Chamber graphs of some geometries related to the Petersen graph

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CHAMBER GRAPHS OF SOME GEOMETRIES RELATED TO THE PETERSEN GRAPH

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In this thesis we study the chamber graphs of the geometries $\Gamma(A_{2n+1})$, $\Gamma(3A_7)$, $\Gamma(L_2(11))$ and $\Gamma(L_2(25))$ which are related to the Petersen graph [4, 13].

In Chapter 2 we look at the chamber graph of $\Gamma(A_{2n+1})$ and see what minimal paths between chambers look like. Chapter 3 finds and proves the diameter of these chamber graphs and we see what two chambers might look like if they are as far apart as possible. We discover the full automorphism group of the chamber graph.

Chapters 4, 5 and 6 focus on the chamber graphs of $\Gamma(L_2(11))$, $\Gamma(L_2(25))$ and $\Gamma(3A_7)$ respectively. We ask questions such as what two chambers look like if they are as far apart as possible, and we find the automorphism groups of the chamber graphs.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Thank you Peter for all your help, and giving me a kick at the right times! And thank you also to the School of Mathematics for funding me.

Thank you all my family and friends. Especially Mum and Dad, because I owe them my existence and much more.

Thank You God for everything You've done for me.

Chapter 1

Introduction

In 1974 Jacques Tits [30] noticed a phenomenon occurring in groups of Lie type which he referred to as a B, N*-pair* of the group. That is, he discovered that every group of Lie type contains two subgroups B and N with certain properties. It was also noticed that groups of Lie type acted in a nice way on things called *buildings*. Over the next few years, it turned out that buildings and B, N-pairs are equivalent, with every B, N-pair corresponding to exactly one building, and each building corresponding to any number of B, N-pairs.

Buildings (or B, N-pairs) are helpful with groups of Lie type and so people have tried to find similar things to "explain" the 26 Sporadic simple groups [3, 4, 5, 13, 14, 23]. The definition of a building has changed over the last few years. Occasionally they are defined as *geometries*. A few of the geometries related to the Sporadic simple groups are those related to the Petersen graph [13, 14] on which this thesis is based.

The Petersen graph is shown in Fig.1.1. Around 1898 Julius Petersen [22] presented

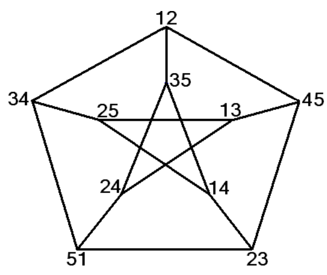


Fig. 1.1. The Petersen graph

this graph as the smallest counter-example to the claim that a bridgeless connected cubic graph has an edge-colouring with three colours. Donald Knuth states that the

Petersen graph is "a remarkable configuration that serves as a counterexample to many optimistic predictions about what might be true for graphs in general" [19]. In 1980 Jonathan Hall [10] discovered there are only three locally Petersen graphs (discussed later). Since then there has been much interest in geometries related to the Petersen graph. For example, all Petersen geometries P^m have been classified (there are only nine) by Ivanov and Shpectorov [17].

Many properties about geometries can be expressed using their chamber graphs. For instance the automorphism group of a chamber graph must contain the automorphism group of the geometry from which it came. Jacques Tits later studied geometries by looking at their chamber graphs, or chamber systems (which are the same as chamber graphs, except adjacencies are labelled by type) [31]. In fact the definition of a building has sometimes been written in terms of chambers [26, 32]. There has been much research into the chamber graphs of geometries [2, 24, 27, 28, 29].
This thesis contains results mainly about the diameter and automorphism group of chamber graphs, and about what pairs of chambers look like if the distance between them is maximal. For example, each of the locally Petersen graphs discovered by Hall gives rise to a geometry of rank 3 by taking vertices, lines and triangles as elements of type 0, 1 and 2 respectively. We will investigate the chamber graphs of each of these geometries, amongst others.

The rest of the introduction is basically a slightly more detailed version of the above.

1.1 B, N-pairs and buildings

Definition 1.1. *The group G is said to have a B, N-*pair *or (G, B, N, S) is said to be a* Tits system *if*

- $G = \langle B, N \rangle$
- $B \cap N \lhd N$ *and the group* $W = N/B \cap N$ *has a set* S *of involution generators.*
- *If* $s \in S, w \in W$ *then* $sBw \subseteq BwB \cup BswB$.
- $sBs \neq B$ *for all* $s \in S$.

The group W is the *Weyl* group of the B, N-pair. The *Weyl distance* of an element

$w \in W$ is the length of a shortest word equal to w using letters $s \in S$.
The definition of a building varies depending on what book you're reading. We will use the following definition: Let S be a simplicial complex of rank n. A sub-simplex of rank n is called a *chamber*. A sub-simplex of rank $n-1$ is called a *panel*. Two chambers C and D of S are said to be *adjacent* if $C \cap D$ is a panel. We call S thick if every panel is contained in three or more chambers. We call S thin if every panel is contained in exactly two chambers.

Definition 1.2. *Let X be a simplicial complex of rank n and A be a set of subcomplexes of X called* apartments. *We call $\Delta = (X, A)$ a building if the following hold:*

- *X is thick and each apartment is thin.*
- *Any two subsimplices of X lie in some common apartment.*
- *The graph of adjacent chambers (the* chamber graph*) of X is connected.*
- *Every subsimplex of X is contained in some chamber of X.*
- *If S and T are both subsimplices of apartments Σ and Σ' then there is a bijection between Σ and Σ' preserving the partial order and fixing $\Sigma \cap \Sigma'$ pointwise.*

Given a building $\Delta = (X, A)$, Aschbacher ([1], Chapter 14) defines an equivalence relation on the elements of X whose equivalence classes are called *types*. Two elements are said to be of the same type if they are in the same equivalence class.

Definition 1.3. *We say g is a type-preserving automorphism of a building $\Delta = (X, A)$ if g is a permutation of the elements of X preserving the partial order, preserving type (fixing each type setwise) and preserving A setwise.*

1.1.1 Correspondence between BN-pairs and buildings

Consider the following situation: Let Δ be a building. Let C be any chamber in any apartment Σ. Let G be a group of type-preserving automorphisms of Δ with the property that given any chamber C' in any apartment Σ', there exists $g \in G$ such that $Cg = C'$ and $\Sigma g = \Sigma'$.

Given the situation above, it can be shown that $N = \mathrm{Stab}_G(\Sigma)$ and $B = \mathrm{Stab}_G(C)$ are a B, N-pair of G ([1], Chapter 14).
Conversely, given a group G with a B, N-pair, we can achieve the building above by

taking chambers to be the right cosets of B, $\Delta = \{\text{right cosets of } P \text{ for all } B \leqslant P < G\}$ with partial order $Pg < P'g'$ if $P'g' \subseteq Pg$, $\Sigma = \{Pn \text{ for all } B \leqslant P < G, n \in N\}$ and apartments are elements of Σ^G where G acts by right multiplication on Δ.

The method just described provides each such G with its unique building $\Delta(G)$, and each Δ with a group G such that $\Delta = \Delta(G)$. Interestingly, any $B \leqslant P \leqslant G$ is exactly generated by B along with some subset of S. It can be seen that statements about the distance between chambers are essentially equivalent to statements about Weyl distance, for a pair of adjacent chambers is of the form $\{C, Cs\}$ for some $s \in S$.

Definition 1.4. *Let $\mathscr{C}$ be a chamber graph whose automorphism group is transitive on the chambers. Let $\Delta_0(c) = \{c\}$ and let $\Delta_i(c)$ be the set of chambers whose distance from c is i. These sets form the* disc structure *of the chamber graph, unique up to isomorphism. We call $\Delta_i(c)$ the i^{th} disc of the chamber graph (from c).*

1.1.2 An example

Definition 1.5. *Let G have a B,N-pair. We usually call B a* Borel subgroup *of G and this is unique up to conjugation. Any subgroup P containing a conjugate of B is called a* parabolic subgroup.

Let $G = GL_n(F)$ act on $V = F^n$ with basis $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, ..., \mathbf{v}_n = \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Then we write the "standard" chamber

$$V_{n-1} = \langle \mathbf{v}_1, ..., \mathbf{v}_{n-1} \rangle \supset ... \supset V_2 = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle \supset V_1 = \langle \mathbf{v}_1 \rangle$$

(of rank $n-1$) whose stabilizer in G is the subgroup composed of all upper-triangular matrices. Usually this is chosen as the Borel subgroup B and the subgroup of monomial matrices is chosen as N.

1.2 The Petersen graph

Definition 1.6. *The Petersen graph is the graph whose vertices are elements of the set $\{\{a, b\} : a, b \in \{1, 2, 3, 4, 5\}\}$. Two vertices are incident if and only if they are disjoint. (See Fig.1.1.)*

Definition 1.7. *Let Ω be an m-set. The graph $T_2(m)$ is the graph whose vertices are 2-subsets of Ω, where two vertices are incident if and only if they share an element. Thus in the complement $\overline{T_2(m)}$ of $T_2(m)$ two vertices are incident if and only if they are disjoint.*

In Fig.1.2 we meet $T_2(5)$ and $T_2(7)$. Notice how we let each straight line represent the set of pairs containing a particular element, with the pairs distance 1 apart around the edge. The graphs $\overline{T_2(5)}$ and $\overline{T_2(7)}$ are the complements of these. Note that $\overline{T_2(5)}$

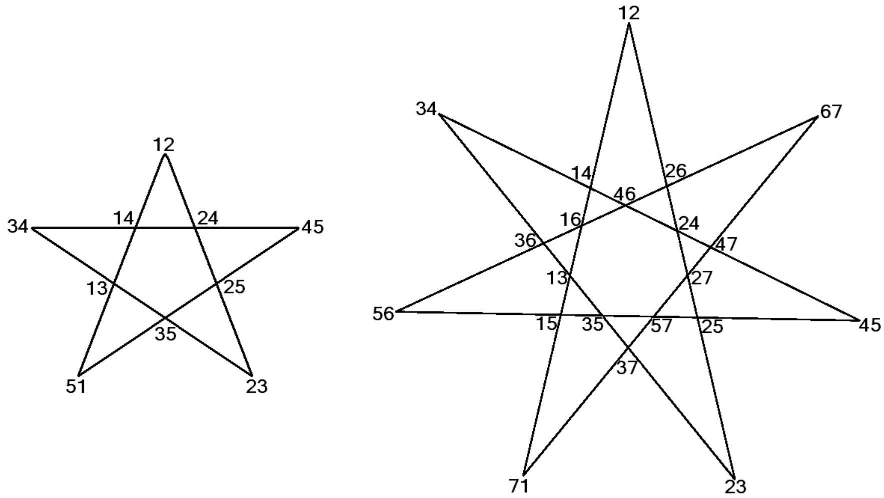


Fig. 1.2. $T_2(5)$ and $T_2(7)$

is the Petersen graph.

The Petersen graph has many interesting properties. For example, it is the unique smallest cubic graph with girth five [11]. In Fig.1.3 we show five ways of drawing the Petersen graph. Fig.1.3 is copied from Geoffrey Exoo [8].

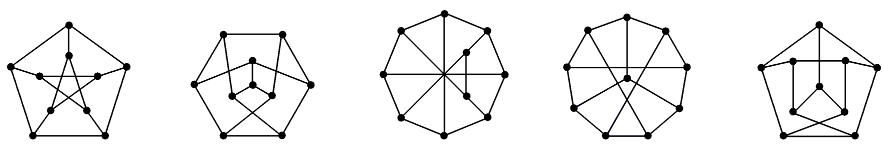


Fig. 1.3. Five ways of drawing the Petersen graph

Definition 1.8. *A graph Γ is called* locally Petersen *if for every $t \in \Gamma$, the graph induced by Γ on the neighbours of t in Γ is isomorphic to the Petersen graph.*

Jonathan Hall [10] showed that there are only three locally Petersen graphs up to isomorphism:

- The graph $\overline{T_2(7)}$

- The Conway-Smith graph.
- The Hall graph.

In fact, the Conway-Smith graph is the graph whose vertices are the elements of type 0 and whose adjacencies are the elements of type 1 described in Chapter 6. This becomes obvious later in Section 6.3.2.

Similarly the Hall graph is the graph whose vertices are the elements of type 0 and whose adjacencies are the elements of type 1 described in Chapter 5. This is shown later in Section 5.6.

1.3 Geometries

Buildings are sometimes expressed as geometries. Definition 1.9 is copied from Ivanov and Shpectorov [13].

Definition 1.9. (Γ, I, t, T) *(or just* Γ*) is called a geometry of rank* n *if:*

- Γ *is a set of vertices with symmetric incidence relation* I*.*
- t *is a map from* Γ *onto* $T = \{0, 1, 2, ..., n-1\}$*.*
- $v \in \Gamma$ *is said to be of* type i *if* $t(v) = i$*. Every maximal clique of* Γ *consists of exactly one element of each type.*

Note that the definition means two vertices of the same type cannot be incident. A clique of Γ is called a flag. A maximal flag is called a *chamber.* Two chambers are i-adjacent if they differ only by their element of type i. A geometry is called thin (respectively, thick) if every flag of order $n-1$ is contained in exactly two chambers (respectively, three or more chambers). The automorphism group $\mathrm{Aut}(\Gamma)$ is the set of all permutations of Γ preserving type and incidence. A geometry is said to be *flag-transitive* if its automorphism group is transitive on flags of the same type. In this thesis we consider only flag-transitive geometries.

Definition 1.10. *Let* F *be a flag of* Γ*. The residue* $(\Gamma_F, I_F, T_F, t_F)$ *(or just* Γ_F*) of* F *is defined by* $\Gamma_F = \{v \in \Gamma | v \notin F$ *and* $F \cup \{v\}$ *is a flag of* $\Gamma\}$*. As one would expect, we define* $T_F = t(\Gamma_F)$ *and* I_F *and* t_F *are the restrictions of* I *and* t *(resp.) to* Γ_F*.*

By flag-transitivity, all rank 2 residues of type $\{i, j\}$ of a geometry are isomorphic. Call this residue Γ_{ij}. Jacques Tits realized that a lot can be said about a geometry if we know its rank 2 residues.

Definition 1.11. *If for every flag F and all distinct $i, j \in T_F$, we have that*

$$\{v \in \Gamma_F | v \textit{ has type } i\} \cup \{v \in \Gamma_F | v \textit{ has type } j\}$$

is connected (with the obvious restriction of I) then we say Γ is strongly connected.

In other words, if $F \cup \{x, y\}$ and $F \cup \{u, v\}$ are chambers then $F \cup \{x, y\}$ is adjacent to another chamber containing F, which is adjacent to another chamber containing F, etc. until we get to $F \cup \{u, v\}$. Notice that if Γ is strongly connected then all its residues must be strongly connected.

Definition 1.12. *Let F be a flag with $T_F = \{i\}$. We say the number of chambers containing F is $q_i + 1$ (unique by flag-transitivity).*

We are now in a position to draw the *basic diagram* of Γ as a graph. This is how we draw: Each vertex represents a type from T. Write q_i underneath i. The edge between i and j is labelled by Γ_{ij}. These are a few of the labels widely used: (although sometimes authors use different notation)

- Let Γ_{ij} have every element of type i incident with every element of type j. Then we draw the empty edge between i and j like so:

- If we have that Γ_{ij} represents a projective plane of order q. That is,

  Any two elements of type i are both incident to exactly one element of type j,

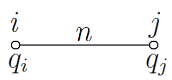
  Any two elements of type j are both incident to exactly one element of type i,

  Each element of type i is incident to exactly $q + 1$ elements of type j,

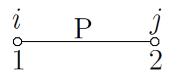
  Each element of type j is incident to exactly $q + 1$ elements of type i,

then we draw:

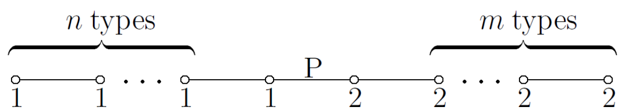
- Let Γ_{ij} represent a generalized n-gon. That is, the girth ($2n$) of the graph Γ_{ij} is twice its diameter (n). Let $n \geqslant 4$. Then we draw



- Let the elements of Γ_{ij} of type i and j be the vertices and edges of the Petersen graph respectively. Then we label the line between them like so:

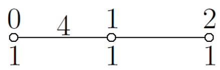


Definition 1.13. *The diagram shown here (of rank $n+m+2$) we will call P_n^m.*

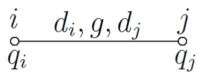


We usually omit writing n or m if it is equal to zero. Thus the diagram mentioned just before we simply call P.

An example: Let Γ be the set of points, lines and faces of a cube. To define I, let a face of the cube be incident to the 4 lines and 4 points around its edge, and let a line be incident to the 2 points at either end of it (here we see some motivation for the word "flag"). Let $T = \{0, 1, 2\}$ with the points, lines and faces having type 0, 1 and 2 respectively. We draw the basic diagram:



More generally, let $\mathrm{dist}(x, y)$ represent the distance between two vertices x and y in the graph Γ_{ij}. Then we draw:

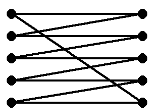


where $\quad d_i = \min\{\max\{\mathrm{dist}(x, y) : y \in \Gamma_{ij}\} : x \in \Gamma_{ij} | x \text{ has type } i\}$

$\quad d_j = \min\{\max\{\mathrm{dist}(x, y) : y \in \Gamma_{ij}\} : x \in \Gamma_{ij} | x \text{ has type } j\}$

$\quad g$ is half the cardinality of a smallest thin connected subgeometry of Γ_{ij}.

Such a subgeometry must have even cardinality $2g$ as it is forced to look like this:



If $d_i = g = d_j = n$ then we simply write n above the edge: This is the diagram for the n-gon. If $n = 3$ then we don't label the edge: This is the diagram for the projective plane. If $n = 2$ then we draw the empty edge: All elements of type i are incident to all elements of type j. If $n = 4$ then a double line is sometimes used.

People have searched for geometries whose automorphisms are sporadic simple groups. For example, let G be a sporadic simple group, let p be a prime dividing $|G|$ and let $B = N_G(P)$ for some Sylow p-subgroup P of G. Ronan and Stroth [25] give a complete list of all systems $\{P_i : i \in \{1, 2, ..., n\}\}$ (where the P_i are subgroups of G) such that:

1) The n subgroups P_i generate G.

2) No $n - 1$ subgroups P_i generate G.

3) The largest normal p-subgroup of each P_i is not the identity.

4) B is contained in a unique maximal subgroup of P_i for each i.

5) If I and J are subsets of $\{1, 2, ..., n - 1, n\}$ then

$$\langle P_i : i \in I\rangle \cap \langle P_i : i \in J\rangle = \langle P_i : i \in I \cap J\rangle$$

These give rise to geometries by our method described in Section 1.1.1. Ronan and Stroth assume 1 to 5 because if G is a group of Lie type over a field of characteristic p then its Borel subgroup and parabolic subgroups satisfy these conditions.

1.4 Petersen geometries

Definition 1.14. *In this thesis we define a* Petersen geometry *to be any flag-transitive geometry with a diagram P_n^m. (See Definition 1.13). However, the definition varies from paper to paper.*

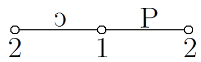
In 1988 Ivanov and Shpectorov showed that any flag-transitive Petersen geometry P^m is one of nine geometries [17]. These are normally called [15, 16]:

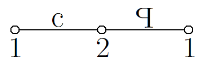
$$\begin{aligned}
\mathscr{G}(A_5) &\quad \text{with diagram } P \\
\mathscr{G}(M_{22}), \mathscr{G}(3 \cdot M_{22}) &\quad \text{with diagram } P^1 \\
\mathscr{G}(M_{23}), \mathscr{G}(Co_2), \mathscr{G}(3^{23} \cdot Co_2), \mathscr{G}(J_4) &\quad \text{with diagram } P^2 \\
\mathscr{G}(BM), \mathscr{G}(3^{4371} \cdot BM) &\quad \text{with diagram } P^3
\end{aligned}$$

So called because any group acting flag-transitively on the geometries with diagram P^2 and P^3 must be the group mentioned in its name. The only groups acting flag-transitively on $\mathscr{G}(A_5)$ are A_5 and S_5. The only groups acting flag-transitively on $\mathscr{G}(M_{22})$ are M_{22} and $\mathrm{Aut}(M_{22})$. The only groups acting flag-transitively on $\mathscr{G}(3M_{22})$

are $3M_{22}$ and $3\mathrm{Aut}(M_{22})$. We will talk more about $\mathscr{G}(M_{22})$ and $\mathscr{G}(M_{23})$ a bit later.

Thomas Meixner [20, 21] has classified all geometries with diagrams





(Where "c" is the rank 2 geometry where elements of type 0 and 1 are vertices and edges of an n-clique respectively. In the cases above $n = 4$.) Only a finite number of geometries have the diagram on the left. Buekenhout [3] introduced "c" in order to generalize the notion of a geometry so that we could find more geometries whose automorphism group is a simple sporadic group.

The Petersen geometries investigated in this thesis are denoted $\Gamma(A_{2n+1})$, $\Gamma(3A_7)$, $\Gamma(L_2(25))$ and $\Gamma(L_2(11))$. All have been described by Buekenhout [4] (page 24) and are described in more detail in Ivanov and Shpectorov's paper [13] (pages 939,944,945). We copy Table 1.1 from page 934 of this paper. (Since then it has been discovered that a few of the geometries in this table are residues of larger geometries. For example, this is true for the geometry of A_{2n+1}, and the geometries of both $L_2(25)$ and $PSp_4(5)$ can be expanded by an infinite series.) Our geometries are those in the top two rows of the table.

1.4.1 A couple of Petersen geometries

Rob Curtis [7] studied the Mathieu groups by considering the 12-dimensional vector space over $\mathbb{F}_2$ generated by the subsets of a 24-set Ω in Fig.1.4, thinking of addition as symmetric difference. This is usually called the *Golay code*. This contains the empty

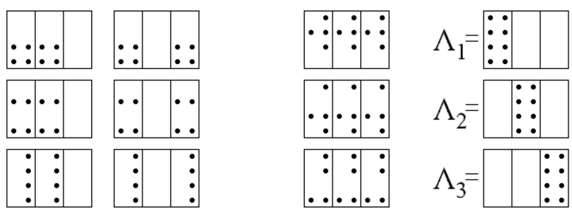
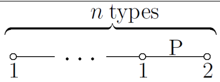
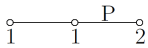
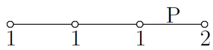
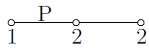
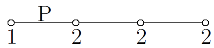
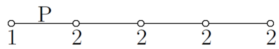
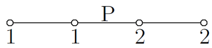
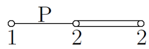
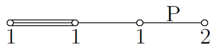
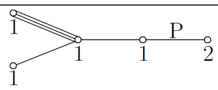


Fig. 1.4

set, Ω, 759 octads (8-sets), 759 octad complements (16-sets) and 2576 dodecads (12-sets). The 759 octads form the unique Steiner system $S(5, 8, 24)$, whose automorphism

A_{2n+1}	
$3A_7, L_2(25), L_2(11)$	
$M_{11}, He, PSp_4(5)$	
$M_{22}, 3M_{22}$	
M_{23}, Co_2, J_4	
F_2	
McL	
HS	
J_1	
$O'S, \widehat{O'S}$	

Tab. 1.1

group in S_{24} is M_{24}. In Fig.1.5 we display Curtis' MOG. Notice the MOG shows us

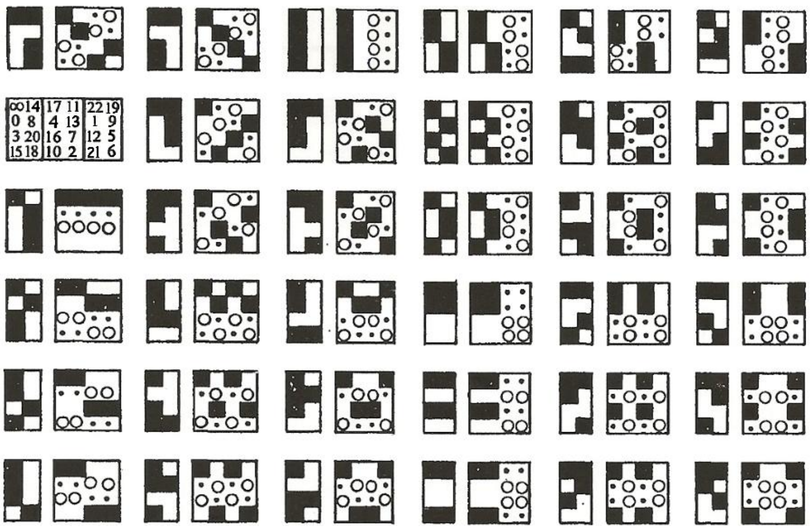


Fig. 1.5. The MOG

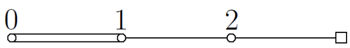
35 copies of Ω, each consisting of a rectangle (Λ_1) and a square ($\Lambda_2 \cup \Lambda_3$). Choose one of the 35 copies of Ω: Then choose either the white squares or black squares in Λ_1 and

choose either the white squares, black squares, dots or circles in $\Lambda_2 \cup \Lambda_3$. This gives us an 8-set. Apply any permutation of the Λ_i from the group generated by

and this gives us one of the 759 octads. All octads may be obtained from the MOG in this way.

A tetrad is a 4-set of Ω. The symmetric difference of two octads lies in the Golay code mentioned earlier. This forces their intersection to have order 0, 4 or 8. Five elements immediately define an octad. Hence a tetrad inside an octad defines a partition of Ω into six disjoint tetrads (two of which make the octad). This partition is called a *sextet*. There are $\binom{24}{4}/6 = 1771$ sextets. A partition of Ω into three disjoint octads is called a trio. It can be shown there are 3795 trios.

Consider the geometry whose elements of 0, 1 and 2 are the octads, trios and sextets of the MOG respectively, where incidence is defined in the obvious way. (An octad is incident to a trio if it is a member of the trio, a trio to a sextet if the tetrads of the sextet can be paired to make the trio, and an octad to a trio if two tetrads from the trio can be paired to make the octad). The group M_{24} is flag-transitive on this geometry. Its chamber graph was investigated by Peter Rowley [28]. It has diagram

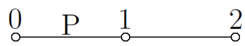


(The square represents something called a "ghost node" [23].)

Consider the following geometry of the group M_{23} fixing $a \in \Omega$: Let the elements of type 0 be the 506 octads not containing a and let the elements of type 1,2 and 3 be the 3795 trios, 1771 sextets and 23 elements $\Omega \backslash \{a\}$ respectively. Elements of type 0,1 and 2 are adjacent to each other in the obvious way (when one is a "sub-partition" of the other). An element b of type 3 is incident to an element p of type $t \neq 3$ if $\{a, b\}$ is a subset of one of the partitions of p (or is disjoint to the octad if $t = 0$). This is a Petersen geometry with diagram:

There is more about this geometry by Ivanov and Shpectorov [14]. The group M_{23} is this geometry's full automorphism group and acts flag-transitively. Its chamber graph was studied by Peter Rowley [27].

Another geometry can be obtained from the one above by taking the residue of type $\{0, 1, 2\}$. Therefore let $b \in \Omega \backslash \{a\}$: Elements of type 0 are octads disjoint from $\{a, b\}$. Elements of type 1 are all trios with $\{a, b\}$ a subset of one of its octads. Elements of type 2 are all sextets with $\{a, b\}$ a subset of one of its tetrads. This geometry has the following diagram:



The subgroup $M_{22} \leqslant M_{24}$ fixing $\{a, b\}$ pointwise acts flag-transitively on this geometry. The automorphism group $\mathrm{Aut}(M_{22})$ of M_{22} is in fact equal to the setwise-stabilizer of $\{a, b\}$ in M_{24}. It turns out $\mathrm{Aut}(M_{22})$ is equal to the group of automorphisms of this geometry [14]. This geometry's chamber graph has also been studied by Peter Rowley.

1.5 Amalgams

For the purposes of this thesis an amalgam of subgroups of a finite group G is a set of subgroups $\{G_0, G_1, ..., G_i\}$ of G which intersect in a certain way. In this thesis we always have $i = 3$ and we represent amalgams as shown in Fig.1.6. This is a shortened

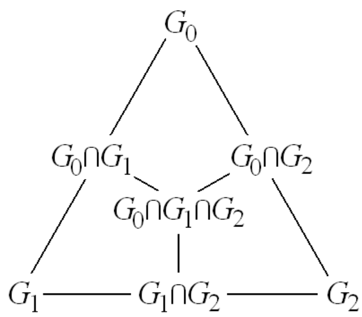


Fig. 1.6

definition [12]. We use amalgams to describe geometries in the following way: Given G_0, G_1 and G_2, elements of type i are the right cosets of G_i for $i \in \{0, 1, 2\}$. Two elements of different type are incident if they have non-empty intersection.

1.6 My results

The main result of this thesis (and the one I am most proud of) is the diameter of the chamber graph of $\Gamma(A_{2n+1})$ (see Theorem 3.38). Chapters 2 and 3 are almost entirely devoted to this. Table 1.2 shows the diameter and full automorphism group of the chamber graph of each geometry studied in this thesis.

Geometry	Diameter of the chamber graph	Full automorphism group of the geometry and its chamber graph
$\Gamma(A_{2n+1})$	$n^2 + \lfloor \frac{2n}{3} \rfloor$	S_{2n+1}
$\Gamma(L_2(11))$	9	$L_2(11)$
$\Gamma(L_2(25))$	18	$P\Sigma L(2,25)$
$\Gamma(3A_7)$	20	$3S_7$

Tab. 1.2

In Chapter 2 we investigate what the chamber graph of $\Gamma(A_{2n+1})$ looks like. Usually, there are many minimal paths between any two chambers A and B. In order to make life easier, we define certain paths which are *ordered from A to B*. It turns out that the distance between two chambers A and B can be derived from something we call their *intersection matrix* $M(A,B)$. Considering the "distance of a matrix" is therefore equivalent to considering the distance between two chambers. In this chapter we discover ways in which we can start minimal paths between chambers with certain intersection matrices, notably Theorems 2.35, 2.42 and 2.43.

In Chapter 3 we use the results proved in Chapter 2 to find a lower bound of the diameter. That is, two chambers which are exactly a certain distance apart. We then focus on finding an upper bound on the diameter. We do this by taking any two chambers A and B and looking at their intersection matrix $M(A,B)$. Our method for "attacking" $M(A,B)$ is to split it into smaller matrices. Then we can state bounds for the smaller matrices by induction. Finally, we find the full automorphism group of the chamber graph.

In Chapters 4, 5 and 6 we do similar things for the chamber graphs of $\Gamma(L_2(11))$, $\Gamma(L_2(25))$ and $\Gamma(3A_7)$ respectively. The geometries of $\Gamma(L_2(25))$ and $\Gamma(3A_7)$ are easily defined by the Hall graph and the Conway-Smith graph respectively, and $\Gamma(L_2(11))$

is basically a description of the Petersen design. (A 2-(11,3,3)-design.) We find nice ways of thinking about these three objects. Chambers of the geometries $\Gamma(L_2(25))$ and $\Gamma(3A_7)$ which are maximal distance apart look interesting: Those of $\Gamma(L_2(25))$ "intersect" in a certain way and those of $\Gamma(3A_7)$ are multiples of each other.

Curiously, the disc structures of the chamber graphs of $\Gamma(A_7)$, $\Gamma(L_2(11))$, $\Gamma(L_2(25))$ and $\Gamma(3A_7)$ are all very similar. (See Table 7.1.) There is probably a very clever reason for this. Some of the results (for example Figures 2.6, 5.1 and 6.1) give us a hint as to why this is.

Chapter 7 consists of a few extra results I have found (or disproved) which do not fit into the other chapters. We end with a conjecture which I am extremely suspicious is true but have not been able to prove.

Chapter 2

The geometries $\Gamma(A_{2n+1})$

First let's define $\Gamma(A_{2n+1})$.

Definition 2.1. *Let $2n+1 \geqslant 3$, $\Omega = \{1, 2, 3, ..., 2n+1\}$ and let S be the set of all 2-sets of Ω. We define Γ^i (the elements of type i) as follows:*

$\Gamma^0 = \{\{s\} | s \in S\} = \{\{\{a, b\}\} | a, b \in \Omega, a \neq b\}$

$\Gamma^1 = \{\{s, t\} | s, t \in S, s \text{ and } t \text{ disjoint }\}$

$\vdots$

$\Gamma^i = \{\{s_1, s_2, s_3, ..., s_{i+1}\} | s_j \in S \text{ and } s_j \text{ and } s_k \text{ disjoint whenever } j \neq k\}$

for $0 \leqslant i \leqslant n-1$. We have $\Gamma(A_{2n+1}) = \bigcup \Gamma^i$, where two elements of different type are incident if one is a subset of the other.

We will show later that the automorphism group of the above geometry is S_{2n+1}. For now it is enough to note that the automorphism group contains S_{2n+1} so this is a flag-transitive geometry. We know that $\Gamma(A_5)$ is the geometry formed by the points and lines of the Petersen graph, shown in Fig.1.1.

2.1 Chambers

Definition 2.2. *Throughout this thesis a chamber of the geometry $\Gamma(A_{2n+1})$ will sometimes be referred to as an n-chamber. In this chapter a chamber is often assumed to be an n-chamber unless stated otherwise.*

Theorem 2.3. $\Gamma(A_{2n+1})$ *contains exactly $\frac{(2n+1)!}{2^n}$ chambers.*

Proof. By Definition 2.1 S_{2n+1} acts transitively on the chambers. The stabilizer of a chamber in S_{2n+1} has order 2^n. $\square$

Definition 2.4. *Let C be an n-chamber whose element of type i is c_i:*

$$c_0 \subset c_1 \subset \ldots \subset c_{n-2} \subset c_{n-1}$$

For $1 \leqslant i \leqslant n-1$, the i^{th} part *$C(i)$ of C is defined as the only element in c_{n-i} which is not in c_{n-i-1}. We define $C(n)$ as the only element in c_0. Define $C(0)$ as the unique element of Ω not contained in any $C(i)$ for $i \geqslant 1$. We do not call $C(0)$ a part of C.*

Definition 2.5. *Let C be an n-chamber. We will write C in the following way:*

$$C = C(1)|C(2)|\ldots|C(n-1)|C(n)$$

It can be tedious to write each $C(i)$ in the form $\{a,b\}$ so for the sake of brevity, we will sometime miss out curly brackets or commas if it doesn't add confusion. On occasion, it is convenient to write $C(0)$ at the beginning, and for some $1 \leqslant x \leqslant n$ we sometimes indicate $C(x)$:

$$C = C(0)|C(1)|C(2)|\ldots|\overset{x}{C(x)}|\ldots|C(n-1)|C(n)$$

For example, in $\Gamma(A_5)$ the chamber

$$C : \{\{2,3\},\{1,4\}\} \supset \{\{2,3\}\}$$

has $C(1) = \{1,4\}, C(2) = \{2,3\}$ and $C(0) = 5$. We can write $C = 14|23$ or $5|14|23$.

In $\Gamma(A_7)$ the chamber

$$D : \{\{3,4\},\{1,2\},\{5,7\}\} \supset \{\{3,4\},\{5,7\}\} \supset \{\{5,7\}\}$$

has $D(1) = \{1,2\}, D(2) = \{3,4\}, D(3) = \{5,7\}$ and $D(0) = 6$. We may write $D = 12|34|\overset{3}{57}$ or $6|12|34|\overset{3}{57}$.

All we are doing is writing the omitted parts in order. Notice that there is no ambiguity: Each chamber is uniquely defined by this notation.

Recall that two adjacent chambers C and D are said to be i-adjacent if they differ only by their elements of type i. What do adjacent chambers look like when written using the notation of Definitions 2.4 and 2.5? This brings us on to Theorem 2.6.

