) The well-known Baer–Suzuki theorem states that if every two conjugates of an element a of a finite group G generate a finite p-subgroup, then a is contained in a normal p-subgroup. Does such a theorem hold in the class of binary finite groups? *A. V. Borovik*

Yes, such a theorem does hold for binary finite groups. By the Baer–Suzuki theorem $\langle a_1, b\rangle$ is a finite p-group for any element $a_1 \in a^G = \{a^g \mid g \in G\}$ and for any p-element $b \in G$. Now induction on n yields that the product $a_1 \cdots a_n$ is a p-element for any $a_1, \dots, a_n \in a^G$. (A. I. Sozutov, *Siberian Math. J.* **41**, no. 3 (2000), 561–562.)

11.20. Suppose we have $[a, b] = [c, d]$ in an absolutely free group, where $a, b, [a, b]$ are basic commutators (in some fixed free generators). If c and d are arbitrary (proper) commutators, does it follow that $a = c$ and $b = d$? *A. Gaglione, D. Spellman*

No, not always. For example, if x_1, x_2, x_3 are free generators, $x_1 < x_2 < x_3$, $a = [[x_2, x_1], x_3]$, $b = [x_2, x_1]$, $c = b^{-1}$, $d = b^{x_3 b}$ (V. G. Bardakov, *Abstracts of the IIIrd Intern. Conf. on Algebra*, Krasnoyarsk, 1993, p. 33 (Russian)).

11.21. Let $\mathfrak{N}_p$ denote the formation of all finite p-groups, for a given prime number p. Is it true that, for every subformation $\mathfrak{F}$ of $\mathfrak{N}_p$, there exists a variety $\mathfrak{M}$ such that $\mathfrak{F} = \mathfrak{N}_p \cap \mathfrak{M}$? *A. F. Vasil'yev*

No, it is not true. There is a natural one-to-one correspondence between formations of finite p-groups and varieties of pro-p-groups: for every variety $\mathfrak{V}$ of pro-p-groups the class of all finite groups in $\mathfrak{V}$ is a formation of p-groups and every formation of p-groups arises in this way. There are continuously many varieties of nilpotent pro-p-groups of class at most 6 (A. N. Zubkov, *Siberian Math. J.*, **29**, no. 3 (1988), 491–494) and only countably many varieties of nilpotent groups of class at most 6. (A. N. Krasil'nikov, *Letter of July, 17th, 1998.*)

11.24. A Fitting class $\mathfrak{F}$ is said to be *local* if there exists a group function f (for definition see (L. A. Shemetkov, *Formations of Finite Groups*, Moscow, Nauka, 1978 (Russian)) such that $f(p)$ is a Fitting class for every prime number p and $\mathfrak{F} = \mathfrak{G}_{\pi(\mathfrak{F})} \bigcap \left(\bigcap_{p \in \pi(\mathfrak{F})} f(p)\mathfrak{N}_p\mathfrak{G}_{p'}\right)$. Is every hereditary Fitting class of finite groups local? *N. T. Vorob'ëv*

No, not every (L. A. Shemetkov, A. F. Vasil'yev, *Abstracts of the Conf. of Mathematicians of Belarus', Part 1*, Grodno, 1992, p. 56 (Russian); S. F. Kamornikov, *Math. Notes*, **55**, no. 6 (1994), 586–588).

11.25. a) Does there exist a local product (different from the class of all finite groups and from the class of all finite soluble groups) of Fitting classes each of which is not local and is not a formation? See the definition of the *product* of Fitting classes in (N. T. Vorob'ev, *Math. Notes*, **43**, No 1–2 (1988), 91–94). *N. T. Vorob'ëv*

Yes, there exists (N. T. Vorob'ev, A. N. Skiba, *Problems in Algebra*, **8**, Gomel', 1995, 55–58 (Russian)).

11.26. Does there exist a group which is not isomorphic to outer automorphism group of a metabelian group with trivial center? *R. Göbel*

No, given any group G there is a metabelian group M with trivial center such that $\operatorname{Out} M \cong G$ (R. Göbel, A. Paras, *J. Pure Appl. Algebra*, **149**, no. 3 (2000) 251–266), and if G is finite or countable then M above can be chosen countable (R. Göbel, A. Paras, in: *Abelian Groups and Modules, Proc. Int. Conf. Dublin, 1998*, Birkhäuser, Basel, 1999, 309–317).

11.27. What are the minimum numbers of generators for groups G satisfying $S \leqslant G \leqslant \operatorname{Aut} S$ where S is a finite simple non-abelian group? *K. Gruenberg*

The number $d(G)$ is found (mod CFSG) for every such a group G. In particular, $d(G) = \max\{2, d(G/S)\}$ and $d(G) \leqslant 3$ (F. Dalla Volta, A. Lucchini, *J. Algebra*, **178**, no. 1 (1995), 194–223).

11.29. f) Let F be a free group and $\mathfrak{f} = \mathbb{Z}F(F-1)$ the augmentation ideal of the integral group ring $\mathbb{Z}F$. For any normal subgroup R of F define the corresponding ideal $\mathfrak{r} = \mathbb{Z}F(R-1) = {}_{\mathrm{id}}(r-1 \mid r \in R)$. One may identify, for instance, $F \cap (1+\mathfrak{rf}) = R'$, where F is naturally imbedded into $\mathbb{Z}F$ and $1+\mathfrak{rf} = \{1+a \mid a \in \mathfrak{rf}\}$. Is the quotient group $(F \cap (1+\mathfrak{r}+\mathfrak{f}^n))/R \cdot \gamma_n(F)$ always abelian? *N. D. Gupta*

Yes, it is (N. D. Gupta, Yu. V. Kuz'min, *J. Pure Appl. Algebra*, **78**, no. 1 (1992), 165–172).

11.33. a) Let $G(q)$ be a simple Chevalley group over a field of order q. Prove that there exists m such that the restriction of every non-one-dimensional complex representation of $G(q^m)$ to $G(q)$ contains all irreducible representations of $G(q)$ as composition factors. *A. E. Zalesskii*

This is proved in (D. Gluck, *J. Algebra*, **155**, no. 2 (1993), 221–237).

11.35. Suppose that H is a finite linear group over $\mathbb{C}$ and h is an element of H of prime order p which is not contained in any abelian normal subgroup. Is it true that h has at least $(p-1)/2$ different eigenvalues? *A. E. Zalesskii*

Yes, it is (G. R. Robinson, *J. Algebra*, **178**, no. 2 (1995), 635–642).

11.37. b) Can the free Burnside group $B(m,n)$, for any m and $n = 2^l \gg 1$, be given by defining relations of the form $v^n = 1$ such that for any natural divisor d of n distinct from n the element v^d is not trivial in $B(m,n)$? *S. V. Ivanov*

Yes, it can (S. V. Ivanov, *Int. J. Algebra Comput.*, **4**, no. 1–2 (1994), 1–308).

11.42. Does there exist a torsion-free group having exactly three conjugacy classes and containing a subgroup of index 2? *A. V. Izosov*

Yes, it exists (A. Minasyan, *Comment. Math. Helv.*, **84** (2009), 259–296).

11.43. For a finite group X, we denote by $k(X)$ the number of its conjugacy classes. Is it true that $k(AB) \leqslant k(A)k(B)$? *L. S. Kazarin*

No, it is not true in general: let $G = \langle a, b \mid a^{30} = b^2 = 1,\ a^b = a^{-1}\rangle \cong D_{60}$ be the dihedral group of order 60, and let $A = \langle a^{10}, ba\rangle \cong D_6$ and $B = \langle a^6, b\rangle \cong D_{10}$. Then $G = AB$, but G has 18 conjugacy classes and A and B only 3 and 4, respectively. From (P. Gallagher, *Math. Z.*, **118** (1970), 175–179) a positive answer follows if A or B is normal. The problem remains open in the case where A and B have coprime orders, see new problem 14.44. (J. Sangroniz, *Letter of December, 17, 1998.*)

11.46. b) Does there exist a finite p-group of nilpotency class greater than 2, with $\mathrm{Aut}\, G = \mathrm{Aut}_c G \cdot \mathrm{Inn}\, G$, where $\mathrm{Aut}_c G$ is the group of central automorphisms of G? *A. Caranti*

Yes, it exists (I. Malinowska, *Rend. Sem. Mat. Padova*, **91** (1994), 265–271).

11.47. Let $\mathscr{L}_d$ be the homogeneous component of degree d in a free Lie algebra $\mathscr{L}$ of rank 2 over the field of order 2. What is the dimension of the fixed point space in $\mathscr{L}_d$ for the automorphism of $\mathscr{L}$ which interchanges two elements of a free generating set of $\mathscr{L}$? *L. G. Kovács*

It is found; R. M. Bryant and R. Stöhr (*Arch. Math.*, **67**, no. 4 (1996), 281–289) confirmed the conjecture from (M. W. Short, *Commun. Algebra*, **23**, no. 8 (1995), 3051–3057).

11.53. (P. Kleidman). Do the sporadic simple groups of Rudvalis Ru, Mathieu M_{22}, and Higman–Sims HS embed into the simple group $E_7(5)$? *V. D. Mazurov*

Yes, they do (P. B. Kleidman, R. A. Wilson, *J. Algebra*, **157**, no. 2 (1993), 316–330).

11.54. Is it true that in the group of coloured braids only the identity braid is a conjugate to its inverse? (For definition see Archive, 10.24.) *G. S. Makanin*

11.55. Is it true that extraction of roots in the group of coloured braids is uniquely determined? *G. S. Makanin*

The answers to both 11.54 and 11.55 are affirmative, since the group K_n of coloured braids is embeddable into the group of those automorphisms of the free group F_n that act trivially modulo the derived subgroup of F_n, and this group is residually in the class of torsion-free nilpotent groups, for which the corresponding assertions are true (V. A. Roman'kov, *Letter of October, 3, 1990*). See also (V. G. Bardakov, *Russ. Acad. Sci. Sbornik Math.*, **76**, no. 1 (1993), 123–153).