Theorem 2.6. *The n-chamber C has exactly $n+1$ neighbours. Of these, $n-1$ are of the form D where for some $x(1 \leqslant x < n)$ we have $D(x) = C(x+1)$, $C(x) = D(x+1)$ and $C(i) = D(i)$ whenever $i \notin \{x, x+1\}$. That is, D looks like:*

$$\begin{gathered} C(2)|C(1)|C(3)|C(4)|...|C(n-1)|C(n) \\ C(1)|C(3)|C(2)|C(4)|...|C(n-1)|C(n) \\ C(1)|C(2)|C(4)|C(3)|...|C(n-1)|C(n) \\ \vdots \\ C(1)|C(2)|C(3)|C(4)|...|C(n)|C(n-1) \end{gathered}$$

The remaining two are of the form D where $C(i) = D(i)$ for $i > 1$.

Proof. Let C and D be x-adjacent. Let C be the chamber

$$c_0 \subset c_1 \subset ... \subset c_{n-2} \subset c_{n-1}$$

If $0 \leqslant x \leqslant n-2$ then we have that $c_{x+1} = c_x \cup C(n-x-1)$ and $c_x = c_{x-1} \cup C(n-x)$. This forces D's element of type x to be $c_{x-1} \cup C(n-x-1)$. This means

$$\begin{aligned} D(n-x) &= C(n-x-1) & \\ C(n-x) &= D(n-x-1) & \\ C(i) &= D(i) & \text{otherwise} \end{aligned}$$

On the other hand if $x = n-1$ then $C(i) = D(i)$ for $i > 1$. Clearly $D(0) \neq C(0)$ and so we have two options for $D(1)$. $\square$

For example, we observe that the chambers adjacent to 12|34|56 are 12|56|34, 34|12|56, 17|34|56 and 27|34|56, shown here:

$$\begin{array}{cccc} \{\{1,2\},\{3,4\},\{5,6\}\} & \{\{1,2\},\{3,4\},\{5,6\}\} & \{\{1,7\},\{3,4\},\{5,6\}\} & \{\{7,2\},\{3,4\},\{5,6\}\} \\ \cup & \cup & \cup & \cup \\ \{\{3,4\},\{5,6\}\} & \{\{1,2\},\{5,6\}\} & \{\{3,4\},\{5,6\}\} & \{\{3,4\},\{5,6\}\} \\ \cup & \cup & \cup & \cup \\ \{\{3,4\}\} & \{\{5,6\}\} & \{\{5,6\}\} & \{\{5,6\}\} \end{array}$$

If we look closely we see that the two type-2-adjacencies are different from the type-0-adjacency and type-1-adjacency.

Definition 2.7. *Let A and B be adjacent chambers. Write A—B. We say that A is a* neighbour *of B. Furthermore, if $A(x) = B(x+1)$ we call the adjacency a $(x, x+1)$-swap. We may say A is a $(x, x+1)$-swapping of B, or $A \underset{x,x+1}{\text{——}} B$. If $A(0) \neq B(0)$ we call the adjacency a 0-swap. We may say A is a 0-swapping of B, or $A \underset{0}{\text{——}} B$.*

For example, $C = 9|12|34|56|78$ is a 1,2-swap of $9|34|12|56|78$

a 2,3-swap of $9|12|56|34|78$

a 3,4-swap of $9|12|34|78|56$

a 0-swap of $1|29|34|56|78$

a 0-swap of $2|19|34|56|78$

Definition 2.8. *An adjacency may be referred to as a* jump *if it is not a 0-swap. (So called because two parts "jump" over each other by Theorem 2.6 and Definition 2.4.)*

2.2 The Chamber Graph

In Fig.2.1 we draw the chamber graph of $\Gamma(A_5)$. Notice how this graph consists of $\binom{5}{2} = 10$ triangles, and treating each of these as a vertex gives us the Petersen graph. We state this more generally in Theorem 2.9.

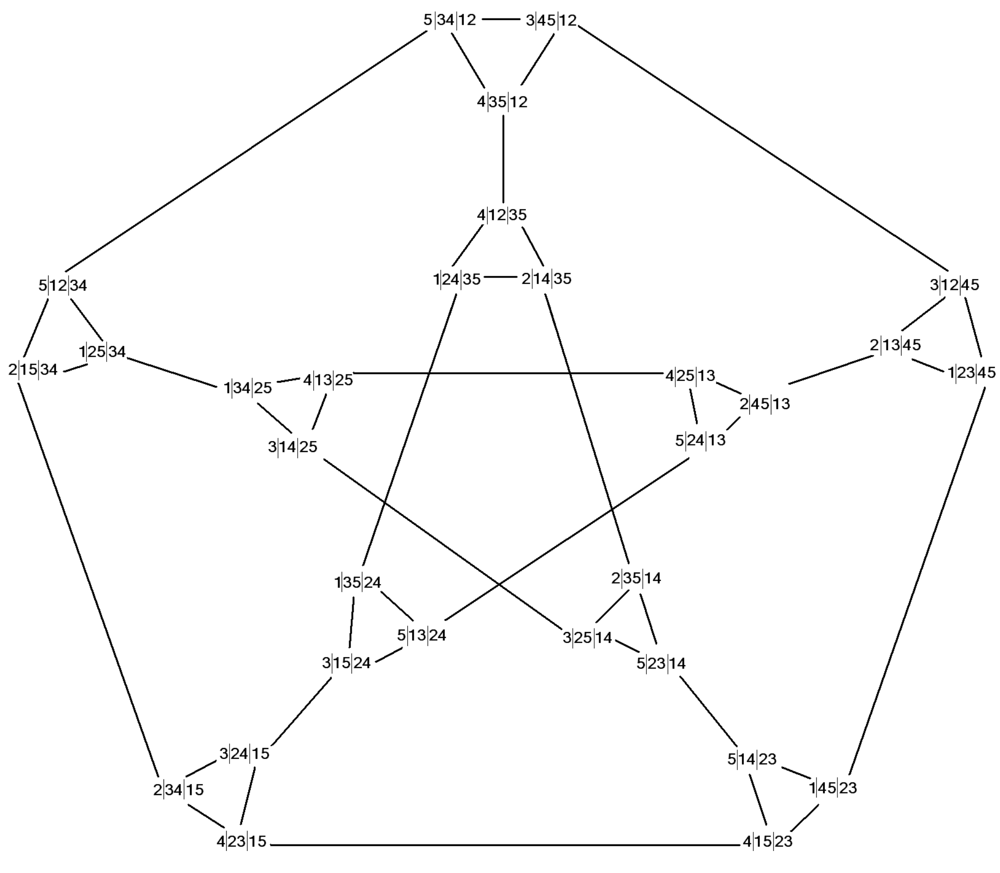


Fig. 2.1. The Chamber Graph of $\Gamma(A_5)$

Theorem 2.9. *Consider the chamber graph of* $\Gamma(A_{2n+1})$. *Let* Λ *be a graph of* $\binom{2n+1}{2}$ *vertices, each labelled by a pair* $\{i, j\}$ *and representing all chambers with* n^{th} *part* $\{i, j\}$.

Let two vertices be joined by an edge whenever a chamber of one is adjacent to a chamber of the other. Then Λ is isomorphic to $\overline{T_2(2n+1)}$. The chambers of a particular vertex are isomorphic to the chamber graph of $\Gamma(A_{2n-1})$.

Proof. The proof follows easily from Theorem 2.6 and Definition 1.7. $\square$

2.3 Intersection matrices

Definition 2.10. *The intersection matrix $M(A, B)$ between n-chambers A and B is defined as the $n \times n$ matrix where*

$$M(A, B)_{ij} = |A(i) \cap B(j)|$$

Recall that the chambers adjacent to 12|34|56 are 12|56|34, 34|12|56, 17|34|56 and 27|34|56. Thus the intersection matrices of 12|34|56 with its neighbours are shown in Fig.2.2.

$$\begin{array}{c|ccc|} & 34 & 12 & 56 \\ \hline 12 & & 2 & \\ 34 & 2 & & \\ 56 & & & 2 \\ \hline \end{array} \qquad \begin{array}{c|ccc|} & 12 & 56 & 34 \\ \hline 12 & 2 & & \\ 34 & & & 2 \\ 56 & & 2 & \\ \hline \end{array} \qquad \begin{array}{c|ccc|} & 17 & 34 & 56 \\ \hline 12 & 1 & & \\ 34 & & 2 & \\ 56 & & & 2 \\ \hline \end{array} \qquad \begin{array}{c|ccc|} & 27 & 34 & 56 \\ \hline 12 & 1 & & \\ 34 & & 2 & \\ 56 & & & 2 \\ \hline \end{array}$$

Fig. 2.2

Lemma 2.11. *Let A and B be n-chambers. Then $M(A, B)$ satisfies the following:*

- *Any entry is either 0,1 or 2.*
- *The sum of the entries in any column is 1 or 2.*
- *The sum of the entries in any row is 1 or 2.*
- *The sum of the entries in $M(A, B)$ is $2n$ or $2n-1$.*

Definition 2.12. *If the entries of the $n \times n$ matrix $M(A, B)$ sum to $2n$ then we call $M(A, B)$* even. *We call it* odd *otherwise. If it is odd then the unique row whose entries sum to 1 we will call the* odd row. *Similarly the unique column whose entries sum to 1 we will call the* odd column.

The uniqueness of the odd row and odd column is obvious by Lemma 2.11.

Lemma 2.13. *Let A and B be n-chambers and $g \in S_{2n+1}$. Then $M(A, B) = M(A^g, B^g)$.*

Proof. The proof is obvious from Definition 2.10. $\square$

We prove Lemma 2.14 for the sake of proving Lemma 2.15.

Lemma 2.14. *Let A be an n-chamber and M be an $n \times n$ matrix satisfying the properties stated in Lemma 2.11. Let $1 \leqslant k < n$ and $(B_j)_{j=k+1}^{n}$ be a sequence such that*
I. *Each B_j is a 2-subset of $\{1, 2, ..., 2n+1\}$.*
II. *B_i and B_j are disjoint unless $i = j$.*
III. *Each B_j intersects A by the required j^{th} column of M. (That is, $M_{ij} = |A(i) \cap B_j|$ for all $1 \leqslant i \leqslant n$.)*
Then there exists B_k such that $(B_j)_{j=k}^{n}$ satisfies I, II *and* III. *Let $G = S_{2n+1}$. If $(C_k, B_{k+1}, ...B_n)$ fulfills* I, II *and* III *then there exists $g \in Stab_G(A)$ such that*
$(C_{k-1}, B_k, ...B_n)^g = (B_{k-1}, B_k, ...B_n)$.

Proof. Let the i^{th} part $A(i)$ of A be $\{x_i, y_i\}$ for all i $(1 \leqslant i \leqslant n)$. $\mathrm{Stab}_G(A) = \langle (x_i, y_i) : 1 \leqslant i \leqslant n \rangle \cong \mathbb{Z}_2^n$. Consider the k^{th} column of M. We have six possible cases: (These are illustrated in Fig. 2.3).

i) $M_{ik} = 2$ for some $1 \leqslant i \leqslant n$. This forces $B_k = A(i)$.

For cases ii and iii let $M_{ik} = 1$ and all other entries of the column be zero.

ii) $M_{ij} = 0$ for all $k < j \leqslant n$. This forces $B_k = \{x_i, A(0)\}$ or $\{y_i, A(0)\}$.

iii) $M_{ij} = 1$ for some $k < j \leqslant n$. Without loss of generality let $x_i \in B_j$. This forces $B(i) = \{y_i, A(0)\}$.

For cases iv, v and vi let $M_{ik} = M_{i'k} = 1$ where $i \neq i'$.

iv) $M_{ij} = M_{i'j} = 0$ for all $k < j \leqslant n$. This forces $B_k = \{x_i, x_{i'}\}$, $\{x_i, y_{i'}\}$, $\{y_i, x_{i'}\}$ or $\{y_i, y_{i'}\}$.

v) $M_{ij} = 1$ for some $k < j \leqslant n$ and $M_{i'j'} = 0$ for all $k < j' \leqslant n$. Without loss of generality let $x_i \in B_j$. This forces $B_k = \{y_i, x_{i'}\}$ or $\{y_i, y_{i'}\}$.

vi) $M_{ij} = M_{i'j'} = 1$ for some $k < j \leqslant n$, $k < j' \leqslant n$. Then without loss of generality let $x_i \in B_j$ and $x_{i'} \in B_{j'}$. This forces $B_k = \{y_i, y_{i'}\}$.

In all six cases we see there are only one, two or four options for B_k. Furthermore, if B_k and C_k are two options then there exists $g \in \mathrm{Stab}_G(A)$ such that $B_k^g = C_k$ and g fixes B_j for $j > k$. $\square$

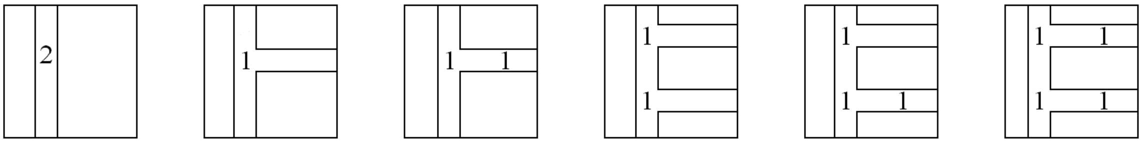


Fig. 2.3. i, ii, iii, iv, v and vi

The idea behind Lemma 2.14 is that, if we want $M = M(A, B)$, each B_j is a possible candidate for $B(j)$.

Lemma 2.15. *Let A be an n-chamber and M be an $n \times n$ matrix satisfying the properties stated in Lemma 2.11. Then the set*

$$\{B : B \textit{ is an } n\textit{-chamber such that } M = M(A, B)\}$$

is nonempty. Let $G = S_{2n+1}$. Then $Stab_G(A)$ is transitive on this set.

Proof. Let the i^{th} part $A(i)$ of A be $\{x_i, y_i\}$ for all i $(1 \leqslant i \leqslant n)$. $\mathrm{Stab}_G(A) = \langle (x_i, y_i) : 1 \leqslant i \leqslant n \rangle \cong \mathbb{Z}_2^n$. We will inductively define sequences

$$(B_k)_{j=k}^n = (B_k, ..., B_n)$$

$(1 \leqslant k \leqslant n)$ such that conditions I, II and III of Lemma 2.14 are satisfied, as well as the following condition:
IV. If $(C_j)_{j=k}^n = (C_k, ..., C_n)$ satisfies I, II and III then there exists $g \in \mathrm{Stab}_G(A)$ such that $(C_k, ..., C_n)^g = (B_k, ..., B_n)$.

To begin the induction, consider the rightmost column of M. Let B_n be a 2-subset of $\{1, 2, ..., 2n+1\}$ which intersects A by this column: That is, let $M_{in} = |A(i) \cap B_n|$ for $1 \leqslant i \leqslant n$. We have three cases:
i): $M_{in} = 2$ for some $1 \leqslant i \leqslant n$. This forces $B_n = A(i)$.
ii): $M_{in} = 1$ for some $1 \leqslant i \leqslant n$ and $M_{jn} = 0$ whenever $i \neq j$. This forces $B_n = \{x_i, A(0)\}$ or $\{y_i, A(0)\}$.
iii): $M_{in} = M_{jn} = 1$ for some $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ where $i \neq j$. This forces $B_n = \{x_i, x_j\}$, $\{x_i, y_j\}$, $\{y_i, x_j\}$ or $\{y_i, y_j\}$.
In all three cases we see that (B_n) is a sequence satisfying I, II and III in Lemma 2.14. IV is also satisfied.

Suppose we have a sequence $(B_j)_{j=k+1}^{n}$ (where $1 \leqslant k < n$) satisfying I, II, III and IV. By Lemma 2.14 there is a sequence $(B_j)_{j=k}^{n}$ satisfying I, II and III. To show it satisfies IV, let $(C_j)_{j=k}^{n}$ be another such sequence. By induction we know there exists $g \in \mathrm{Stab}_G(A)$ such that $(C_k, C_{k+1} ..., C_n)^g = (C_k^g, B_{k+1}, ..., B_n)$. By Lemma 2.14 there exists $h \in \mathrm{Stab}_G(A)$ such that $(C_k^g, B_{k+1}, ..., B_n)^h = (B_k, B_{k+1}, ..., B_n)$.

This is true up to $k = 1$. Define the n-chamber B by $B(i) = B_i$ for $1 \leqslant i \leqslant n$. For any other chamber C satisfying $M = M(A, C)$, we see there is some element of S_{2n+1} fixing A taking C to B. □

Lemma 2.16. *Let the two matrices $M = M(A, B)$ and M' differ only by their leftmost column. Then there is a 0-swapping B' of B such that $M' = M(A, B')$.*

Proof. For $2 \leqslant j \leqslant n$ define $B_j = B(j)$. We see that A, M' and $(B_2, ..., B_n)$ satisfy conditions I, II and III in Lemma 2.14. Hence there is a B_1 such that A, M' and $(B_1, B_2, ..., B_n)$ also satisfy I, II and III. Define the chamber B' by $B'(j) = B_j$ for $1 \leqslant j \leqslant n$. Then we have that $M(A, B') = M'$ and $B(j) = B'(j)$ whenever $j \geqslant 2$. Hence $B \underset{0}{\text{---}} B'$. □

Let's see what happens to $M(A, B)$ if we replace B with one of its neighbours:

Lemma 2.17. *Let A and B be chambers. Then the intersection matrices of A with any neighbour of B can be obtained from $M(A, B)$ by either:*

- *Swapping two adjacent columns* *(a jump) or*
- *Changing the entries in the leftmost column to anything that satisfies the four conditions of Lemma 2.11.* *(a 0-swap).*

Similarly we can see how B intersects a neighbour of A by either,

- *Swapping two adjacent rows* *(a jump) or*
- *Changing the entries in the top row to anything that satisfies the four conditions of Lemma 2.11.* *(a 0-swap).*

Proof. Recall Theorem 2.6 and Definition 2.7. The claim about jumps is obvious. By Lemma 2.15, if a matrix M' satisfies the four conditions stated in Lemma 2.11 and differs from $M(A, B)$ only by its leftmost column, then by Lemma 2.16 there exists a chamber $B' \underset{0}{\text{---}} B$ such that $M' = M(A, B')$. □

Note that any change made by a 0-swap always looks like something happening in Fig.2.4. (We write a zero in the entry defined by the odd row and odd column if possible.)

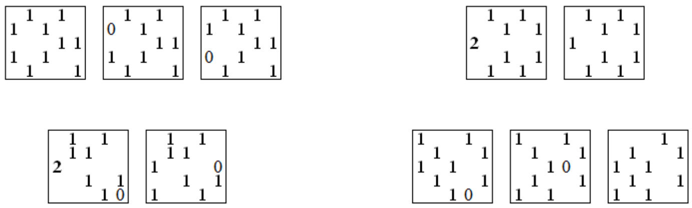


Fig. 2.4. Any 0-swap looks like one of these.

Theorem 2.18. *Let the chambers A, B, C, D satisfy $M(A, B) = M(C, D)$. Then dist$(A, B) =$dist(C, D).*

Proof. We know S_{2n+1} acts on the chamber graph, preserving distance. There exists $g \in S_{2n+1}$ such that $C^g = A$. We have dist(C, D) =dist(A, D^g). By Lemma 2.13 we know $M(A, B) = M(C, D) = M(A, D^g)$. By Lemma 2.15 there is a permutation fixing A taking D^g to B. Therefore there exists some element of S_{2n+1} taking C to A and D to B. Hence dist(A, B) =dist(C, D). □

2.4 Matrix distance

Following Theorem 2.18 we can now say that any intersection matrix M has its own unique distance.

Definition 2.19. *The distance dist(M) of an $n \times n$ matrix M satisfying the properties stated in Lemma 2.11 is the distance between any two n-chambers A and B such that $M = M(A, B)$.*

Indeed, for the rest of the thesis we will talk about the distance of a matrix as much, if not more, as the distance between two chambers.

In Fig.2.5 we represent the distance of all chambers of $\Gamma(A_5)$ from some original chamber 12|34. We do the same for $\Gamma(A_7)$ in Fig.2.6. It is wise at this point to stare at Figures 2.5 and 2.6 for a few minutes to get an idea of how the jumps and 0-swaps change our matrices.

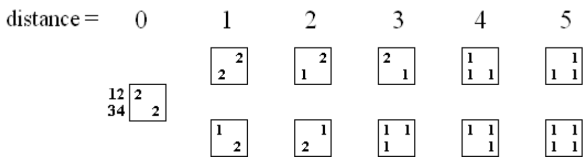


Fig. 2.5

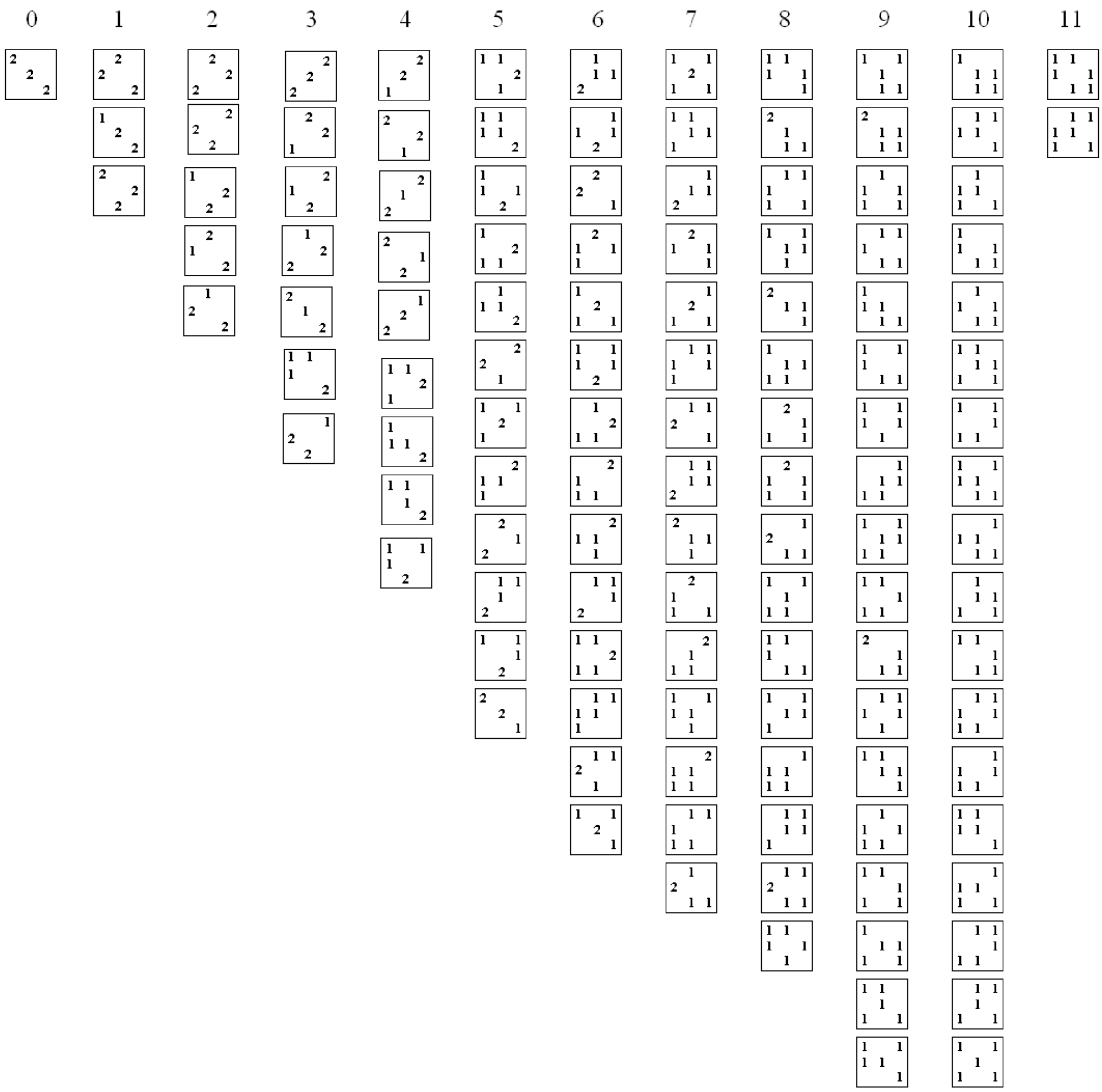


Fig. 2.6

2.5 Minimal paths

How do we find minimal paths between chambers? We are now in a position to state that if we want to find a minimal path between chambers A and B, one method is to see how few of the operations described in Lemma 2.17 we can apply to $M(A,B)$ until we have diagonal twos. How can we pick out a quickest way? We state the next definition bearing in mind Definitions 2.7 and 2.8.

Definition 2.20. *Given two chambers A and B, we say $M(A,B)$ has* unjumpable rows *if there is no $C \underset{x,x+1}{\text{———}} B$ such that dist$(A,C) =$dist$(A,B)-1$. We say $M(A,B)$ has* unjumpable columns *if there is no $C \underset{x,x+1}{\text{———}} A$ such that dist$(C,B) =$dist$(A,B)-1$. We say $M(A,B)$ is* unjumpable *if it has unjumpable rows and unjumpable columns.*

2.5.1 Ordered and similar paths

Notation Let $x>0$ and $y>0$. Let the chambers A and B satisfy $A(x)=B(y)$ and $\text{dist}(A,B)=|x-y|$. Then A and B are joined by a minimal path consisting only of $|x-y|$ jumps. We write $A \underset{x,}{\text{——}} \cdots \underset{,y}{\text{——}} B$ or $B \underset{y,}{\text{——}} \cdots \underset{,x}{\text{——}} A$. For example, if

$A=$ 1,2|3,4|5,6|7,8|9,10|11,12|13,14

$B=$ 3,4|5,6|7,8|9,10|11,12|13,14|1,2

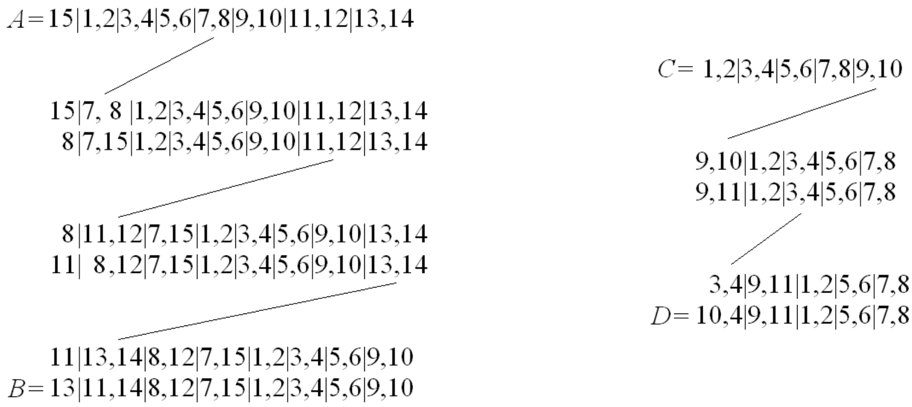
we might write $A \underset{1,}{\text{——}} \cdots \underset{,7}{\text{——}} B$.

Lemma 2.21. *Let $1 \leqslant i < j \leqslant n$. If $A \underset{i,}{\text{——}} \cdots \underset{,j}{\text{——}} B$ then*

$$\begin{aligned} A(k) &= B(k) && \text{if } k<i \text{ or } k>j \\ A(k) &= B(k-1) && \text{if } i<k \leqslant j \\ A(i) &= B(j) \end{aligned}$$

Definition 2.22. *This is a recursive definition:*
A minimal path P of length $l \geqslant 1$ joining A and B is ordered from A to B *if there is a chamber $B' \in P$ such that $B \underset{0}{\text{——}} B' \underset{1,}{\text{——}} \cdots \underset{,l}{\text{——}} A$. (If $l=1$ then $B'=A$.)*
A minimal path P is ordered from A to B if there is a chamber $C \in P$ such that P is ordered from A to C and ordered from C to B.

For example, the following minimal paths are ordered from A to B and C to D (of length 17 and 8 respectively):



Definition 2.23. *Let P be a minimal path of length l joining chambers A and B. Write P as the sequence of chambers $(P_i)_{i=0}^{l}$ where $P_0 = A$, $P_l = B$ and P_i is adjacent to P_{i+1}. We define the subsequence X of P by letting $P_i \in X$ if and only if P_i is a 0-swap of some chamber in P. We define* the sequence of 0-swaps of P from A to B *to be the sequence $(X_i(1))$.*

For example, the sequence of 0-swaps of the above path from A to B and C to D is $(\{7,8\},\{7,15\},\{11,12\},\{8,12\},\{13,14\},\{11,14\})$ and $(\{9,10\},\{9,11\},\{3,4\},\{10,4\})$ respectively.

Definition 2.24. *Let P and P' be two minimal paths between A and B. We say P and P' are similar if the sequence of 0-swaps of P from A to B is equal to the sequence of 0-swaps of P' from A to B.*

Definition 2.24 is the same if we interchange A and B. Thus similarity is well-defined and is an equivalence relation on minimal paths. It is a hard-looking definition for what is actually very intuitive. For example, the minimal path P is similar to Q but not R in Fig.2.7. This is because the sequences of 0-swaps (from 12|34|56|78 to 49|32|78|56) are

$$P, Q : (\{1,2\},\{1,9\},\{3,4\},\{3,2\},\{1,9\},\{4,9\}) \qquad R : (\{3,4\},\{4,9\},\{1,2\},\{3,2\})$$

When we say P and Q are similar we are saying they are equal "mod jumps".

Definition 2.25. *A chamber C is said to be* in-between *A and B if dist(A,C) + dist(C,B) = dist(A,B). Let P be a minimal path joining chambers A and B. Let $C, D \in P$. The path* following P from C to D *is the subset*

$$P_{C \text{ to } D} = \{E \in P : E \text{ is in-between } C \text{ and } D\}$$

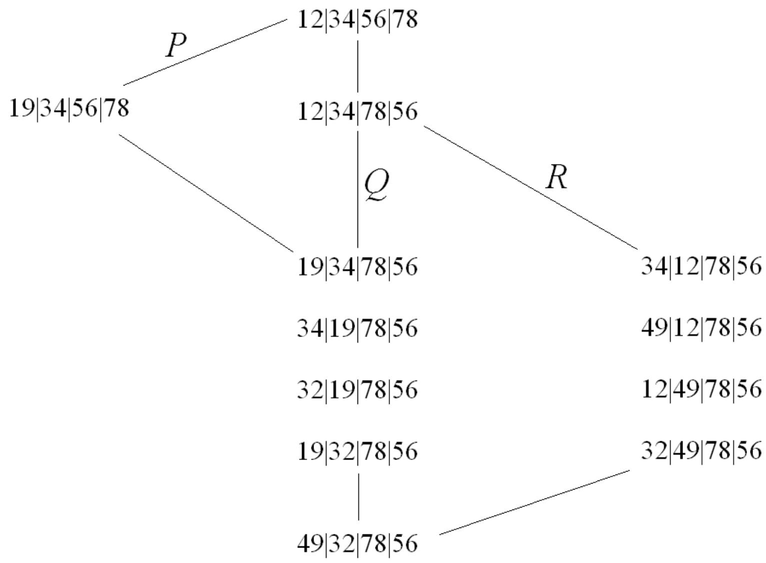


Fig. 2.7

2.5.2 Some commuting diagrams

In this thesis we will use the following definition of what it means for a commuting diagram to hold.

Definition 2.26. *We say that the following commuting diagram* holds *if the existence of any three of the chambers in it imply the existence of a fourth satisfying it.*

$$\begin{tikzcd} A \arrow[r, dash] \arrow[d, dash] & B \arrow[d, dash] \\ C \arrow[r, dash] & D \end{tikzcd}$$

Lemma 2.27. *Let $x \geqslant 1$, $y \geqslant 1$ and $\{x, x+1, y, y+1\}$ be a 4-set. Then the diagram on the left holds. Let $z \geqslant 2$. Then the diagram on the right holds:*

$$\begin{tikzcd} A \arrow[r, dash, "x{,}x+1"'] \arrow[d, dash, "y{,}y+1"] & B \arrow[d, dash, "y{,}y+1"] \\ C \arrow[r, dash, "x{,}x+1"'] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, dash, "0"'] \arrow[d, dash, "z{,}z+1"] & B \arrow[d, dash, "z{,}z+1"] \\ C \arrow[r, dash, "0"'] & D \end{tikzcd}$$

Proof. We prove the first claim. Let A, B and D satisfy the claims made about them in the left diagram. Then

$$\begin{aligned} A(i) &= D(i) && \text{if } i \notin \{x, x+1, y, y+1\} \\ A(x) &= D(x+1) && A(y) = D(y+1) \\ D(x) &= A(x+1) && D(y) = A(y+1) \end{aligned}$$

This is still achieved if we interchange the $(x, x+1)$-swap and the $(y, y+1)$-swap so the existence of C follows. A similar argument proves the existence of A, B or D given that the other three exist. Now we prove the second claim. Let $A \underset{0}{\text{——}} B \underset{z,z+1}{\text{————}} D$. This implies

$$\begin{aligned} A(i) &= D(i) \text{ if } i \notin \{0, 1, z, z+1\} \\ A(z) &= D(z+1) \\ D(z) &= A(z+1) \end{aligned}$$

Therefore if $C \underset{z,z+1}{\text{————}} A$ we have that $C(i) = D(i)$ whenever $i \notin \{0, 1\}$. A similar argument proves the existence of A, B or D given that the other three exist. $\square$

Lemma 2.28. *Let $1 \leqslant i < j \leqslant n$. If $\{x, x+1\}$ and $\{i, i+1, ..., j-1, j\}$ are disjoint then the diagram on the left holds. If $i < x < j$ then the diagram on the right holds:*

$$\begin{tikzcd} A \arrow[r, dash, "i,"'] \arrow[d, dash, "x{,}x+1"'] & \cdots \arrow[r, dash, ",j"'] & B \arrow[d, dash, "x{,}x+1"'] \\ C \arrow[r, dash, "i,"'] & \cdots \arrow[r, dash, ",j"'] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, dash, "i,"'] \arrow[d, dash, "x{,}x+1"'] & \cdots \arrow[r, dash, ",j"'] & B \arrow[d, dash, "x-1{,}x"'] \\ C \arrow[r, dash, "i,"'] & \cdots \arrow[r, dash, ",j"'] & D \end{tikzcd}$$

Proof. First note that if $A \underset{i,}{\text{——}} \cdots \underset{,j}{\text{——}} B$ then by Lemma 2.21,

$$\begin{aligned} A(k) &= B(k) && \text{if } k < i \text{ or } k > j \\ A(k) &= B(k-1) && \text{if } i < k \leqslant j \\ A(i) &= B(j) \end{aligned}$$

If $\{x, x+1\}$ and $\{i, i+1, ..., j-1, j\}$ are disjoint the diagram on the left holds by Lemma 2.27. Let $i < x < j$. Assume the existence of A, B and C. Let $A(x) = \{a, b\}$ and $A(x+1) = \{c, d\}$. Then $C = A^{(a,c)(b,d)}$. It follows that $C \underset{i,}{\text{——}} \cdots \underset{,j}{\text{——}} D = B^{(a,c)(b,d)}$. We know that $B(x-1) = \{a, b\}$ and $B(x) = \{c, d\}$. Therefore $B \underset{x-1,x}{\text{————}} D$. A similar argument proves the existence of A, B or C given that the other three exist. $\square$

Lemma 2.29. *Let $i \geqslant 1$ and $x \geqslant 2$. If $x > i$ then the diagram on the left holds. If $x < i$ then the diagram on the right holds.*

$$\begin{tikzcd} A \arrow[r, dash, "0"'] \arrow[d, dash, "x{,}x+1"'] & A' \arrow[r, dash, "1,"'] & \cdots \arrow[r, dash, ",i"'] & B \arrow[d, dash, "x{,}x+1"'] \\ C \arrow[r, dash, "0"'] & C' \arrow[r, dash, "1,"'] & \cdots \arrow[r, dash, ",i"'] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, dash, "0"'] \arrow[d, dash, "x{,}x+1"'] & A' \arrow[r, dash, "1,"'] & \cdots \arrow[r, dash, ",i"'] & B \arrow[d, dash, "x-1{,}x"'] \\ C \arrow[r, dash, "0"'] & C' \arrow[r, dash, "1,"'] & \cdots \arrow[r, dash, ",i"'] & D \end{tikzcd}$$

Proof. The result follows from Lemmas 2.27 and 2.28. $\square$

Lemma 2.30. *Let $1 \leqslant k < x \leqslant n$. Then this commuting diagram holds:*

$$\begin{tikzcd}
A \arrow[r, dash, "0"'] \arrow[d, dash, "k+1,"] & A' \arrow[r, dash, "1,"'] & \cdots \arrow[r, dash, ",x+1"'] & B \arrow[d, dash, "k,"] \\
\vdots \arrow[d, dash, ",x+1"] & & & \vdots \arrow[d, dash, ",x"] \\
C \arrow[r, dash, "0"'] & C' \arrow[r, dash, "1,"'] & \cdots \arrow[r, dash, ",x+1"'] & D
\end{tikzcd}$$

Proof. This can easily be checked using the right-hand diagram of Lemma 2.29. □

2.6 Ways of starting minimal paths

For the rest of this chapter let a, b, c, d and e be distinct elements of $\{1, 2, ..., 2n, 2n+1\}$. That is, $|\{a, b, c, d, e\}| = 5$.

Lemma 2.31. *Let X be an equivalence class of similar minimal paths from A to B. There exists a path $P \in X$ and a chamber $C \in P$ such that $P_{A \text{ to } C}$ is ordered from A to C and $P_{C \text{ to } B}$ consists only of jumps. Furthermore, C and $P_{A \text{ to } C}$ are defined by X.*

Proof. First we prove uniqueness: If we have another path $P' \in X$ and a chamber $C' \in P'$ satisfying the theorem, then having P and P' similar and ordered means $C = C'$ and $P_{A \text{ to } C} = P'_{A \text{ to } C'}$ by Definitions 2.22, 2.23 and 2.24. Therefore we need only prove existence. We do this by induction on the distance between A and B. If $\mathrm{dist}(A, B) = 1$ then the result is clearly true. Therefore assume the result is true for distance less than L and let $\mathrm{dist}(A, B) = L$.

Let $P \in X$ and $A' \in P$ be adjacent to A. By induction, we may assume P is ordered from A' to some chamber C and consists only of jumps from C to B. Furthermore, there exist chambers D and D' such that

$$P \; : \; A \frac{\quad}{} A' \frac{\quad}{i,} \cdots \frac{\quad}{,1} D \frac{\quad}{0} D' \cdots\cdots\cdots C \overbrace{\cdots\cdots\cdots}^{\text{jumps}} B$$

We consider two cases:

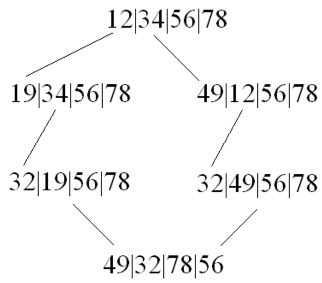
Case I. $A \frac{\quad}{0} A'$. Then we must have that $i \geqslant 2$ and so P is ordered from A to C.

Case II. $A \frac{\qquad}{x,x+1} A'$. If $i = x$ then P is ordered from A to C. If $i = x + 1$ then we have a contradiction. Therefore assume $i \notin \{x, x + 1\}$. By Lemma 2.29 there is a path

$$Q \ : \ A \underset{i,}{\text{——}} \cdots \underset{,1}{\text{——}} E \underset{0}{\text{——}} E' \text{——} D' \cdots\cdots\cdots C \overbrace{\cdots\cdots\cdots}^{\text{jumps}} B$$

similar to P, where $E' \underset{x,x+1}{\text{——}} D'$ (if $x > i$) or $E' \underset{x+1,x+2}{\text{——}} D'$ (if $x < i-1$). By induction the result holds for Q. □

In the proof above we are "applying all of X's 0-swaps to A as quickly as we can". For example, if we apply this method to Fig.2.7 we get:



Lemma 2.32. *Let A and B be n-chambers. Let $B = a|bc|\cdots$ and no 0-swapping of B lie in-between A and B. Let the chamber $C \neq B$ be in-between A and B. If $C = a|bc|\cdots$ then no 0-swapping of C lies in-between A and C.*

Proof. Without loss of generality suppose the chamber $C^{(a,b)} \underset{0}{\text{——}} C$ lies in-between A and C. Then $\text{dist}(B, C) = \text{dist}(B^{(a,b)}, C^{(a,b)})$, giving us $\text{dist}(A, B^{(a,b)}) < \text{dist}(A, B)$. This is a contradiction as $B^{(a,b)} \underset{0}{\text{——}} B$. □

Lemma 2.33. *Suppose $a \in B(u)$ (or $a = B(u)$ if $u = 0$) and $B(v) = \{b, c\}$ where $0 \leqslant u < v \leqslant n$. If $\{b, c\}$ is not a part of the chamber A then any minimal path P in-between A and B is similar to some path containing a subpath:*

$$B \cdots\cdots\cdots C \underset{x+1,}{\text{——}} \cdots \underset{,1}{\text{——}} D \underset{0}{\text{——}} E \cdots\cdots\cdots A$$

where $a \in C(x)$ (or $a = C(x)$ if $x = 0$) and $C(x+1) = \{b, c\}$ for some $x \geqslant 0$.

Proof. We prove by induction on $\text{dist}(A, B)$. We may claim it is true for distance 0. Assume it is true for distance less than L and let $\text{dist}(A, B) = L$. By Lemma 2.31 we may assume P is ordered from B to some B' and consists only of jumps between B' and A. Assume $A = B'$, or the result follows by induction using that $\text{dist}(B', B) < L$. If P of the form

$$B \underset{y,}{\text{——}} \cdots \underset{,1}{\text{——}} A' \underset{0}{\text{——}} A$$

then we must have $y = v$ and the theorem is proved. Therefore P is of the form

$$B \underset{y,}{\text{———}} \cdots \underset{,1}{\text{———}} F' \underset{0}{\text{———}} F \cdots\cdots\cdots A$$

We assume $P_{F \text{ to } A}$ is not similar to any minimal path having the subpath claimed. Therefore by induction we must have $F'(1) = \{b, c\}$ and $F(1) \neq \{b, c\}$, and so $y = v$. (Even if $y = 1$ this still proves the theorem). $\square$

Lemma 2.34. *Let A and B be n-chambers. Let $B = a|bc|\cdots$ and no 0-swapping of B lie in-between A and B. If $\{b, c\}$ is not a part of A then any minimal path between A and B is similar to a minimal path containing a subpath:*

$$B \cdots\cdots\cdots C \underset{x+1,}{\text{———}} \cdots \underset{,1}{\text{———}} D \underset{0}{\text{———}} E \cdots\cdots\cdots A$$

where C is of the form $d|\cdots|ae|\overset{x+1}{bc}|\cdots$ for some $1 \leqslant x < n$. The following are true:

(1) *We have dist$(A, D) <$ dist$(A, B) - 2x$.*

(2) *We have dist$(A, D^{(b,d)}) \leqslant$ dist(A, D).*

(3) *We have dist$(A, D) \leqslant$ dist$(A, D^{(b,d)(b,e)})$.*

Proof. By Lemma 2.33 we know such a path must exist where $x \geqslant 0$. Suppose $x = 0$. We must have $B \neq C$. This leads to a contradiction by Lemma 2.32. Hence $x \geqslant 1$. We now prove (1), (2) and (3):

(1) It is clear that $\text{dist}(B, C) \geqslant x + 1$ and $\text{dist}(C, D) \geqslant x$.

(2) We know this because either $E = D^{(b,d)}$, or else D, $D^{(b,d)}$ and E are all 0-swappings of each other. Thus we cannot have $\text{dist}(A, E) < \text{dist}(A, D) < \text{dist}(A, D^{(b,d)})$.

(3) Consider this path of length $x + 1$:

$$C \underset{x+1,x}{\text{———}} C^{(a,c)(b,e)} \underset{x,}{\text{———}} \cdots \underset{,1}{\text{———}} D \underset{0}{\text{———}} D^{(b,d)}$$

Hence there is a path P of length x joining $C^{(a,c)}$ and $D^{(b,d)(b,e)}$. Note that the distance between A and B is $\text{dist}(B, C) + x + \text{dist}(D, A)$. Using P we see there is a path

$$Q: \quad B \underset{0}{\text{———}} B^{(a,c)} \cdots\cdots\cdots C^{(a,c)} \cdots\cdots\cdots D^{(b,d)(b,e)} \cdots\cdots\cdots A$$

whose length is at most $1 + \text{dist}(B, C) + x + \text{dist}(D^{(b,d)(b,e)}, A)$. Therefore if (3) is false it follows that Q is minimal. This is a contradiction as Q contains a 0-swapping of B. $\square$

We are now in a position where we can prove Theorem 2.35.

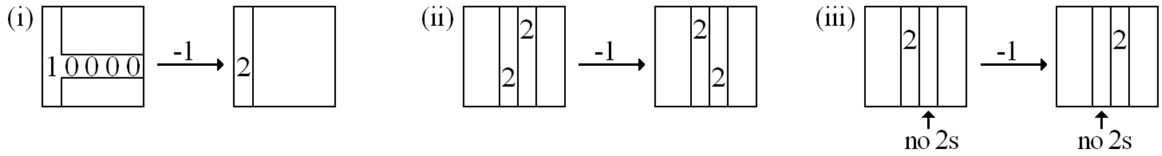
Theorem 2.35. *The following are true: (Let $i \neq 0$.)*

(i): *Let $A(i) = \{a, b\}$ and $B = a|bc|\cdots$. Then the chamber $B^{(a,c)} \underset{0}{\text{———}} B$ with first part $\{a, b\}$ is in-between A and B.*

(ii): *Let $A(x) = B(i)$ and $A(y) = B(i+1)$ where $y < x$. Then the chamber $C \underset{i,i+1}{\text{———}} B$ is in-between A and B.*

(iii): *Let $B(i) = A(j)$ and $B(i+1) \neq A(k)$ for all k $(1 \leqslant k \leqslant n)$. Then the chamber $C \underset{i,i+1}{\text{———}} B$ is in-between A and B.*

We illustrate the theorem here:



Proof. We prove (i), (ii) and (iii) by induction on the distance of $M = M(A, B)$. We may claim they are true for distance 0. We assume they are true for distance less than L and that M has distance L.

Proof of (i) for distance L. The matrix $M(A, B)$ has an odd row whose leftmost entry is 1. We can make the following assumptions:

• The odd row of M is the top row. That is, $i = 1$: If $i \neq 1$ then A has a neighbour A' such that $A'(j) = \{a, b\}$ for some $1 \leqslant j \leqslant n$, and $\mathrm{dist}(A', B) = L - 1$. This means, by induction, that $B^{(a,c)}$ lies in-between A' and B.

• There is no chamber $B' \underset{0}{\text{———}} B$ in-between A and B: If $B' = B^{(a,b)}$ then by Lemma 2.13 the matrix $M = M(A^{(a,b)}, B^{(a,b)}) = M(A, B^{(a,b)})$ has length L. The only other option for B' is $B^{(a,c)}$, as required.

Therefore by Lemma 2.34 there is a minimal path between A and B of the form

$$B \cdots\cdots\cdots C \underset{x+1,}{\text{———}} \cdots \underset{,1}{\text{———}} D \underset{0}{\text{———}} E \cdots\cdots\cdots A$$

where C is of the form $d|\cdots|ae|\overset{x+1}{bc}|\cdots$ and D is of the form $d|bc|\cdots|\overset{x+1}{ae}|\cdots$ for some $1 \leqslant x < n$.

We have $D^{(b,d)} = b|dc|\cdots|\overset{x+1}{ae}|\cdots$. There exists F such that

$$D^{(b,d)} \underset{x+1,}{\text{———}} \cdots \underset{,1}{\text{———}} F \underset{0}{\text{———}} F^{(b,e)} \underset{1,}{\text{———}} \cdots \underset{,x+1}{\text{———}} D^{(b,d)(b,e)}$$

This is illustrated in Fig.2.8. Let θ be the number of entries within the leftmost x columns of $M(A, D^{(b,d)})$ that are equal to 2. We know the following by induction: (We are allowed to use induction by combining parts (1) and (2) of Lemma 2.34).

$$\begin{aligned} \mathrm{dist}(A, F) &\leqslant \mathrm{dist}(A, D^{(b,d)}) + x - 2\theta && \text{by (iii)} \\ \mathrm{dist}(A, F^{(b,e)}) &= \mathrm{dist}(A, F) - 1 && \text{by (i)} \\ \mathrm{dist}(A, D^{(b,d)(b,e)}) &= \mathrm{dist}(A, F^{(b,e)}) - x + 2\theta && \text{by (ii) and (iii)} \end{aligned}$$

It follows that $\mathrm{dist}(A, D^{(b,d)(b,e)}) < \mathrm{dist}(A, D^{(b,d)})$. This contradicts Lemma 2.34 (2)(3)

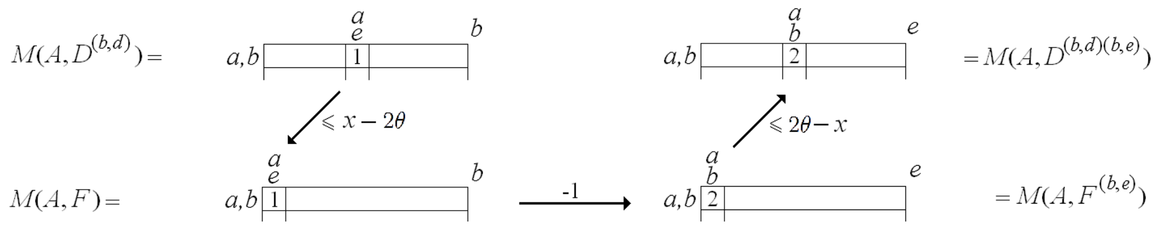


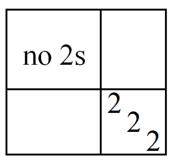
Fig. 2.8

and proves (i) for matrices of distance L.

Proof of (ii) for distance L. Let $A(x) = B(i) = \{a, b\}$ and $A(y) = B(i+1) = \{c, d\}$. There exists D——A such that the matrix $M(D, B)$ is of length $L - 1$. Using (i) we know that D has both parts $\{a, b\}$ and $\{c, d\}$. There are two possibilities: The first is that $x = y + 1$ and $D = A^{(a,c)(b,d)}$. In that case, $M(D, B) = M(D^{(a,c)(b,d)}, B^{(a,c)(b,d)}) = M(A, C)$ and we are done. The second is that $D(x) \neq A(y)$ and $D(y) \neq A(x)$. In that case, (ii) holds by induction.

Proof of (iii) for distance L. There exists D——A such that the matrix $M(D, B)$ is of length $L - 1$. Using (i) we know that D has the part $B(i)$. There are two possibilities: The first is that we still have $B(i + 1) \neq D(k)$ for all k. In that case, we are done by induction using (iii). The second is that $B(i + 1) = D(1)$. In that case, we are done by induction using (ii). $\square$

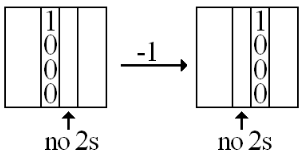
Theorem 2.35 tells us that any unjumpable intersection matrix with entries equal to 2 is of the form



Lemma 2.36. *We add another statement to our collection in Theorem 2.35.*

(iv)*: Let* $A = a|bc|\cdots$*. Suppose* $i \geqslant 1$*,* $B(i) = \{a, b\}$ *and* $B(i+1)$ *is not equal to any part of* A*. Then* $C \underset{i,i+1}{\text{———}} B$ *is in-between* A *and* B*.*

We illustrate the lemma here:

(iv)


Proof. Theorem 2.35 (i) tells us that $D = A^{(a,c)} \underset{0}{\text{——}} A$ is in-between A and B. Note that the top entry of the i^{th} column of $M(D, B)$ is equal to 2, and the $i + 1^{\text{th}}$ column still contains no entries equal to 2. Therefore by Theorem 2.35 (iii) we know C is in-between B and D. □

Lemma 2.37. *Let* P *be a minimal path. Let there be some chamber in* P *with a part* $\{a, b\}$*. The subset of* P *consisting of chambers with part* $\{a, b\}$ *is itself a minimal path.*

Proof. For a contradiction, suppose A, B and C are chambers of P where A and C have a part $\{a, b\}$ and B does not, yet B is in-between them and is adjacent to C:

$$P: \quad \cdots\cdots\cdots A \cdots\cdots\cdots B \underset{0}{\text{——}} C \cdots\cdots\cdots$$

This is obviously a contradiction as Theorem 2.35 (i) states that C is in-between A and B. □

Lemma 2.38. *Let* $\{a, a'\}$ *(but not* $\{b, b'\}$*) be a part of* A *and let* $\{b, b'\}$ *(but not* $\{a, a'\}$*) be a part of* B *where* a*,* a'*,* b *and* b' *are distinct. Let* P *be a minimal path in-between* A *and* B *containing a chamber which has both parts* $\{a, a'\}$ *and* $\{b, b'\}$*. Then* P *contains two chambers* U *and* V*, where* $U = \cdots|\overset{x}{a}a'|bb'|\cdots$ *and* $V = \cdots|\overset{x}{b}b'|aa'|\cdots$ *where*

$$P: \quad B \cdots\cdots\cdots U \underset{x,x+1}{\text{————}} V \cdots\cdots\cdots A$$

Proof. By Lemma 2.37 the following two sets are subpaths of P:

$$P_{\{a,a'\}} = \{\text{Chambers in } P \text{ with a part } \{a, a'\}\}$$
$$P_{\{b,b'\}} = \{\text{Chambers in } P \text{ with a part } \{b, b'\}\}$$

Neither of these subpaths are equal to P and their intersection is nonempty. We prove by induction on the length of P. If P has length 0 we may claim the theorem holds. Assume the theorem holds up to distance $L-1$ and let P have distance L. Let $B' \in P$ be adjacent to B. By induction, the theorem holds unless $B \underset{0}{\text{---}} B' = aa'|\cdots|bb'|\cdots$. By similar reasoning, assume there exists $A' \in P$ where $A \underset{0}{\text{---}} A' = bb'|\cdots|aa'|\cdots$. All chambers of P in-between A' and B' have both parts $\{a, a'\}$ and $\{b, b'\}$. Let U be the closest chamber of P to A' such that $U(x) = \{a, a'\}$ and $U(y) = \{b, b'\}$ where $x < y$. Then we must have $y = x + 1$ and $U \underset{x,x+1}{\text{———}} V \in P$. □

Recall Definition 2.23. We prove Lemma 2.39 for the sake of proving Lemma 2.40.

Lemma 2.39. *Let* $\{A(i) : 1 \leqslant i \leqslant n\} = \{A'(i) : 1 \leqslant i \leqslant n\}$ *and* $\{B(i) : 1 \leqslant i \leqslant n\} = \{B'(i) : 1 \leqslant i \leqslant n\}$. *Let P (respectively, P') be a minimal path in-between A and B (respectively, A' and B'). Let the sequence of 0-swaps of P from A to B be equal to the sequence of 0-swaps of P' from A' to B'. If P contains a chamber C having parts $C(x)$ and $C(y)$ then P' contains a chamber C' having parts $C'(x') = C(x)$ and $C'(y') = C(y)$.*

Proof. We prove by induction on the length of the sequence S of 0-swaps of P from A to B. Note that $|S|$ is even. If $|S| = 0$ then $\{C(i) : 1 \leqslant i \leqslant n\}$ is the same for all $C \in P \cup P'$ and we are done. Assume the result is true up to length $2L-2$ and let $|S| = 2L$.

Let P and P' be of the form

$$A \cdots\cdots\cdots D \underset{0}{\text{---}} E \cdots\cdots\cdots B \quad \text{and} \quad A' \cdots\cdots\cdots D' \underset{0}{\text{---}} E' \cdots\cdots\cdots B'$$

respectively, where P (respectively, P') consists only of jumps between A and D (respectively, A' and D'). Thus $S = (D(1), E(1), ...) = (D'(1), E'(1), ...)$. By induction, the lemma is true for $P_{A \text{ to } D}$ and $P'_{A' \text{ to } D'}$ and is true for $P_{E \text{ to } B}$ and $P'_{E' \text{ to } B'}$. □

Lemma 2.40. *Let P and P' be similar minimal paths joining A and B. If P contains a chamber C having parts $C(x)$ and $C(y)$ then P' contains a chamber C' having parts $C'(x') = C(x)$ and $C'(y') = C(y)$.*

Proof. The lemma follows from Lemma 2.39. □

Lemma 2.41 is a slight variation of Lemma 2.34

Lemma 2.41. *Let $A = a|bd'|\cdots$ and $B = a|bc|\cdots$ $(d' \neq c)$. Let no 0-swapping of A or B lie in-between A and B. There is a minimal path between A and B of the form:*

$$B\cdots\cdots\cdots C\underset{x+1,}{\text{———}}\cdots\underset{,1}{\text{———}}D\underset{0}{\text{——}}E\cdots\cdots\cdots A$$

such that C is of the form $d|\cdots|ae|\overset{x+1}{bc}|\cdots$ for some $1 \leqslant x < n$ and the following are satisfied:

(1) *We have dist$(A, D) <$ dist$(A, B) - 2x$.*

(2) *We have dist$(A, D^{(b,d)}) \leqslant$ dist(A, D).*

(3) *We have dist$(A, D) \leqslant$ dist$(A, D^{(b,d)(b,e)})$.*

(4) *The leftmost $x + 1$ columns of $M(A, C)$ contain no entries equal to 2.*

(That is, $C(i)$ is not a part of A whenever $1 \leqslant i \leqslant x + 1$).

(5) *dist$(C^{(a,c)}, D^{(b,e)}) = x - 1$.*

(6) $|\{a, b, c, d, e\}| = |\{a, b, c, d', e\}| = 5$ *(but it is possible that $d = d'$).*

Proof. The existence of such a path satisfying (1), (2) and (3) follows automatically from Lemma 2.34. (In fact they are implied). It remains to prove we may assume (4), (5) and (6):

(4): Suppose the leftmost $x + 1$ columns of $M(A, C)$ contain θ entries equal to 2. Consider the set

$$\{i : 1 \leqslant i \leqslant x + 1 | C(i) \in \{A(j) : 1 \leqslant j \leqslant n\}\}$$

of order θ, not containing x or $x + 1$. Let $\theta > 0$. Then $x \geqslant 2$. Let k be the largest element of this set. By Lemma 2.21 we see that $E(k + 1)$ is a part of A and $E(i)$ is not whenever $k + 2 \leqslant i \leqslant x + 1$. Therefore by Theorem 2.35 (iii) we have a minimal path of the form

$$B\cdots\cdots\cdots C\underset{x+1,}{\text{———}}\cdots\underset{,1}{\text{———}}D\underset{0}{\text{——}}E\underset{k+1,}{\text{———}}\cdots\underset{,x+1}{\text{———}}E'\cdots\cdots\cdots A$$

Therefore by Lemma 2.30 there is a minimal path of the form

$$B\cdots\cdots\cdots C\underset{k,}{\text{———}}\cdots\underset{,x}{\text{———}}C'\underset{x+1,x}{\text{———}}C''\underset{x,}{\text{———}}\cdots\underset{,1}{\text{———}}D'\underset{0}{\text{——}}E'\cdots\cdots\cdots A$$

where $x \geqslant 2$, $C'' = d|\cdots|ae|\overset{x}{bc}|C(k)|\cdots$ and $\theta - 1$ of the leftmost x columns of $M(A, C'')$ are equal to 2. Hence we may use induction to assume $\theta = 0$.

(5): Consider our path between C and D of length x:

$$C\underset{x+1,x}{\text{———}}C^{(a,c)(b,e)}\underset{x,}{\text{———}}\cdots\underset{,1}{\text{———}}D$$

Hence there is a minimal path of length $x-1$ joining $C^{(a,c)}$ and $D^{(b,e)}$.

(6): Note that $\{a,d'\}$ cannot be a part of C. Otherwise by Theorem 2.35 (i) there is a 0-swapping $A^{(a,b)}$ of A in-between A and B. Therefore $d' \neq e$. $\square$

We are now in a position to prove Theorem 2.42. Before this, let's look back at Figures 2.5 and 2.6 and make some observations. Let $M(A,B)$ be one of the following four matrices:

$\boxed{\begin{array}{cc}1&1\\1&1\end{array}}$ distance 5 $\quad$ $\boxed{\begin{array}{cc}0&1\\1&1\end{array}}$ distance 5 $\quad$ $\boxed{\begin{array}{ccc}1&1&\\1&&1\\&1&1\end{array}}$ distance 11 $\quad$ $\boxed{\begin{array}{ccc}0&1&\\1&&1\\&1&1\end{array}}$ distance 10

In each case there exists $A'\underset{0}{\text{——}}A$ and $B'\underset{0}{\text{——}}B$ such that $M(A',B')$ looks like one of the following:

$\boxed{\begin{array}{cc}2&\\&1\end{array}}$ distance 3 $\quad$ $\boxed{\begin{array}{ccc}2&&\\&&1\\&1&1\end{array}}$ distance 9

Notice that in the three left cases the distance of the matrix has gone down by 2. Thus A' and B' both lie on a minimal path joining A and B. In the other case, however, the distance has gone down by only 1. Thus A' and B' do not lie in-between A and B. However, we find that every minimal path joining A and B contains some chamber C with both parts $C(i) = A(1)$ and $C(j) = B(1)$.

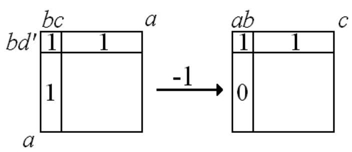
Theorem 2.42. *The following are true:*

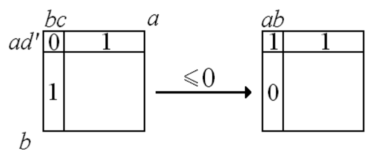
(v): *Let $A = a|bd'|\cdots$ and $B = a|bc|\cdots$ $(d' \neq c)$. Then the chamber $B^{(a,c)}\underset{0}{\text{——}}B$ is in-between A and B.*

(vi): *Let $A = b|ad'|\cdots$ and $B = a|bc|\cdots$ $(d' \neq c)$. Then $B^{(a,c)}\underset{0}{\text{——}}B$. Either:*

- *dist$(A, B^{(a,c)}) = $ dist$(A,B) - 1$, or*
- *dist$(A, B^{(a,c)}) = $ dist(A,B) and every minimal path between A and B contains a chamber C with both parts $A(1) = \{a,d'\}$ and $B(1) = \{b,c\}$.*

We illustrate the theorem here:

(v) 

(vi) 

Proof. We prove (v) and (vi) by induction on the distance of $M = M(A,B)$. We may claim they are true for distance 0. Assume they are both true for distance less than L and that M has distance L.

Proof of (v) for distance L. We can make some assumptions:

- There is no chamber $B' \underset{0}{\text{——}} B$ in-between A and B: If $B' = B^{(a,b)}$ then the result follows by induction using (vi). The only other option for B' is $B^{(a,c)}$, as required.
- There is no chamber $A' \underset{0}{\text{——}} A$ in-between A and B: If $A' = A^{(a,b)}$ then $B^{(a,b)}$ is in-between A and B which cannot happen by our first assumption. If $A' = A^{(a,d')} = d'|ab|\cdots$ then by Theorem 2.35 (i) $B^{(a,c)}$ is in-between A' and B, as required.

By Lemma 2.41 we may assume there is a minimal path between A and B of the form

$$B \cdots\cdots\cdots C \underset{x+1,}{\text{———}} \cdots \underset{,1}{\text{——}} D \underset{0}{\text{——}} E \cdots\cdots\cdots A$$

where C is of the form $d|\cdots|ae|\overset{x+1}{bc}|\cdots$, D is of the form $d|bc|\cdots|\overset{x+1}{ae}|\cdots$ where $x \geqslant 1$, and the lemma's five points (1), (2), (3), (4), (5) and (6) are satisfied.

The chamber $D^{(b,d)} = b|dc|\cdots|\overset{x+1}{ae}|\cdots$ is a 0-swapping of D. Let $\text{dist}(A, D^{(b,d)}) = \text{dist}(A, D) - z_1$ where z_1 =0 or 1 by Lemma 2.41 (2).

There exists F such that

$$D^{(b,d)} \underset{x+1,}{\text{———}} \cdots \underset{,1}{\text{——}} F \underset{0}{\text{——}} F^{(b,e)} \underset{1,}{\text{——}} \cdots \underset{,x+1}{\text{———}} D^{(b,d)(b,e)}$$

Recall by Lemma 2.41 (4) that $M(A, C)$ has no entries equal to 2 in its leftmost $x+1$ columns. The same goes for $M(A, D)$. For $M(A, D^{(b,d)})$ we have two cases:

Case I: The pair $\{d, c\}$ is a part of A. Then $z_1 = 1$ by Theorem 2.35 (i). In that case the leftmost x columns of $M(A, D^{(b,d)})$ contain exactly one entry (in the leftmost column) equal to 2. We know the following by induction, Theorem 2.35 and Lemma 2.36: (This is illustrated in Fig 2.9. We are allowed to use induction by combining (1) and (2) of Lemma 2.41.)

$$\begin{aligned} \text{dist}(A, F) &\leqslant \text{dist}(A, D^{(b,d)}) + x - 2 && \text{by (iii)} \\ \text{dist}(A, F^{(b,e)}) &\leqslant \text{dist}(A, F) && \text{by (vi)} \\ \text{dist}(A, D^{(b,d)(b,e)}) &= \text{dist}(A, F^{(b,e)}) - x + 2 && \text{by (iii) and (iv)} \end{aligned}$$

It follows that $\mathrm{dist}(A, D^{(b,d)(b,e)}) \leqslant \mathrm{dist}(A, D^{(b,d)}) < \mathrm{dist}(A, D)$. This is a contradiction by Lemma 2.41 (3) so we may discount Case I.

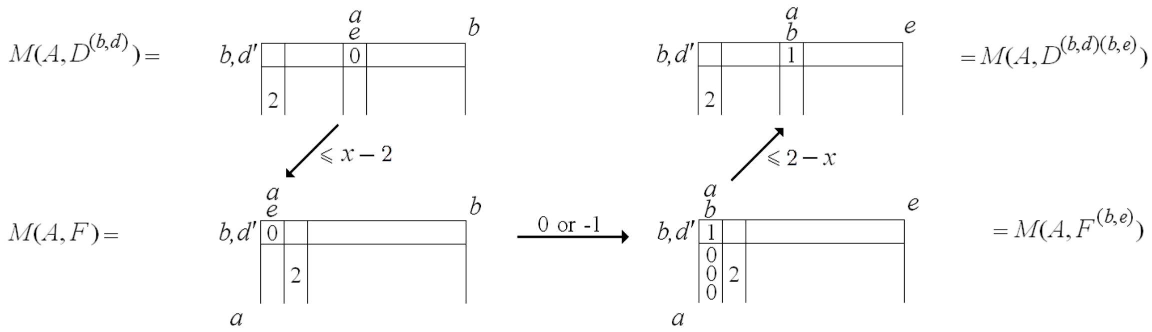


Fig. 2.9. Case I.

Case II: The pair $\{d, c\}$ is not a part of A. In that case the leftmost x columns of $M(A, D^{(b,d)})$ contain no entries equal to 2. We know the following by induction, Theorem 2.35 and Lemma 2.36: (This is illustrated in Fig 2.10. We are allowed to use induction by combining (1) and (2) of Lemma 2.41.)

$$\begin{aligned}
\mathrm{dist}(A, F) &= \mathrm{dist}(A, D^{(b,d)}) + z_2 && \text{where } z_2 \leqslant x \\
\mathrm{dist}(A, F^{(b,e)}) &= \mathrm{dist}(A, F) - z_3 && \text{where } z_3 = 0 \text{ or } 1 && \text{by (vi)} \\
\mathrm{dist}(A, D^{(b,d)(b,e)}) &= \mathrm{dist}(A, F^{(b,e)}) - x && && \text{by (iv)}
\end{aligned}$$

It follows that $\mathrm{dist}(A, D^{(b,d)(b,e)}) = \mathrm{dist}(A, D^{(b,d)}) + z_2 - z_3 - x$. We now have two

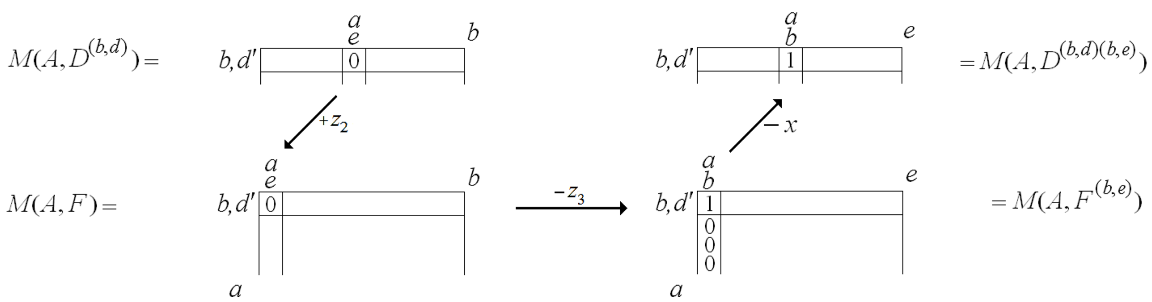


Fig. 2.10. Case II.

subcases:

Case II(i) $-z_1 + z_2 - z_3 < x$

Case II(ii) $-z_1 + z_2 - z_3 = x$

If we have Case II(i) then $\mathrm{dist}(A, D^{(b,d)(b,e)}) < \mathrm{dist}(A, D)$, contradicting Lemma 2.41. Therefore assume Case II(ii). We have $z_3 = 0$. But by induction (vi) that means all minimal paths between F and A contain a chamber with both parts $\{b, d'\}$ and $\{a, e\}$. Therefore by Lemma 2.38 all minimal paths between F and A contain a pair of adjacent chambers $U = ...|bd'|ae|...$ and $V = ...|ae|bd'|...$ (where V is in-between U and A).

Using $z_2 = x$ we see that $D^{(b,d)}$ lies on a minimal path between A and F. Using $z_1 = 0$ we may assume $E = D^{(c,d)}$ lies on this path. Hence $D^{(c,d)}$ lies on a minimal path between A and B, and U and V must lie on this path, in-between $D^{(c,d)}$ and A. Therefore we have a minimal path

$$B \cdots\cdots C \underset{x+1,}{\text{———}} \cdots\cdots \underset{,1}{\text{——}} D \underset{0}{\text{——}} D^{(c,d)} \cdots\cdots U \text{——} V \cdots\cdots A$$

of length $L = \text{dist}(B, C) + x + \text{dist}(D, U) + 1 + \text{dist}(V, A)$. But we can find another path of this length joining A and B:

$$B \underset{0}{\text{——}} B^{(a,c)} \cdots\cdots\cdots C^{(a,c)} \cdots\cdots\cdots D^{(b,e)} \cdots\cdots\cdots U^{(b,e)} \cdots\cdots\cdots A^{(a,d')} \underset{0}{\text{——}} A$$

Using Lemma 2.41 (5), this path has length $1 + \text{dist}(B, C) + (x - 1) + \text{dist}(D, U) + \text{dist}(U^{(b,e)}, A^{(a,d')}) + 1$. Using that $\text{dist}(U^{(b,e)}, A^{(a,d')}) = \text{dist}(U^{(b,e)(a,d')}, A) = \text{dist}(V, A)$, we see this is a minimal path containing $B^{(a,c)}$.