11.57. An *upper composition factor* of a group G is a composition factor of some finite quotient of G. Is there any restriction on the set of non-abelian upper composition factors of a finitely generated group? *A. Mann, D. Segal*

There are no restrictions: any set of non-abelian finite simple groups can be the set of upper composition factors of a 63-generator group; if we allow the group to have also abelian upper composition factors, the number of generators can be reduced to 3 (D. Segal, *Proc. London Math. Soc. (3)*, **82** (2001), 597–613).

11.64. Let $\pi(G)$ denote the set of prime divisors of the order of a finite group G. Are there only finitely many finite simple groups G, different from alternating groups, which have a proper subgroup H such that $\pi(H) = \pi(G)$? *V. S. Monakhov*

No, there are infinitely many such groups. If $G = S_{4k}(2^s)$ and $H \cong \Omega^-_{4k}(2^s)$, then $\pi(G) = \pi(H)$ (V. I. Zenkov, *Letter of March, 10, 1994.*)

11.68. Can every fully ordered group be embedded in a fully ordered group (continuing the given order) with only 3 classes of conjugate elements? *B. Neumann*

No, not every (V. V. Bludov, *Algebra and Logic*, **44**, no. 6 (2005), 370–380).

11.74. Let G be a non-elementary hyperbolic group and let G^n be the subgroup generated by the nth powers of the elements of G.

a) (M. Gromov). Is it true that G/G^n is infinite for some $n = n(G)$?

b) Is it true that $\bigcap_{n=1}^{\infty} G^n = \{1\}$? *A. Yu. Ol'shanskiĭ*

a) Yes, it is; b) Yes, it is (S. V. Ivanov, A. Yu. Ol'shanskii, *Trans. Amer. Math. Soc.*, **348**, N 6 (1996), 2091–2138).

11.75. Let us consider the class of groups with n generators and m relators. A subclass of this class is called *dense* if the ratio of the number of presentations of the form $\langle a_1, \ldots, a_n \mid R_1, \ldots, R_m \rangle$ (where $|R_i| = d_i$) for groups from this subclass to the number of all such presentations converges to 1 when $d_1 + \cdots + d_m$ tends to infinity. Prove that for every $k < m$ and for any n the subclass of groups all of whose k-generator subgroups are free is dense. *A. Yu. Ol'shanskiĭ*

This is proved (G. N. Arzhantseva, A. Yu. Ol'shanskiĭ, *Math. Notes*, **59**, no. 4 (1996), 350–355).

11.79. Let G be a finite group of automorphisms of an infinite field F of characteristic p. Taking integral powers of the elements of F and the action of G define the action of the group ring $\mathbb{Z}G$ of G on the multiplicative group of F. Is it true that any subfield of F that contains the images of all elements of F under the action of some fixed element of $\mathbb{Z}G \setminus p\mathbb{Z}G$ contains infinitely many G-invariant elements of F? *K. N. Ponomarëv*

Yes (K. N. Ponomarëv, *Siber. Math. J.*, **33** (1992), 1094–1099).

11.82. Let R be the normal closure of an element r in a free group F with the natural length function and suppose that s is an element of minimal length in R. Is it true that s is conjugate to one of the following elements: r, r^{-1}, $[r, f]$, $[r^{-1}, f]$ for some $f \in F$? *V. N. Remeslennikov*

No, not always (J. McCool, *Glasgow Math. J.*, **43**, no. 1 (2001), 123–124).

11.91. Prove that a hereditary formation $\mathfrak{F}$ of finite soluble groups is local if every finite soluble non-simple minimal non-$\mathfrak{F}$-group is a Shmidt group (that is, a non-nilpotent finite group all of whose proper subgroups are nilpotent). *V. N. Semenchuk*

This is proved (A. N. Skiba, *Dokl. Akad. Nauk Belorus. SSR*, **34**, no. 11 (1990), 982–985 (Russian)).

11.93. Is the variety of all lattices generated by the block lattices (see Archive, 9.62) of finite groups? *D. M. Smirnov*

No, it is not (D. M. Smirnov, *Siberian Math. J.*, **33**, no. 4 (1992), 663–668).

11.97. Are there only finitely many finite simple groups with a given set of all different values of irreducible characters on a single element? *S. P. Strunkov*

No; all complex irreducible characters of the groups $L_2(2^m)$, $m \geqslant 2$, take the values 0, ± 1 on involutions (V. D. Mazurov).

11.98. b) Let r be the number of conjugacy classes of elements in a finite (simple) group G. Is it true that $|G| \leqslant \exp(r)$? *S. P. Strunkov*

No, this is not true: the group M_{22} has order 443520 and contains 12 conjugacy classes (T. Plunkett, *Letter of May, 9, 2000*).

11.101. Does there exist a Golod group (see 9.76) with finite centre? *A. V. Timofeenko*

Yes, it exists (V. A. Sereda, A. I. Sozutov, *Algebra and Logic*, **45**, no. 2 (2006), 134–138).

11.103. Is a 2-group satisfying the minimum condition for centralizers necessarily locally finite? *J. Wilson*

Yes, it is (F. O. Wagner, *J. Algebra*, **217**, no. 2 (1999), 448–460).

11.104. Let G be a finite group of order $p^a \cdot q^b \cdots$, where $p, q, \ldots$ are distinct primes. Introduce distinct variables $x_p, x_q, \ldots$ corresponding to $p, q, \ldots$. Define functions f, φ from the lattice of subgroups of G to the polynomial ring $\mathbb{Z}[x_p, x_q, \ldots]$ as follows:

(1) if H has order $p^\alpha \cdot q^\beta \cdots$, then $f(H) = x_p^\alpha \cdot x_q^\beta \cdots$;

(2) for all $H \leqslant G$, we have $\sum\limits_{K \leqslant H} \varphi(K) = f(H)$.

Then $f(G), \varphi(G)$ may be called the *order* and *Eulerian polynomials* of G. Substituting $p^m, q^m, \ldots$ for $x_p, x_q, \ldots$ in these polynomials we get the mth power of the order of G and the number of ordered m-tuples of elements that generate G respectively.

It is known that if G is p-solvable, then $\varphi(G)$ is a product of a polynomial in x_p and a polynomial in the remaining variables. Consequently, if G is solvable, $\varphi(G)$ is the product of a polynomial in x_p by a polynomial in x_q by $\ldots$. Are the converses of these statements true a) for solvable groups? b) for p-solvable groups? *G. E. Wall*

Yes, the converses are true: a) for solvable groups (E. Detomi, A. Lucchini, *J. London Math. Soc. (2)*, **70** (2004), 165–181); b) for p-solvable groups (E. Damian, A. Lucchini, *Commun. Algebra*, **35** (2007), 3451–3472).

11.105. a) Let $\mathfrak{V}$ be a variety of groups. Its relatively free group of given rank has a presentation F/N, where F is absolutely free of the same rank and N fully invariant in F. The associated Lie ring $\mathscr{L}(F/N)$ has a presentation L/J, where L is the free Lie ring of the same rank and J an ideal of L. Is J always fully invariant in L? *G. E. Wall*

No, not always; the ideal J is not fully invariant for $F/(F^2)^4$, that is, for $\mathfrak{V} = \mathfrak{B}_4\mathfrak{B}_2$ (D. Groves, *J. Algebra*, **211**, no. 1 (1999), 15–25).

11.106. Can every periodic group be embedded in a simple periodic group? *R. Phillips*

Yes (A. Yu. Ol'shanskii, *Ukrain. Math. J.*, **44** (1992), 761–763).

11.108. Is every locally finite simple group absolutely simple? A group G is said to be *absolutely simple* if the only composition series of G is $\{1, G\}$. For equivalent formulations see (R. E. Phillips, *Rend. Sem. Mat. Univ. Padova*, **79** (1988), 213–220). *R. Phillips*

No, not every (U. Meierfrankenfeld, in: *Proc. Int. Conf. Finite and Locally Finite Groups, Istanbul, 1994*, Klüwer, 1995, 189–212).

11.110. Is it possible to embed $SL(2,\mathbb{Q})$ in the multiplicative group of some division ring? *B. Hartley*

No, it is not (W. Dicks, B. Hartley, *Commun. Algebra*, **19** (1991), 1919–1943).

11.112. b) Let $L = L(K(p))$ be the associated Lie ring of a free countably generated group $K(p)$ of the Kostrikin variety of locally finite groups of a given prime exponent p. Is it true that all identities of L follow from multilinear identities of L? *E. I. Khukhro*

No, there are two relations of weight 29 which hold in $L(K(7))$ (in two generators, of multiweights $(14,15)$ and $(15,14)$) that are not consequences of multilinear relations (E. O'Brien, M. R. Vaughan-Lee, *Int. J. Algebra Comput.*, **12** (2002), 575–592; M. F. Newman, M. R. Vaughan-Lee, *Electron. Res. Announc. Amer. Math. Soc.*, **4**, no. 1 (1998), 1–3).

11.126. Do there exist a constant h and a function f with the following property: if a finite soluble group G admits an automorphism φ of order 4 such that $|C_G(\varphi)| \leqslant m$, then G has a normal series $1 \leqslant M \leqslant N \leqslant G$ such that the index $|G:N|$ does not exceed $f(m)$, the group N/M is nilpotent of class $\leqslant 2$, and the group M is nilpotent of class $\leqslant h$? *P. V. Shumyatskiĭ*

Yes, they exist (N. Yu. Makarenko, E. I. Khukhro, *Algebra and Logic*, **45** (2006), 326–343).

11.128. A group G is said to be a *K-group* if for every subgroup $A \leqslant G$ there exists a subgroup $B \leqslant G$ such that $A \cap B = 1$ and $\langle A, B\rangle = G$. Is it true that normal subgroups of K-groups are also K-groups? *M. Emaldi*

No, it is not (V. N. Obraztsov, *J. Austral. Math. Soc. (Ser. A)*, **61**, no. 2 (1996), 267–288).

12.1. b) (A. A. Bovdi). H. Bass (*Topology*, **4**, no. 4 (1966), 391–400) has constructed explicitly a proper subgroup of finite index in the group of units of the integer group ring of a finite cyclic group. Construct analogous subgroups for finite abelian non-cyclic groups. *R. Zh. Aleev*

They are constructed: for p-groups, $p \neq 2$, in (K. Hoechsmann, J. Ritter, *J. Pure Appl. Algebra*, **68**, no. 3 (1990), 325–333); for the general case of groups of central units of integral group rings of arbitrary (not necessarily abelian) finite groups in (R. Zh. Aleev, *Mat. Tr. Novosibirsk Inst. Math.*, **3**, no. 1 (2000), 3–37 (Russian)).

12.2. Is it true that every non-discrete group topology (in an abelian group) can be strengthened up to a maximal complete group topology? *V. I. Arnautov*

No, not always, since under the assumption of the Continuum Hypothesis there exist non-complete maximal topologies (V. I. Arnautov, E. G. Zelenyuk, *Ukrain. Math. J.*, **43**, no. 1 (1991), 15–20).

12.5. Does there exist a countable non-trivial filter in the lattice of quasivarieties of metabelian torsion-free groups? *A. I. Budkin*

No, there is no such filter (S. V. Lenyuk, *Siberian Math. J.*, **39**, no. 1 (1998), 57–62).

12.7. Is it true that every radical hereditary formation of finite groups is a composition one? *A. F. Vasil'ev*

No, it is not (S. F. Kamornikov, *Math. Notes*, **55**, no. 6 (1994), 586–588).

12.10. (P. Neumann). Can the free group on two generators be embedded in Sym ($\mathbb{N}$) so that the image of every non-identity element has only a finite number of orbits?

A. M. W. Glass

Yes, it can (H. D. Macpherson, in: *Ordered groups and infinite permutation groups, Partially based on Conf. Luminy, France, 1993* (*Mathematics and its Applications*, **354**), Kluwer, Dordrecht, 1996, 221–230).

12.14. If T is a countable theory, does there exist a model $\mathfrak{A}$ of T such that the theory of Aut ($\mathfrak{A}$) is undecidable? *M. Giraudet, A. M. W. Glass*

Yes, moreover, every first order theory having infinite models has a model whose automorphism group has undecidable existential theory (V. V. Bludov, M. Giraudet, A. M. W. Glass, G. Sabbagh, in *Models, Modules and Abelian Groups*, de Gruyter, Berlin, 2008, 325–328).

12.22. a) Let $\Delta(G)$ be the augmentation ideal of the integer group ring of an arbitrary group G. Then $D_n(G) = G \cap (1 + \Delta^n(G))$ contains the nth lower central subgroup $\gamma_n(G)$ of G. Is it true that $D_n(G)/\gamma_n(G)$ is central in $G/\gamma_n(G)$?

N. D. Gupta, Yu. V. Kuz'min

No, not always; moreover, $D_n(G)/\gamma_n(G)$ need not be contained in any term of the upper central series of $G/\gamma_n(G)$ with fixed number (N. D. Gupta, Yu. V. Kuz'min, *J. Pure Appl. Algebra*, **104**, no. 1 (1995), 191–197).

12.24. Given a ring R with identity, the automorphisms of $R[[x]]$ sending x to $x\left(1 + \sum_{i=1}^{\infty} a_i x^i\right)$, $a_i \in R$, form a group $N(R)$. We know that $N(\mathbb{Z})$ contains a copy of the free group F_2 of rank 2 and, from work of A. Weiss, that $N(\mathbb{Z}/p\mathbb{Z})$ contains a copy of every finite p-group (but not of $\mathbb{Z}_{p^\infty}$), p a prime. Does $N(\mathbb{Z}/p\mathbb{Z})$ contain a copy of F_2? *D. L. Johnson*

Yes, it does (R. Camina, *J. Algebra*, **196**, no. 1 (1997), 101–113). I. B. Fesenko noted that this fact might have been known to specialists in the theory of fields of norms back in 1985 (J.-P. Wintenberger, J.-M. Fontaine, F. Laubie); however a proof based on this theory first appeared only in (I. B. Fesenko, *Preprint*, Nottingham Univ., 1998).

12.25. Let G be a finite group acting irreducibly on a vector space V. An orbit α^G for $\alpha \in V$ is said to be *p-regular* if the stabilizer of α in G is a p'-subgroup. Does G have a regular orbit on V if it has a p-regular orbit for every prime p? *Jiping Zhang*

No, not always. For example, let $H = \mathbb{A}_4 \wr \mathbb{Z}_5$, $V = O_2(H)$, and G be a complement to V in H, G acting on V by conjugation. (V. I. Zenkov, *Letter of February, 11, 1994*.)

12.26. (Shi Shengming). Is it true that a finite p-soluble group G has a p-block of defect zero if and only if there exists an element $x \in O_{p'}(G)$ such that $C_G(x)$ is a p'-subgroup? *Jiping Zhang*

No, it is not. The group $3^2.GL_2(3)$ has a 2-block of defect 0, but the centralizer of every element from $O(G)$ has even order (V. I. Zenkov, in: *Trudy Inst. Matem. i Mekh. UrO RAN*, **3** (1995), 36–40 (Russian)).

12.31. For relatively free groups G, prove or disprove the following conjecture of P. Hall: if a word v takes only finitely many values on G then the verbal subgroup vG is finite. *S. V. Ivanov*

The conjecture is disproved (A. Storozhev, *Proc. Amer. Math. Soc.*, **124**, no. 10 (1996), 2953–2954).

12.36. Let p be a prime, V an n-dimensional vector space over the field of p elements, and let G be a subgroup of $GL(V)$. Let $S = S[V^*]$ be the symmetric algebra on V^*, the dual of V. Let $T = S^G$ be the ring of invariants and let b_m be the dimension of the homogeneous component of degree m. Then the Poincaré series $\sum\limits_{m\geqslant 0} b_m t^m$ is a rational function with a Laurent power series expansion $\sum\limits_{i\geqslant -n} a_i(1-t)^i$ about $t = 1$ where $a_{-n} = \dfrac{1}{|G|}$.

Conjecture: $a_{-n+1} = \dfrac{r}{2|G|}$ where $r = \sum\limits_{W}((p-1)\alpha_W + s_W - 1)$, the sum is taken over all maximal subspaces W of V, and α_W, s_W are defined by $|G_W| = p^{\alpha_W} \cdot s_W$ where $p \nmid s_W$ and G_W denotes the pointwise stabilizer of W.

D. Carlisle, P. H. Kropholler

The conjecture is proved (D. J. Benson, W. W. Crawley-Boevey, *Bull. London Math. Soc.*, **27**, no. 5 (1995), 435–440); another proof based on the Grothendieck–Riemann–Roch Theorem was later obtained in (A. Neeman, *Comment. Math. Helv.*, **70**, no. 3 (1995), 339–349).

12.39. (W. J. Shi). Must a finite group and a finite simple group be isomorphic if they have equal orders and the same set of orders of elements? *A. S. Kondratiev*

Yes, they must (M. C. Xu, W. J. Shi, *Algebra Colloq.*, **10** (2003), 427–443; A.V. Vasil'ev, M.A. Grechkoseeva, V.D. Mazurov, *Algebra Logic*, **48** (2009), 385–409).

12.42. Describe the automorphisms of the Sylow p-subgroup of a Chevalley group of normal type over $\mathbb{Z}/p^m\mathbb{Z}$, $m \geqslant 2$, where p is a prime. *V. M. Levchuk*

Described in (S. G. Kolesnikov, *Algebra and Logic*, **43**, no. 1 (2004), 17–33); *Izv. Gomel' Univ.*, **36**, no. 3 (2006), 137–146 (Russian); *J. Math. Sci., New York*, **152** (2008), 220–246).

12.44. (P. Hall). Is there a non-trivial group which is isomorphic with every proper extension of itself by itself? *J. C. Lennox*

Yes, such groups exist of cardinality any regular cardinal (R. Göbel, S. Shelah, *Math. Proc. Cambridge Philos. Society (2)*, **134**, no. 1 (2003), 23–31).

12.45. (P. Hall). Must a non-trivial group, which is isomorphic to each of its non-trivial normal subgroups, be either free of infinite rank, simple, or infinite cyclic? (Lennox, Smith and Wiegold, 1992, have shown that a finitely generated group of this kind which has a proper normal subgroup of finite index is infinite cyclic.) *J. C. Lennox*

No (V. N. Obraztsov, *Proc. London Math. Soc.*, **75** (1997), 79–98).

12.46. Let F be the nonabelian free group on two generators x, y. For $a, b \in \mathbb{C}$, $|a| = |b| = 1$, let $\vartheta_{a,b}$ be the automorphism of $\mathbb{C}F$ defined by $\vartheta_{a,b}(x) = ax$, $\vartheta_{a,b}(y) = by$. Given $0 \neq \alpha \in \mathbb{C}F$, can we always find $a, b \in \mathbb{C} \setminus \{1\}$ with $|a| = |b| = 1$ such that $\alpha\mathbb{C}F \cap \vartheta_{a,b}(\alpha)\mathbb{C}F \neq 0$? *P. A. Linnell*

No, not always (A. V. Tushev, *Ukrain. Mat. J.*, **47**, no. 4 (1995), 571–572).

12.47. Let k be a field, let p be a prime, and let G be the Wreath product $\mathbb{Z}_p \wr \mathbb{Z}$ (so the base group has exponent p). Does kG have a classical quotient ring? (i. e. do the non-zero-divisors of kG form an Ore set?) *P. A. Linnell*

No, it does not (P. A. Linnell, W. Lück, T. Schick, in: *High-dimensional manifold topology. Proc. of the school, ICTP (Trieste, 2001)*, World Scientific, River Edge, NJ, 2003, 315–321).