Proof of (vi) for distance L. First we want to show that if we have a minimal path P between A and B containing a chamber C with both parts $\{a, d'\}$ and $\{b, c\}$ then $\text{dist}(A, B^{(a,c)}) \leqslant \text{dist}(A, B)$: By Lemma 2.38 there exist adjacent chambers $D = \cdots |ad'|bc| \cdots$ and $D^{(d',b)(a,c)}$ lying on P, where $D^{(d',b)(a,c)}$ is in-between D and A. Since $\text{dist}(B, D) = \text{dist}(B^{(d',b)(a,c)}, D^{(d',b)(a,c)})$, we must have that $\text{dist}(A, B^{(d',b)(a,c)}) < \text{dist}(A, B)$. However, $\text{dist}(A, B^{(d',b)(a,c)}) = \text{dist}(A^{(d',b)}, B^{(a,c)})$. Using that $A^{(d',b)}$ is a neighbour of A, we have $\text{dist}(A, B^{(a,c)}) \leqslant \text{dist}(A, B)$, as required.

All that remains is to assume there is a minimal path P which does not contain a chamber with both parts $\{a, d'\}$ and $\{b, c\}$, and show that $B^{(a,c)}$ is in-between A and B as a result. We may assume that any 0-swapping B' of B is not in-between A and B: If $B' = B^{(a,b)}$ then the result follows by induction using (v). The only other option for B' is $B^{(a,c)}$, as required.

By Lemmas 2.34 and 2.40 we may assume P contains the subpath:

$$B \cdots\cdots\cdots C \underset{x+1,}{\text{———}} \cdots \underset{,1}{\text{——}} D \underset{0}{\text{——}} E \cdots\cdots\cdots A$$

where C is of the form $d|\cdots|ae|\overset{x+1}{bc}|\cdots$ and D is of the form $d|bc|\cdots|\overset{x+1}{ae}|\cdots$ for some $x \geqslant 1$. We may assume $d' \neq e$, or else C has both parts $\{b, c\}$ and $\{a, d'\}$.

We have $D^{(b,d)} = b|dc|\cdots|\overset{x+1}{ae}|\cdots$. There exists F such that:

$$D^{(d,b)} \underset{x+1,}{\text{———}} \cdots \underset{,1}{\text{———}} F \underset{0}{\text{——}} F^{(b,e)} \underset{1,}{\text{———}} \cdots \underset{,x+1}{\text{———}} D^{(b,d)(b,e)}$$

This is illustrated in Fig.2.11. Let θ be the number of entries within the leftmost x columns of $M(A, D^{(d,b)})$ that are equal to 2. We know the following by induction, Theorem 2.35 and Lemma 2.36: (We are allowed to use induction by combining (1) and (2) of Lemma 2.34.)

$$\begin{aligned}
\operatorname{dist}(A, F) &\leqslant \operatorname{dist}(A, D^{(d,b)}) + x - 2\theta && \text{by (iii)}\\
\operatorname{dist}(A, F^{(b,e)}) &= \operatorname{dist}(A, F) - 1 && \text{by (v)}\\
\operatorname{dist}(A, D^{(b,d)(b,e)}) &\leqslant \operatorname{dist}(A, F^{(b,e)}) - x + 2\theta && \text{by (iv)}
\end{aligned}$$

Hence $\operatorname{dist}(A, D^{(b,d)(b,e)}) < \operatorname{dist}(A, D^{(d,b)})$. This contradicts Lemma 2.34 (2)(3) and

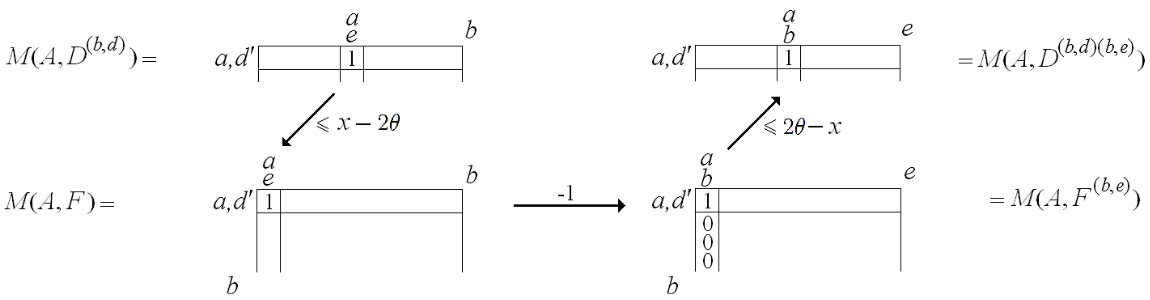


Fig. 2.11

proves (vi) for distance L. □

Theorem 2.43. *Let M be an $n \times n$ intersection matrix with top row odd, leftmost column odd, and $M_{12} = 1$. In other words, let $M = M(A, B)$ where $A = b|ad|\ldots$ and $B = a|bc|de|\ldots$. Then either:*

- $B^{(a,c)} \underset{0}{\text{——}} B$ *lies in-between A and B, or*
- $B' \underset{1,2}{\text{——}} B$ *lies in-between A and B, or both.*

Proof. By Theorem 2.42(vi) we have that either $B^{(a,c)}$ is in-between A and B, or a minimal path between A and B contains some chamber C with both parts $\{a, d\}$ and $\{b, c\}$. Assume the latter. The leftmost column of $M(C, B)$ contains a 2 and the column next to it does not. By Theorem 2.35 (iii) we have that $B' \underset{1,2}{\text{——}} B$ lies in-between B and C. $\square$

Lemma 2.44. *Let $1 \leqslant p < n$. Let A and B be n-chambers and let A' and B' be p-chambers. Let $A(i) = A'(i)$ and $B(i) = B'(i)$ when $i \leqslant p$. Let $A(i) = B(i)$ if $i > p$. Then dist$(A, B) =$ dist(A', B').*

Proof. First we prove that $\operatorname{dist}(A, B) \leqslant \operatorname{dist}(A', B')$. Let C' be any p-chamber adjacent to B'. Then define the n-chamber C by $C(i) = C'(i)$ if $i \leqslant p$ and $C(i) = B(i)$ otherwise. Clearly B and C are adjacent. Hence for every minimal path from B' to A' there is a path of the same length from B to a chamber D such that $D(i) = A'(i)$ whenever $i \leqslant p$ and $D(i) = B(i)$ otherwise. Hence $D = A$.
It remains to prove that $\operatorname{dist}(A, B) \geqslant \operatorname{dist}(A', B')$. Let P be a minimal path in-between A and B. By Theorem 2.35 ((ii) and (iii)) we see that every chamber $C \in P$ must have $C(i) = A(i) = B(i)$ when $i > p$. Therefore if $C \in P$ is adjacent to B define the p-chamber C' by $C'(i) = C(i)$. Clearly B' and C' are adjacent. Hence for every minimal path joining A and B there is a path of the same length from A' to B'. $\square$

Lemma 2.45. *Let $1 \leqslant p < n$. Let M be an $n \times n$ intersection matrix such that $M = P \oplus 2I_{n-p}$, where P is a $p \times p$ intersection matrix. Then dist$(M) =$ dist(P).*

Proof. Our matrix M looks like

$$\begin{array}{|c|c|} \hline P & \\ \hline & \begin{smallmatrix} 2 & & \\ & 2 & \\ & & 2 \end{smallmatrix} \\ \hline \end{array}$$

By Lemma 2.15 $M = M(A, B)$ where $A(i) = B(i)$ when $i > p$. Therefore define the p-chambers A' and B' where $A'(i) = A(i)$ and $B'(i) = B(i)$ for $1 \leqslant i \leqslant p$. The intersection matrix $M(A', B')$ is P. By Lemma 2.44 $\operatorname{dist}(A, B) = \operatorname{dist}(A', B')$. $\square$

Chapter 3

The diameter and automorphism group of $\Gamma(A_{2n+1})$

Definition 3.1. *Let* $\operatorname{diam}(n)$ *be the diameter of the chamber graph of* $\Gamma(A_{2n+1})$*. Let* $\operatorname{diam}_{odd}(n)$ *be the largest possible distance of an odd* $n \times n$ *intersection matrix.*

Clearly $\operatorname{diam}_{\text{odd}}(n) \leqslant \operatorname{diam}(n)$. We prove a lower bound for both in the next Section.

3.1 A lower bound of the diameter

Theorem 3.2.

$$diam(n) \geqslant \begin{cases} n^2 + \frac{2n}{3} & \text{if } n \equiv 0 \bmod 3 \\ n^2 + \frac{2n}{3} - \frac{2}{3} & \text{if } n \equiv 1 \bmod 3 \\ n^2 + \frac{2n}{3} - \frac{1}{3} & \text{if } n \equiv 2 \bmod 3 \end{cases}$$

$$diam_{odd}(n) \geqslant \begin{cases} n^2 + \frac{2n}{3} - 1 & \text{if } n \equiv 0 \bmod 3 \\ n^2 + \frac{2n}{3} - \frac{2}{3} & \text{if } n \equiv 1 \bmod 3 \\ n^2 + \frac{2n}{3} - \frac{1}{3} & \text{if } n \equiv 2 \bmod 3 \end{cases}$$

Alternatively, $diam(n) \geqslant n^2 + \lfloor \frac{2n}{3} \rfloor$ *and* $diam_{odd}(n) \geqslant n^2 + \lfloor \frac{2n-1}{3} \rfloor$*. Notice our claims differ only when* $n \equiv 0 \bmod 3$*.*

Proof. The theorem is true if $n = 1$. We prove by induction on n, defining an even $n \times n$ matrix E_n and an odd $n \times n$ matrix O_n having the distances claimed for each $n \geqslant 2$. We do so for $n = 2, 3$ and 4:

$E_2 = \begin{bmatrix}1&1\\1&1\end{bmatrix}$ $O_2 = \begin{bmatrix}0&1\\1&1\end{bmatrix}$ $E_3 = \begin{bmatrix}1&1&\\1&&1\\&1&1\end{bmatrix}$ $O_3 = \begin{bmatrix}0&1&\\1&&1\\&1&1\end{bmatrix}$ $E_4 = \begin{bmatrix}1&1&&\\1&&1&\\&1&&1\\&&1&1\end{bmatrix}$ $O_4 = \begin{bmatrix}1&&&\\&1&1&\\&1&&1\\&&1&1\end{bmatrix}$

This is enough to show the Theorem is true for $n = 2, 3$ and 4 using Figures 2.5 and 2.6, Theorems 2.35 and 2.42, and Lemma 2.45. Assume the theorem is true for order less than n. Now we prove it for matrices of order $n > 4$. First we define E_n as the block diagonal matrix $E_{n-3} \oplus E_3$. For example,

$$\begin{bmatrix}1&1&&&&&&\\1&1&&&&&&\\&&1&1&&&&\\&&1&&1&&&\\&&&1&1&&&\\&&&&&1&1&\\&&&&&1&&1\\&&&&&&1&1\end{bmatrix} \quad \begin{bmatrix}1&1&&&&&&&\\1&&1&&&&&&\\&1&1&&&&&&\\&&&1&1&&&&\\&&&1&&1&&&\\&&&&1&1&&&\\&&&&&&1&1&\\&&&&&&1&&1\\&&&&&&&1&1\end{bmatrix} \quad \begin{bmatrix}1&1&&&&&&&&\\1&&1&&&&&&&\\&1&&1&&&&&&\\&&1&1&&&&&&\\&&&&1&1&&&&\\&&&&1&&1&&&\\&&&&&1&1&&&\\&&&&&&&1&1&\\&&&&&&&1&&1\\&&&&&&&&1&1\end{bmatrix}$$

Next we define O_n as the block diagonal matrix $O_{n-3} \oplus E_3$. For example,

$$\begin{bmatrix}0&1&&&&&&\\1&1&&&&&&\\&&1&1&&&&\\&&1&&1&&&\\&&&1&1&&&\\&&&&&1&1&\\&&&&&1&&1\\&&&&&&1&1\end{bmatrix} \quad \begin{bmatrix}0&1&&&&&&&\\1&&1&&&&&&\\&1&1&&&&&&\\&&&1&1&&&&\\&&&1&&1&&&\\&&&&1&1&&&\\&&&&&&1&1&\\&&&&&&1&&1\\&&&&&&&1&1\end{bmatrix} \quad \begin{bmatrix}1&&&&&&&&&\\&1&1&&&&&&&\\&1&&1&&&&&&\\&&1&1&&&&&&\\&&&&1&1&&&&\\&&&&1&&1&&&\\&&&&&1&1&&&\\&&&&&&&1&1&\\&&&&&&&1&&1\\&&&&&&&&1&1\end{bmatrix}$$

We need to prove that E_n has the distance claimed. By Theorems 2.35 and 2.42 (applying (v), then (i), then (iii)) and Lemma 2.45 we see that dist(E_n) =dist$(O_{n-1}) + 2n$. This proves our claim for diam(n). It only remains to show that O_n has the distance claimed. Let $O_n = M(A, B)$. We have three cases:

$n \equiv 1 \bmod 3$. Applying Theorem 2.35 ((i) then (iii)) and Lemma 2.45 we have dist$(O_n) = 2n - 1+$dist(E_{n-1}). This gives the required result.

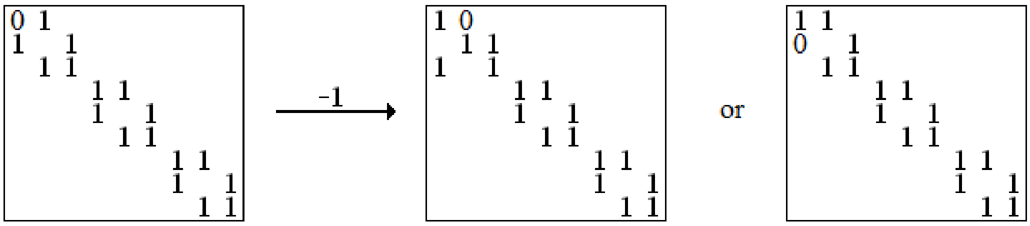
$n \equiv 2 \bmod 3$. Then by Theorem 2.43 there exists either $B' \underset{0}{\text{——}} B$ or $B'' \underset{1,2}{\text{——}} B$ in-between A and B.

$$\begin{bmatrix}0&1&&&&&&\\1&1&&&&&&\\&&1&1&&&&\\&&1&&1&&&\\&&&1&1&&&\\&&&&&1&1&\\&&&&&1&&1\\&&&&&&1&1\end{bmatrix} \xrightarrow{-1} \begin{bmatrix}1&0&&&&&&\\1&1&&&&&&\\&&1&1&&&&\\&&1&&1&&&\\&&&1&1&&&\\&&&&&1&1&\\&&&&&1&&1\\&&&&&&1&1\end{bmatrix} \text{ or } \begin{bmatrix}1&1&&&&&&\\0&1&&&&&&\\&&1&1&&&&\\&&1&&1&&&\\&&&1&1&&&\\&&&&&1&1&\\&&&&&1&&1\\&&&&&&1&1\end{bmatrix}$$

$M(A, B')$ is the transpose of $M(A, B'')$ so both are shorter than O_n. Applying Theorem 2.35 ((i) then (iii)) and Lemma 2.45 we see that the distance between A and B is

$\mathrm{dist}(O_{n-1}) + 2n$. This gives the required result.

$n \equiv 0 \bmod 3$. Then by Theorem 2.43 there exists either $B' \underset{0}{\text{——}} B$ or $B'' \underset{1,2}{\text{——}} B$ in-between A and B.



By applying Theorem 2.35 ((i) then (iii)) to $M(A, B')$ and $M(A, B'')$ we get two matrices: One is $O_n \oplus 2I$, and the other (as we saw in the previous case $n \equiv 2 \bmod 3$) is shorter than $O_n \oplus 2I$. It follows that B'' lies in-between A and B whereas B' does not. We see that the distance between A and B is $\mathrm{dist}(O_{n-1}) + 2n - 1$ by Lemma 2.45. This gives the required result. $\square$

3.2 Sets of parts, AB-sets, and AB-sequences

We have diam(1)=1, diam(2)=5, diam(3)=11, diam(4)=18 and diam(5)=28. This can be shown using Magma. All the intersection matrices of chambers in $\Gamma(A_9)$ distance 18 apart are displayed in Fig.3.1. All the intersection matrices of chambers in $\Gamma(A_{11})$ distance 28 apart are shown in Fig.3.2.

3.2.1 Introducing sets of parts

Definition 3.3. *Let A be an n-chamber. We say X is a set of parts of A if every element of X is equal to some part $A(i) \neq A(0)$ of A. If X has order x we may call X an x-set of parts of A.*

For example, $\{\{3, 4\}, \{7, 8\}\}$ is a 2-set of parts of $1, 2|3, 4|5, 6|7, 8|9, 10$.

Definition 3.4. *Let X be an x-set of parts of a chamber A. We say A is X-first if every $A(i) \in X$ has $1 \leqslant i \leqslant x$. Similarly, we say A is X-last if every $A(i) \in X$ has $n - x < i \leqslant n$.*

For example, $1, 2|3, 4|5, 6|7, 8|9, 10$ is $\{\{1, 2\}, \{3, 4\}\}$-first and $\{\{7, 8\}, \{9, 10\}\}$-last.

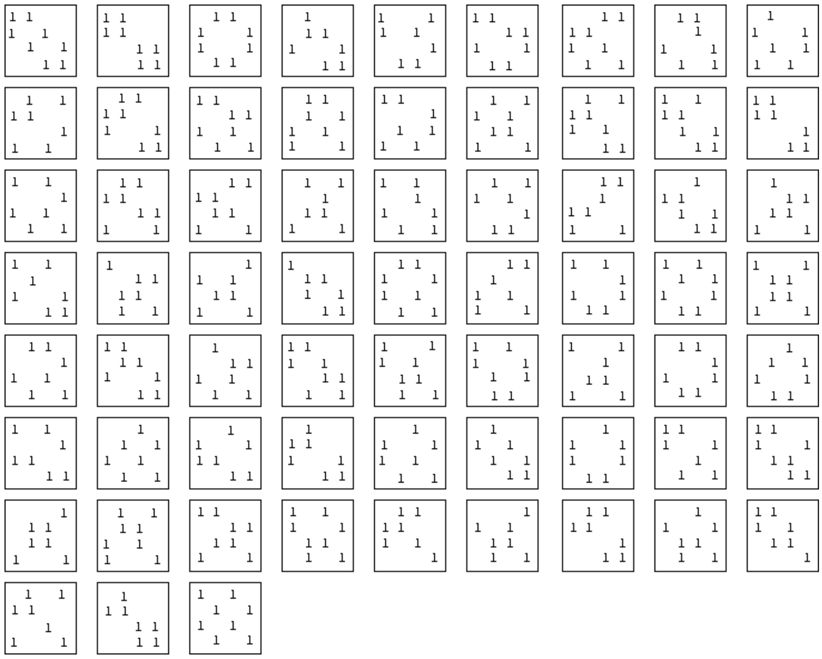


Fig. 3.1. All 4×4 intersection matrices of length 18

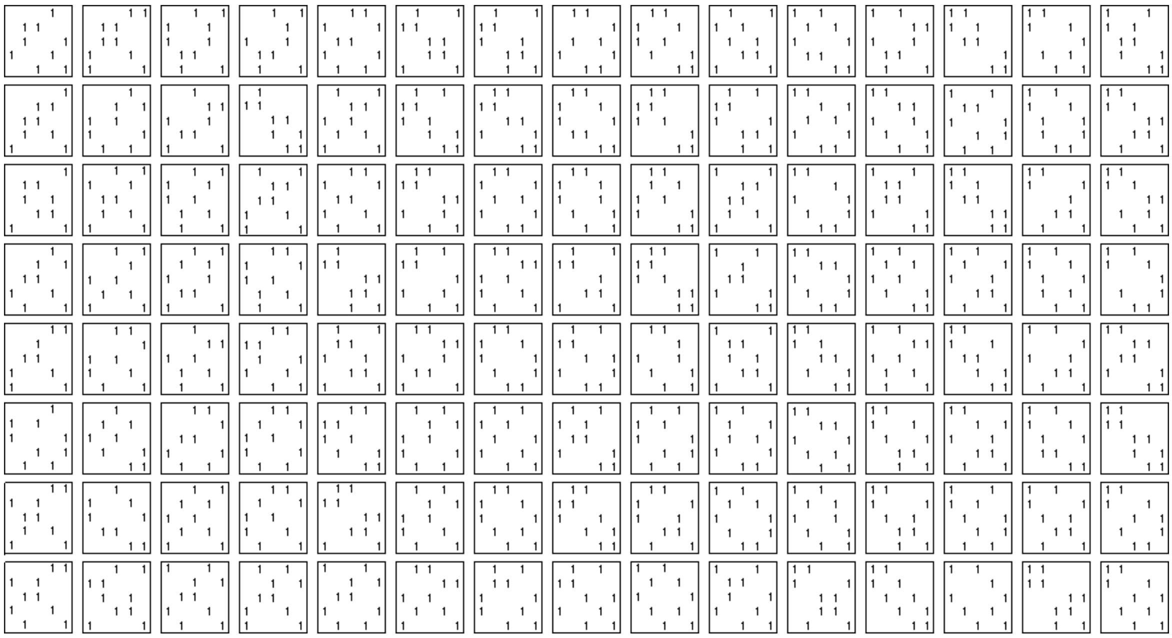


Fig. 3.2. All 5×5 intersection matrices of length 28

Definition 3.5. *Let A be a chamber and X a set of parts of A, we define the* mix of X in A *to be*

$$mix(X, A) = \sum_{A(i) \in X} |\{j : 1 \leqslant j < i \text{ and } A(j) \notin X\}|$$

Equivalently, $mix(X, A)$ is the minimum distance between A and an X-first chamber B (this equivalence is proved in Lemma 3.7).

For example,

$$\begin{aligned} \text{mix}(\ \{\{3,4\},\{7,8\}\} \quad &, \ 1,2|3,4|5,6|7,8|9,10\ ) &= 1+2 \\ \text{mix}(\ \{\{5,6\},\{9,10\}\} \quad &, \ 1,2|3,4|5,6|7,8|9,10\ ) &= 2+3 \end{aligned}$$

Lemma 3.6. *Let A be an n-chamber and X an x-set of parts of A where $n = x + y$. Then mix(X, A) cannot exceed xy.*

Proof. By considering Definition 3.5 we see that the maximum value of $\text{mix}(X, A)$ is xy, when A is X-last. □

Lemma 3.7. *Let A be an n-chamber and X an x-set of parts of A. Let $n = x + y$. There is a unique X-first chamber B distance mix(X, A) from A, and this is the closest X-first chamber to A. Similarly, there is a unique X-last chamber C distance $xy -$ mix(X, A) from A, and this is the closest X-last chamber to A.*

Proof. First note that if X is a set of parts of two chambers, joined by a minimal path P, then X is a set of parts of every chamber in P by Lemma 2.37. Also note that if D and E are adjacent and X is a set of parts of both of them then $\text{mix}(X, D)$ and $\text{mix}(X, E)$ differ by at most 1. First we prove the claim about B. We do this by induction on $\text{mix}(X, A)$. The result is clearly true if $\text{mix}(X, A) = 0$. If it is true for $\text{mix}(X, A) \leqslant k - 1$ then it follows for $\text{mix}(X, A) = k$. A similar argument proves the claim about C by induction on $xy - \text{mix}(X, A)$. □

For example, let $C = 1,2|3,4|5,6|7,8|9,10$ and $X = \{\{5,6\},\{7,8\}\}$. The nearest X-first chamber to C is distance $\text{mix}(X, C)$ from C.

3.2.2 Introducing AB-sets

Definition 3.8. *Let A and B be n-chambers. We define $G(A, B)$ as the undirected graph with coloured edges, whose vertices are elements of the set $\{1, 2, 3, ..., 2n+1\}$ and whose red (respectively, blue) edges are the parts $A(i)$ (respectively, $B(i)$) for $1 \leqslant i \leqslant n$.*

The graph $G(A, B)$ consists of n red edges and n blue edges. The connected components look like

We always have exactly one connected component of odd order greater than or equal to 1, which contains $A(0)$ and $B(0)$. We prove this formally later.

Definition 3.9. *The set of vertices of a particular connected component of the graph $G(A, B)$ is called an AB-set. A set is called even (respectively, odd) if its order is even (respectively, odd).*

Consider the matrix $M(A, B)$. Each entry equal to 1 in $M(A, B)$ represents an element in the same AB-set as an element represented by an entry equal to 1 in the same row or column. For example, see Fig 3.3.

Fig. 3.3. Three AB-sets of order 4, 5 and 6

3.2.3 Introducing AB-sequences

Definition 3.10. *A non-repeating sequence $S = (S_1, ..., S_L)$ of length L whose elements are vertices of $G(A, B)$ is called an AB-sequence if each element in the sequence is incident to the element before and after it (if these exist). A sequence is called even (respectively, odd) if its order is even (respectively, odd).*

Notice AB-sequences are "ordered bits of AB-sets". For example, let $A = 1, 2|3, 4$ $|5, 6|7, 8|9, 10$ and $B = 1, 9|7, 11|6, 8|2, 3|4, 10$. Then $(7, 8)$ and $(1, 2, 3, 4, 10)$ are AB-sequences, illustrated in Fig.3.4.

Lemma 3.11. *Let A and B be chambers. Let B and B' be connected by a path of jumps. A sequence S is an AB-sequence if and only if it is an AB'-sequence.*

Proof. It is enough to note that $G(A, B) = G(A', B')$. $\square$

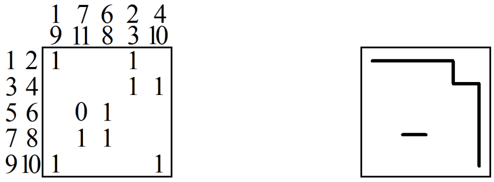


Fig. 3.4

Definition 3.12. *Let C be an n-chamber and let S be a subset of $\{1, 2, 3, ..., 2n + 1\}$ or sequence of elements from $\{1, 2, 3, ..., 2n + 1\}$. Define*

$$C_S = \{C(i) : 1 \leqslant i \leqslant n | C(i) \subseteq S\}$$

(If S is a sequence, write $C(i) \subseteq S$ to mean "Both elements of $C(i)$ appear in S".)

For example, if S is either of the AB-sequences in Fig.3.4 then we shade the rows of A_S and columns of B_S here:

Lemma 3.13. *The only way a blue edge $\{a, b\}$ and a red edge $\{a, b\}$ can both appear in $G(A, B)$ is if for some i and j we have $A(i) = B(j) = \{a, b\}$. The only sequences containing a and b are (a), (b), (a, b) and (b, a).*

Lemma 3.14. *Let $S = (S_1, S_2, ..., S_{L-1}, S_L)$ be an AB-sequence of length $L \geqslant 3$.*

- *If $\{S_1, S_2\}$ and $\{S_{L-1}, S_L\}$ are the same colour then L is even.*
- *If $\{S_1, S_2\}$ and $\{S_{L-1}, S_L\}$ are different colours then L is odd.*

Proof. The theorem is true for $L = 3$ and inductively follows for all higher L. □

Lemma 3.15. *Let S be an AB-sequence of odd length $2p + 1$. Then $|A_S| = |B_S| = p$.*

Proof. By Lemmas 3.13 and 3.14 the sequence S must be composed of exactly p red edges (parts of A) and p blue edges (parts of B). □

Definition 3.16. *We say an AB-sequence S is* maximal *if there is no AB-sequence T such that S is a proper subsequence of T.*

Note that each maximal AB-sequence corresponds to a particular AB-set.

Lemma 3.17. *Let $S = (S_i)_{i=1}^{L} = (S_1, ..., S_L)$ be a maximal AB-sequence of length $L \geqslant 1$. Either $\{S_1, S_L\}$ is a part of A or B, or $\{S_1, S_L\} = \{A(0), B(0)\}$.*

Proof. Consider $G(A, B)$. The lemma follows from the fact that either $2n$ vertices have valency 2 and the other has valency 0, or else $2n-1$ vertices have valency 2 and the others have valency 1. □

Lemma 3.18. *Let A and B be n-chambers. If $A(0) = B(0)$ there is a unique maximal AB-sequence of odd order, namely $(A(0)) = (B(0))$. If $A(0) \neq B(0)$ there are exactly two maximal AB-sequences of odd order. These are of the form $(A(0), ..., B(0))$ and $(B(0), ..., A(0))$.*

Proof. Let $A(0) = B(0)$. In that case a maximal AB-sequence $S = (S_1, ..., S_L) \neq (A(0))$ has length $L > 1$ and cannot contain $A(0)$. Suppose $L \geqslant 3$. By Lemma 3.17 we have that $\{S_1, S_L\}$ is an edge. Therefore $\{S_1, S_2\}$ and $\{S_{L-1}, S_L\}$ are both edges of the same colour. It follows that L is even by Lemma 3.14.
Let $A(0) \neq B(0)$. A vertex in $G(A, B)$ has valency at most 2. Therefore there is exactly one maximal AB-sequence of the form $(A(0), ..., B(0))$ and exactly one of the form $(A(0), ..., B(0))$. Any other maximal AB-sequence $S = (S_1, ..., S_L)$ has length $L \geqslant 2$ and cannot contain $A(0)$ or $B(0)$. By Lemmas 3.14 and 3.17 L is even. □

Lemma 3.19. *Let A and B be n-chambers. There is exactly one AB-set of odd order. This contains $A(0)$ and $B(0)$.*

Proof. Using that any vertex of $G(A, B)$ has valency at most 2, it can be shown that each maximal AB-sequence corresponds to exactly one AB-set, and each AB-set corresponds to one or more maximal AB-sequences. The result follows from Lemma 3.18. □

Lemma 3.20. *Let C be a chamber and S be a sequence. Let $g \in S_{2n+1}$. Then $mix(C_S, C) = mix(C^g{}_{S^g}, C^g)$.*

Proof. It is enough to notice that $C(i) \in C_S$ if and only if $C^g(i) \in C^g{}_{S^g}$. □

Lemma 3.21. *Let S be an AB-sequence. Let $g \in S_{2n+1}$. Then S^g is an A^gB^g-sequence.*

Proof. Obvious when $L = 1$. Let $L \geqslant 2$. Without loss of generality let $S = (S_i)_{i=1}^L$ be composed of the red edges $\{S_{2i}, S_{2i+1}\}$ from A and the blue edges $\{S_{2i-1}, S_{2i}\}$ from B. Then S^g is composed of the red edges $\{S_{2i}, S_{2i+1}\}^g$ from A^g and the blue edges $\{S_{2i-1}, S_{2i}\}^g$ from B^g. □

Lemma 3.22. *Let S be an AB-sequence. Let $g \in S_{2n+1}$ fix each element in S. Then:*

- *S is an A^gB-sequence, an AB^g-sequence and an A^gB^g-sequence.*
- *$A_S = A^g_S$ and $B_S = B^g_S$.*
- *$mix(S, A^g_S) = mix(S, A_S)$ and $mix(S, B^g_S) = mix(S, B_S)$.*

Proof. Obvious when $L = 1$. Let $L \geqslant 2$. Without loss of generality let $S = (S_i)_{i=1}^L$ be composed of the red edges $\{S_{2i}, S_{2i+1}\}$ from A and the blue edges $\{S_{2i-1}, S_{2i}\}$ from B. Then S is composed of exactly the same red edges $\{S_{2i}, S_{2i+1}\}^g$ from A or A^g and exactly the same blue edges $\{S_{2i-1}, S_{2i}\}^g$ from B or B^g. □

Lemma 3.23. *Let A and B be chambers. Suppose that:*

- *$A = e|ab| \cdots$ and $B(0) = e$.*
- *$S = (S_i)_{i=1}^L = (a, b, ...)$ is an AB-sequence of odd length $L \geqslant 3$.*

Then:

- *The sequence $S^{(a,e)} = (e, b, ...)$ is an $A^{(a,e)}B$-sequence.*
- *$B_S = B_{S^{(a,e)}}$ and $mix(B_{S^{(a,e)}}, B) = mix(B_S, B)$*
- *$mix(A_S, B) = mix({A^{(a,e)}}_{S^{(a,e)}}, A^{(a,e)})$*

Proof. The shorter sequence $T = (S_i)_{i=2}^L = (b, ...)$ is an AB-sequence which does not contain a or e. By Lemma 3.22 T is an $A^{(a,e)}B$-sequence. Using $A^{(a,e)}(1) = \{e, b\}$ we see that $(e, b, ...) = S^{(a,e)}$ is an $A^{(a,e)}B$-sequence. By Lemma 3.15 we have $|B_{S^{(a,e)}}| = p$ so $B_S = B_{S^{(a,e)}}$ and therefore $\text{mix}(B_{S^{(a,e)}}, B) = \text{mix}(B_S, B)$. By Lemma 3.20 we have $\text{mix}({A^{(a,e)}}_{S^{(a,e)}}, A^{(a,e)}) = \text{mix}(A_S, A)$. □

Lemma 3.24. *Let S be an AB-sequence of length $2p+1$. Let $B(1) \notin B_S$. Let B' be a 0-swapping of B. Then S is an AB'-sequence, $B'_S = B_S$ and $mix(B_S, B) = mix(B'_S, B')$.*

Proof. Let $S = (S_i)_{i=1}^{2p+1}$. If $p = 0$ then the result is obvious. Let $p \geqslant 1$. Recall Lemma 3.15: Without loss of generality let $S = (S_i)_{i=1}^L$ be composed of the p red edges $\{S_{2i}, S_{2i+1}\}$ from A and the p blue edges $\{S_{2i-1}, S_{2i}\} \neq B(1)$ from B. These blue edges are also parts of B'. Hence S is an AB'-sequence. By Lemma 3.15 $|B'_S| = p$ so S is composed of exactly the same blue edges $\{S_{2i-1}, S_{2i}\} \neq B'(1)$ from B'. □

Lemma 3.25. *Let $S = (a, b, ..., c, d)$ be an AB-sequence of length $2p+1$, where $A(x) = \{a, b\}$ and $B(y) = \{c, d\}$. Suppose every $A(x') \in A_S$ has $x \leqslant x'$ and every $B(y') \in B_S$*

has $y \leqslant y'$. Consider the subsequence $T = (b, ..., c)$ of S which is two elements shorter. We have

$$mix(A_T, A) = mix(A_S, A) + p - x \qquad mix(B_T, B) = mix(B_S, B) + p - y$$

Proof. By Lemma 3.15 we have $|A_S| = |B_S| = p$. We have $A_S = A_T \cup \{A(x)\}$ and $B_S = B_T \cup \{B(y)\}$. Hence by Definition 3.5,

$$\text{mix}(A_S, A) = \sum_{A(i) \in A_S} |\{j : 1 \leqslant j < i | A(j) \notin A_S\}|$$

$$\text{mix}(A_T, A) = \sum_{A(i) \in A_T} |\{j : 1 \leqslant j < i | A(j) \notin A_T\}|$$

This gives $\text{mix}(A_T, A) = \text{mix}(A_S, A) + (p-1) - (x-1)$. A similar argument applies to $\text{mix}(B_T, B)$. □

3.3 Split intersection matrices

Definition 3.26. *We say the $n \times n$ intersection matrix M is split into P and Q if there exist p and q (where $1 \leqslant p < n$, $1 \leqslant q < n$ and $p + q = n$) satisfying all of the following:*

We have $M_{ij} = 0$ whenever $i > q$ and $j > p$

We have that P is the $p \times p$ intersection matrix defined by $P_{i,j} = M_{i+q,j}$

We have that Q is the $q \times q$ intersection matrix defined by $Q_{i,j} = M_{i,j+p}$

Definition 3.26 is illustrated in Fig.3.5. Recall Definition 2.12. Notice that if M is split into P and Q then M is even if and only if P and Q are both even. Also notice that M has exactly one non-zero entry outside P or Q if and only if P and Q are both odd. We prove this formally in Lemma 3.28.