12.49. Construct all non-split extensions of elementary abelian 2-groups V by $H = PSL_2(q)$ for which H acts irreducibly on V. *V. D. Mazurov*

They are constructed (V. P. Burichenko, *Algebra and Logic*, **39** (2000), 160–183).

12.65. Let $\mathscr{P} = (P_0, P_1, P_2)$ be a parabolic system in a finite group G, belonging to the C_3 Coxeter diagram $\underset{1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{2}{\circ}\!=\!\!=\!\!=\!\underset{3}{\circ}$ and let the Borel subgroup have index at least 3 in P_0 and P_1. It is known that, if we furthermore assume that the chamber system of $\mathscr{P}$ is geometric and that the projective planes arising as $\{0,1\}$-residues from $\mathscr{P}$ are desarguesian, then either $G^{(\infty)}$ is a Chevalley group of type C_3 or B_3 or $G = A_7$. Can we obtain the same conclusion in general, without assuming the previous two hypothesis? *A. Pasini*

Yes, we can (S. Yoshiara, *J. Algebraic Combin.*, **5**, no. 3 (1996), 251–284).

12.70. Let p be a prime number, F a free pro-p-group of finite rank, and $\Delta \neq 1$ an automorphism of F whose order is a power of p. Is the rank of $\mathrm{Fix}_F(\Delta) = \{x \in F \mid \Delta(x) = x\}$ finite? If the order of Δ is prime to p, then $\mathrm{Fix}_F(\Delta)$ has infinite rank (W. Herfort, L. Ribes, *Proc. Amer. Math. Soc.*, **108** (1990), 287–295). *L. Ribes*

Yes, it is finite (C. Scheiderer, *Proc. Amer. Math. Soc.*, **127**, no. 3 (1999), 695–700). Moreover, in (W. N. Herfort, L. Ribes, P. A. Zalesskii, *Forum Math.*, **11**, no. 1 (1999), 49–61) it is proved that if P is a finite p-group of automorphisms of a free pro-p group F of arbitrary rank, then the group of fixed points of P is a free factor of F and the latter result has been extended in (P. A. Zalesskii, *J. Reine Angew. Math.*, **572** (2004), 97–110) to the situation of a free pro-p group of arbitrary rank.

12.71. Let $d(G)$ denote the smallest cardinality of a generating set of the group G. Let A and B be finite groups. Is there a finite group G such that $A, B \leqslant G$, $G = \langle A, B\rangle$, and $d(G) = d(A) + d(B)$? The corresponding question has negative answer in the class of solvable groups (L. G. Kovács, H.-S. Sim, *Indag. Math.*, **2**, no. 2 (1991), 229–232). *L. Ribes*

No, not always (A. Lucchini, *J. Group Theory*, **4**, no. 1 (2001), 53–58).

12.74. Let $\mathfrak{F}$ be a non-primary one-generator composition formation of finite groups. Is it true that if $\mathfrak{F} = \mathfrak{M}\mathfrak{H}$ and the formations $\mathfrak{M}$ and $\mathfrak{H}$ are non-trivial, then $\mathfrak{M}$ is a composition formation? *A. N. Skiba*

This is true if $\mathfrak{F} \neq \mathfrak{H}$ (A. N. Skiba, *Algebra of formations*, Belaruskaja Navuka, Minsk, 1997, p. 144 (Russian)). In general the answer is negative (W. Guo, *Commun. Algebra*, **28**, no. 10 (2000), 4767–4782).

12.76. Is every group generated by a set of 3-transpositions locally finite? A *set of 3-transpositions* is, by definition, a normal set of involutions such that the orders of their pairwise products are at most 3. *A. I. Sozutov*

Yes, it is (H. Cuypers, J. I. Hall, in: *Groups, combinatorics and geometry. Proc. L. M. S. Durham symp., July 5–15, 1990* (*London Math. Soc. Lecture Note Ser.*, **165**), Cambridge Univ. Press, 1992, 121–138).

12.78. (M. J. Curran). a) Does there exist a group of order p^6 (here p is a prime number), whose automorphism group has also order p^6?

b) Is it true that for $p \equiv 1 \pmod 3$, the smallest order of a p-group that is the automorphism group of a p-group is p^7?

c) The same question for $p = 3$ with 3^9 replacing p^7. *A. I. Starostin*

a) No, there is no such a group; b) yes, it is true; c) no, it is not true (S. Yu, G. Ban, J. Zhang, *Algebra Colloq.*, **3**, no. 2 (1996), 97–106).

12.80. (K. W. Roggenkamp). a) Is it true that the number of p-blocks of defect 0 of a finite group G is equal to the number of the conjugacy classes of elements $g \in G$ such that the number of solutions of the equation $[x, y] = g$ in G is not divisible by p? *S. P. Strunkov*

No; for example, if $p = 2$ and $G = \mathbb{Z}_3 \times \mathbb{S}_3$ (L. Barker, *Letter of May, 27, 1996.*)

12.82. Find all pairs (n, r) such that the symmetric group $\mathbb{S}_n$ contains a maximal subgroup isomorphic to $\mathbb{S}_r$. *V. I. Sushchanskiĭ*

They are found (B. Newton, B. Benesh, *J. Algebra*, **304**, no. 2 (2006), 1108–1113).

12.84. (Well-known problem). Is it true that if there exist two non-isomorphic groups with the given set of orders of the elements, then there are infinitely many groups with this set of orders of the elements? *S. A. Syskin*

No, it is not (V. D. Mazurov, *Algebra and Logic*, **33**, no. 1 (1994), 49–55).

12.90. Let G be a finitely generated soluble minimax group and let H be a finitely generated residually finite group which has precisely the same finite images as G. Must H be a minimax group? *J. S. Wilson*

Yes, it must (A. Mann, D. Segal, *Proc. London Math. Soc. (3)*, **61** (1990), 529–545; see also A. V. Tushev, *Math. Notes*, **56**, no. 5 (1994), 1190–1192).

12.91. Every metabelian group belonging to a Fitting class of (finite) supersoluble groups is nilpotent. Does the following generalization also hold: Every group belonging to a Fitting class of supersoluble groups has only central minimal normal subgroups? *H. Heineken*

No, it does not (H. Heineken, *Rend. Sem. Mat. Univ. Padova*, **98** (1997), 241–251).

12.93. Let $N \rightarrowtail G \twoheadrightarrow Q$ be an extension of nilpotent groups, with Q finitely generated, which splits at every prime. Does the extension split? This is known to be true if N is finite or commutative. *P. Hilton*

No, not always (K. Lorensen, *Math. Proc. Cambridge Philos. Soc.*, **123**, no. 2 (1998), 213–215).

12.96. Find a non-empty Fitting class $\mathfrak{F}$ and a non-soluble finite group G such that G has no $\mathfrak{F}$-injectors. *L. A. Shemetkov*

Found by E. Salomon (Mainz Univ., unpublished); his example is presented in § 7.1 of (A. Ballester-Bolinches, L. M. Ezquerro, *Classes of finite groups*, Springer, 2006).

12.97. Let $\mathfrak{F}$ be the formation of all finite groups all of whose composition factors are isomorphic to some fixed simple non-abelian group T. Prove that $\mathfrak{F}$ is indecomposable into a product of two non-trivial subformations. *L. A. Shemetkov*

It is proved (O. V. Mel'nikov, *Problems in Algebra*, **9**, Gomel', 1996, 42–47 (Russian)).

12.98. Let F be a free group of finite rank, R its recursively defined normal subgroup. Is it true that

a) the word problem for F/R is soluble if and only if it is soluble for $F/[R, R]$?

b) the conjugacy problem for F/R is soluble if and only if it is soluble for $F/[R, R]$?

c) the conjugacy problem for $F/[R, R]$ is soluble if the word problem is soluble for $F/[R, R]$? *V. E. Shpil'rain*

a) Yes, it is; b) No, it is not; c) No, it is not (M. I. Anokhin, *Math. Notes*, **61**, no. 1–2 (1997), 3–8).

12.102. Is every proper factor-group of a group of Golod (see 9.76) residually finite? *V. P. Shunkov*

No, not every (L. Hammoudi, *Algebra Colloq.*, **5** (1998), 371–376).

13.10. Is there a function $f: \mathbb{N} \to \mathbb{N}$ such that, for every soluble group G of derived length k generated by a set A, the validity of the identity $x^4 = 1$ on each subgroup generated by at most $f(k)$ elements of A implies that G is a group of exponent 4? *V. V. Bludov*

No, there is not (G. S. Deryabina, A. N. Krasil'nikov, *Siberian Math. J.*, **44**, no. 1 (2003), 58–60).

13.11. Is a torsion-free group almost polycyclic if it has a finite set of generators $a_1, \dots, a_n$ such that every element of the group has a unique presentation in the form $a_1^{k_1} \cdots a_n^{k_n}$, where $k_1, \dots, k_n \in \mathbb{Z}$? *V. V. Bludov*

No, not always (A. Muranov, *Trans. Amer. Math. Soc.*, **359** (2007), 3609–3645).

13.16. Is every locally nilpotent group with minimum condition on centralizers hypercentral? *F. O Wagner*

Yes, it is (V. V. Bludov, *Algebra and Logic*, **37**, no. 3 (1998), 151–156).

13.18. Let F be a finitely generated non-Abelian free group and let G be the Cartesian (unrestricted) product of countable infinity of copies of F. Must the Abelianization G/G' of G be torsion-free? *A. M. Gaglione, D. Spellman*

No, it may contain elements of order 2 (O. Kharlampovich, A. Myasnikov, in: *Knots, braids, and mapping class groups — papers dedicated to Joan S. Birman (New York, 1998)*, Amer. Math. Soc., Providence, RI, 2001, 77–83).