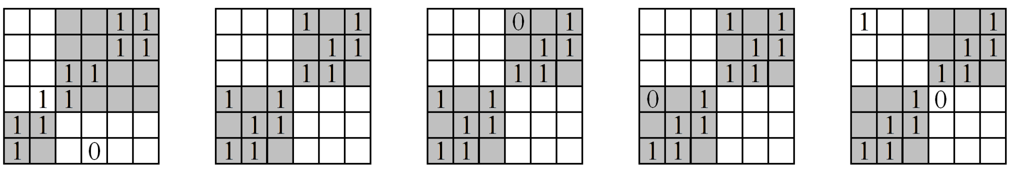


Fig. 3.5

Lemma 3.27. *Let M be an $n \times n$ intersection matrix. Let $n = p + q$ where $(1 \leqslant p \leqslant n-1)$. Let $M_{ij} = 0$ whenever both $i > q$ and $j > p$ are satisfied. Then M is split into a $p \times p$ matrix P and a $q \times q$ matrix Q.*

Proof. Define the $p \times p$ matrix P and the $q \times q$ matrix Q by $P_{i,j} = M_{i+q,j}$ and $Q_{i,j} = M_{i,j+p}$ respectively. Any entry M_{ij} of M belongs to one of the four quadrants in Fig 3.6. Now $\sum_{i>q,j>p} M_{ij} = 0$. The bottom p rows must sum to $2p-1$ or $2p$.

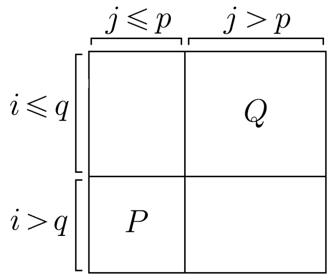


Fig. 3.6

This forces the entries of P to sum to $2p-1$ or $2p$. Each row of P must each sum to 1 or 2, and each column must sum to 2 or less. This fulfills the requirements of Lemma 2.11. Therefore by Lemma 2.15 we see P is an intersection matrix. A similar argument applies to Q. $\square$

Lemma 3.28. *Let P be a $p \times p$ intersection matrix and Q a $q \times q$ intersection matrix. There is exactly one intersection matrix M which is split into P and Q.*
Furthermore, if P or Q is even then M has no non-zero entries outside P or Q. If P and Q are odd then M has exactly one non-zero entry equal to 1 outside P or Q. This is the entry defined by the odd column of P and odd row of Q.

Proof. Let $n = p + q$. We are forced to define $M_{ij} = 0$ whenever $i > q$ and $j > p$. We are forced to define $M_{ij} = P_{i-q,j}$ whenever $i > q$ and $j \leqslant p$. If P is even then $\sum_{i>q,j\leqslant p} M_{ij} = 2p$. As the leftmost p rows can only sum to $2p$ or $2p-1$, this forces $\sum_{i\leqslant q,j\leqslant p} M_{ij} = 0$. The remaining entries of M are defined by Q.
A similar argument can be used when Q is even. Therefore assume both P and Q are odd. We have $\sum_{i>q,j>p} M_{ij} = 0$, $\sum_{i>q,j\leqslant p} M_{ij} = 2p-1$ and $\sum_{i\leqslant q,j>p} M_{ij} = 2q-1$. This forces $\sum_{i\leqslant q,j\leqslant p} M_{ij} = 1$ or 2. This sum cannot equal 2 because then the leftmost p columns sum to more than $2p$. Therefore it must equal 1. Due to the uniqueness of the odd column of P and the odd row of Q, there is only one entry which can equal 1. $\square$

Lemma 3.29. *Let $M = M(A,B)$ be split into a $p \times p$ matrix P and a $q \times q$ matrix Q. Let C and D be p-chambers such that $P = M(C,D)$. Let $D' — D$ and $P' = M(C,D')$.*

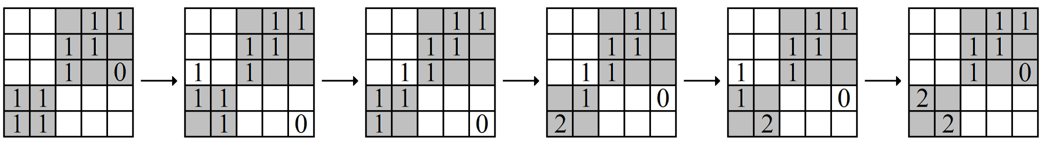
Then there is a matrix $M' = M(A, B')$ split into P' and Q for some $B' \text{——} B$.

Proof. Suppose $D' \underset{i,i+1}{\text{——}} D$. Then let $B' \underset{i,i+1}{\text{——}} B$. It follows that $M(A, B')$ is split into P' and Q. Suppose $D' \underset{0}{\text{——}} D$. By Lemma 3.28 there is a unique matrix M' split into P' and Q. The leftmost q columns of M and M' are identical. Now P and P' differ only by their leftmost column. In particular, any column (except the leftmost) of P is odd if and only if the same column in P' is odd. Hence M and M' differ only by their leftmost column. By Lemma 2.16 there exists a chamber $B' \underset{0}{\text{——}} B$ such that $M' = M(A, B')$. ☐

Lemma 3.30. *If an $n \times n$ intersection matrix $M = M(A, B)$ is split into a $p \times p$ matrix P and a $q \times q$ matrix Q, then dist$(M) \leqslant$ dist$(P) +$ dist$(Q) + pq$.*

Proof. We prove by induction on $\operatorname{dist}(P)$. If $\operatorname{dist}(P) = 0$ then the distance between A and B is equal to pq by Theorem 2.35 (ii) and Lemma 2.45. Therefore assume the theorem is true whenever $\operatorname{dist}(P) < L$ and let $\operatorname{dist}(P) = L \neq 0$.
By Lemma 3.29 we see that B is adjacent to a chamber B' such that $M(A, B')$ is split into P' and Q where $\operatorname{dist}(P') = \operatorname{dist}(P) - 1$. By induction, $\operatorname{dist}(A, B') \leqslant \operatorname{dist}(P') + \operatorname{dist}(Q) + pq$. ☐

An example of Lemma 3.30 working is shown here:



Lemma 3.31. *Let $M = M(A, B)$ be an $n \times n$ matrix. Let S be an AB-sequence of odd length $2p + 1$ where $1 \leqslant p \leqslant n - 1$. Let A be A_S-last and B be B_S-first.*

- *If $B(0) \in S$ then M is split into an odd $p \times p$ matrix and another matrix.*
- *If $B(0) \in S$ and $A(0) \notin S$ then M is split into an odd $p \times p$ matrix and an odd matrix.*

Proof. Let $B(0) \in S$. Without loss of generality let $S = (1, 2, ..., 2p, 2p+1)$ and assume $B(0) = 1$, otherwise we have $B(0) = 2p + 1$ and the proof is similar. By Lemma 3.15 we have $|A_S| = |B_S| = p$. Let $n = p + q$. The bottom p rows represent parts

$$\{A(i) : q < i \leqslant n\} = \{\{1, 2\}, \{3, 4\}, ..., \{2p - 1, 2p\}\}$$

so their entries must sum to $2p-1$ using $B(0) = 1$. The leftmost p columns represent parts

$$\{B(i) : 1 \leqslant i \leqslant p\} = \{\{2,3\},\{4,5\},...,\{2p,2p+1\}\}$$

Hence $\sum_{i>q,j\leqslant p} M_{ij} = 2p-1$ and $\sum_{i>q,j>p} M_{ij} = 0$. Thus by Lemma 3.27 M is split into a $p \times p$ matrix P and a $q \times q$ matrix Q. The matrix P is odd. Let $A(0) \notin S$. Then $A(0) \neq \{2p+1\}$. This means $2p+1 \in A(i)$ for some $i \leqslant p$. This gives $M_{ij} = 1$ for some $1 \leqslant i \leqslant q$, $1 \leqslant j \leqslant p$. Hence by Lemma 3.28 P and Q are both odd. $\square$

3.4 An upper bound for diam and $\mathrm{diam}_{\mathrm{odd}}$

3.4.1 A few lemmas

These lemmas will prepare us for Theorem 3.37.

Lemma 3.32. *Let X be a set of parts of the n-chamber C. Let $|X| = p$ and $C(1) \notin X$. Let $C \underset{1,}{\rule{2em}{0.4pt}} \cdots \underset{,n}{\rule{2em}{0.4pt}} D$. Then $mix(X, D) = mix(X, C) - p$.*

Proof. By Lemma 2.21 we have $D(i) = C(i+1)$ whenever $i \leqslant n-1$ and $D(n) = C(1)$. By Definition 3.5 we get $\mathrm{mix}(X, D) = \mathrm{mix}(X, C) - p$. $\square$

Lemma 3.33. *Let A and B be n-chambers. Let $A(n) = B(n) = \{2n, 2n+1\}$. Let S be an AB-sequence not equal to $(2n)$, $(2n+1)$, $(2n, 2n+1)$ or $(2n+1, 2n)$. Define the $(n-1)$-chambers C and D by $C(i) = A(i)$ and $D(i) = B(i)$ for $0 \leqslant i \leqslant n-1$. Then $dist(A, B) = dist(C, D)$. Furthermore, S is a CD-sequence, $mix(C_S, C) = mix(A_S, A)$ and $mix(D_S, D) = mix(B_S, B)$.*

Proof. We have $\mathrm{dist}(A, B) = \mathrm{dist}(C, D)$ by Lemma 2.44. Without loss of generality let $S = (S_i)_{i=1}^{L}$ be composed of red edges $\{S_{2i}, S_{2i+1}\} \neq A(n)$ from A and blue edges $\{S_{2i-1}, S_{2i}\} \neq B(n)$ from B. Hence S is a CD-sequence. $A(n) \notin A_S$ and $B(n) \notin B_S$ and so $A_S = C_S$ and $B_S = D_S$. $\square$

Lemma 3.34. *Let A and B be chambers. Let $S = (a, b, ..., c, d)$ be an AB-sequence of length $2p+1$. Let $B(1) = \{a, b\}$. Let $A(1) \notin A_S$ and $A(2) = \{c, d\}$. Let $T = (b, ..., c)$ be a subsequence of S of length $2p-1$. T is an AB-sequence. If $mix(A_S, A) > mix(B_S, B)$ then $mix(A_T, A) \geqslant mix(B_T, B)$.*

Proof. By Lemma 3.25 we have $\text{mix}(A_T, A) = \text{mix}(A_S, A) + p - 2$ and $\text{mix}(B_T, B) = \text{mix}(B_S, B) + p - 1$. □

Lemma 3.35. *Let A and B be chambers. Let $S = (a, b, ..., c, d)$ be an AB-sequence of length $2p + 1$. Let $A(1) = \{a, b\}$ and $B(1) = \{c, d\}$. Let $mix(A_S, A) \geqslant mix(B_S, B)$. Consider the subsequence $T = (b, ..., c)$ of S of length $2p - 1$. T is an AB-sequence and $mix(A_T, A) \geqslant mix(B_T, B)$.*

Proof. By Lemma 3.25 we have $\text{mix}(A_T, A) = \text{mix}(A_S, A) + p - 1$ and $\text{mix}(B_T, B) = \text{mix}(B_S, B) + p - 1$. □

Before embarking on Lemma 3.36 it is useful to remind ourselves of Definition 3.4.

Lemma 3.36. *Let $M = M(A, B)$ be an $n \times n$ matrix. Let S be an AB-sequence of odd length $2p + 1$ where $1 \leqslant p \leqslant n - 1$. Let $n = p + q$.*
If $B(0) \in S$ then

$$dist(A, B) \leqslant mix(B_S, B) - mix(A_S, A) + diam_{odd}(p) + diam(q) + 2pq$$

If $B(0) \in S$ and $A(0) \notin S$ then

$$dist(A, B) \leqslant mix(B_S, B) - mix(A_S, A) + diam_{odd}(p) + diam_{odd}(q) + 2pq$$

Proof. Let C be the closest A_S-last chamber to A and D be the closest B_S-first chamber to B. By Lemma 3.7 we have

$$\text{dist}(A, C) = pq - \text{mix}(A_S, A) \qquad \text{dist}(B, D) = \text{mix}(B_S, B)$$

Now $A_S = C_S$, $B_S = D_S$ and S is a CD-sequence. The chamber C is C_S-last and the chamber D is D_S-first. If $D(0) = B(0) \in S$ then $M(C, D)$ is split into an odd $p \times p$ matrix P and a $q \times q$ matrix Q by Lemma 3.31. This implies $\text{dist}(C, D) \leqslant \text{diam}_{\text{odd}}(p) + \text{diam}(q) + pq$ by Lemma 3.30. If $D(0) = B(0) \in S$ and $C(0) = A(0) \notin S$ then both P and Q are odd by Lemma 3.31. This implies $\text{dist}(C, D) \leqslant \text{diam}_{\text{odd}}(p) + \text{diam}_{\text{odd}}(q) + pq$ by Lemma 3.30. This gives the required result. □

3.4.2 The upper bound

For the rest of this chapter let a, b, c, d, e and f be distinct elements of $\{1, 2, ..., 2n+1\}$. That is, $|\{a, b, c, d, e, f\}| = 6$.

Theorem 3.37.

$$diam_{odd}(n) \leqslant \begin{cases} n^2 + \frac{2n}{3} - 1 & \textit{if } n \equiv 0 \bmod 3 \\ n^2 + \frac{2n}{3} - \frac{2}{3} & \textit{if } n \equiv 1 \bmod 3 \\ n^2 + \frac{2n}{3} - \frac{1}{3} & \textit{if } n \equiv 2 \bmod 3 \end{cases}$$

$$diam(n) \leqslant \begin{cases} n^2 + \frac{2n}{3} & \textit{if } n \equiv 0 \bmod 3 \\ n^2 + \frac{2n}{3} - \frac{2}{3} & \textit{if } n \equiv 1 \bmod 3 \\ n^2 + \frac{2n}{3} - \frac{1}{3} & \textit{if } n \equiv 2 \bmod 3 \end{cases}$$

Proof. We prove by induction on n. The theorem is clearly true for $n = 1, 2$ and 3 by Figures 2.5 and 2.6. Therefore assume the result holds for $n < k$. We prove it for $n = k > 3$. Let $M = M(A, B)$ be an $n \times n$ matrix. By induction we may assume no entries are equal to 2: For if M is even (respectively, odd) and $M_{ij} = 2$ we have a path of length $(n - i) + (n - j) + \text{diam}(n - 1) \leqslant 2n - 2 + \text{diam}(n - 1)$ (respectively, $2n - 2 + \text{diam}_{\text{odd}}(n - 1)$) joining A and B due to Lemma 2.45. Our proof consists of two parts. We will first prove the claim about $\text{diam}_{\text{odd}}(n)$ and then the claim about $\text{diam}(n)$.

Proof that $\mathbf{diam_{odd}}(n) \leqslant n^2 + \lfloor \frac{2n-1}{3} \rfloor$ for $n \times n$ matrices

Let M be odd. Recall Definition 3.9. We have four cases:

Case I. There is more than one AB-set.

Case II. The odd column of M is not the leftmost column.

Case III. The odd column of M is the leftmost column and $M_{21} = 0$.

Case IV. The odd column of M is the leftmost column and $M_{21} = 1$.

Case I. There is more than one AB-set.

By Lemma 3.18 let S be the unique AB-sequence $(A(0), ..., B(0))$ of length $2p + 1$. Clearly $1 \leqslant p \leqslant n - 1$. Let $n = p + q$. We have two subcases:

Case I(i): We have $\text{mix}(A_S, A) \geqslant \text{mix}(B_S, B)$. Using $B(0) \in S$ and Lemma 3.36,

$$\text{dist}(A, B) \leqslant \text{mix}(B_S, B) - \text{mix}(A_S, A) + \text{diam}_{\text{odd}}(p) + \text{diam}(q) + 2pq$$

This gives us the required result in all nine cases. (There are three options for n mod 3 and three for p mod 3.)

Case I(*ii*): We have $\text{mix}(A_S, A) < \text{mix}(B_S, B)$. Using $A(0) \in S$ and Lemma 3.36,

$$\text{dist}(A, B) \leqslant \text{mix}(A_S, A) - \text{mix}(B_S, B) + \text{diam}_{\text{odd}}(p) + \text{diam}(q) + 2pq$$

This completes Case I.

An example is illustrated in Fig.3.7. (A_S and B_S are shaded.)

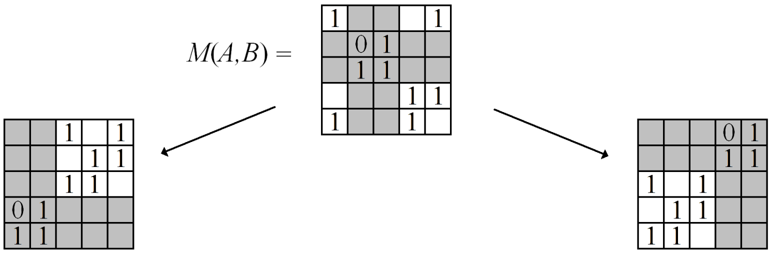


Fig. 3.7. Case I(i) (left) and I(ii) (right)

During cases II, III and IV we assume M has only one AB-set. In particular, by Lemma 3.18, we assume any AB-sequence is a subsequence of the only two maximal AB-sequences $(A(0), ..., B(0))$ and $(B(0), ..., A(0))$, each of length $2n + 1$.

Case II. The odd column of M is not the leftmost column.

Let $B(1) = \{a, b\}$ and A have parts $\{a, c\}$ and $\{b, d\}$. Without loss of generality, there exists a unique AB-sequence $S = (A(0), ..., d, b)$ not containing a, c or $B(0)$. Let $|S| = 2p + 1$ using Lemma 3.14 where $1 \leqslant p \leqslant n - 1$. Let $n = p + q$. We have two subcases:

Case II(*i*): We have $\text{mix}(A_S, A) \leqslant \text{mix}(B_S, B)$. Using $A(0) \in S$, $B(0) \notin S$ and Lemma 3.36,

$$\text{dist}(A, B) \leqslant \text{mix}(A_S, A) - \text{mix}(B_S, B) + \text{diam}_{\text{odd}}(p) + \text{diam}_{\text{odd}}(q) + 2pq$$

This proves the result for Case II(i).

Case II(*ii*): We have $\text{mix}(A_S, A) > \text{mix}(B_S, B)$. We know $B(0)$ cannot be a, b or d. Consider the chamber $B^{(b,B(0))} \underset{0}{\text{———}} B$. By Lemma 3.24 we have that S is an $AB^{(b,B(0))}$-sequence and $\text{mix}({B^{(b,B(0))}}_S, B^{(b,B(0))}) = \text{mix}(B_S, B)$. Using $A(0) \in S$, $B^{(b,B(0))}(0) \in S$ and Lemma 3.36,

$$\text{dist}(A, B^{(b,B(0))}) \leqslant \text{mix}({B^{(b,B(0))}}_S, B^{(b,B(0))}) - \text{mix}(A_S, A) + \text{diam}_{\text{odd}}(p) + \text{diam}(q) + 2pq$$

This gives us

$$\text{dist}(A, B) \leqslant 1 + \text{mix}(B_S, B) - \text{mix}(A_S, A) + \text{diam}_{\text{odd}}(p) + \text{diam}(q) + 2pq$$

This proves the theorem for Case II(ii).

An example is illustrated in Fig.3.8. (A_S and B_S are shaded.)

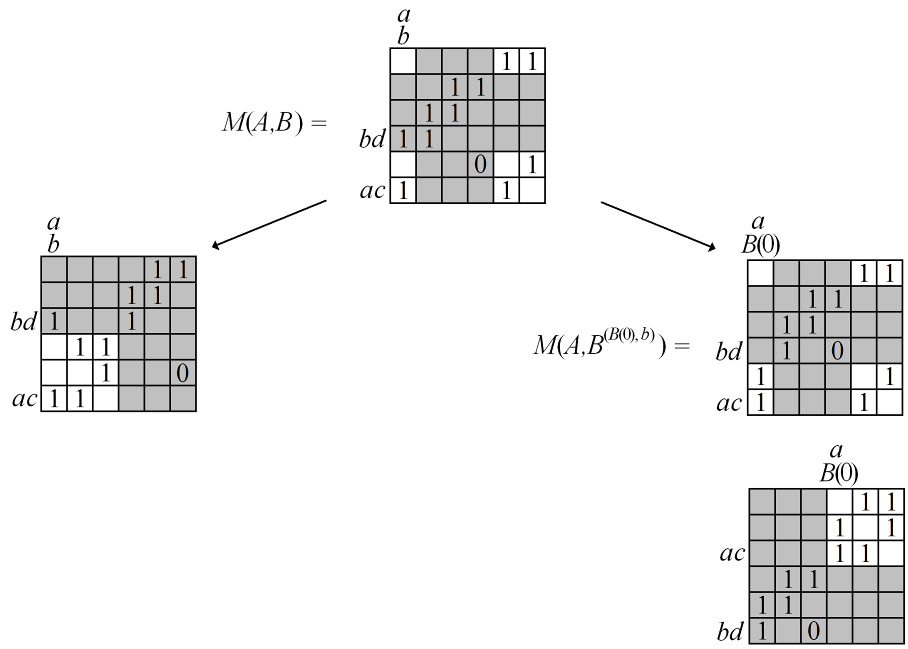


Fig. 3.8. Case II(i) (left) and II(ii) (right)

Case III. The odd column of M is the leftmost column and $M_{21} = 0$.

We may assume that the odd row is the top one, or we can apply the Case II argument to the transpose $M(B, A)$ of M. Therefore $M_{11} = 0$ to avoid more than one AB-set. Therefore let $A(1) = \{a, B(0)\}$ and $A(2) = \{b, c\}$. Let $d \in A(x)$ for some $3 \leqslant x \leqslant n$. Let $B(1) = \{d, A(0)\}$. Without loss of generality, there exists a unique AB-sequence $S = (A(0), d, ..., b, c)$ not containing a or $B(0)$. Let $|S| = 2p + 1$ by Lemma 3.14. Clearly $2 \leqslant p \leqslant n - 1$. Let $n = p + q$. We have two subcases:

Case III(i): We have $\text{mix}(A_S, A) \leqslant \text{mix}(B_S, B)$. Using $A(0) \in S$, $B(0) \notin S$ and Lemma 3.36,

$$\text{dist}(A, B) \leqslant \text{mix}(A_S, A) - \text{mix}(B_S, B) + \text{diam}_{\text{odd}}(p) + \text{diam}_{\text{odd}}(q) + 2pq$$

This proves the Theorem for Case III(i).

Case III(ii): We have $\text{mix}(A_S, A) > \text{mix}(B_S, B)$.

Consider the subsequence $T = (d, ..., b)$ of S of length $2p - 1$. By Lemma 3.34 T is an AB-sequence and $\text{mix}(A_T, A) \geqslant \text{mix}(B_T, B)$. We have $A(1) \notin A_T$ and $B(1) \notin B_T$. Consider the chambers

$$A^{(A(0),a)} \underset{0}{\text{———}} A \qquad\qquad B^{(B(0),d)} \underset{0}{\text{———}} B$$

Using Lemma 3.24 we see that T is an $A^{(A(0),a)}B^{(B(0),d)}$-sequence, where

$$\text{mix}(A^{(A(0),a)}{}_T, A^{(A(0),a)}) \geqslant \text{mix}(B^{(B(0),d)}{}_T, B^{(B(0),d)})$$

Now $A^{(A(0),a)}(1) = B^{(B(0),d)}(1) = \{A(0), B(0)\}$, which is not contained in $A^{(A(0),a)}{}_T$ or $B^{(B(0),d)}{}_T$ as T contains neither $A(0)$ or $B(0)$.

Consider the chambers A' and B':

$$A^{(A(0),a)} \underset{1,}{\text{———}} \cdots \underset{,n}{\text{———}} A' \qquad B^{(B(0),d)} \underset{1,}{\text{———}} \cdots \underset{,n}{\text{———}} B'$$

T is an $A'B'$-sequence by Lemma 3.11. By Lemmas 3.15 and 3.32 we have that $\text{mix}(A'_T, A') \geqslant \text{mix}(B'_T, B')$. We have $A'(n) = B'(n) = \{A(0), B(0)\}$.

Define the $(n-1)$-chambers C and D by $C(i) = A'(i)$ and $D(i) = B'(i)$ for $i \leqslant n-1$. By Lemma 3.33 $\text{dist}(A', B') = \text{dist}(C, D)$ and T is a CD-sequence with $\text{mix}(C_T, C) \geqslant \text{mix}(D_T, D)$.

We have $C(0) = A^{(A(0),a)}(0) = a \notin T$ and $D(0) = B^{(B(0),d)}(0) = d \in T$ so by Lemma 3.36,

$$\text{dist}(C, D) \leqslant \text{mix}(D_T, D) - \text{mix}(C_T, C) + \text{diam}_{\text{odd}}(p-1) + \text{diam}_{\text{odd}}(q) + 2(p-1)q$$

It can be checked that $\text{diam}(p) = \text{diam}_{\text{odd}}(p-1) + 2p$. Hence

$$\text{dist}(A', B') = \text{dist}(C, D) \leqslant \text{diam}(p) + \text{diam}_{\text{odd}}(q) + 2pq - 2p - 2q$$

Using $n = p + q$ and our paths of length n from A to A' and B to B', this proves the Theorem for Case III(ii).

An example is illustrated in Fig.3.9. (A_S, B_S, C_T and D_T are shaded.)

Case IV. The odd column of M is the leftmost column and $M_{21} = 1$.

We may assume the top row is odd and $M_{12} = 1$, or we simply apply the Case II or III argument to the transpose $M(B, A)$. Let $M_{22} = 0$ to avoid more than one AB-set. Therefore let $A = a|bc|de|\cdots$ and $B = c|ad|bf|\cdots$. We know there exists a chamber $C = e|ad|bc|...$ distance 2 from A and a chamber $D = f|ad|bc|...$ distance 3 from B. Hence, using $C(0) \neq D(0)$ and Lemma 2.45, $\text{dist}(A, B) \leqslant 5 + 4(n-2) + \text{diam}_{\text{odd}}(n-2)$.

This proves the result for Cases I,II,III and IV.

An example is illustrated in Fig.3.10.

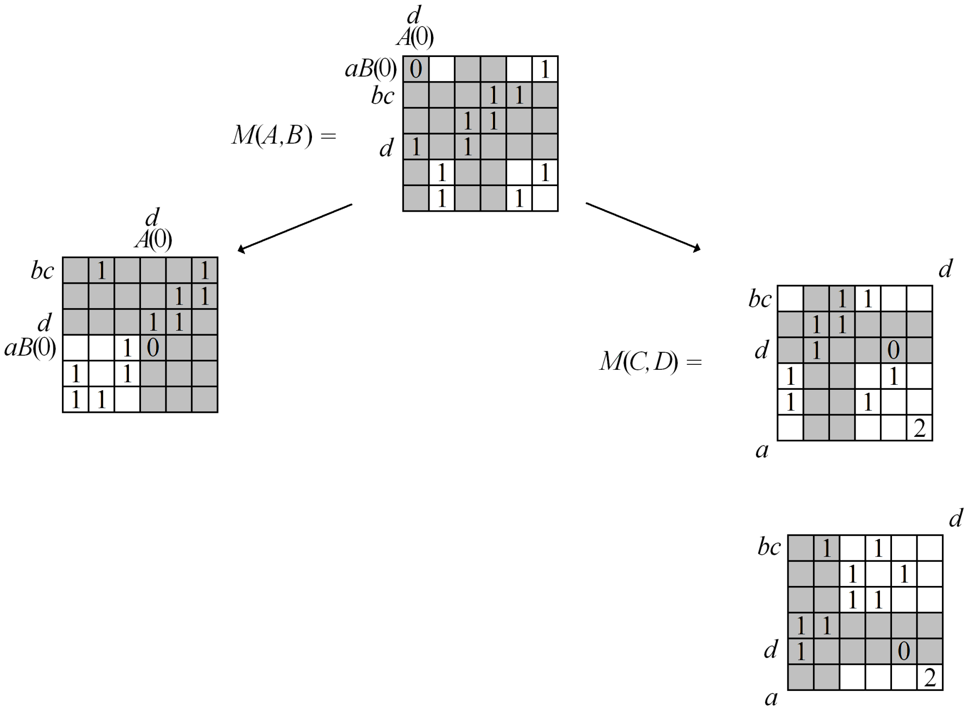


Fig. 3.9. Case III(i) (left) and III(ii) (right)

Fig. 3.10. Case IV

Proof that diam$(n) \leqslant n^2 + \lfloor \frac{2n}{3} \rfloor$ for $n \times n$ matrices

We may assume M is even. We know there are two AB-sets or more as $\{A(0)\} = \{B(0)\}$ is an AB-set. We have three cases:

Case I. There are more than two AB-sets.

Case II. We have $M_{11} = 1$.

Case III. We have $M_{11} = 0$.

Case I. There are more than two AB-sets.

By Lemma 3.18 choose a maximal AB-sequence S of order $2p$ where $1 \leqslant p \leqslant n-1$. Let $n = p + q$. Without loss of generality let $S = (1, 2, 3, ..., 2p)$. We have $A(0) \notin S$ and $B(0) \notin S$. We have $|A_S| = |B_S| = p$. Without loss of generality let $\text{mix}(A_S, A) \leqslant \text{mix}(B_S, B)$. Let C be the closest A_S-first chamber to A and D the closest B_S-last chamber to B.

$$\mathrm{dist}(A, C) = \mathrm{mix}(A_S, A) \qquad \mathrm{dist}(B, D) = pq - \mathrm{mix}(B_S, B)$$

Consider $M(C, D)$. Without loss of generality let the top p rows represent parts

$$\{C(i) : 1 \leqslant i \leqslant p\} = \{\{1, 2\}, \{3, 4\}, ..., \{2p - 1, 2p\}\}$$

and let the rightmost p columns represent parts

$$\{D(i) : q < i \leqslant n\} = \{\{2, 3\}, \{4, 5\}, ..., \{2p, 1\}\}$$

Clearly the intersection of these rows and columns sums to $\sum_{i\leqslant p,j>q} M(C, D)_{ij} = 2p$. Therefore $\sum_{i>p,j>q} M(C, D)_{ij} = 0$. Hence by Lemma 3.27 $M(C, D)$ is split into a $q \times q$ matrix and a $p \times p$ matrix. By Lemma 3.30 its distance cannot exceed $\mathrm{diam}(p) + \mathrm{diam}(q) + pq$. Therefore,

$$\mathrm{dist}(A, B) \leqslant \mathrm{mix}(A_S, A) - \mathrm{mix}(B_S, B) + \mathrm{diam}(p) + \mathrm{diam}(q) + 2pq$$

This gives the required result. During Cases II and III we may assume there are only two AB-sets.

Case II. We have $M_{11} = 1$.

Let $A = a|bc|\cdots$ and $B = a|bd|\cdots$. Let $A \underset{0}{\text{---}} C = c|ab|\cdots$ and $B \underset{0}{\text{---}} D = d|ab|\cdots$. Note $C(0) \neq D(0)$. By applying Theorem 2.35(iii) and Lemma 2.45 we know there is a path of length less than or equal to $2 + 2(n - 1) + \mathrm{diam_{odd}}(n - 1)$ joining A and B, giving us our result.

Case III. We have $M_{11} = 0$.

Let $A(1) = \{a, b\}$, $B(1) = \{c, d\}$ and $A(0) = B(0) = e$. Without loss of generality, there exists a unique AB-sequence $S = (a, b, ..., c, d)$. Let $|S| = 2p + 1$ where $2 \leqslant p \leqslant n - 1$ using Lemma 3.14. Let $n = p + q$.

Without loss of generality let $\mathrm{mix}(A_S, A) \geqslant \mathrm{mix}(B_S, B)$. Consider $A^{(a,e)} \underset{0}{\text{---}} A$. By Lemma 3.23 the sequence $S^{(a,e)} = (e, b, ..., c, d)$ is an $A^{(a,e)}B$-sequence where $\mathrm{mix}({A^{(a,e)}}_{S^{(a,e)}}, A^{(a,e)}) \geqslant \mathrm{mix}(B_{S^{(a,e)}}, B)$. We have $A^{(a,e)}(0) \notin S^{(a,e)}$ and $B(0) \in S^{(a,e)}$ so by Lemma 3.36,

$$\mathrm{dist}(A^{(a,e)}, B) \leqslant \mathrm{mix}(B_{S^{(a,e)}}, B) - \mathrm{mix}({A^{(a,e)}}_{S^{(a,e)}}, A^{(a,e)}) + \mathrm{diam_{odd}}(p) + \mathrm{diam_{odd}}(q) + 2pq$$

Therefore,

$$\mathrm{dist}(A, B) \leqslant 1 + \mathrm{diam}_{\mathrm{odd}}(p) + \mathrm{diam}_{\mathrm{odd}}(q) + 2pq$$

This gives the required result in every case except when $n \equiv 1 \bmod 3$ and $p \equiv q \equiv 2 \bmod 3$. Therefore assume that this is the case. Consider the subsequence $T = (b, ..., c)$ of S which has length $2p - 1$ and does not contain a, d or e. T is an AB-sequence. By Lemma 3.35 we know that $\mathrm{mix}(A_T, A) \geqslant \mathrm{mix}(B_T, B)$.

Consider $B^{(c,e)} \underset{0}{\text{——}} B$. By Lemma 3.24 T is an $AB^{(c,e)}$-sequence and $\mathrm{mix}(A_T, A) \geqslant \mathrm{mix}({B^{(c,e)}}_T, B^{(c,e)})$. We have $A(0) \notin T$ and $B^{(c,e)}(0) \in T$ so by Lemma 3.36,

$$\begin{aligned}\mathrm{dist}(A, B^{(c,e)}) \leqslant\ & \mathrm{mix}({B^{(c,e)}}_T, B^{(c,e)}) - \mathrm{mix}(A_T, A) + \\ & \mathrm{diam}_{\mathrm{odd}}(p-1) + \mathrm{diam}_{\mathrm{odd}}(q+1) + 2(p-1)(q+1)\end{aligned}$$

Therefore,

$$\mathrm{dist}(A, B) \leqslant 1 + \mathrm{diam}_{\mathrm{odd}}(p-1) + \mathrm{diam}_{\mathrm{odd}}(q+1) + 2(p-1)(q+1)$$

where $p - 1 \equiv 1 \bmod 3$ and $q + 1 \equiv 0 \bmod 3$. This completes the proof for Case IV. An example is illustrated in Fig 3.11 (using S on the top, T on the bottom). $\square$

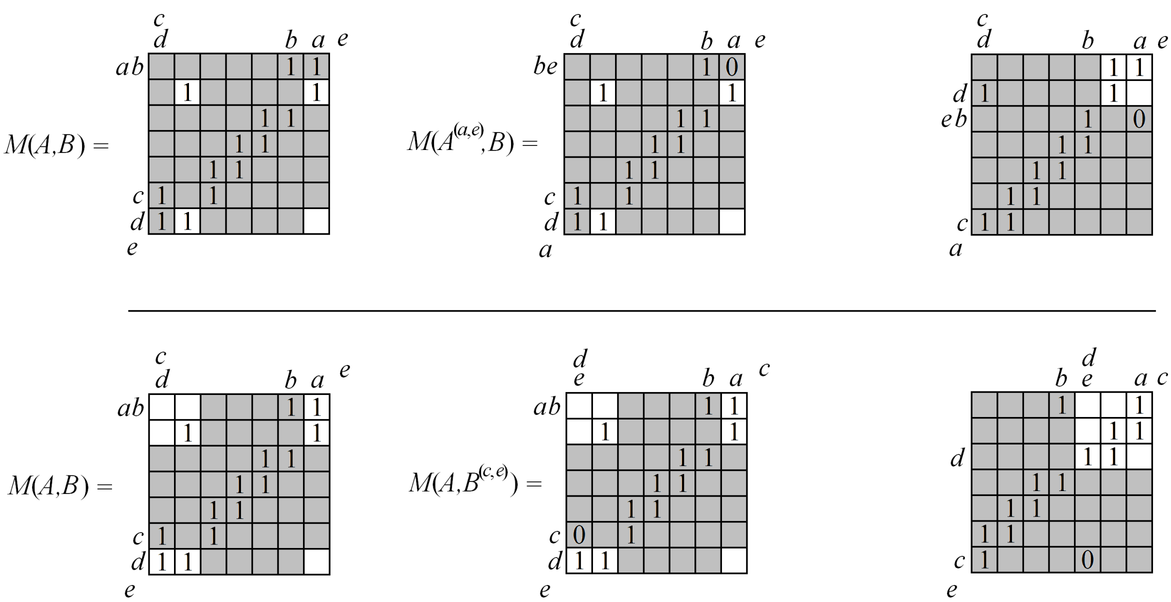


Fig. 3.11. Case III

The following theorem immediately arises from Theorems 3.2 and 3.37.