13.21. a) Is there an infinite finitely generated residually finite p-group, in which the order $|g|$ of an arbitrary element g does not exceed $f(\delta(g))$, where $\delta(g)$ is the length of g with respect to a fixed set of generators and $f(n)$ is a function growing at $n \to \infty$ slower than any power function n^λ, $\lambda > 0$? *R. I. Grigorchuk*

Yes, there is (A. V. Rozhkov, *Dokl. Math.*, **58**, no. 2 (1998), 234–237).

13.28. (D. M. Evans). A permutation group on an infinite set is *cofinitary* if its non-identity elements fix only finitely many points. Is it true that a closed cofinitary permutation group is locally compact (in the topology of pointwise convergence)?

P. J. Cameron

No, not always (G. Hjorth, *J. Algebra*, **200**, no. 2 (1998), 439–448).

13.33. (F. Gross). Is a normal subgroup of a finite D_π-group (see Archive, 3.62) always a D_π-group? *V. D. Mazurov*

Yes, it is mod CFSG (E. P. Vdovin, D. O. Revin, *in: Ischia Group Theory 2004* (*Contemp. Math.*, **402**), Amer. Math. Soc., 2006, 229–263).

13.34. (I. D. Macdonald). If the identity $[x, y]^n = 1$ holds on a group, is the derived subgroup of the group periodic? *V. D. Mazurov*

No, not always (G. S. Deryabina, P. A. Kozhevnikov, *Commun. Algebra*, **27**, no. 9 (1999), 4525–4530; S. I. Adyan, *Dokl. Math.*, **62**, no. 2 (2000), 174–176).

13.36. b) (A. Lubotzky, A. Shalev). For a finitely generated pro-p-group G set $a_n(G) = \dim_{\mathbb{F}_p} I^n/I^{n+1}$, where I is the augmentation ideal of the group ring $\mathbb{F}_p[[G]]$. We define the *growth* of G to be the growth of the sequence $\{a_n(G)\}_{n\in\mathbb{N}}$. Is the growth of G exponential if G contains a finitely generated closed subgroup of exponential growth? *O. V. Mel'nikov*

b) Not necessarily. Let G be the Nottingham group over $\mathbb{F}_p$. The growth of $c_n(G) := \log_p |G : \omega_n(G)|$, where $\{\omega_n(G)\}$ is the Zassenhaus filtration, is linear, so by Quillen's theorem the growth of G is subexponential. But G has a 2-generator free pro-p subgroup (R. Camina, *J. Algebra*, **196** (1997), 101–113) with exponential growth. (M. Ershov, *Letter of 19.10.2009.*)

13.38. Let $G = F/R$ be a pro-p-group with one defining relation, where R is the normal subgroup of a free pro-p-group F generated by a single element $r \in F^p[F, F]$.

a) Suppose that $r = t^p$ for some $t \in F$; can G contain a Demushkin group as a subgroup?

b) Do there exist two pro-p-groups $G_1 \supset G_2$ with one defining relation, where G_1 has elements of finite order, while the subgroup G_2 is torsion-free? *O. V. Mel'nikov*

a) Yes, it can. b) Yes, they exist. The pro-p-group with presentation $\langle a, b \mid [a, b]^p = 1\rangle$ contains the Demushkin group $\langle x_1, \ldots, x_{2g} \mid \prod_{i=1}^{g} [x_{2i-1}, x_{2i}] = 1\rangle$ for some $g > 1$. (O. V. Mel'nikov, *Letter of February, 28, 1999*).

13.40. A group G is said to be ω-*residually free* if, for every finite set of non-trivial elements of G, there is a homomorphism of G into a free group such that the images of all these elements are non-trivial. Is every finitely-generated ω-residually free group embeddable in a free $\mathbb{Z}[x]$-group? *A. G. Myasnikov, V. N. Remeslennikov*

Yes, it is (O. Kharlampovich, A. Myasnikov, *J. Algebra*, **200** (1998), 472–570).

13.46. Can every uncountable abelian group of finite odd exponent be partitioned into two subsets so that neither of them contains cosets of infinite subgroups? Among countable abelian groups, such partitions exist for groups with finitely many involutions. *I. V. Protasov*

Yes, it can (E. G. Zelenyuk, *Math. Notes*, **67** (2000), 599–602).

13.47. Can every countable abelian group with finitely many involutions be partitioned into two subsets that are dense in every group topology? *I. V. Protasov*

Yes, it can (E. G. Zelenyuk, *Ukrain. Math. J.*, **51**, no. 1 (1999), 44–50).

13.54. b) Is it true that every group is embeddable in the kernel of some Frobenius group (see Archive, 6.53)? *A. I. Sozutov*

Yes, it is true (V. V. Bludov, *Siberian Math. J.*, **38**, no. 6 (1997), 1054–1056).

13.56. (A. Shalev). Let G be a finite p-group of sectional rank r, and φ an automorphism of G having exactly m fixed points. Is the derived length of G bounded by a function depending on r and m only? *E. I. Khukhro*

Yes, it is (A. Jaikin-Zapirain, *Israel J. Math.*, **129** (2002), 209–220).

13.58. Let φ be an automorphism of prime order p of a nilpotent (periodic) group G such that $C_G(\varphi)$ is a group of finite sectional rank r. Does G possess a normal subgroup N which is nilpotent of class bounded by a function of p only and is such that G/N is a group of finite sectional rank bounded in terms of r and p?

This was proved for $p = 2$ in (P. Shumyatsky, *Arch. Math.*, **71** (1998), 425–432). *E. I. Khukhro*

Yes, it does (E. I. Khukhro, *J. London Math. Soc.*, **77** (2008), 130–148).

13.61. We call a metric space *narrow* if it is quasiisometric to a subset of the real line, and *wide* otherwise. Let G be a group with the finite set of generators A, and let $\Gamma = \Gamma(G, A)$ be the Cayley graph of G with the natural metric. Suppose that, after deleting any narrow subset L from Γ, at most two connected components of the graph $\Gamma \setminus L$ can be wide, and there exists at least one such a subset L yielding exactly two wide components in $\Gamma \setminus L$. Is it true that Γ is quasiisometric to an Euclidean or a hyperbolic plane? *V. A. Churkin*

No, it is not true in general (O. V. Bogopol'skii, *Preprint*, Novosibirsk, 1998 (Russian)). See also new problem 14.98.

13.62. Let U and V be non-cyclic subgroups of a free group. Does the inclusion $[U, U] \leqslant [V, V]$ imply that $U \leqslant V$? *V. P. Shaptala*

No, it does not (M. J. Dunwoody, *Arch. Math.*, **16** (1965), 153–157).

13.63. Let $\pi_e(G)$ denote the set of orders of elements of a group G. For $\Gamma \subseteq \mathbb{N}$ let $h(\Gamma)$ denote the number of non-isomorphic finite groups G with $\pi_e(G) = \Gamma$. Is there a number k such that, for every Γ, either $h(\Gamma) \leqslant k$, or $h(\Gamma) = \infty$? *W. J. Shi*

No, there is no such number: $h(\pi_e(L_3(7^{3^r}))) = r+1$ for any $r \geqslant 0$ (A. V. Zavarnitsine, *J. Group Theory*, **7**, no. 1 (2004), 81–97).

13.66. Let F be a

a) free;

b) free metabelian

group of finite rank. Let M denote the set of all endomorphisms of F with non-cyclic images. Can one choose two elements $g, h \in F$ such that, for every $\varphi, \psi \in M$, equalities $\varphi(g) = \psi(g)$ and $\varphi(h) = \psi(h)$ imply that $\varphi = \psi$, that is, the endomorphisms in M are uniquely determined by their values at g and h? *V. E. Shpil'rain*

a) Yes, one can (D. Lee, *J. Algebra*, **247** (2002), no. 2, 509–540).

b) No, not always (E. I. Timoshenko, *Math. Notes*, **62**, no. 5–6 (1998), 767–770).

14.10. b) Find an explicit embedding of $\mathbb{Q}$ in a finitely generated group; such a group exists by Theorem IV in (G. Higman, B. H. Neumann, H. Neumann, *J. London Math. Soc.*, **24** (1949), 247–254). *P. de la Harpe*

Found (V. H. Mikaelian, *Int. J. Math. Math. Sci.*, **2005** (2005), 2119–2123).

14.13. a) By definition the *commutator length* of an element z of the derived subgroup of a group G is the least possible number of commutators from G whose product is equal to z. Does there exist a simple group on which the commutator length is not bounded? *V. G. Bardakov*

Simple groups with this property were constructed in (J. Barge, É. Ghys, *Math. Ann.*, **294** (1992), 235–265), and finitely generated simple ones in (A. Muranov, *Int. J. Algebra Comput.*, **17** (2007), 607–659).

14.25. Let qG denote the quasivariety generated by a group G. Is it true that there exists a finitely generated group G such that the set of proper maximal subquasivarieties in qG is infinite? *A. I. Budkin*

Yes, it is (V. V. Bludov, *Algebra and Logic*, **41**, no. 1 (2002), 1–8).

14.27. Let Γ be a group generated by a finite set S. Assume that there exists a nested sequence $F_1 \subset F_2 \subset \cdots$ of finite subsets of Γ such that (i) $F_k \neq F_{k+1}$ for all $k \geqslant 1$, (ii) $\Gamma = \bigcup\limits_{k\geqslant 1} F_k$, (iii) $\lim\limits_{k\to\infty} \dfrac{|\partial F_k|}{|F_k|} = 0$, where, by definition, $\partial F_k = \{\, \gamma \in \Gamma \setminus F_k \mid \text{there exists } s \in S \text{ such that } \gamma s \in F_k \,\}$, and (iv) there exist constants $c \geqslant 0$, $d \geqslant 1$ such that $|F_k| \leqslant ck^d$ for all $k \geqslant 1$. Does it follow that Γ has polynomial growth? *A. G. Vaillant*

No, not always. These properties are enjoyed by every group of intermediate growth (V. G. Bardakov, *Algebra and Logic*, **40**, no. 1 (2001), 12–16).

14.33. Does there exist a finitely presented pro-p-group (p being a prime) which contains an isomorphic copy of every countably based pro-p-group? *R. I. Grigorchuk*

Yes, it exists: the Nottingham group over $\mathbb{F}_p$ for $p > 2$, which was shown to be finitely presented in (M. V. Ershov, *J. London Math. Soc.*, **71** (2005), 362–378). (M. Ershov, *Letter of 19.10.2009.*)

14.34. By definition, a locally compact group has the *Kazhdan T-property* if the trivial representation is an isolated point in the natural topological space of unitary representations of the group. Does there exist a profinite group with two dense discrete subgroups one of which is amenable, and the other has the Kazhdan T-property?

See (A. Lubotzky, *Discrete groups, expanding graphs and invariant measures* (*Progress in Mathematics, Boston, Mass.*, **125**), Birkhäuser, Basel, 1994) for further motivation. *R. I. Grigorchuk, A. Lubotzky*

Yes, it exists (M. Kassabov, *Invent. Math.*, **170** (2007), 297–326; M. Ershov, A. Jaikin-Zapirain, *Invent. Math.*, **179**, 303–347).

14.48. If an equation over a free group F has no solution in F, is there a finite quotient of F in which the equation has no solution? *L. Comerford*

No, not always (T. Coulbois, A. Khelif, *Proc. Amer. Math. Soc.*, **127**, no. 4 (1999), 963–965).

14.49. Is $SL_3(\mathbb{Z})$ a factor group of the modular group $PSL_2(\mathbb{Z})$? Since the latter is isomorphic to the free product of two cyclic groups of orders 2 and 3, the question asks if $SL_3(\mathbb{Z})$ can be generated by two elements of orders 2 and 3. *M. Conder*

No, it is not (M. C. Tamburini, P. Zucca, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, **11**, no. 1 (2000), 5–7; Ya. N. Nuzhin, *Math. Notes*, **70**, no. 1–2 (2001), 71–78). M. Conder has shown, however, that $SL_3(\mathbb{Z})$ has a subgroup of index 57 that is a factor group of the modular group $PSL_2(\mathbb{Z})$. Also it has been shown in (M. C. Tamburini, J. S. Wilson, N. Gavioli, *J. Algebra*, **168** (1994), 353–370) that $SL_d(\mathbb{Z})$ is a factor group of the modular group $PSL_2(\mathbb{Z})$ for all $d \geqslant 28$.

14.50. (Z. I. Borevich). A subgroup A of a group G is said to be *paranormal* (respectively, *polynormal*) if $A^x \leqslant \langle A^u \mid u \in \langle A, A^x \rangle \rangle$ (respectively, $A^x \leqslant \big\langle A^u \mid u \in A^{\langle x \rangle} \big\rangle$) for any $x \in G$. Is every polynormal subgroup of a finite group paranormal? *A. S. Kondratiev*

Not always (V. I. Mysovskikh, *Dokl. Math.*, **60** (1999), 71–72).

14.52. It is known that if a finitely generated group is residually torsion-free nilpotent, then the group is residually finite p-group, for every prime p. Is the converse true? *Yu. V. Kuz'min*

Not always (B. Hartley, in: *Symposia Mathematica, Bologna, Vol. XVII, Convegno sui Gruppi Infiniti, INDAM, Roma, 1973*, Academic Press, London, 1976, 225–234).

14.55. a) Prove that the Nottingham group $J = N(\mathbb{Z}/p\mathbb{Z})$ (as defined in Archive, 12.24) is finitely presented for $p > 2$. *C. R. Leedham-Green*

Proved for $p > 2$ (M. V. Ershov, *J. London Math. Soc.*, **71** (2005), 362–378).

14.58. b) Suppose that A is a periodic group of regular automorphisms of an abelian group. Is A finite if A is generated by elements of order 3? *V. D. Mazurov*

Yes, it is (A. Kh. Zhurtov, *Siberian Math. J.*, **41**, no. 2 (2000), 268–275).

14.60. Suppose that H is a non-trivial normal subgroup of a finite group G such that the factor-group G/H is isomorphic to one of the simple groups $L_n(q)$, $n \geqslant 3$. Is it true that G has an element whose order is distinct from the order of any element in G/H? *V. D. Mazurov*

This is true for $n \neq 4$ (A. V. Zavarnitsine, *Siberian Math. J.*, **49** (2008), 246–256), and false for $n = 4$ (A. V. Zavarnitsine, *Siberian Electron. Math. Rep.*, **5** (2008), 68–74).

14.62. Suppose that H is a non-soluble normal subgroup of a finite group G. Does there always exist a maximal soluble subgroup S of H such that $G = H \cdot N_G(S)$? *V. S. Monakhov*

Yes, it always exists mod CFSG (V. I. Zenkov, V. S. Monakhov, D. O. Revin, *Algebra and Logic*, **43**, no. 2 (2004), 102–108).

14.63. What are the composition factors of non-soluble finite groups all of whose normalizers of Sylow subgroups are 2-nilpotent, in particular, supersoluble? *V. S. Monakhov*

Described (L. S. Kazarin, A. A. Volochkov, *Mathematics in Yaroslavl' Univ.: Coll. of Surveys to 30-th Anniv. of Math. Fac.*, Yaroslavl', 2006, 243–255 (Russian)).

14.66. (Well-known problem). Let G be a finite soluble group, $\pi(G)$ the set of primes dividing the order of G, and $\nu(G)$ the maximum number of primes dividing the order of some element. Does there exist a linear bound for $|\pi(G)|$ in terms of $\nu(G)$? *A. Moretó*

Yes, it does (T. M. Keller, *J. Algebra*, **178**, no. 2 (1995), 643–652).

14.71. Consider a free group F of finite rank and an arbitrary group G. Define the *G-closure* $\mathrm{cl}_G(T)$ of any subset $T \subseteq F$ as the intersection of the kernels of all those homomorphisms $\mu : F \to G$ of F into G that vanish on T: $\mathrm{cl}_G(T) = \bigcap\{\mathrm{Ker}\,\mu \mid \mu : F \to G;\ T \subseteq \mathrm{Ker}\,\mu\}$. Groups G and H are called *geometrically equivalent* if for every free group F and every subset $T \subset F$ the G- and H-closures of T coincide: $\mathrm{cl}_G(T) = \mathrm{cl}_H(T)$. It is easy to see that if G and H are geometrically equivalent then they have the same quasiidentities. Is it true that if two groups have the same quasiidentities then they are geometrically equivalent? This is true for nilpotent groups. *B. I. Plotkin*

No, not always (V. V. Bludov, *Abstracts of the 7th Int. Conf. Groups and Group Rings*, Suprasĺ, Poland, 1999, p. 6; R. Göbel, S. Shelah, *Proc. Amer. Math. Soc.*, **130** (2002), 673–674 (electronic); A. G. Myasnikov, V. N. Remeslennikov, *J. Algebra*, **234**, no. 1 (2000), 225–276). The latter paper contains also necessary and sufficient conditions for geometrical equivalence.

14.77. Let p be a prime number and X a finite set of powers of p containing 1. Is it true that X is the set of all lengths of the conjugacy classes of some finite p-group? *J. Sangroniz*

Yes, it is (J. Cossey, T. Hawkes, *Proc. Amer. Math. Soc.*, **128**, no. 1 (2000), 49–51).

14.80. Is the lattice of all totally local formations of finite groups modular? The definition of a *totally local* formation see in (L. A. Shemetkov, A. N. Skiba, *Formatsii algebraicheskikh system*, Moscow, Nauka, 1989 (Russian)). *A. N. Skiba, L. A. Shemetkov*

Yes, even distributive (V. G. Safonov, *Commun. Algebra*, **35** (2007), 3495–3502).

14.86. Does there exist an infinite locally nilpotent p-group that is equal to its commutator subgroup and in which every proper subgroup is nilpotent? *J. Wiegold*

No, it does not exist (A. O. Asar, *J. Lond. Math. Soc. (2)*, **61**, no. 2 (2000), 412–422).

14.88. We say that an element u of a group G is a *test element* if for any endomorphism φ of G the equality $\varphi(u) = u$ implies that φ is an automorphism of G. Does a free soluble group of rank 2 and derived length $d > 2$ have any test elements? *B. Fine, V. Shpilrain*

Yes, it does (E. I. Timoshenko, *Algebra and Logic*, **45**, no. 4 (2006), 254–260.)

14.92. (I. D. Macdonald). Every finite p-group has at least $p-1$ conjugacy classes of maximum size. I. D. Macdonald (*Proc. Edinburgh Math. Soc.*, **26** (1983), 233–239) constructed groups of order 2^n for any $n \geqslant 7$ with just one conjugacy class of maximum size. Are there any examples with exactly $p-1$ conjugacy classes of maximum size for odd p? *G. Fernández–Alcober*

Yes, there are, for $p = 3$ (A. Jaikin-Zapirain, M. F. Newman, E. A. O'Brien, *Israel J. Math.*, **172**, no. 1 (2009), 119–123).

14.96. Suppose that a finite p-group P admits an automorphism of order p^n having exactly p^m fixed points. By (E. I. Khukhro, *Russ. Acad. Sci. Sbornik Math.*, **80** (1995), 435–444) then P has a subgroup of index bounded in terms of p, n and m which is soluble of derived length bounded in terms of p^n. Is it true that P has also a subgroup of index bounded in terms of p, n and m which is soluble of derived length bounded in terms of m? There are positive answers in the cases of $m = 1$ (S. McKay, *Quart. J. Math. Oxford, Ser. (2)*, **38** (1987), 489–502; I. Kiming, *Math. Scand.*, **62** (1988), 153–172) and $n = 1$ (Yu. A. Medvedev, see Archive, 10.68). *E. I. Khukhro*

Yes, it is true (A. Jaikin-Zapirain, *Adv. Math.*, **153**, no. 2 (2000), 391–402).