Theorem 3.38. *Let* $n \geqslant 2$*.*

$$diam(n) = \begin{cases} n^2 + \frac{2n}{3} & \text{if } n \equiv 0 \bmod 3 \\ n^2 + \frac{2n}{3} - \frac{2}{3} & \text{if } n \equiv 1 \bmod 3 \\ n^2 + \frac{2n}{3} - \frac{1}{3} & \text{if } n \equiv 2 \bmod 3 \end{cases}$$

$$diam_{odd}(n) = \begin{cases} n^2 + \frac{2n}{3} - 1 & \text{if } n \equiv 0 \bmod 3 \\ n^2 + \frac{2n}{3} - \frac{2}{3} & \text{if } n \equiv 1 \bmod 3 \\ n^2 + \frac{2n}{3} - \frac{1}{3} & \text{if } n \equiv 2 \bmod 3 \end{cases}$$

In other words, $\mathrm{diam}(n) = n^2 + \lfloor \frac{2n}{3} \rfloor$ and $\mathrm{diam}_{\mathrm{odd}}(n) = n^2 + \lfloor \frac{2n-1}{3} \rfloor$. Notice our claims differ only when $n \equiv 0 \bmod 3$.

3.5 The automorphism group of the chamber graph

Lemma 3.39 applies to geometries in general and is used in other chapters.

Lemma 3.39. *Suppose* Γ *is a flag-transitive geometry of rank* n *whose types are 0, 1, 2, ...,* $n-1$*. Suppose each element of type* i *is a* $(i+1)$*-set, and two elements of different type are incident if one is a subset of the other. If the chamber graph has valency* $n+1$*, then:*

(1) *Any automorphism of the chamber graph preserves adjacency type.*

Consider the subgraph of the chamber graph consisting of all chambers with a particular element of type 0. If this subgraph is connected, then:

(2) *Any nontrivial automorphism of the chamber graph acts as a nontrivial permutation on the points (elements of type 0).*

Proof. First prove (1). Let C be a chamber and denote its element of type i by $\{C_k : 1 \leqslant k \leqslant i+1\}$. C is of the form:

$$\{C_1\} \subset \{C_1, C_2\} \subset \{C_1, C_2, C_3\} \subset \cdots \subset \{C_1, ..., C_{n-1}\} \subset \{C_1, ..., C_n\}$$

Therefore C has exactly one i-adjacent neighbour if $i \neq n-1$ and must have two $(n-1)$-adjacent neighbours. Thus $(n-1)$-adjacencies form cliques of size 3. Observe that no subgraph of the chamber graph is a clique of size 3 unless its chambers are $(n-1)$-adjacent. Thus $(n-1)$-adjacencies are preserved.

Let $i \neq j$. Let A be i-adjacent to B, and B be j-adjacent to C. This is a minimal path of length two. The following are true:

• If $i \neq j-1$ and $i \neq j+1$ then there exists another minimal path between A and C, for if A is j-adjacent to B' it follows that B' is i-adjacent to C.

• If $i = j-1$ or $i = j+1$ then there is only one minimal path joining A and C.

Hence, starting from $(n-1)$, we see by induction that i-adjacencies are preserved for $0 \leqslant i \leqslant n-1$.

To prove (2), suppose two chambers A and B have $A_i = B_j$. Then there is a chamber C distance $i-1$ from A with $C_1 = A_i$. Similarly there is a chamber D distance $j-1$ from B with $D_1 = B_j$. There exists some (not necessarily minimal) path between C and D such that every chamber E on this path has $E_1 = A_i = B_j$. Any automorphism g takes this path to another path of the same type. Hence, $A_i^g = B_j^g$. □

The other lemmas in this section apply only to our geometries $\Gamma(A_{2n+1})$.

Lemma 3.40. *Let g be an automorphism of the chamber graph of $\Gamma(A_{2n+1})$. Then $M(A,B)_{ij} = 2$ if and only if $M(A^g, B^g)_{ij} = 2$.*

Proof. This follows from Definition 2.4 and Lemma 3.39 (2). □

Lemma 3.41. *If $M(A,B)$ is odd then there is a path P:*

$$B \underset{i,}{\text{———}} \cdots \underset{,1}{\text{———}} C \underset{0}{\text{———}} D$$

where $M(A,D)$ contains a 2 in its leftmost column and $M(A,C)$ does not. If $M(A,B)$ is even there is no such path.

Proof. Let $M(A,B)$ be odd. Then it has a row whose entries sum to 1. Let the i^{th} entry along this row be equal to 1. Then the existence of such a path is obvious. Let $M(A,B)$ be even. Every row sums to 2. We see that it is impossible to construct the above path. □

Lemma 3.42. *Let g be an automorphism of the chamber graph of $\Gamma(A_{2n+1})$. Then $M(A,B)$ is odd if and only if $M(A^g,B^g)$ is odd.*

Proof. Observe by Lemma 3.39 (1) that g takes the path described in Lemma 3.41 to another path of the same type. The lemma follows easily from Lemmas 3.40 and 3.41. □

Theorem 3.43. *The automorphism group* G *of the chamber graph of* $\Gamma(A_{2n+1})$ *is* S_{2n+1}.

Proof. We know that G contains a subgroup $S \cong S_{2n+1}$. For a contradiction suppose $S < G$. By transitivity of S, there exists $g \in G \backslash S$ fixing some chamber C. Recall Theorem 2.3. We will inductively prove that g acts as a permutation on each of the following sets of chambers

$$T_1, T_2, T_3, ..., T_x, ..., T_{\frac{(2n+1)!}{2^n}}$$

for $1 \leqslant x \leqslant \frac{(2n+1)!}{2^n}$ where $T_1 = \{C\}$, T_x is connected and has order x and $T_x \subset T_{x+1}$. Clearly g acts as a permutation on T_1. Assume g acts as a permutation $h \in S$ on T_x. We may assume g fixes every chamber in T_x (or else consider $gh^{-1} \notin S$). Consider a chamber $B \in T_x$ adjacent to some chamber $B' \notin T_x$.

If $B \underset{i,i+1}{\text{———}} B'$ then g fixes B' by Lemma 3.39 and we can define $T_{x+1} = T_x \cup \{B'\}$. Therefore let $B(1) = \{1, 2\}, B(0) = 3$ and $B' = B^{(1,3)}$. The three chambers B, B' and $B^{(2,3)}$ form a clique. If g fixes B' then we define $T_{x+1} = T_x \cup \{B'\}$. Therefore assume g swaps $B^{(1,3)}$ and $B^{(2,3)}$. Using that T_x is connected we have only two cases:
Case I : There exists a chamber $A \in T_x$ with $A(0) = 1$ or 2.
Case II: Every chamber in T_x has a part equal to $\{1, 2\}$.

Case I : Without loss of generality let $A(0) = 2$. Notice that $M(A, B^{(1,3)})$ is odd and $M(A, B^{(2,3)})$ is even. By Lemma 3.42, $B^{(1,3)}$ and $B^{(2,3)}$ cannot be swapped by g.
Case II: Let $T_{x+1} = T_x \cup \{B'\}$. Then g acts as the permutation $(1, 2)$ on T_{x+1}.

Hence by induction we see g acts as a permutation on $T_{\frac{(2n+1)!}{2^n}}$ as required. □

Chapter 4

The Petersen geometry $\Gamma(L_2(11))$

Consider the amalgam of subgroups of $G = L_2(11)$ shown in Fig.4.1. There are 330

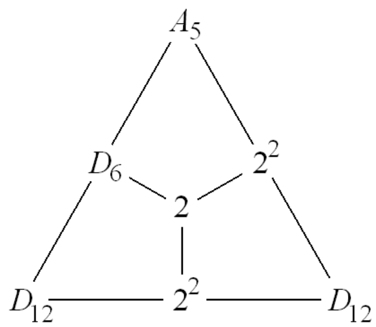


Fig. 4.1

chambers (flags of type $\{0,1,2\}$) and G is transitive on these. There are 110 flags of type $\{0,1\}$, 165 flags of type $\{0,2\}$ and 165 flags of type $\{1,2\}$. The group $L_2(11)$ has exactly one conjugacy class of subgroups isomorphic to D_{12} and two conjugacy classes of subgroups isomorphic to A_5.

4.1 The group $G = L_2(11)$ acting on 11 elements

Ivanov and Shpectorov give a definition of this geometry [13] (page 942). Let $L_2(11)$ act faithfully on the 11-set $\Omega = \{1, 2, ..., 11\}$. Let the elements of type 0 be the eleven 1-subsets of Ω. Let the elements of type 1 be the fifty-five 2-subsets of Ω. The involutions of $L_2(11)$ form a single conjugacy class of length 55 with cycle type 2^4. Let the elements of type 2 be the fifty-five 3-subsets of Ω fixed by such an involution. Two elements of different type are adjacent if one is contained in the other.

4.1.1 The chamber graph

We can see that the elements of type 2 form a $2-(11,3,3)$-design whose automorphism group in S_{11} is isomorphic to $L_2(11)$ [18] (page 8). It is straightforward for Magma to compute the chamber graph. We show the disc sizes in Table 7.1. The chamber graph has valency 4 and diameter 9. Let C be the chamber $\{a\} \subset \{a,b\} \subset \{a,b,c\}$. Then there are thirty-two chambers $\{x\} \subset \{x,y\} \subset \{x,y,z\}$ distance 9 from C: Thirty with $\{a,b,c\} \cap \{x,y,z\} = \varnothing$ and two with $\{a,b,c\} \cap \{x,y,z\} = \{c\} = \{z\}$. The stabilizer $\mathrm{Stab}_G(C)$ has sixteen orbits on these thirty-two chambers, each of length two.

Theorem 4.1. *The automorphism group of the chamber graph of* $\Gamma(L_2(11))$ *is* $L_2(11)$.

Proof. Each chamber has four neighbours. It can be seen by Magma that the subgraph of chambers with a particular element of type 0 is connected. By Lemma 3.39 we see that any nontrivial automorphism of the chamber graph of $\Gamma(L_2(11))$ is a nontrivial permutation on the eleven points. Khosrovshahi and Tayfeh-Rezaie [18] (page 8) show that the largest possible automorphism group of a $2-(11,3,3)$-design has order 660. □

4.2 The group $G = L_2(11)$ acting on 12 elements

Let $G = L_2(11) = \{\{+M, -M\} : M \in SL_2(11)\}$. Let $\Omega = \mathbb{F}_{11} \backslash \{0\}$. Then we denote $\Omega^2 = \{1,3,4,5,9\}$ and $\Omega^5 = \{1,10\}$. The group G acts on the projective space $\underline{V} = \{0,1,2,3,4,5,6,7,8,9,10,\infty\}$ where

$$\begin{aligned} \lambda = &\quad \{\text{elements of the form } \tbinom{\lambda i}{i}\} \quad \text{for all } \lambda \in \mathbb{F}_{11}, \text{ and} \\ \infty = &\quad \{\text{elements of the form } \tbinom{i}{0}\}. \end{aligned}$$

We define the following permutations (acting on the right) of $\underline{V}$ (using $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$):

$$\tau : x \to -\frac{1}{x} \qquad \times m : x \to mx \qquad + c : x \to x + c$$

where $m, c \in F$ and $m \neq 0$. We will mostly consider G as the group of permutations

$$G := \langle\, \tau\ ,\ \times 4\ ,\ +1\, \rangle$$

We are allowed to do this by a paper of Conway [6]. We use the following notation for the four groups

$$M = \langle \times 2 \rangle \qquad M_2 = \langle \times 4 \rangle \qquad M_5 = \{\times 1, \times 10\} \qquad C = \langle +1 \rangle$$

of order 10, 5, 2 and 11 respectively. The order of the group MC is 110. We define

$$Y := \langle\ MC\ ,\ \tau\ \rangle$$

of order 1320. We have that G is a normal subgroup of Y of index 2. In fact, G is all even permutations of Y. Thus M_2 is a subgroup of G, whereas M_5 is not.

4.3 Elements of type 0

There are two conjugacy classes of subgroups isomorphic to A_5 in $L_2(11)$, each of length 11. Consider the eleven permutations of the form $\times 2 + c$. Each one fixes ∞ and $-c$ and acts as a 10-cycle on the other elements. For each permutation $\times 2 + c$ we construct two block systems. We do this by defining blocks of the form $\{-c, \infty\}$ and $\{\lambda, 2\lambda + c\}$. These block systems are:

$\{0, c\}$	$\{1, 2\}$	$\{c, 3c\}$	$\{2, 4\}$
$\{3c, 7c\}$	$\{4, 8\}$	$\{7c, 4c\}$	$\{8, 5\}$
$\{4c, 9c\}$	$\{5, 10\}$	$\{9c, 8c\}$	$\{10, 9\}$
$\{8c, 6c\}$	$\{9, 7\}$	$\{6c, 2c\}$	$\{7, 3\}$
$\{2c, 5c\}$	$\{3, 6\}$	$\{5c, 0\}$	$\{6, 1\}$
$\{-c, \infty\}$	$\{\infty, 0\}$	$\{-c, \infty\}$	$\{\infty, 0\}$
when $c \neq 0$		when $c \neq 0$	

The stabilizer of each of these twenty-two block systems is a subgroup A_5. The group M_2C acts transitively on each conjugacy class of subgroups of type 0 by conjugation. Therefore define B_0 as the block system above containing $\{0, \infty\}$ and $\{1, 2\}$. We have $\mathrm{Stab}_{MC}(B_0) = M_2$. We define G_0 as the stabilizer in G of B_0. We may label our eleven elements of type 0 as the blocks $B_c = B_0 + c$ (containing $\{c, \infty\}$).

4.4 Elements of type 1 and 2

Consider the pair $\{B_0, B_3\}$. Its stabilizer G_1 in G is isomorphic to D_{12}. Now G is 2-transitive on elements of type 0. Therefore we label elements of type 1 by pairs $\{B_i, B_j\}$.

Let G_2 be the stabilizer in G of the triple $\{B_0, B_3, B_{10}\}$.

$\{1, 2\}$	$\{4, 5\}$	$\{0, 1\}$
$\{4, 8\}$	$\{7, 0\}$	$\{3, 7\}$
$\{5, 10\}$	$\{8, 2\}$	$\{4, 9\}$
$\{9, 7\}$	$\{1, 10\}$	$\{8, 6\}$
$\{3, 6\}$	$\{6, 9\}$	$\{2, 5\}$
$\{\infty, 0\}$	$\{\infty, 3\}$	$\{\infty, 10\}$

This triple has an orbit of length 55 under G. There are $\binom{11}{3} - 55 = 110$ remaining triples which themselves form an orbit under G. One of is these is $\{B_0, B_1, B_{10}\}$.

$$B_1 = \{\ \{2, 3\}, \{5, 9\}, \{6, 0\}, \{10, 8\}, \{4, 7\}, \{\infty, 1\}\ \}$$

Notice that $\{9, 7\} \in B_0$, $\{4, 7\} \in B_1$ and $\{4, 9\} \in B_{10}$. Also $\{0, \infty\} \in B_0$, $\{\infty, 1\} \in B_1$ and $\{0, 1\} \in B_{10}$. We can find a pair from each B_i such that their union is a 3-set. This is not possible with $\{B_0, B_3, B_{10}\}$. In fact the 55 elements of $\{B_0, B_3, B_{10}\}^{M_2C}$ are all triples without this property, and the remaining 110 triples are those with the property. Label elements of type 2 by the 55 triples without this property.

There are 55 subgroups of G isomorphic to D_{12}, all conjugate in G. All G's elements of order 6 are of the form $\{+\begin{pmatrix} x & y \\ \frac{x(5-x)-1}{y} & 5-x \end{pmatrix}, -\begin{pmatrix} x & y \\ \frac{x(5-x)-1}{y} & 5-x \end{pmatrix}\}$ where $x, y \in \mathbb{F}_{11}, y \neq 0$. This element acts as the permutation

$$(0, \tfrac{y}{5-x}, \tfrac{y}{7-x}, \tfrac{y}{8-x}, \tfrac{y}{9-x}, \tfrac{y}{0-x})(\tfrac{y}{4-x}, \tfrac{y}{2-x}, \tfrac{y}{10-x}, \tfrac{y}{6-x}, \tfrac{y}{3-x}, \tfrac{y}{1-x})$$

Let a subgroup D_{12} in G contain elements g and g^{-1} of order 6. Then the subgroup is the stabilizer in G of each of the fourteen block systems shown in Fig.4.2.

4.5 Incidence of this geometry

It can be verified by Magma that G_0, G_1 and G_2 form the amalgam required. By flag-transitivity, two elements of the geometry are adjacent if and only if they are of

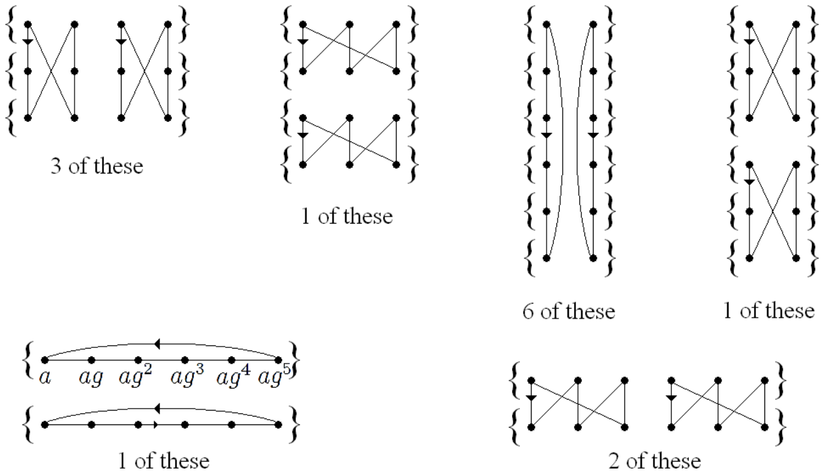


Fig. 4.2

different type and one is contained in the other.

4.5.1 A way of labelling with pairs (a, b)

The group C is transitive on the elements of type 0 and M_2C is transitive on the elements of type 1 and 2. Therefore

We may label elements of type 0 by $\{(1,c)_0 : c \in \mathbb{F}_{11}\}$

We may label elements of type 1 by $\{(m,c)_1 : m \in \Omega^2, c \in \mathbb{F}_{11}\}$

We may label elements of type 2 by $\{(m,c)_2 : m \in \Omega^2, c \in \mathbb{F}_{11}\}$

where $(m,c)_i$ represents $\times m + c$ applied to the element of type i described earlier. For example $(3,1)_1$ represents $\{B_0, B_3\} \times 3 + 1$.

If we use the above notation, then (by Magma):

$(1,b)_0$ is adjacent to $(c,d)_1$ if and only if $\frac{d-b}{c} \in \{0,8\}$.

$(1,b)_0$ is adjacent to $(c,d)_2$ if and only if $\frac{d-b}{c} \in \{0,1,8\}$.

$(a,b)_1$ is adjacent to $(c,d)_2$ if and only if $c = a$ and $d = b$, or

$c = 3a$ and $d = b + c$, or

$c = 9a$ and $d = b + c$.

4.6 Magma code

4.6.1 On 11 elements

```
/*
```

```
www-ATLAS of Group Representations.
L2(11) represented as permutations on 11 elements.
*/
G<x,y>:=PermutationGroup<11|\[
1,10,4,3,9,7,6,8,5,2,11]
,\[
2,11,5,4,10,8,7,9,6,3,1]
>;
print "Group G is L2(11) < Sym(11)";

///////////////////////////////////////////////////////////////////Define elements of type 0,1 and 2
type0:={1..11};
type1:=Subsets(type0,2);
G!(1,5)(3,8)(4,10)(7,9) in G;
x:={2,6,11};
type2:={x^g:g in G};

////////////////////////////////////////////////////////////////////////Define an original chamber
original_chamber:={ {2} , {2,6} , {2,6,11} };
chambers:={original_chamber^g:g in G};     #chambers;  /// 330

/////////////////////////////////////////////////////////////////////////////////////////Find discs
All_discs:=[];
disc0:={original_chamber};     previous_discs:=disc0;

for n in {1..9} do
    discn:={x:x in (chambers sdiff previous_discs) |#{d:d in previous_discs|#(x meet d) eq 2} ne 0
           };
    Include(~All_discs,discn);
    previous_discs:=previous_discs join discn;
end for;
```

4.6.2 On 12 elements

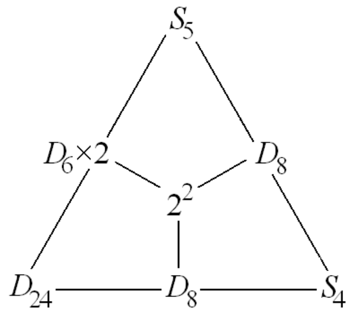
```
//////////////////////// writing infinity as 11, let S be all permutations of the projective space
S:=Sym( {0,1,2,3,4,5,6,7,8,9,10,11} );
// now we define G as a subgroup of S
add1:=   S!(1,2,3,4,5,6,7,8,9,10,0);
times2:= S!(1,2,4,8,5,10,9,7,3,6); minus:=times2^5;
inverse:= S!(0,11)(2,6)(3,4)(5,9)(7,8);

C :=sub< S | add1    >;
M :=sub< S | times2  >;     MC :=sub< S | M ,C >;
M2:=sub< S | times2^2>;     M2C:=sub< S | M2,C >;
G :=sub< S | M2,C, minus*inverse >;
Y :=sub< S | M ,C, minus*inverse >;

////// define the block systems of type 0,1 and 2
```

```
B0:={ {1,2},{4,8},{5,10},{9,7},{3,6},{0,11} };
B1:=B0^add1;    B3:=B0^(add1^3);    B10:=B0^(add1^10);

////////////////////////////////////////////////////////////////////////////////Define subgroups
G0:=sub < G | {g: g in G |  B0        ^g eq  B0         } > ;
G1:=sub < G | {g: g in G | {B0,B3}    ^g eq {B0,B3}     } > ;
G2:=sub < G | {g: g in G | {B0,B3,B10}^g eq {B0,B3,B10} } > ;

////////////////////////////////////////////////////////////////Make sure these form the amalgam
H:=CyclicGroup(2); H2:=DirectProduct(H,H);
IsIsomorphic(G0               ,Alt(5)         );
IsIsomorphic(G1               ,DihedralGroup(6));
IsIsomorphic(G2               ,DihedralGroup(6));
IsIsomorphic(G0 meet G1       ,DihedralGroup(3));
IsIsomorphic(G0 meet G2       ,H2             );
IsIsomorphic(G1 meet G2       ,H2             );
IsIsomorphic(G0 meet G1 meet G2,H             );

//////////////////////////////////////////////////////////////////Labelling elements of type 0,1,2
type0:={B0^g:g in G};                  #type0; /// 11
type1:=Subsets(type0,2);               #type1; /// 55
type2:={ {B0,B3,B10}^g : g in G };     #type2; /// 55

///////////////////////////////////////////////////////////////////////G has 2 orbits on triples
#{ {B0,B1,B10}^g : g in G };                    /// 110

////////////////////////////It only remains to back up our claim that this is a Petersen geometry
///////////////////////////////////////////////////////////////find elements of type 1 adjacent to B0
for m in M2 do for c in C do
    if {g*m*c : g in G1} meet {g: g in G0} ne {} then
        print [1^m,0^c];
    end if;
end for; end for;

///////////////////////////////////////////////////////////////find elements of type 2 adjacent to B0
for m in M2 do for c in C do
    if {g*m*c : g in G2} meet {g: g in G0} ne {} then
        print [1^m,0^c];
    end if;
end for; end for;

///////////////////////////////////////////////////////////////find elements of type 2 adjacent to B1
for m in M2 do for c in C do
    if {g*m*c : g in G2} meet {g: g in G1} ne {} then
        print [1^m,0^c];
    end if;
end for; end for;
```

Chapter 5

The Petersen geometry $\Gamma(L_2(25))$

Consider this amalgam of $L_2(25) = \{\ \{+M, -M\} : M \in SL_2(25)\}$:



This gives rise to a flag-transitive geometry consisting of 65 elements of type 0, 325 elements of type 1, 325 elements of type 2, 650 flags of type $\{0,1\}$, 975 of type $\{0,2\}$, 975 of type $\{1,2\}$, and 1950 chambers.

5.1 The Field of order 25

In this chapter we will write the field $\mathbb{F}_{25}$ as the set

$$\{a + b\sqrt{2} \quad : \quad a, b \in \mathbb{F}_5\}$$

and we define the operations $+$ and $\times$ by

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2}$$
$$(a + b\sqrt{2}) \times (c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}$$

Let $\Omega = \mathbb{F}_{25} \setminus \{0\}$. Then:

We see Ω^2 is the 12-set $\{a + ab\sqrt{2} \,:\, a \in \{1,2,3,4\}, b \in \{0,1,4\}\}$.

We see Ω^3 is the 8-set $\{a + b\sqrt{2} \,:\, a, b \in \mathbb{F}_5, ab = 0\} \setminus \{0\}$.

We see Ω^4 is the 6-set $\{\pm 1, \pm(2 + 2\sqrt{2}), \pm(2 + 3\sqrt{2})\}$.

We see Ω^6 is the 4-set $\{1, 2, 3, 4\}$.

Let $f = 1 + 2\sqrt{2}$. Powers of f are listed here:

$f^0 = 1$	$f^6 = 3$	$f^{12} = 4$	$f^{18} = 2$
$f = 1 + 2\sqrt{2}$	$f^7 = 3 + \sqrt{2}$	$f^{13} = 4 + 3\sqrt{2}$	$f^{19} = 2 + 4\sqrt{2}$
$f^2 = 4 + 4\sqrt{2}$	$f^8 = 2 + 2\sqrt{2}$	$f^{14} = 1 + \sqrt{2}$	$f^{20} = 3 + 3\sqrt{2}$
$f^3 = 2\sqrt{2}$	$f^9 = \sqrt{2}$	$f^{15} = 3\sqrt{2}$	$f^{21} = 4\sqrt{2}$
$f^4 = 3 + 2\sqrt{2}$	$f^{10} = 4 + \sqrt{2}$	$f^{16} = 2 + 3\sqrt{2}$	$f^{22} = 1 + 4\sqrt{2}$
$f^5 = 1 + 3\sqrt{2}$	$f^{11} = 3 + 4\sqrt{2}$	$f^{17} = 4 + 2\sqrt{2}$	$f^{23} = 2 + \sqrt{2}$

There are only two automorphisms of this field: The identity and the map $a+b\sqrt{2} \longrightarrow a - b\sqrt{2}$.

5.2 The Group $G = L_2(25)$

G acts on the projective space $\underline{V} = \{0, f^0, f^1, f^2, ..., f^{22}, f^{23}, \infty\}$, where

$$\begin{aligned} \lambda = \;& \{\text{elements of the form } \tbinom{\lambda i}{i}\} \quad \text{for all } \lambda \in \underline{V}\setminus\{\infty\}, \text{ and} \\ \infty = \;& \{\text{elements of the form } \tbinom{i}{0}\}. \end{aligned}$$

We define the following permutations acting on the right of $\underline{V}$ (using $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$):

$$\tau : x \to \frac{1}{x} \qquad \times m : x \to mx \qquad + c : x \to x + c$$

for $m, c \in \mathbb{F}_{25}$, $m \notin \{0, \infty\}$, $c \neq \infty$. We define the groups

$$M = \langle \times f \rangle \qquad\qquad C = \langle +1, +\sqrt{2} \rangle$$

of order 24 and 25 respectively. Note that M has six proper subgroups $\langle \times f^2 \rangle$, $\langle \times f^3 \rangle$, $\langle \times f^4 \rangle$, $\langle \times f^6 \rangle$, $\langle \times f^8 \rangle$, $\langle \times f^{12} \rangle$ and C also has six. The group MC has order $24 \times 25 = 600$. We will mostly consider G as the group of permutations

$$G := \langle\ \tau\ ,\ \times f^2\ ,\ C\ \rangle$$

G is in fact the group of all even permutations of Y, where

$$Y := \langle\ MC\ ,\ \tau\ \rangle$$

of order 15600.

5.3 Elements of type 0

There are two conjugacy classes of subgroups S_5 in G, each of length 65. Let $P = \{0, 1, 2, 3, 4, \infty\}$. We have $\mathrm{Stab}_G(P) = \langle \times f^6, +1, \tau \rangle$. This is a subgroup S_5 acting on the 5-set

$$\{\{t, \infty\}, \{t+1, t+4\}, \{t+2, t+3\}\} \ : t \in \{0, 1, 2, 3, 4\} \ \}$$

Denote the field automorphism $a + b\sqrt{2} \longrightarrow \overline{a + b\sqrt{2}} = a - b\sqrt{2}$. The set $Q = \{z/\overline{z} : z \in \Omega\}$ is a group by multiplication of order 6. Define G_0:

$$Q = \{f^0, f^4, f^8, f^{12}, f^{16}, f^{20}\} \qquad G_0 = \mathrm{Stab}_G(Q)$$

Subgroups S_5 in G have two orbits under conjugation by MC. One orbit consists of 30 subgroups stabilizing elements of P^{MC}. The other consists of 100 subgroups stabilizing elements of Q^{MC}. Under conjugation by G we end up with two orbits of length 65, one containing G_0 and the other containing $\mathrm{Stab}_G(P)$. These two partitions cut each other $15^2.50^2$. Labelling the 65 cosets of G_0 in G is made difficult by the fact that no coset takes Q to P. Therefore we will label the cosets of G_0 by

$f^iP + c$ where $i \in \{1, 3, 5\}$ and $c \in \{0, 1, 2, 3, 4\}$.
$f^iQ + c$ where $i \in \{0, 2\}$ and $c \in \mathbb{F}_{25}$.

This is how we label our elements of type 0.
(The stabilizer in G of any 6-set fixed by some element $g \in G$ of order 6 is S_5.)

5.4 Elements of type 1

Let $G_1 \cong D_{24}$ be the stabilizer in G of $\{Q, 2Q\}$. Notice the two elements of this pair are disjoint. Cosets of G_1 in G may be labelled by elements of $\{Q, 2Q\}^G$. By Magma, we see that this is in fact all 325 pairs of disjoint elements of type 0.

There are $\frac{26 \times 25}{2} = 325$ subgroups D_{24} in G. These form a single conjugacy class in G. Let $P_1 = \{0, \infty\}$ and $Q_1 = \{1, 4\}$. The stabilizer of P_1 in G acts as D_{24} on Ω, while stabilizing the set $\{0, \infty\}$. The group G is 2-transitive so each subgroup is the stabilizer in G of some pair $\{a,b\}$. The stabilizer of Q_1 consists of the twenty-four elements of the form $\pm\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$ and $\pm\left(\begin{smallmatrix} -a & b \\ -b & a \end{smallmatrix}\right)$ in G.

5.5 Elements of type 2

Let $G_2 \cong S_4$ be the stabilizer in G of $\{Q, 2Q, \sqrt{2}P\}$. Note that any two elements of this triple are disjoint. The 325 cosets of G_2 in G may be labelled by elements of $\{Q, 2Q, \sqrt{2}P\}^G$. By Magma, we see that this is in fact all 325 triples of type 0 such that any of its two elements are disjoint.

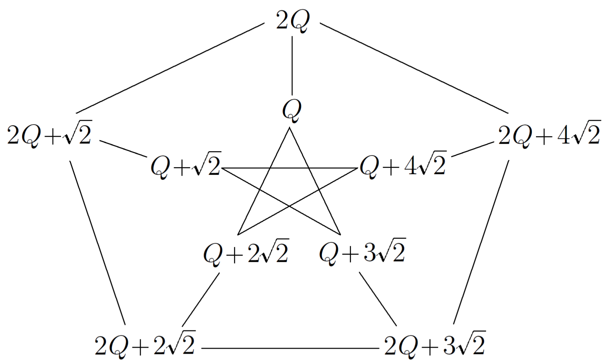
There is a single conjugacy class of subgroups S_4 in G. One subgroup is the stabilizer in G of $\{\{0,\infty\},\{1,4\},\{2,3\}\}$. Recall that this is one of the five elements of the set mentioned in Section 5.3. This subgroup consists of all maps $x \longmapsto \lambda x$, all maps $x \longmapsto \frac{\lambda}{x}$ and all maps $x \longmapsto \lambda\frac{x-c}{x+c}$ where $\lambda, c \in \{1,2,3,4\}$.

Let $a,b,c,d,e,f,x,y \in \underline{V}$. Interestingly, it turns out that a subgroup $D_{24} \leqslant G$ stabilizing $\{x,y\}$ intersects a subgroup $S_4 \leqslant G$ stabilizing $\{\{a,b\},\{c,d\},\{e,f\}\}$ by D_8 (as required in our amalgam) if and only if $\{x,y\} \in \{\{a,b\},\{c,d\},\{e,f\}\}$.

5.6 A way of describing this geometry

One amalgam satisfying our requirements is $\{G_0, G_1, G_2\}$. The group G is flag-transitive. We have already labelled our elements of type 0 by elements of Q^G. Label the 325 elements of type 1 by the disjoint pairs of elements of type 0 and label the 325 elements of type 2 by the disjoint triples of elements of type 0. Two elements of different type are incident if one is contained in the other.

We wish to show that the graph whose vertices are elements of type 0 and whose adjacencies are elements of type 1 is locally Petersen. To do this, observe that $\{Q, 2Q\}$, $\{Q, Q+2\sqrt{2}\}$, $\{\sqrt{2}P, Q\}$ and $\{\sqrt{2}P, Q+\sqrt{2}\}$ are elements of type 1. The neighbours of $\sqrt{2}P$ are:



Let i be odd and j be even. Elements of type 1 are all of the form

$\{f^iP+a, f^jQ+b\}$ where $a-b \in f^iP$	150 of these
$\{f^jQ+a, f^jQ+b\}$ where $a-b \in f^{j+3}Q$	150 of these
$\{\ \ Q+a,\ \ 2Q+b\}$ where $a=b$	25 of these

5.7 Chambers

Clearly we have six chambers for each element of type 2. From now on we will write the chamber

$$\{Q\} \subset \{Q, 2Q\} \subset \{Q, 2Q, \sqrt{2}P\}$$

using the notation $\sqrt{2}P|2Q|Q$. Its neighbours in the chamber graph are $\sqrt{2}P|Q|2Q$ (0-adjacent), $2Q|\sqrt{2}P|Q$ (1-adjacent) and $fP|2Q|Q$ and $f^5P|2Q|Q$ (both 2-adjacent).

Definition 5.1. *Let C_t be C's element of type t. Let $C(3)$ be the only element of C_0, let $C(2)$ be the element of C_1 which is not $C(3)$, and let $C(1)$ be the element of C_2 which is not $C(2)$ or $C(3)$. The intersection matrix $M(C,D)$ between chambers C and D is defined by $M(C,D)_{ij} = |C(i) \cap D(j)|$.*

We know our chamber graph has valency 4. We have just shown that if two elements are distance 2 apart then they intersect by one of:

$$\boxed{\begin{matrix} 6 & & \\ & & 6 \\ & 6 & \end{matrix}} \qquad \boxed{\begin{matrix} & 6 & \\ 6 & & \\ & & 6 \end{matrix}} \qquad \boxed{\begin{matrix} 2 & & \\ & 6 & \\ & & 6 \end{matrix}}$$

Using Magma it can be checked that if $M(A,B) = M(C,D)$ then $\mathrm{dist}(A,B) = \mathrm{dist}(C,D)$ but I have been unable to give a computer-free proof. The disc sizes are shown in Table 7.1. Magma also shows an element of type 0 can only intersect an element of type 2 by $(1,1,1)$, $(0,2,2)$, $(0,0,6)$, $(1,2,2)$, $(0,0,2)$ or $(1,1,2)$. It turns out the distance between C and D is 18 if and only if $M(C, D)$ is the matrix shown on the far right of Fig.5.1.