14.102. c) (V. Lin). Let B_n be the braid group on n strings, and let $n > 4$. Does B_n have proper non-abelian torsion-free factor-groups?

Comment of 2001: S. P. Humphries, (*Int. J. Algebra Comput.*, **11**, no. 3 (2001), 363–373) has constructed a representation of B_n which is shown to provide torsion-free non-abelian factor groups of B_n as well as of $[B_n, B_n]$ for $n < 7$. It is likely that the same representation should work for other values of n as well. *V. Shpilrain*

Yes, it does; moreover, every braid group B_n is residually torsion-free nilpotent-by-finite (P. Linnell, T. Schick, *J. Amer. Math. Soc.*, **20**, no. 4 (2007), 1003–1051).

14.103. Let H be a proper subgroup of a group G and let elements $a, b \in H$ have distinct prime orders p, q. Suppose that, for every $g \in G \setminus H$, the subgroup $\langle a, b^g\rangle$ is a finite Frobenius group with complement of order pq. Does the subgroup generated by the union of the kernels of all Frobenius subgroups of G with complement $\langle a\rangle$ intersect $\langle a\rangle$ trivially? The case where all groups $\langle a, b^g\rangle$, $g \in G$, are finite is of special interest. *V. P. Shunkov*

No, in general case not necessarily. As a counter-example one can take $G = \langle x, y, z, a, b \mid x^7 = y^7 = [x, y, y] = [x, y, x] = a^3 = b^2 = [a, b] = 1,\ \ x^a = x^2,\ \ y^a = y^2,\ \ x^b = x^{-1},\ \ y^b = y^{-1}\rangle$ with $H = \langle a, b\rangle$. Analogous examples exist also for $p = 2$ and every odd prime q (A. I. Sozutov, *Letter of 2002*).

14.104. An infinite group G is called a *monster of the first kind* if it has elements of order > 2 and for any such an element a and for any proper subgroup H of G, there is an element g in $G \setminus H$, such that $\langle a,\, a^g\rangle = G$. A. Yu. Ol'shanskiĭ showed that there are continuously many monsters of the first kind (see Archive, 6.63). Does there exist, for any such a monster, a torsion-free group which is a central extension of a cyclic group by the given monster? *V. P. Shunkov*

No, not always, since for such an extension to exist it is necessary that every finite subgroup of the monster is cyclic, but this is not always true (A. I. Sozutov, *Letter of November, 20, 2001*).

15.4. Is it true that large growth implies non-amenability? More precisely, consider a number $\epsilon > 0$, an integer $k \geqslant 2$, a group Γ generated by a set S of k elements, and the corresponding exponential growth rate $\omega(\Gamma, S)$ defined as in 14.7. For ϵ small enough, does the inequality $\omega(\Gamma, S) \geqslant 2k - 1 - \epsilon$ imply that Γ is non-amenable?

If $\omega(\Gamma, S) = 2k - 1$, it is easy to show that Γ is free on S, and in particular non-amenable; see Section 2 in (R. I. Grigorchuk, P. de la Harpe, *J. Dynam. Control Syst.*, **3**, no. 1 (1997), 51–89). *P. de la Harpe*

No, it does not. Counterexamples can be found even in the classes of abelian-by-nilpotent and metabelian-by-finite groups. (G. N. Arzhantseva, V. S. Guba, L. Guyot, *J. Group Theory*, **8** (2005), 389–394).

15.8. a) (S. M. Ulam). Consider the usual compact group $SO(3)$ of all rotations of a 3-dimensional Euclidean space, and let G denote this group viewed as a discrete group. Can G act non-trivially on a countable set? *P. de la Harpe*

Yes, it can (S. Thomas, *J. Group Theory*, **2** (1999), 401–434); another proof is in (Yu. L. Ershov, V. A. Churkin, *Dokl. Math.*, **70**, no. 3 (2004), 896–898).

15.10. (Yu. I. Merzlyakov). Is the group of all automorphisms of the free group F_n that act trivially on $F_n/[F_n, F_n]$ linear for $n \geqslant 3$? *V. G. Bardakov*

No, it is not: for $n \geqslant 5$ (A. Pettet, *Cohomology of some subgroups of the automorphism group of a free group*, Ph.D. Thesis, 2006), and for $n \geqslant 3$ (V. G. Bardakov, R. Mikhailov, *Commun. Algebra*, **36**, no. 4 (2008), 1489–1499).

15.14. Do there exist finitely generated branch groups (see 15.12)

a) that are non-amenable?

c) that contain F_2?

d) that have exponential growth? *L. Bartholdi, R. I. Grigorchuk, Z. Šuniḱ*

a), c), d) Such groups do exist (S. Sidki, J. S. Wilson, *Arch. Math.*, **80**, no. 5 (2003), 458–463).

15.24. Suppose that a finite p-group G has a subgroup of exponent p and order p^n. Is it true that if p is sufficiently large relative to n, then G contains a normal subgroup of exponent p and order p^n?

J. L. Alperin and G. Glauberman (*J. Algebra*, **203**, no. 2 (1998), 533–566) proved that if a finite p-group contains an elementary abelian subgroup of order p^n, then it contains a normal elementary abelian subgroup of the same order provided $p > 4n-7$, and the analogue for arbitrary abelian subgroups is proved in (G. Glauberman, *J. Algebra*, **272** (2004), 128–153). *Ya. G. Berkovich*

Yes, it is true if $p > n$. By induction, G contains a subgroup H of exponent p and order p^n which is normal in a maximal subgroup M of G. Then $H \leqslant \zeta_{p-1}(M)$. The elements of order p of $\zeta_{p-1}(M)$ constitute a normal subgroup of G, which contains H. (A. Mann, *Letter of 1 October 2002.*)

15.25. A finite group G is said to be *rational* if every irreducible character of G takes only rational values. Are the Sylow 2-subgroups of the symmetric groups $\mathbb{S}_{2^n}$ rational? *Ya. G. Berkovich*

Yes, they are. A Sylow 2-subgroup T_n of $\mathbb{S}_{2^n}$ is the wreath product of the one for $\mathbb{S}_{2^{n-1}}$ with the group C of order 2. If T_n is rational, then the wreath product of T_n with C is also rational by Corollary 70 in (D. Kletzing *Structure and representations of Q-groups* (Lecture Notes in Math., **1084**), Springer, Berlin, 1984). The result follows by induction. (A. Mann, *Letter of 1 October 2002*).

15.27. Is it possible that $\operatorname{Aut} G \cong \operatorname{Aut} H$ for a finite p-group G of order > 2 and a proper subgroup $H < G$? *Ya. G. Berkovich*

Yes, it is possible (T. Li, *Arch. Math.*, **92** (2009), 287–290) for $|G| = 2|H| = 32$.

15.34. Is any free product of linearly ordered groups with an amalgamated subgroup right-orderable? *V. V. Bludov*

Yes, it is (V. V. Bludov, A. M. W. Glass, *Math. Proc. Cambridge Philos. Soc.*, **146** (2009), 591–601).

15.39. Axiomatizing the basic properties of subnormal subgroups, we say that a functor τ associating with every finite group G some non-empty set $\tau(G)$ of its subgroups is an *ETP-functor* if

1) $\tau(A)^{\varphi} \subseteq \tau(B)$ and $\tau(B)^{\varphi^{-1}} \subseteq \tau(A)$ for any epimorphism $\varphi : A \longmapsto B$, as well as $\{H \cap R \mid R \in \tau(G)\} \subseteq \tau(H)$ for any subgroup $H \leqslant G$;

2) $\tau(H) \subseteq \tau(G)$ for any subgroup $H \in \tau(G)$;

3) $\tau(G)$ is a sublattice of the lattice of all subgroups of G.

Let τ be an ETP-functor. Does there exist a hereditary formation $\mathfrak{F}$ such that $\tau(G)$ coincides with the set of all $\mathfrak{F}$-subnormal subgroups in any finite group G? This is true for finite soluble groups (A. F. Vasil'ev, S. F. Kamornikov, *Siberian Math. J.*, **42**, no. 1 (2001), 25–33). *A. F. Vasil'ev, S. F. Kamornikov*

Not always (S. F. Kamornikov, *Math. Notes*, **89**, no. 3–4 (2011), 340–348).

15.43. Let G be a finite group of order n.

a) Is it true that $|\mathrm{Aut}\, G| \geqslant \varphi(n)$ where φ is Euler's function?

b) Is it true that G is cyclic if $|\mathrm{Aut}\, G| = \varphi(n)$? *M. Deaconescu*

Both questions have negative answers; moreover, $|\mathrm{Aut}\, G|/\varphi(|G|)$ can be made arbitrarily small (J. N. Bray, R. A. Wilson, *Bull. London Math. Soc.*, **37**, no. 3 (2005), 381–385).

15.44. b) Is the ring of invariants $K[M(n)^m]^{GL(n)}$ Cohen–Macaulay in all characteristics? (Here $M(n)^m$ is the direct sum of m copies of the space of $n \times n$ matrices.) *A. N. Zubkov*

Yes, it is (M. Hashimoto, *Math. Z.*, **236** (2001), 605–623).

15.60. Is it true that any finitely generated p'-isolated subgroup of a free group is separable in the class of finite p-groups? It is easy to see that this is true for cyclic subgroups. *D. I. Moldavanskiĭ*

No, it is not (V. G. Bardakov, *Siberian Math. J.*, **45**, no. 3 (2004), 416–419).