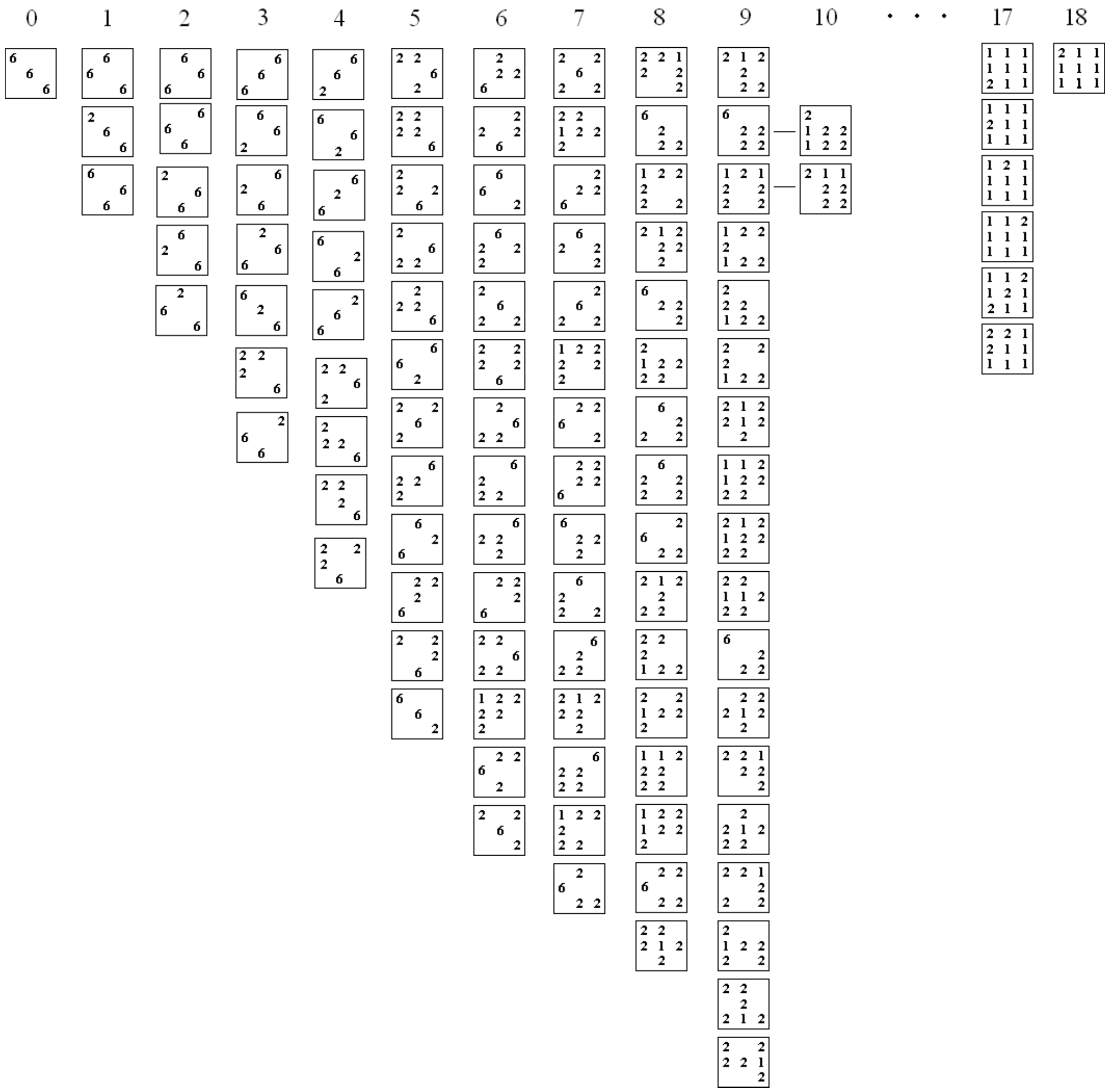


Fig. 5.1

5.7.1 The automorphism group

It has been shown [9, 10] that the automorphism group of the locally Petersen graph whose vertices are elements of type 0 and lines are elements of type 1 is $P\Sigma L(2, 25)$, the group of order 15600 generated by $L_2(25)$ along with the field automorphism $a + b\sqrt{2} \longrightarrow a - b\sqrt{2}$.

Theorem 5.2. *The automorphism group of the chamber graph of* $\Gamma(L_2(25))$ *is* $P\Sigma L(2, 25)$.

Proof. Each chamber has four neighbours. Because the Hall graph is locally Petersen, we know that the subgraph of chambers with a particular element of type 0 is connected. By Lemma 3.39 the automorphism group of the chamber graph of $\Gamma(L_2(25))$ is a permutation group on the points. Elements of type 1 and 2 are defined by their incident elements of type 0. □

Recall that we are allowed to assign each intersection matrix $M(C, D)$ the distance $\mathrm{dist}(C, D)$. Let C be a chamber and M an intersection matrix such that $\mathrm{dist}(M) \notin \{11, 12, 13, 14, 15\}$. It can be shown by Magma that the set of chambers $\{D : M(C, D) = M\}$ is an orbit under $\mathrm{Stab}_{P\Sigma L(2,25)}(C)$. In total, there are 279 intersection matrices and 288 such orbits. The difference between matrices and orbits in discs 11, 12, 13, 14, 15 is 1, 2, 3, 2, 1 respectively.

5.8 Magma code

We denote $a + b\sqrt{2}$ by `[a,b]` and ∞ by `[5,5]`.

```
F:={[a,b]:a in {0..4},b in {0..4}}; F:=F join {[5,5]}; S:=SymmetricGroup(F);
add1:=S!
([0,0],[1,0],[2,0],[3,0],[4,0])
([0,1],[1,1],[2,1],[3,1],[4,1])
([0,2],[1,2],[2,2],[3,2],[4,2])
([0,3],[1,3],[2,3],[3,3],[4,3])
([0,4],[1,4],[2,4],[3,4],[4,4]);
addroot2:=S!
([0,0],[0,1],[0,2],[0,3],[0,4])
([1,0],[1,1],[1,2],[1,3],[1,4])
([2,0],[2,1],[2,2],[2,3],[2,4])
([3,0],[3,1],[3,2],[3,3],[3,4])
([4,0],[4,1],[4,2],[4,3],[4,4]);

timesf:=S!([1,0],[1,2],[4,4],[0,2],[3,2],[1,3],[3,0],[3,1],[2,2],[0,1],[4,1],[3,4],
           [4,0],[4,3],[1,1],[0,3],[2,3],[4,2],[2,0],[2,4],[3,3],[0,4],[1,4],[2,1]);

inverse:=S!([1,3],[2,4])([3,3],[3,2])([3,1],[4,2])([0,2],[0,4])([2,3],[2,2])([3,4],[4,3])
           ([1,2],[2,1])([2,0],[3,0])([1,4],[4,4])([4,1],[1,1])([0,3],[0,1])([0,0],[5,5]);

outer:=S!([1,1],[1,4])([1,2],[1,3])
         ([2,1],[2,4])([2,2],[2,3])
         ([3,1],[3,4])([3,2],[3,3])
         ([4,1],[4,4])([4,2],[4,3])
         ([0,1],[0,4])([0,2],[0,3]);
```

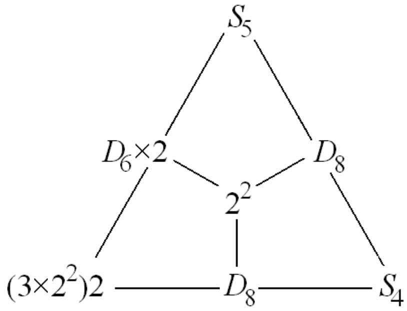
```
G:=sub<S|add1,timesf^2,inverse>;    Y:=sub<S|G,outer>;
M :=sub<S|timesf>;
C :=sub<S|add1,addroot2>;    MC :=sub<S|M ,C>;
M2:=sub<G|timesf^2 >;        M2C:=sub<S|M2,C>;

//////////////////////////////////////////////////////////////////////////////////////P and Q
P:={[0,0],[1,0],[2,0],[3,0],[4,0],[5,5]};
Q:={[1,0],[3,3],[2,3],[3,2],[4,0],[2,2]};

/////////////////////////////////////////////////////////////////////////////Define an amalgam
c0:={Q^(timesf^2)               };   G0:=sub<G|{g:g in G|c0^g eq c0}>;
c1:={Q^(timesf^2),Q             };   G1:=sub<G|{g:g in G|c1^g eq c1}>;
c2:={Q^(timesf^2),Q,P^(timesf^3)};   G2:=sub<G|{g:g in G|c2^g eq c2}>;

/////////////////////////////////////////////////////////////////////Make sure it is an amalgam
H:=CyclicGroup(2);
IsIsomorphic(G0, Sym(5)            );
IsIsomorphic(G1, DihedralGroup(12) );
IsIsomorphic(G2, Sym(4)            );
IsIsomorphic(G0 meet G1, DirectProduct(DihedralGroup(3),H));
IsIsomorphic(G0 meet G2, DihedralGroup(4)                 );
IsIsomorphic(G1 meet G2, DihedralGroup(4)                 );
IsIsomorphic(G0 meet G1 meet G2, DirectProduct(H,H)       );

//////////////////////////////Work out disjoint pairs and disjoint triples of elements of type 0
type0:={Q^g:g in G};
#type0;             ///// There are 65 elements of type 0

disjointpairs:={{x,y}:x in type0,y in type0|#(x meet y) eq 0};
#disjointpairs;     ///// There are 325 disjoint pairs from type0

disjointtriples:={x join y:x in disjointpairs,y in disjointpairs|(x sdiff y) in disjointpairs};
#disjointtriples;   ///// There are 325 disjoint triples from type0

/////////////////////////////////////////It turns out this is a convenient way to label cosets
c2 in disjointtriples; ///// true
#(c0^G);    /// 65
#(c1^G);    /// 325 so must be equal to disjointpairs.   We can label cosets this way.
#(c2^G);    /// 325 so must be equal to disjointtriples. We can label cosets this way.

//////////////////////////////////////////////////////////////////////////////////////Chambers
c:=[P^(timesf^3),Q^(timesf^2),Q];
chambers:={c^g:g in G};
c^(timesf^12) eq c;
c^inverse     eq c;
c^outer       eq c;

//////////////////////////////////////////////////////////////////////Construct discs from c
All_discs:=[];
```

```
///////////////////////////////////////////////////////////////////////////////// the 0th disc
disc0:={c};
previous_discs:=disc0;

///////////////////////////////////////////////////////////////////////////////// the nth disc
for n in {1..18} do
   disc_n:={d: d in (chambers sdiff previous_discs) | [d[2],d[1],d[3]] in previous_discs or
                                                      [d[1],d[3],d[2]] in previous_discs or
                                 #{x:x in previous_discs| x[2] eq d[2] and x[3] eq d[3]} gt 0
            };
   Include(~All_discs , disc_n );
   previous_discs:=previous_discs join disc_n;
end for;

//////////////////////////////////////////////////////////////////Work out matrices for each disc
AllMatrices:=[];
for disc in All_discs do
   matrices:={Matrix(IntegerRing(),3,3,[#(c[1] meet d[1]),#(c[1] meet d[2]),#(c[1] meet d[3]),
                                        #(c[2] meet d[1]),#(c[2] meet d[2]),#(c[2] meet d[3]),
                                        #(c[3] meet d[1]),#(c[3] meet d[2]),#(c[3] meet d[3])])
             :d in disc };
   Include(~AllMatrices , matrices);
end for;
```

Chapter 6

The Petersen geometry $\Gamma(3A_7)$

Consider the following amalgam of $3A_7$:



We use Rob Wilson's definition of $3A_7$ [34]. Let $\omega = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Thus $\overline{\omega} = \omega^2$. Consider the nine vectors given by

$$(2,0,0,0,0,0) \qquad (0,0,1,1,1,1) \qquad (0,1,0,1,\omega,\overline{\omega})$$

and their multiples by ω and $\overline{\omega}$. Consider the group $G_2 \cong S_4$ generated by the following coordinate permutations of order 2 and 3:

Let Ω_{63} be the union of orbits of the nine vectors above under this group. Then Ω_{63} is a set of sixty-three vectors. We define $3A_7$ as the set of symmetries in $GL_6(\mathbb{C})$ of Ω_{63}.

6.1 Introducing Z, Ω_{21} and Ω_7

The center of $3A_7$ is the group $Z = \{I, \omega I, \overline{\omega} I\}$. Thus Ω_{63} may be partitioned into 21 Z-orbits, each of length 3. Let Ω_{21} be all of these. Let $\underline{1}$ be the following 6-subset of

Ω_{21}:

$$\begin{aligned}\underline{1} = \{ \quad & (2,0,0,0,0,0)^Z, \quad (0,0,1,1,1,1)^Z, \\ & (0,1,0,1,\omega,\overline{\omega})^Z, \quad (0,1,1,0,\overline{\omega},\omega)^Z, \\ & (0,1,\overline{\omega},\omega,1,0)^Z, \quad (0,1,\omega,\overline{\omega},0,1)^Z \quad \}\end{aligned}$$

That is, the set containing $(2,0,0,0,0,0)^Z$ along with the five other Z-orbits (not containing vectors with a coordinate 2) whose dot product with $(2,0,0,0,0,0)^Z$ is zero. Under G, $\underline{1}$ has an orbit Ω_7 of order 7, consising of

$$\begin{aligned}\underline{\infty} = \{ \quad & (2,0,0,0,0,0)^Z, \quad (0,0,0,2,0,0)^Z, \\ & (0,2,0,0,0,0)^Z, \quad (0,0,0,0,2,0)^Z, \\ & (0,0,2,0,0,0)^Z, \quad (0,0,0,0,0,2)^Z \quad \}\end{aligned}$$

and $\underline{i}$ $(1 \leqslant i \leqslant 6)$, where $\underline{i}$ contains exactly one Z-orbit containing a vector whose ith coordinate is 2. Then G acts as A_7 on $\underline{1}, ..., \underline{6}$ and $\underline{\infty}$. Note that each element in Ω_{21} is the intersection of a unique pair $\underline{i}$ and $\underline{j}$.

6.2 Elements of $\Gamma(A_7)$

We look back briefly at the elements of $\Gamma(A_7)$. A subgroup of type 0 is $\mathrm{Stab}_{A_7}\{1,2\} \cong S_5$ acting on the 5-set $\{3, ..., 7\}$. A subgroup of type 2 is $\mathrm{Stab}_{A_7}\{\{1,2\},\{3,4\},\{5,6\}\} \cong S_4$ acting on the four elements

$$\left\{\begin{array}{cc} \boxed{21} & 12 \\ 34 & \boxed{43} \\ 56 & \boxed{65} \end{array}\right\} \quad \left\{\begin{array}{cc} 12 & \boxed{21} \\ \boxed{43} & 34 \\ 56 & \boxed{65} \end{array}\right\} \quad \left\{\begin{array}{cc} 12 & \boxed{21} \\ 34 & \boxed{43} \\ \boxed{65} & 56 \end{array}\right\} \quad \left\{\begin{array}{cc} 12 & \boxed{21} \\ 34 & \boxed{43} \\ 56 & \boxed{65} \end{array}\right\}$$

(We draw squares to emphasize when the two numbers have been swapped). A subgroup of type 1 is $\mathrm{Stab}_{A_7}\{\{1,2\},\{3,4\}\} \cong (3 \times 2^2)2$, as we see here:

(1)	(1, 2) (5, 6)
(1, 3)(2, 4)	(1, 3, 2, 4) (5, 6)
(1, 2)(3, 4)	(3, 4) (5, 6)
(1, 4)(2, 3)	(1, 4, 2, 3) (5, 6)
(5, 6, 7)	(1, 2) (6, 7)
(1, 2)(3, 4) (5, 6, 7)	(3, 4) (6, 7)
(1, 4)(2, 3) (5, 6, 7)	(1, 4, 2, 3) (6, 7)
(1, 3)(2, 4) (5, 6, 7)	(1, 3, 2, 4) (6, 7)
(5, 7, 6)	(1, 2) (5, 7)
(1, 3)(2, 4) (5, 7, 6)	(1, 3, 2, 4) (5, 7)
(1, 2)(3, 4) (5, 7, 6)	(3, 4) (5, 7)
(1, 4)(2, 3) (5, 7, 6)	(1, 4, 2, 3) (5, 7)

Hence $\Gamma(A_7)$ consists of $\binom{7}{2} = 21$ elements of type 0, $\frac{\binom{7}{2}\binom{5}{2}}{2} = 105$ elements of type 1, and $\frac{\binom{7}{2}\binom{5}{2}\binom{3}{2}}{3!} = 105$ elements of type 2.

6.3 Elements of $\Gamma(3A_7)$

Our group $3A_7$ contains exactly one conjugacy class of subgroups S_5, one conjugacy class of subgroups $(3\times 2^2)2$ (satisfying $D_6\times 2 < (3\times 2^2)2$, as required in our amalgam), and four conjugacy classes of subgroups S_4. As S_4 and S_5 have only the identity as their center and $(3\times 2^2)2$ has a center of order 2, these must intersect Z trivially.

6.3.1 Elements of type 0

There are 21 subgroups isomorphic to S_5 in $3A_7$. The pointwise stabilizer in G of $x \in \Omega_{21}$ is S_5, for x is the intersection of a unique pair $\underline{i}$ and $\underline{j}$. Thus the stabilizer acts as S_5 on $\Omega_7\backslash\{\underline{i},\underline{j}\}$. We may represent our 63 elements of type 0 by elements $x \in \Omega_{63}$.

6.3.2 Elements of type 1

We may represent the 315 cosets of a subgroup $(3\times 2^2)2 \leqslant 3A_7$ by all pairs $\{x, y\}$ where $x, y \in \Omega_{63}$ such that there is some permutation in G swapping x and y, and no $\underline{i}$ contains both x and y. We define two elements of $3A_7$ of order 2 and 3 respectively:

$$M = \frac{1}{2}\begin{pmatrix} 2&0&0&0&0&0\\ 0&0&1&1&1&1\\ 0&1&0&1&\omega&\overline{\omega}\\ 0&1&1&0&\overline{\omega}&\omega\\ 0&1&\overline{\omega}&\omega&1&0\\ 0&1&\omega&\overline{\omega}&0&1 \end{pmatrix} \qquad N = \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&\overline{\omega}&0\\ 0&0&0&0&0&\omega \end{pmatrix}$$

We compute all the elements of type 1. Recall the group of coordinate permutations G_2. All 2-subsets of $(1,1,1,1,0,0)^{G_2}$ are of type 1. This gives us 9 elements under ZG_2.

The stabilizer $\mathrm{Stab}_{G_2}(\{(0,1,0,1,\omega,\overline{\omega}),(1,0,1,0,\omega,\overline{\omega})\})$ has order 4, so this gives us 18 elements under ZG_2.

$$\begin{aligned}
(0,0,0,0,0,2)M &= (0,1,\omega,\overline{\omega},0,1) & (0,0,0,0,0,2)N &= \omega(0,0,0,0,0,2)\\
(0,0,1,1,1,1)M &= (0,2,0,0,0,0) & (0,0,0,0,2,0)N &= \overline{\omega}(0,0,0,0,2,0)\\
(1,1,1,1,0,0)M &= (1,1,1,1,0,0) & (0,0,1,1,1,1)N &= (1,0,0,1,\overline{\omega},\omega)\\
(0,0,2,0,0,0)M &= (0,1,0,1,\omega,\overline{\omega}) & (\omega,\overline{\omega},0,1,0,1)N &= \omega(0,1,\omega,\overline{\omega},0,1)\\
& & (\omega,\overline{\omega},1,0,1,0)N &= \overline{\omega}(\omega,\overline{\omega},1,0,1,0)
\end{aligned}$$

Using M:

$\mathrm{Stab}_{G_2}\{(0,0,0,0,0,2),(0,1,\omega,\overline{\omega},0,1)\}$ is the identity, giving us 72 elements.

$\mathrm{Stab}_{G_2}\{(0,2,0,0,0,0),(1,1,1,1,0,0)\}$ has order 2, giving us 36 elements.

$\mathrm{Stab}_{G_2}\{(1,1,1,1,0,0),(0,1,0,1,\omega,\overline{\omega})\}$ has order 2, giving us 36 elements.

Using N:

$\mathrm{Stab}_{G_2}\{\omega(0,0,0,0,0,2),(1,0,0,1,\overline{\omega},\omega)\}$ has order 2, giving us 36 elements.

$\mathrm{Stab}_{G_2}\{\overline{\omega}(0,0,0,0,2,0),(1,0,0,1,\overline{\omega},\omega)\}$ has order 2, giving us 36 elements.

$\mathrm{Stab}_{G_2}\{\omega(0,1,\omega,\overline{\omega},0,1),\overline{\omega}(\omega,\overline{\omega},1,0,1,0)\}$ is the identity. This gives us 72 elements.

Thus we may define our elements of type 1 as all pairs $\{u,v\}$ where $u,v \in \Omega_{63}$ such that:

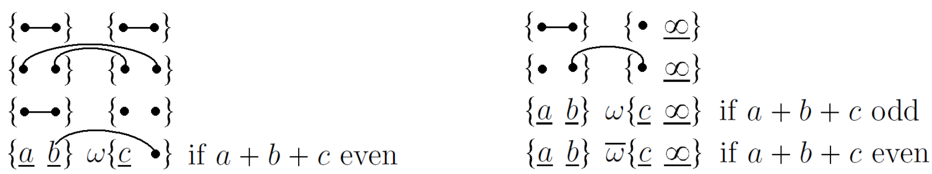
- Every coordinate of $u+v$ is an integer multiple of 1, ω or $\overline{\omega}$.
- Every coordinate of $u+v$ is non-zero or there is a coordinate equal to 3, 3ω, or $3\overline{\omega}$.

Another way of labelling

Using the 1-to-1 corresspondence between Ω_{21} and pairs $\{\underline{i},\underline{j}\} \subset \Omega_7$ we may label the elements of Ω_{63} by $\{\underline{1},\underline{2}\} = (0,0,1,1,1,1)$, $\{\underline{1},\underline{3}\} = (0,1,0,1,\omega,\overline{\omega})$, $\{\underline{1},\underline{\infty}\} = (2,0,0,0,0,0)$ and their images under ZG_2. Re-labelling by this notation gives us the following elements of type 1:

$$\begin{array}{ll}
\{\underline{3},\underline{4}\},\ \ \{\underline{5},\underline{6}\} & (1,1,0,0,1,1)\ ,\ \ (1,1,1,1,0,0) \\
\{\underline{1},\underline{3}\},\ \ \{\underline{2},\underline{4}\} & (0,1,0,1,\omega,\overline{\omega}),\ \ (1,0,1,0,\omega,\overline{\omega}) \\
\{\underline{5},\underline{6}\},\ \ \{\underline{1},\underline{3}\} & (1,1,1,1,0,0)\ ,\ \ (0,1,0,1,\omega,\overline{\omega}) \\
\{\underline{1},\underline{5}\},\omega\{\underline{4},\underline{6}\} & (0,1,\omega,\overline{\omega},0,1),\ \omega(\omega,\overline{\omega},1,0,1,0) \\
 & \\
\{\underline{5},\underline{6}\},\ \ \{\underline{1},\underline{\infty}\} & (1,1,1,1,0,0)\ ,\ \ (2,0,0,0,0,0) \\
\{\underline{1},\underline{5}\},\ \ \{\underline{6},\underline{\infty}\} & (0,1,\omega,\overline{\omega},0,1),\ \ (0,0,0,0,0,2) \\
\{\underline{2},\underline{3}\},\overline{\omega}\{\underline{5},\underline{\infty}\} & (1,0,0,1,\overline{\omega},\omega),\ \overline{\omega}(0,0,0,0,2,0) \\
\{\underline{2},\underline{3}\},\omega\{\underline{6},\underline{\infty}\} & (1,0,0,1,\overline{\omega},\omega),\ \omega(0,0,0,0,0,2)
\end{array}$$

along with their images under ZG_2. That is, the following and their multiples by ω and $\overline{\omega}$: (let dots be elements of $\{1,2,3,4,5,6\} = \{a,b,c,d,e,f\}$ and write $a \bullet\!\!-\!\!\bullet b$ if and only if $\{a,b\} = \{2k-1,2k\}$)



6.3.3 Elements of type 2

We may represent the 315 cosets of the subgroup $G_2 \cong S_4$ by triples $\{u, v, w\}$ where $u, v, w \in \Omega_{63}$ such that any 2-subset of the triple is an element of type 1. Elements of type 2 are the orbits of the following under ZG_2:

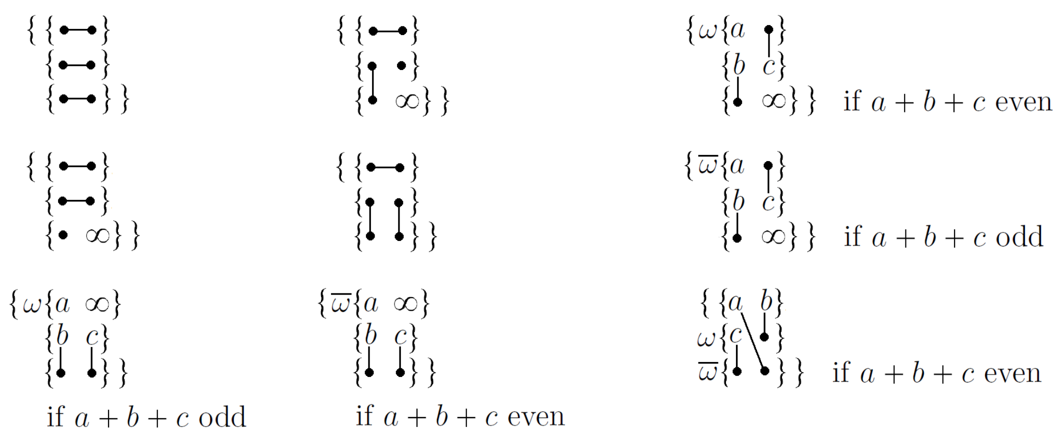
$$\begin{array}{lll}
\{\,(0,0,1,1,1,1), & \{\,(0,0,1,1,1,1), & \{\omega(1,0,1,0,\omega,\overline{\omega}), \\
\;(1,1,0,0,1,1), & \;(\omega,\overline{\omega},0,1,0,1), & \;(\omega,\overline{\omega},0,1,0,1), \\
\;(1,1,1,1,0,0)\,\} & \;(0,0,0,0,0,2)\,\} & \;(0,0,0,0,0,2)\,\} \\[1em]
\{\,(0,0,1,1,1,1), & \{\,(0,0,1,1,1,1), & \{\overline{\omega}(0,1,1,0,\overline{\omega},\omega), \\
\;(1,1,0,0,1,1), & \;(\omega,\overline{\omega},0,1,0,1), & \;(\omega,\overline{\omega},0,1,0,1), \\
\;(0,0,0,0,0,2)\,\} & \;(\omega,\overline{\omega},1,0,1,0)\,\} & \;(0,0,0,0,0,2)\,\} \\[1em]
\{\omega(2,0,0,0,0,0), & \{\overline{\omega}(0,2,0,0,0,0), & \{\;(\omega,\overline{\omega},0,1,0,1), \\
\;(\omega,\overline{\omega},0,1,0,1), & \;(\omega,\overline{\omega},0,1,0,1), & \;\omega(1,0\;\omega,\overline{\omega},1,0), \\
\;(\omega,\overline{\omega},1,0,1,0)\,\} & \;(\omega,\overline{\omega},1,0,1,0)\,\} & \;\overline{\omega}(0,1,1,0,\overline{\omega},\omega)\,\}
\end{array}$$

Another way of labelling

If we use the notation mentioned earlier,

$$\begin{array}{lll}
\{\,\{\underline{1},\underline{2}\}, & \{\,\{\underline{1},\underline{2}\}, & \{\omega\{\underline{2},\underline{4}\}, \\
\;\{\underline{3},\underline{4}\}, & \;\{\underline{3},\underline{5}\}, & \;\{\underline{3},\underline{5}\}, \\
\;\{\underline{5},\underline{6}\}\,\} & \;\{\underline{6},\underline{\infty}\}\,\} & \;\{\underline{6},\underline{\infty}\}\,\} \\[1em]
\{\,\{\underline{1},\underline{2}\}, & \{\,\{\underline{1},\underline{2}\}, & \{\overline{\omega}\{\underline{1},\underline{4}\}, \\
\;\{\underline{3},\underline{4}\}, & \;\{\underline{3},\underline{5}\}, & \;\{\underline{3},\underline{5}\}, \\
\;\{\underline{6},\underline{\infty}\}\,\} & \;\{\underline{4},\underline{6}\}\,\} & \;\{\underline{6},\underline{\infty}\}\,\} \\[1em]
\{\omega\{\underline{1},\underline{\infty}\}, & \{\overline{\omega}\{\underline{2},\underline{\infty}\}, & \{\,\{\underline{3},\underline{5}\}, \\
\;\{\underline{3},\underline{5}\}, & \;\{\underline{3},\underline{5}\}, & \;\omega\{\underline{2},\underline{6}\}, \\
\;\{\underline{4},\underline{6}\}\,\} & \;\{\underline{4},\underline{6}\}\,\} & \;\overline{\omega}\{\underline{1},\underline{4}\}\,\}
\end{array}$$

and their orbits under ZG_2. Let dots be elements of $\{a, b, c, d, e, f, g\} = \{1, ..., 6, \infty\}$. Write $a \bullet\!\!-\!\!\bullet b$ if and only if $\{a, b\} = \{2k-1, 2k\}$. Then the elements of type 2 are



and their multiples by ω and $\overline{\omega}$.

Two elements in the geometry $\Gamma(3A_7)$ are incident if they are of different types and one is contained in the other.

6.4 Chambers

We have 6 chambers for each element of type 2. The disc structure of the chamber graph from some chamber C is shown in Table 7.1. Interestingly, the diameter is 20 and the 20th disc is $\{\omega C, \overline{\omega} C\}$. From now on we omit the underlining when writing elements of Ω_7.

Definition 6.1. *Write the chamber*

$$C: \ \{\omega^z\{e,f\}\} \subset \{\omega^y\{c,d\}, \omega^z\{e,f\}\} \subset \{\omega^x\{a,b\}, \omega^y\{c,d\}, \omega^z\{e,f\}\}$$

as $C = \omega^x ab|\omega^y cd|\omega^z ef$*.*

Lemma 6.2. *If* $\Omega_7 = \{a,b\} \cup \{c,d\} \cup \{e,f\} \cup \{g\}$ *then*

- *There exist* x, y *and* z *such that* $C = \omega^x ab|\omega^y cd|\omega^z ef$ *is a chamber. If, respectively,* x', y' *and* z' *fulfill the same property then* $x - x' \equiv y - y' \equiv z - z' \bmod 3$*.*
- *C has 4 neighbours:* *The chamber* $\omega^u ag|\omega^y cd|\omega^z ef$ *(2-adjacent)*
  The chamber $\omega^v bg|\omega^y cd|\omega^z ef$ *(2-adjacent)*
  The chamber $\omega^y cd|\omega^x ab|\omega^z ef$ *(1-adjacent)*
  The chamber $\omega^x ab|\omega^z ef|\omega^y cd$ *(0-adjacent)*

Proof. The Lemma is clearly true if we take $\{a,b\} = \{1,2\}$, $\{c,d\} = \{3,4\}$ and $\{e,f\} = \{5,6\}$. By flag-transitivity, we see it must be true for all possible $\{a,b\}$, $\{c,d\}$ and $\{e,f\}$. □

In Definition 6.3 we factor out vertices of the chamber graph of $\Gamma(3A_7)$ by multiples of 1, ω and $\overline{\omega}$.

Definition 6.3. *Let* T *be the graph whose vertices are the triples* $\{C, \omega C, \overline{\omega} C\}$ *for all chambers* C*, two vertices adjacent if and only if an element of one is adjacent to an element of the other in the chamber graph of* $\Gamma(3A_7)$*. This means two triples are* *adjacent if and only if each chamber in one is adjacent to exactly one chamber in the other, all three adjacencies being of the same type. Therefore we label the edges of T uniquely with type 0,1 or 2.*

Lemma 6.4. *The graph T of Definition 6.3 is isomorphic to the chamber graph of $\Gamma(A_7)$. The isomorphism preserves the type of adjacency.*

Proof. By Lemma 6.2 there is a 1-to-1 correspondence between the vertices of T and the chambers $ab|cd|ef$ from $\Gamma(A_7)$. Vertices are adjacent if and only if their corresponding chambers are adjacent, and adjacency type is preserved by Definition 6.3. □

The next lemma is similar to Theorem 2.9.

Lemma 6.5. *There are 63 subgraphs of the chamber graph of $\Gamma(3A_7)$ isomorphic to the chamber graph of $\Gamma(A_5)$. Each consists of all chambers $\omega^- - -|\omega^- - -|\omega^x ij$ for some $\omega^x\{i,j\}$.*

Proof. This can be verified for the case $\{i,j\} = \{2k-1, 2k\}$ by looking back at Section 6.3.3. All other cases follow by flag-transitivity. □

For the rest of this Section let $C = 12|34|56$. From Section 6.3.3 recall $ab|cd|ef$ is a chamber whenever $\{a,b\}$, $\{c,d\}$ or $\{e,f\}$ is of the form $\{\bullet\!\!-\!\!\!-\!\!\bullet\}$. Therefore, looking back at Fig.2.6, we see the chambers distance less than 6 from C are all of the form $ab|cd|ef$ and behave exactly like chambers of $\Gamma(A_7)$. This is illustrated in discs 1 to 5 of Fig.6.1.

We have $C = 12|34|56$. To understand Fig.6.1 completely we need Definition 6.6.

Definition 6.6. *Let $D = \omega^x d_1 d_2|\omega^y d_3 d_4|\omega^z d_5 d_6$. The intersection matrix $M(C,D)$ is defined by $M(C,D)_{ij} = |\{2i-1, 2i\} \cap \{d_{2j-1}, d_{2j}\}|$, with ω^x written above the first column, ω^y written above the second and ω^z written above the third. For example, letting $D = 2\infty|\omega 35|\omega 46$ gives $M(C,D) =$*

$$\begin{array}{c} \begin{array}{ccc} & \omega & \omega \end{array} \\ \boxed{\begin{array}{ccc} 1 & & \\ & 1 & 1 \\ & 1 & 1 \end{array}} \end{array}$$

We usually omit writing ω^0. A matrix is called odd if the sum of its entries is odd. It is called even otherwise.

Lemma 6.7. *The distance between C and D can be determined from $M(C,D)$.*

Proof. It is enough to show that $H = \mathrm{Stab}_{3A_7}(C) = \langle(1,2)(3,4),(3,4)(5,6)\rangle$ is transitive on the set X of chambers E satisfying $M(C,E) = M(C,D) = M$. Looking back at Section 6.3.3 we see that the number of different powers of ω written along the top of M is determined by whether M contains a 2 and is even or odd, for we have four cases and an example of each is shown here (let $\{x,y,z\} = \{1,\omega,\overline{\omega}\}$):

$$\begin{array}{c} x\ x\ x \\ \boxed{\begin{array}{ccc} & 1 & 1 \\ 2 & & \\ & 1 & 1 \end{array}} \end{array} \qquad \begin{array}{c} x\ y\ y \\ \boxed{\begin{array}{ccc} & 1 & 1 \\ 1 & 1 & \\ 1 & & \end{array}} \end{array} \qquad \begin{array}{c} x\ y\ y \\ \boxed{\begin{array}{ccc} 1 & & \\ & 1 & 1 \\ & 1 & 1 \end{array}} \end{array} \qquad \begin{array}{c} x\ y\ z \\ \boxed{\begin{array}{ccc} 1 & & 1 \\ & 1 & 1 \\ 1 & 1 & \end{array}} \end{array}$$

If M contains an entry equal to 2 then H is certainly transitive on X and we are done. Therefore consider the three remaining cases where every entry of M is 0 or 1: In each we see that there are six possible ways of writing the powers of ω at the top: Specifically, there are two ways modulo multiplication by some power of ω. It follows that X has order $\frac{8}{2} = 4$ and H is transitive on X. $\square$

Theorem 6.8. *The diameter of the chamber graph of* $\Gamma(3A_7)$ *is 20. Two chambers A and B are distance 20 apart if and only if $A = \omega^{\pm 1}B$.*

Proof. This is a combinatoric sketch-proof. In Fig.6.1 we show all intersection matrices of chambers distance $\leqslant 10$ from C. Now the first disc containing a chamber C', such that $\omega C'$ or $\overline{\omega}C'$ can be found in the same disc or the previous disc, is the tenth disc. This is enough to show that the distance between C and $\omega^{\pm 1}C$ is 20 (by flag-transitivity C is "as close to halfway as possible" along a minimal path from some chamber to its multiple by ω). Now we need to show that the distance between any other chamber and C is $\leqslant 19$. To do this, notice that some matrices in Fig.6.1 are shaded (with distances $\leqslant 14$). Each of these shows a matrix $M(C,D)$ where D is of the form $\omega^- - -|\omega^- - -|\omega^x ij$. This is true for every $\omega^x\{i,j\}$ except $x = \pm 1$ and $\{i,j\} = \{5,6\}$. Therefore, unless E is of the form $\omega^- - -|\omega^- - -|\omega^{\pm 1}56$, the distance between C and E is $\leqslant 14 + 5 = 19$ (using Lemma 6.5 and that the diameter of the chamber graph of $\Gamma(A_5)$ is 5).

Therefore assume E is of the form $\omega^- - -|\omega^- - -|\omega^{\pm 1}56$. Then from Fig.6.1 we see that the only matrices left to consider are $X = \begin{array}{c} \substack{\overline{\omega}\\ \omega}\ \substack{\overline{\omega}\\ \omega}\ \substack{\overline{\omega}\\ \omega} \\ \boxed{\begin{array}{ccc} & 2 & \\ 2 & & \\ & & 2 \end{array}} \end{array}$ and $Y = \begin{array}{c} \substack{\overline{\omega}\\ \omega}\ \substack{\overline{\omega}\\ \omega}\ \substack{\overline{\omega}\\ \omega} \\ \boxed{\begin{array}{ccc} 2 & & \\ & 2 & \\ & & 2 \end{array}} \end{array}$. If $M(C,D) = X$ then we draw a minimal path in Fig.6.1 between C and D of length 19. $\square$

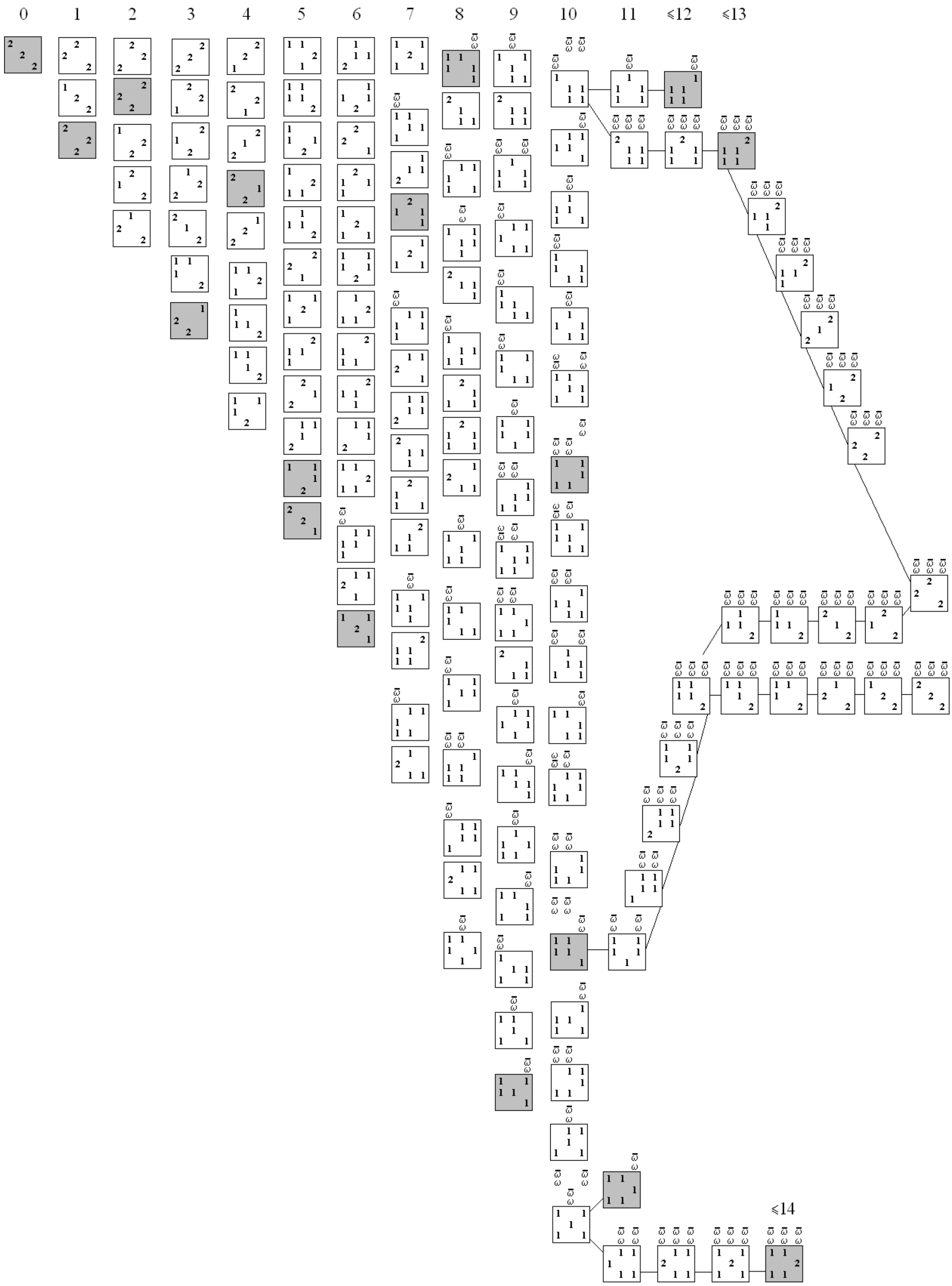


Fig. 6.1. Making things look more complicated than they need to be!

6.5 The automorphism group of $Ch(\Gamma(3A_7))$

Let G be the automorphism group of the chamber graph of $\Gamma(3A_7)$.

Lemma 6.9. *The triples* $\{C, \omega C, \varpi C\}$ *are* G*-blocks.*

Proof. Obvious by Theorem 6.8. $\square$

Lemma 6.10. *The group* G *preserves the type of adjacency between chambers.*

Proof. Each chamber has four neighbours. The result follows using Lemma 3.39. $\square$

Lemma 6.11. *The subgroup* H *of* G *fixing every triple setwise is* Z*.*

Proof. Let $h \in H$. Two triples are adjacent if and only if they can be written as $\{A, \omega A, \varpi A\}$ and $\{B, \omega B, \varpi B\}$ such that $\omega^i A$ is adjacent to $\omega^i B$ for each i. Therefore the cycle type of the action of h on each triple $\{C, \omega C, \varpi C\}$ is identical. We need to show h cannot be composed of 2-cycles.

Let $C = 12|34|56$ and assume h acts as $(\omega C, \varpi C)$ on C^Z. Let $D = \omega 1\infty|35|46$. By Fig.6.1 we see $\mathrm{dist}(C, D) = \mathrm{dist}(C, \varpi D) = 10$, whereas $\mathrm{dist}(C, \omega D) \geqslant 11$. Therefore h acts as $(D, \varpi D)$ on D^Z. Now there is a chamber $F = 12|35|46$ type-2-adjacent to D, lying in-between C and D. The chamber F^h is a multiple of F by some power of ω and is type-2-adjacent to ϖD. By our assumption this chamber must lie in-between C and ϖD, but this is not true by Fig.6.1. $\square$

Theorem 6.12. *The automorphism group* G *of the chamber graph of* $\Gamma(3A_7)$ *is* $3S_7$*.*

Proof. By Theorem 3.43 and Lemmas 6.4 and 6.9 we know that G acts as either A_7 or S_7 on T. To prove it is S_7, consider the following automorphism g acting as (1,2) on T:

$$\{a,b\}^g = \{a^{(1,2)}, b^{(1,2)}\} \qquad \omega\{a,b\}^g = \varpi\{a^{(1,2)}, b^{(1,2)}\} \qquad \varpi\{a,b\}^g = \omega\{a^{(1,2)}, b^{(1,2)}\}$$

To show this is an automorphism, it suffices to show that the adjacencies in Section 6.3.2 are preserved. $\square$

6.6 Magma code for $\Gamma(3A_7)$

```
/*
www-ATLAS of Group Representations.
3.A7 represented as permutations on 63 elements.
*/

G<x,y>:=PermutationGroup<63|\[
2,4,8,1,13,16,19,10,23,3,25,27,14,5,29,18,9,6,20,7,15,33,17,38,26,
11,28,12,21,35,45,41,36,43,44,22,51,39,24,37,46,31,49,30,42,32,58,47,34,50,
40,52,62,53,55,56,57,48,59,63,60,54,61]
,\[
3,6,9,11,1,17,2,13,15,14,23,4,30,31,5,19,21,20,34,35,7,8,29,10,27,
28,42,43,12,37,40,16,18,47,48,50,22,52,25,24,26,53,54,36,38,57,32,33,46,60,
51,61,39,41,44,45,63,58,49,55,56,62,59]
>;
print "Group G is 3.A7 < Sym(63)";

////////////////////////////////////////////////////////////We define the 7-set on which G acts
set7:={
      {
        { 9, 17, 23 },
        { 50, 52, 57 },
        { 36, 38, 46 },
        { 22, 32, 39 },
        { 1, 2, 4 },
        { 24, 33, 41 }
    },{                     /// <-- these two defined by {1,2,4}
        { 44, 45, 49 },
        { 55, 56, 59 },
        { 31, 35, 43 },
        { 30, 34, 42 },
        { 1, 2, 4 },
        { 15, 21, 29 }
    },{
        { 44, 45, 49 },
        { 36, 38, 46 },
        { 40, 48, 54 },
        { 3, 6, 11 },
        { 37, 47, 53 },
        { 5, 7, 12 }
    },{                     /// <-- these two defined by {3,6,11}
        { 50, 52, 57 },
        { 60, 61, 63 },
        { 8, 16, 25 },
        { 3, 6, 11 },
        { 10, 18, 26 },
        { 15, 21, 29 }
    },{
```

```
        { 9, 17, 23 },
        { 14, 20, 28 },
        { 60, 61, 63 },
        { 55, 56, 59 },
        { 13, 19, 27 },
        { 5, 7, 12 }
    },{                     /// <-- these two defined by {14,20,28}
        { 51, 58, 62 },
        { 14, 20, 28 },
        { 40, 48, 54 },
        { 31, 35, 43 },
        { 10, 18, 26 },
        { 24, 33, 41 }
    },{
        { 51, 58, 62 },
        { 30, 34, 42 },
        { 22, 32, 39 },
        { 8, 16, 25 },
        { 37, 47, 53 },
        { 13, 19, 27 }
    }
};

set7^G eq {set7};

//////////////////////////////////////////////////////////////////////////////Find an amalgam
G0:=Stabilizer(G,1);
G1:=Stabilizer(G,{1,3});
G2:=Stabilizer(G,{1,3,20});

A7:=Alt(7); k:={{1,2},{3,4}};
H:=CyclicGroup(2);
D6:=DihedralGroup(3);
D8:=DihedralGroup(4);

IsIsomorphic(G0                ,Sym(5)                      );
IsIsomorphic(G1                ,sub<A7|{g:g in A7|k^g eq k}>);
IsIsomorphic(G2                ,Sym(4)                      );
IsIsomorphic(G0 meet G1        ,DirectProduct(D6,H)         );
IsIsomorphic(G0 meet G2        ,D8                          );
IsIsomorphic(G1 meet G2        ,D8                          );
IsIsomorphic(G0 meet G1 meet G2 ,DirectProduct(H,H)          );

/////////////////////////////////////////////////////we can use {1,3} but not {1,6} or {1,11}
#Stabilizer(G,{1, 6});   //// is 12
#Stabilizer(G,{1,11});   //// is 12
///////////////////////////////////////////////////we can use {3,20} but not {3,14} or {3,28}
#Stabilizer(G,{1, 6});   //// is 12
#Stabilizer(G,{1,11});   //// is 12
```

```
////////////////////////////////////////////////////////////////what's different about {1,3}?
//////// there is some g swapping 1 and 3:
#{g: g in G | 1^g eq 3  and  3^g eq 1};  ///not zero
#{g: g in G | 1^g eq 6  and  6^g eq 1};  ///zero
#{g: g in G | 1^g eq 11 and 11^g eq 1};  ///zero

///////////////////////////////////////////////////////////////what's different about {3,20}?
//////// there is some g swapping 3 and 20:
#{g: g in G | 3^g eq 20  and  20^g eq 3};  ///not zero
#{g: g in G | 3^g eq 14  and  14^g eq 3};  ///zero
#{g: g in G | 3^g eq 28  and  28^g eq 3};  ///zero

////////////////////////////////////////////////////////////////////////////////Chamber graph
V:={[1,3,20]^g:g in G};
E:={ {C,D} : C in V, D in V| C ne D and
(  (C[1] eq D[1] and C[2] eq D[3] and C[3] eq D[2]) or
   (C[1] eq D[2] and C[2] eq D[1] and C[3] eq D[3]) or
   (C[2] eq D[2] and C[3] eq D[3])       )
};

G,V,E := Graph< V| E >; Diameter(G);
x:=Representative(V);
[ #y : y in DistancePartition(x) ];
```

Chapter 7

Other results

7.1 Disc Structures

Recall Definition 1.4. In Fig.7.1 we look at the disc sizes of the chamber graphs of some of the geometries in this thesis, worked out using Magma. Most of these can be found in Peter Rowley's paper [29].

Disc	1	2	3	4	5	6	7	8	9	10	11
$\Gamma(L_2(11))$	4	8	15	26	42	58	76	68	32		
$\Gamma(A_7)$	4	8	15	26	42	58	76	104	136	144	16
$\Gamma(3A_7)$	4	8	15	26	42	58	76	104	136	176	192
$\Gamma(L_2(25))$	4	8	15	26	42	58	76	104	136	176	192
Disc	12	13	14	15	16	17	18	19	20		
$\Gamma(L_2(25))$	216	256	256	232	100	44	8				
$\Gamma(3A_7)$	216	256	256	192	68	36	18	8	2		

Tab. 7.1. Disc sizes of the chamber graphs of some geometries in this thesis.

7.2 A few extra results

Recall Definition 2.20. Fig.7.1 shows us all unjumpable $n \times n$ intersection matrices not containing a 2 for $n = 2, 3$ and 4.

Before proving Theorems 7.1 and 7.2 it is wise to go back and remind ourselves of the notation in Definitions 2.7 and 2.22, as well as Lemma 2.31 and Theorem 2.35.

Theorem 7.1. *Let $M = M(A, B)$ be an $n \times n$ intersection matrix with $M_{n1} = M_{n2} =$*

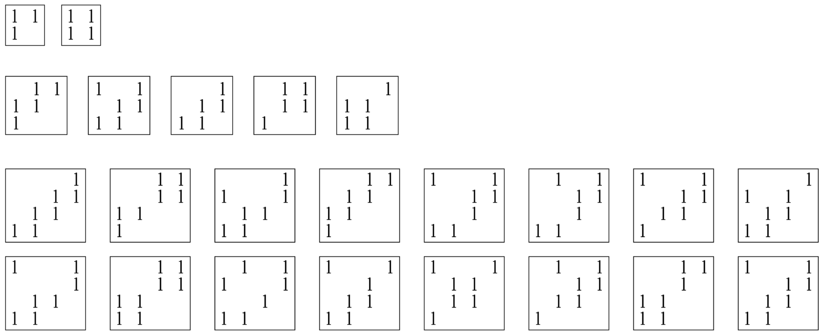


Fig. 7.1

1. *That is, let* $A(n) = \{a, b\}$, $B(1) = \{a, c\}$ *and* $B(2) = \{b, d\}$. *Consider the chamber* C *where* $B \underset{0}{\text{——}} B' \underset{1,2}{\text{——}} B'' \underset{0}{\text{——}} C$ *and* $C(1) = \{a, b\}$. *The chamber* C *is in-between* A *and* B.

Proof. We prove by induction on n and then $\text{dist}(M)$. The result is true for n =1 and 2 by inspection. Assume it is true for matrices of order less than n. We now prove it for matrices of order n. We will do this by induction on the distance of $M(A, B)$. We may claim it is true for distance 0. Assume it is true for $n \times n$ matrices of distance less than L and let M be an $n \times n$ matrix of distance L. We may assume M contains no entries equal to 2 by Theorem 2.35(iii) and Lemma 2.45 or the result follows for M by induction.

If there exists a chamber $A' \text{——} A$ in-between A and B with $A'(n) = A(n)$ then the result is true by induction. Therefore by Lemma 2.31 assume a minimal path from A to B begins by $A \underset{n,}{\text{——}} \cdots \underset{,1}{\text{——}} D$ where $D(1) = \{a, b\}$. Using Theorem 2.35(i) we know that $\text{dist}(D, C) \leqslant \text{dist}(D, B) + 1 + 1 - 1$. Then by Theorem 2.35(iii) we have that $\text{dist}(A, C) = \text{dist}(D, C) - n + 1$. We are working with $n \geqslant 3$ so the result follows by induction. The proof is illustrated in Fig.7.2. □

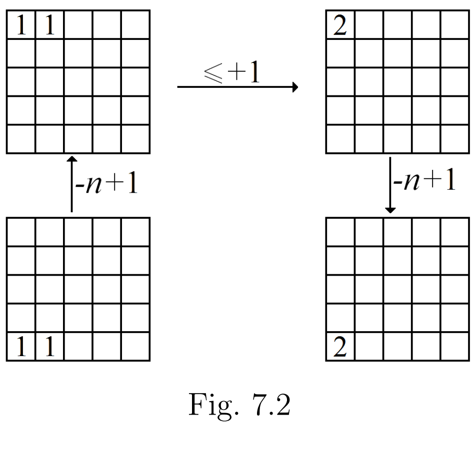


Fig. 7.2

Theorem 7.2. *Let $M(A, B)$ be an $n \times n$ matrix whose bottom row is odd. That is, $A(n) = \{a, b\}$, $a \in B(i)$ and $B(0) = b$. There exists $C \underset{0}{\text{———}} C' \underset{1,}{\text{———}} \cdots \underset{,i}{\text{———}} B$ where $C(1) = \{a, b\}$. The chamber C is in-between A and B.*

Proof. We prove by induction on n and then $\text{dist}(M)$. The result is true for n =1 and 2 by inspection. Assume it is true for matrices of order less than n. We now prove it for matrices of order n. We will do this by induction on the distance of $M(A, B)$. We may claim it is true for distance 0. Assume it is true for $n \times n$ matrices of distance less than L and let M be an $n \times n$ matrix of distance L. We may assume M contains no entries equal to 2 by Theorem 2.35(iii) and Lemma 2.45 or the result follows for M by induction.

If there exists a chamber $A' \text{———} A$ in-between A and B with $A'(n) = A(n)$ then the result is true by induction. Therefore by Lemma 2.31 assume a minimal path from A to B begins by $A \underset{n,}{\text{———}} \cdots \underset{,1}{\text{———}} D$ where $D(1) = \{a, b\}$. Using Theorem 2.35(i) we know that $\text{dist}(D, C) \leqslant \text{dist}(D, B) + (i - 1) - 1$. Then by Theorem 2.35(iii) we have that $\text{dist}(A, C) = \text{dist}(D, C) - (n - 1) \leqslant L - 2n + i \leqslant L - i$. Yet B is distance i from C, so C is in-between A and B. The proof is illustrated in Fig.7.3. □

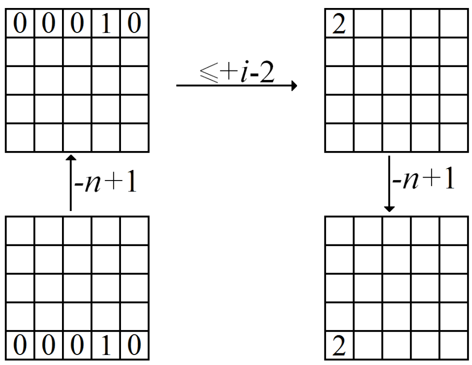


Fig. 7.3

Let C be a chamber with no part $\{a, b\}$. It is easy to see that there is a unique minimal path joining C to the unique nearest chamber D to C having a part $\{a, b\}$. Moreover, this path is ordered from C to D (recall Definition 2.22).

The next theorem is a generalization of Theorem 7.2. It is helpful now to remind ourselves of Lemma 2.31 and Theorem 2.35.

Theorem 7.3. *Let $M = M(A, B)$ be an $n \times n$ intersection matrix with its ith row odd ($M_{ij} = 1$ and $M_{ik} = 0$ for all $k \neq j$) such that $M_{gh} = 0$ if $g \geqslant i$ and $h \leqslant j$. The unique closest chamber C to B with first part $A(i)$ is in-between A and B.*

Proof. We prove by induction on n and then $\mathrm{dist}(M)$. The result is true for n =1 and 2 by inspection. Assume it is true for matrices of order less than n. We now prove it for matrices of order n. We will do this by induction on the distance of $M(A, B)$. We may claim it is true for distance 0. Assume it is true for $n \times n$ matrices of distance less than L and let M be an $n \times n$ matrix of distance L. We may assume M contains no entries equal to 2 by Theorem 2.35(iii) and Lemma 2.45 or the result follows for M by induction.

By Lemma 2.31 the theorem is proved unless $A\underset{i,}{\text{---}}\cdots\underset{,1}{\text{---}}D$ begins a minimal path from A to B, where $D(1) = \{a, b\}$. By Theorem 2.35(i) $\mathrm{dist}(D, C) \leqslant \mathrm{dist}(D, B) + j - 2$. By Theorem 2.35(iii) A lies in-between C and D. Hence $\mathrm{dist}(A, C) \leqslant \mathrm{dist}(A, B) - 2i + j$. Our reasoning is illustrated in Fig.7.4. If $i \geqslant j$ then $\mathrm{dist}(A, C) \leqslant \mathrm{dist}(A, B) - j$,

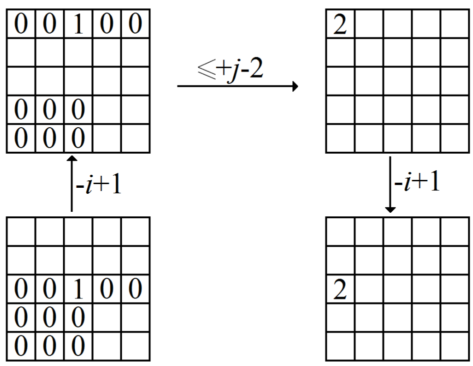


Fig. 7.4

which proves our theorem. Therefore assume $i < j$. But then we have a contradiction: For the leftmost $j-1$ rows of M must sum to $2j-2$ or $2j-3$. But all non-zero entries of these columns are contained in the top $i-1$ rows, which must sum to $2i-2$. If $i < j$ this is impossible. $\square$

Theorem 7.4. *Let A and B be n-chambers. There is only one minimal path between chambers A and B if and only if:*

i) $A\underset{i,}{\text{---}}\cdots\underset{,j}{\text{---}}B$

ii) $A\underset{i,}{\text{---}}\cdots\underset{,1}{\text{---}}C\underset{0}{\text{---}}D\underset{1,}{\text{---}}\cdots\underset{,j}{\text{---}}B.$

iii) $M(A, B) = \boxed{\begin{smallmatrix}1&1\\1&1\end{smallmatrix}} \oplus 2I_{n-2}$ *or* $\boxed{\begin{smallmatrix}1&1\\1&\end{smallmatrix}} \oplus 2I_{n-2}.$

iv) $M(A,C) = \boxed{\begin{smallmatrix}1&1\\ &1\end{smallmatrix}} \oplus 2I_{n-2}$ *where* $C \underset{2,}{\text{—}} \cdots \underset{,i}{\text{—}} B$ *for some* i $(2 \leqslant i \leqslant n)$.

v) $M(C,B) = \boxed{\begin{smallmatrix}1& \\1&1\end{smallmatrix}} \oplus 2I_{n-2}$ *where* $C \underset{2,}{\text{—}} \cdots \underset{,i}{\text{—}} A$ *for some* i $(2 \leqslant i \leqslant n)$.

We show some examples in Fig.7.5.

i) $\begin{bmatrix}2&&&&\\&&&2&\\&2&&&\\&&2&&\\&&&&2\end{bmatrix}$ *ii)* $\begin{bmatrix}2&&&&\\&&&1&\\&2&&&\\&&2&&\\&&&&2\end{bmatrix}$ *iii)* $\begin{bmatrix}1&1&&&\\1&&&&\\&&2&&\\&&&2&\\&&&&2\end{bmatrix}$ $\begin{bmatrix}1&1&&&\\1&1&&&\\&&2&&\\&&&2&\\&&&&2\end{bmatrix}$

iv) $\begin{bmatrix}1&&&1&\\&&&1&\\&2&&&\\&&2&&\\&&&&2\end{bmatrix}$ *v)* $\begin{bmatrix}1&&&&\\&&2&&\\&&&2&\\1&1&&&\\&&&&2\end{bmatrix}$

Fig. 7.5

Proof. The theorem can be proved by simple inspection if $n = 1$ or 2. Therefore assume $n \geqslant 3$. It is easy to prove that i, ii, iii, iv or v implies only one minimal path by looking at Fig.2.5 and considering Theorem 2.35. It remains to prove that if we do not have i, ii, iii, iv or v then there is more than one minimal path. We do this by induction on the distance of $M = M(A,B)$. The result is clearly true for matrices of distance 1. Therefore assume the result is true for matrices of distance less than L and let $\text{dist}(M) = L$. We consider the following cases:

The matrix M has a 2 in every row. Then we must have i.

The matrix M has a 2 in exactly $n-1$ rows. Then we must have ii.

The matrix M has a 2 in every row and column except two rows (a and a') and two columns (b and b'). Let N be the 2×2 submatrix of M defined by a, a', b and b'.

If $N = \boxed{\begin{smallmatrix} &1\\1&1\end{smallmatrix}}$ then there is more than one minimal path. (See Fig.2.5)

If $N = \boxed{\begin{smallmatrix}1&1\\1& \end{smallmatrix}}$ or $\boxed{\begin{smallmatrix}1&1\\1&1\end{smallmatrix}}$ then we can find more than one minimal path, unless $a = b = 1$ and $a' = b' = 2$. This gives us iii.

If $N = \boxed{\begin{smallmatrix}1&1\\ &1\end{smallmatrix}}$ then there is more than one minimal path, unless $a = 1$, $a' = 2$ and $b = 1$. This gives us iv.

If $N = \boxed{\begin{smallmatrix}1& \\1&1\end{smallmatrix}}$ then we have v by similar reasoning.

Lastly, consider the case where M has three rows or more without entries equal to

2. By induction it easily follows that there is more than one minimal path between A and B. $\square$

7.3 Two (probably false) conjectures.

In this section we state two conjectures which I tried to prove for a long time. The first one I disproved and the second one is very probably false.

First (definitely false) conjecture: Every unjumpable (recall Definition 2.20) $n \times n$ intersection matrix $M = M(A, B)$ has either:

i) $M_{n1} = 1$ and the bottom row is odd, or

ii) $M_{n1} = M_{n2} = 1$.

The idea behind this is that for every matrix there is some row which we can put a 2 in as fast as possible and doing so begins a minimal path by Theorems 7.1 and 7.2. The conjecture is true for $n = 1, 2, 3$ and 4 as we can see by Fig.7.1. However for $n = 5$ we have the counter-example shown in Fig.7.6. This is an unjumpable 5×5 intersection matrix which disproves the conjecture.

$$\boxed{\begin{array}{ccccc} 1 & & & 1 & \\ & & 1 & & 1 \\ & 1 & & & 1 \\ 1 & & & 1 & \\ & 1 & 1 & & \end{array}}$$

Fig. 7.6

The second conjecture is a generalization of the first:

Second (probably false) conjecture: Every unjumpable intersection matrix $M = M(A, B)$ has either:

i) $M(A, B)$ is split into a $p \times p$ matrix P and a $q \times q$ matrix Q.

ii) $M(A, C)$ is split into a $p \times p$ matrix P and a $q \times q$ matrix Q for some chamber C in-between A and B where $B \underset{0}{\text{———}} B' \underset{1,}{\text{———}} \cdots \underset{,p+1}{\text{———}} C$.

Informally, the second is claiming that an unjumpable matrix looks like a matrix from either Fig 3.5 or Fig 7.7. The first false conjecture is the special case $p = 1$.

Due to Theorems 2.35, 2.42 and 7.1, it only takes a matter of time to find the distances of the (almost certainly unjumpable) matrices shown in Fig.7.8.

Therefore I am extremely suspicious that the matrix shown in Fig.7.9 is unjumpable

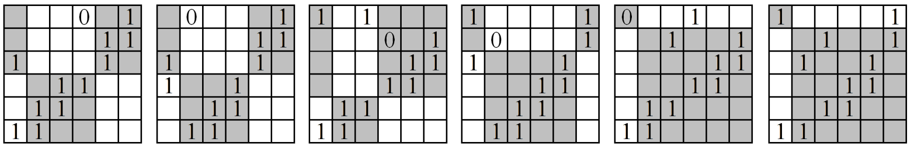
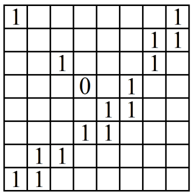
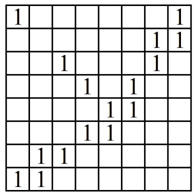


Fig. 7.7



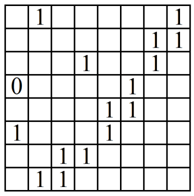
distance 49



distance 47

Fig. 7.8

of length 51. This would disprove the second conjecture. I have not proved this, and it is hard for today's computer to show using brute force. If the question is of interest in a few years, however, computers then should easily be able to find the answer. I



distance 51

Fig. 7.9

am eager to state these conjectures because I spent about a year trying to prove them. Writing them here might stop someone else spending so long.

7.4 A conjecture

Definition 7.5. *Let $M = M(A, B)$ be an $n \times n$ matrix. Let p and q be integers such that $1 \leqslant p$, $1 \leqslant q$, $p + q = n$ and $M_{ij} = 0$ whenever we have both $i \leqslant p$ and $j > p$. Let P be the submatrix of M defined by the leftmost p columns and top p rows, and Q be the submatrix of M defined by the bottom q rows and rightmost q columns. Then M is* big-split *into P and Q.*

In Fig.7.10 we show examples for p =1, 2, 3 and 4.

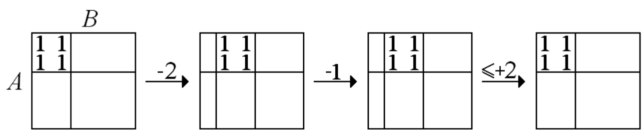
$$\begin{array}{|c|c|c|c|c|}\hline 1 & & 0 & & \\ \hline & & 1 & 1 & \\ \hline 1 & & & & 1 \\ \hline & 1 & & 1 & \\ \hline & 1 & & & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & & & \\ \hline 0 & 1 & & & \\ \hline & & 1 & 1 & \\ \hline & & 1 & & 1 \\ \hline & & & 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 1 & & 1 & & \\ \hline & 1 & & & \\ \hline 1 & & 1 & & \\ \hline & & & 1 & 1 \\ \hline & & & 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & & & \\ \hline & 1 & 1 & & \\ \hline & & 1 & 1 & \\ \hline 1 & & & 1 & \\ \hline & & & & 1 \\ \hline \end{array}$$

Fig. 7.10

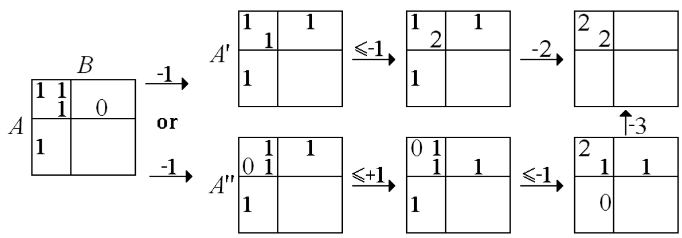
Conjecture: If M is big-split into P and Q then $\mathrm{dist}(M) = \mathrm{dist}(P) + \mathrm{dist}(Q) + 2pq$.

It is obvious that $\mathrm{dist}(M)$ is less than or equal to this by Lemma 3.30. I am convinced that equality is true, but I have only been able to prove this for $p = 1$ (Theorem 2.35) and $q = 1$ (Theorem 7.2):

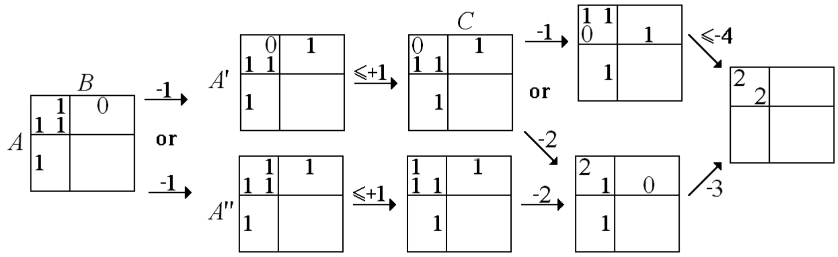
We give the following "sketch-proof" for $p = 2$ (although I am convinced a more beautiful proof exists). We use induction on n and then $\mathrm{dist}(M)$. Let the claim be true for matrices of order $k < n$ as well as $n \times n$ matrices of distance less than d. Let $M = M(A, B)$ be an $n \times n$ matrix of distance d.
If $P = \begin{smallmatrix}1 & \\ 1 & 1\end{smallmatrix}$ or $\begin{smallmatrix}1 & 1\\ 1 & \end{smallmatrix}$ then we are done by Theorem 2.35(i).
If $P = \begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}$ then we may assume the only way to start a minimal path from B to A is $B \underset{3,}{\text{—}} \cdots \underset{,1}{\text{—}} C \underset{0}{\text{—}} D$. Let $D \underset{1,}{\text{—}} \cdots \underset{,3}{\text{—}} E$. It is obvious that $M(A, E)$ is shorter than $M(A, B)$ so our result applies by induction. Yet $M(A, E)$ is big-split into P and Q', where Q' must be one shorter than Q. The result follows:



If $P = \begin{smallmatrix}1 & 1\\ & 1\end{smallmatrix}$ then we may assume the only adjacent chamber to A lying in-between A and B is a 0-swap of A, or we may use the argument used in the case of $P = \begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}$. Either it is A' or A'' (both shown below). In both cases the result follows by Theorem 2.42.



If $P = \boxed{\begin{smallmatrix} & 1\\ 1 & 1\end{smallmatrix}}$ then we may assume the only adjacent chamber to A lying in-between A and B is a 0-swap of A, or we may use the argument used in the case of $P = \boxed{\begin{smallmatrix} 1 & 1\\ 1 & 1\end{smallmatrix}}$. Either it is A' or A'' (both shown below). If it is A'' then we are done by Theorem 2.42. If it is A' then let $C \frac{}{1,2} B$. By Theorem 2.43 either there exists $D \frac{}{0} A'$ or there exists $E \frac{}{1,2} A'$ lying in-between A' and C. In either case the result follows.



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