15.63. b) Let F_n be the free group of finite rank n on the free generators $x_1, \dots, x_n$. An element $u \in F_n$ is called *positive* if u belongs to the semigroup generated by the x_i. An element $u \in F_n$ is called *potentially positive* if $\alpha(u)$ is positive for some automorphism α of F_n. Finally, $u \in F_n$ is called *stably potentially positive* if it is potentially positive as an element of F_m for some $m \geqslant n$. Are there stably potentially positive elements that are not potentially positive? *A. G. Myasnikov, V. E. Shpilrain*

No, there are none (A. Clark, R. Goldstein, *Commun. Algebra*, **33**, no. 11 (2005), 4097–4104).

15.75. a) Does there exist a sequence of identities in two variables $u_1 = 1, u_2 = 1, \dots$ with the following properties: 1) each of these identities implies the next one, and 2) an arbitrary finite group is soluble if and only if it satisfies one of the identities $u_n = 1$? *B. I. Plotkin*

Yes, such sequences exist (T. Bandman, F. Grunewald, G.-M. Greuel, B. Kunyavskii, G. Pfister, E. Plotkin, *Compositio Math.*, **142** (2006), 734–764; J. N. Bray, J. S. Wilson, R. A. Wilson, *Bull. Lond. Math. Soc.*, **37** (2005) 179–186).

15.76. a) If Θ is a variety of groups, then let Θ^0 denote the category of all free groups of finite rank in Θ. It is proved (G. Mashevitzky, B. Plotkin, E. Plotkin *J. Algebra*, **282** (2004), 490–512) that if Θ is the variety of all groups, then every automorphism of the category Θ^0 is an inner one. The same is true if Θ is the variety of all abelian groups. Is this true for the variety of nilpotent groups of class 2?

An automorphism φ of a category is called *inner* if it is isomorphic to the identity automorphism. Let $s : 1 \to \varphi$ be a function defining this isomorphism. Then for every object A we have an isomorphism $s_A : A \to \varphi(A)$ and for any morphism of objects $\mu : A \to B$ we have $\varphi(\mu) = s_B \mu s_A^{-1}$. *B. I. Plotkin*

Yes, it is, even for the variety of nilpotent groups of any class n (A. Tsurkov, *Int. J. Algebra Comput.*, **17** (2007), 1273–1281).

15.81. Let G be a finite non-supersoluble group. Is it true that G has a non-cyclic Sylow subgroup P such that some maximal subgroup of P has no proper complement in G? *A. N. Skiba*

Yes, it is (Wang, Yanming, Wei, Huaquan, *Sci. China Ser. A*, **47** (2004), no. 1, 96–103).

15.86. A group G is called *discriminating* if for any finite set of nontrivial elements of the direct square $G\times G$ there is a homomorphism $G\times G \to G$ which does not annihilate any of them (G. Baumslag, A. G. Myasnikov, V. N. Remeslennikov). A group G is called *squarelike* if G is universally equivalent (in the sense of first order logic) to a discriminating group (B. Fine, A. M. Gaglione, A. G. Myasnikov, D. Spellman). Must every squarelike group be elementarily equivalent to a discriminating group? *D. Spellman*

Yes, it must (O. Belegradek, *J. Group Theory*, **7**, no. 4 (2004), 521–532; B. Fine, A. M. Gaglione, D. Spellman, *Archiv Math.*, **83**, no. 2 (2004), 106–112).

16.8. The *width* $w(G')$ of the derived subgroup G' of a finite non-abelian group G is the smallest positive integer m such that every element of G' is a product of $\leqslant m$ commutators. Is it true that the maximum value of the ratio $w(G')/|G|$ is 1/6 (attained at the symmetric group $\mathbb{S}_3$)? *V. G. Bardakov*

Yes, it is (T. Bonner, *J. Algebra*, **320** (2008), 3165–3171).

16.12. Given a finite p-group G, we define a Φ-*extension* of G as any finite p-group H containing a normal subgroup N of order p such that $H/N \cong G$ and $N \leqslant \Phi(H)$. Is it true that for every finite p-group G there exists an infinite sequence $G = G_1, G_2, \ldots$ such that G_{i+1} is a Φ-extension of G_i for all $i = 1, 2, \ldots$? *Ya. G. Berkovich*

Yes, it is (S. F. Kamornikov, *Izv. Gomel' Univ.*, **5** (2008), 200–201 (Russian)).

16.17. Is it true that a non-abelian simple group cannot contain Engel elements other than the identity element? *V. V. Bludov*

No, it can: since an involution is an Engel element in any 2-group, counterexamples are provided by non-abelian simple 2-groups; see Archive, 4.74(a). (V. V. Bludov, *Letter of 30 March 2006.*)

16.35. Is every finitely presented soluble group nilpotent-by-nilpotent-by-finite?
J. R. J. Groves

No: for a prime p the group $G = \langle a, b, c, d \mid b^a = b^p,\ \ c^a = c^p,\ \ d^a = d,\ \ c^b = c,\ d^c = dd^b,\ \ [d, d^b] = 1,\ \ d^p = 1\rangle$ is soluble of derived length 3, but is not metanilpotent-by-finite. (V. V. Bludov, *Letter of 27 April 2007.*)

16.37. Let G be a solvable rational finite group with an extra-special Sylow 2-subgroup. Is it true that either G is a 2-nilpotent group, or there is a normal subgroup E of G such that G/E is an extension of a normal 3-subgroup by an elementary abelian 2-group?
M. R. Darafsheh

Yes, it is (M. R. Darafsheh, H. Sharifi, *Extracta Math.*, **22**, no. 1 (2007), 83–91).

16.43. Is there a partition of the group $\bigoplus_{\omega_1}(\mathbb{Z}/3\mathbb{Z})$ into three subsets whose complements do not contain cosets modulo infinite subgroups? (There is a partition into two such subsets.)
E. G. Zelenyuk

Yes, moreover, every infinite abelian group with finitely many involutions can be partitioned into infinitely many subsets such that every coset modulo an infinite subgroup meets each subset of the partition (Y. Zelenyuk, *J. Combin. Theory (A)*, **115** (2008), 331–339).

16.55. (Well-known problem). Let V be a faithful absolutely irreducible module for a finite group G. Is it true that $\dim H^1(G, V) \leqslant 2$?
V. D. Mazurov

No, it is not (L. L. Scott, *J. Algebra*, **260** (2003), 416–425; see also J. N. Bray, R. A. Wilson, *J. Group Theory*, **11** (2008), 669–673).

16.57. Suppose that $\omega(G) = \omega(L_2(7)) = \{1, 2, 3, 4, 7\}$. Is $G \cong L_2(7)$? This is true for finite G.
V. D. Mazurov

Yes, it is (D. V. Lytkina, A. A. Kuznetsov, *Siberian Electr. Math. Rep.*, **4** (2007), 136–140; `http://semr.math.nsc.ru`).

16.61. A subgroup H of a group G is *fully invariant* if $\vartheta(H) \leqslant H$ for every endomorphism ϑ of G. Let G be a finite group such that G has a fully invariant subgroup of order d for every d dividing $|G|$. Must G be cyclic?
D. MacHale

No: take $G = C_p \times C_{p^2}$, where C_p, C_{p^2} are cyclic p-groups of orders p, p^2 (A. Abdollahi, *Letter of 4 March 2009*).

16.62. Let G be a group such that every $\alpha \in \operatorname{Aut} G$ fixes every conjugacy class of G (setwise). Must $\operatorname{Aut} G = \operatorname{Inn} G$?
D. MacHale

Not necessarily: the paper (H. Heineken, *Arch. Math.* (*Basel*), **33** (1979/80), no. 6, 497–503), in particular, produces finite p-groups all of whose automorphisms preserve all the conjugacy classes, while by Gaschütz' theorem every finite p-group has outer automorphisms (A. Mann, *Letter of 20 April 2006*).

16.65. Does there exist a finitely presented residually torsion-free nilpotent group with a free presentation $G = F/R$ such that the group $F/[F, R]$ is not residually nilpotent?
R. Mikhailov, I. B. S. Passi

Yes, it exists (V. V. Bludov, *J. Group Theory*, **12**, no. 4 (2009), 579–590).

16.67. *Conjecture:* Given any integer k, there exists an integer $n_0 = n_0(k)$ such that if $n \geqslant n_0$ then the symmetric group of degree n has at least k different ordinary irreducible characters of equal degrees. *A. Moretó*

This is proved (D. Craven, *Proc. London Math. Soc.*, **96** (2008), 26–50).

16.75. Can a non-abelian one-relator group be the group of all automorphisms of some group? *M. V. Neshchadim*

Yes, it can: in (D. J. Collins, *Proc. London Math. Soc. (3)*, **36** (1978), no. 3, 480–493) it was proved that for integers r, s such that $(r, s) = 1$, $|r| \neq |s|$, and $r - s$ is even, the group $G = \langle a, b \mid a^{-1}b^r a = b^s \rangle$ is isomorphic to $\mathrm{Aut}(\mathrm{Aut}(G))$ (V. A. Churkin, *Letter of 5 April 2006*).

16.106. Let $\pi_e(G)$ denote the set of orders of elements of a group G, and $h(\Gamma)$ the number of non-isomorphic finite groups G with $\pi_e(G) = \Gamma$. Do there exist two finite groups G_1, G_2 such that $\pi_e(G_1) = \pi_e(G_2)$, $h(\pi_e(G_1)) < \infty$, and neither of the two groups G_1, G_2 is isomorphic to a subgroup or a quotient of a normal subgroup of the other? *W. J. Shi*

Yes, there exist: for example, $G_1 = L_{15}(2^{60}).3$ and $G_2 = L_{15}(2^{60}).5$ (M. A. Grechkoseeva, *Algebra and Logic*, **47**, no. 4 (2008), 229–241).

16.109. Is there a polynomial time algorithm for solving the word problem in the group $\mathrm{Aut}\, F_n$ (with respect to some particular finite presentation), where F_n is the free group of rank $n \geqslant 2$? *V. E. Shpilrain*

Yes, there is (S. Schleimer, *Comment. Math. Helv.*, **83** (2008), 741–765).

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