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K-THEORY AND LIMIT-OF-DISCRETE-SERIES FOR THE UNIVERSAL COVER OF $\mathrm{SL}_2(\mathbb{R})$

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Abstract. The unitary principal series of the universal cover of $\mathrm{SL}_2(\mathbb{R})$ admits a limit-of-discrete-series. We show how this representation leads to an explicit K-cycle which generates K_1 of the reduced C^*-algebra.

1. Introduction

Let G denote the universal cover of $\mathrm{SL}_2(\mathbb{R})$. Then G is a connected Lie group with Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. Let $\mathfrak{A} = C_r^*(G)$ denote the reduced C^*-algebra of G. It is known (see, for example [CEN]) that, in the sense of K-theory of C^*-algebras, we have

$$K_0\,\mathfrak{A} = 0, \qquad K_1\,\mathfrak{A} = \mathbb{Z}. \tag{1}$$

In this article we relate this result to the representation theory of G.

The group G has the following properties:
– G is a 3-dimensional connected Lie group
– G has infinite centre isomorphic to $\mathbb{Z}$
– the maximal compact subgroup of G is trivial
– G is non-linear, i.e. it is not a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$.

The fact that it is a non-linear group with infinite centre places it outside the range of much classical representation theory, due to Harish-Chandra and others. However, the Plancherel formula was established by Pukánszky [P]. In the reduced dual of G, there is one very special representation, which is in the unitary principal series of G *and* is the direct sum of two elements in the discrete series. We will call this representation the *limit-of-discrete-series* of G. This representation factors through the quotient group $\mathrm{SL}_2(\mathbb{R})$, and becomes the well-known limit-of-discrete-series for $\mathrm{SL}_2(\mathbb{R})$.

The limit-of-discrete series for $\mathrm{SL}_2(\mathbb{R})$ is the induced representation

$$\pi := \mathrm{Ind}_B^{\mathrm{SL}_2(\mathbb{R})}\chi \tag{2}$$

where χ is the unique quadratic character

$$\chi : \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix} \mapsto \begin{cases} 1 & \text{when } x > 0 \\ -1 & \text{when } x < 0 \end{cases}$$

of the standard Borel subgroup B of $\mathrm{SL}_2(\mathbb{R})$. The representation π splits as the direct sum of two irreducible representations:

$$\pi = \pi^+ \oplus \pi^-$$

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The representations π^+ and π^- are not in the discrete series; but their characters θ^+, θ^- are not identically zero on the elliptic set – they have this feature in common with the discrete series of $\mathrm{SL}_2(\mathbb{R})$.

The formal Fourier transform $\widehat{D}$ of the Dirac operator D breaks up as the direct sum of multiplication operators on complex Hermitian line bundles. The multiplication is by real-valued scalar functions. With one exception, the scalar functions stay away from 0, i.e. they remain either positive or negative, the corresponding Kasparov triples are degenerate, and they make no contribution to K-theory.

We show in this article how the limit-of-discrete-series for G allows one to construct a certain complex hermitian line bundle L on the real line $\{q : q \in \mathbb{R}\}$ which realises a generator of $KK^1(\mathbb{C}, C_0(\mathbb{R}))$. We define a $\widehat{D}$-invariant complex Hermitian line bundle L as follows.

$$L_q := \begin{cases} \mathbb{C}\begin{pmatrix} f_\ell \\ -f_{-\ell} \end{pmatrix} & \text{for } q \leq 1/4, q = \ell(1-\ell) \\ \\ \mathbb{C}\begin{pmatrix} f_{1/2} \\ -f_{-1/2} \end{pmatrix} & \text{for } q \geq 1/4 \end{cases}$$

For $q \leq 1/4$ the fibre L_q is spanned by a spinor made from the *lowest weight vector* of the discrete series $D(\ell, +)$ and the *highest weight vector* of the discrete series $D(\ell, -)$ with $q = \ell(1-\ell)$. For $q \geq 1/4$ the fibre L_q is spanned by a spinor made from two vectors of weight $1/2$ and $-1/2$. Since $\ell = 1/2$ when $q = 1/4$, these two line bundles can be glued together to form a line bundle L over $\mathbb{R}$. The C_0-sections of L are spinor fields vanishing at infinity.

On these spinor fields, the operator $\widehat{D}$ is multiplication by a function ω_q for which $\omega_q \to \pm\infty$ as $q \to \mp\infty$. This implies that the unbounded Kasparov triple $[C_0(\mathbb{R}, L), 1, \omega]$ represents a generator of $KK^1(\mathbb{C}, C_0(\mathbb{R}))$, and thence a generator of $K_1(\mathfrak{A})$.

In §2, we give the structure theorem for $\mathfrak{A}$ via the compact-operator-valued Fourier transform.

In §3, we describe in detail the construction of the triple $[C_0(\mathbb{R}, L), 1, \omega]$.

In §4, we give, for completeness, a self-contained proof of (1).

2. Reduced C^*-algebra

We begin with the Plancherel formula of Pukánszky [P] for the universal cover of $\mathrm{SL}_2(\mathbb{R})$.

Theorem 2.1. *The following representations enter into the Plancherel formula:*

$$\text{Principal series}: \qquad \{(T(q,\tau) : q \geq 1/4,\ 0 \leq \tau \leq 1\}, \qquad \Omega = q$$

$$\text{Discrete series}: \qquad D(\ell, +), \quad D(\ell, -), \quad \ell \geq 1/2, \qquad \Omega = \ell(1-\ell)$$

where Ω is the Casimir operator. For every test function f on G, smooth with compact support, we have

$$f(e) = \int_0^\infty \int_0^1 \sigma[\Re \tanh \pi(\sigma + i\tau)]\Theta(\sigma,\tau)(f)d\tau d\sigma + \int_{1/2}^\infty (\ell - 1/2)\Theta(\ell)(f)d\ell$$

where the Harish-Chandra characters are

$$\Theta(\sigma,\tau)(f) = \text{trace} \int_G T(\sigma,\tau)(g)f(g)dg$$

$$\Theta(\ell)(f) = \text{trace} \int_G (D(\ell,+) \oplus D(\ell,-))(g)f(g)dg$$

and $\sigma = \sqrt{q - 1/4}$.

This is a *measure-theoretic* statement. We need a more precise statement in topology.

Note that $\Omega = 1/4$ at each of the following points;

$$T(1/4, 1/2), \quad D(1/2, +), \quad D(\ell, -).$$

We will define the *parameter space* $\mathcal{Z}$ to be the union of the sets

$$\{q \in \mathbb{R} : q \leq 1/4\}$$

$$\{(q, \tau) \in \mathbb{R} \times \mathbb{R} : q \geq 1/4, 0 \leq \tau \leq 1\}$$

with identification of the point $1/4$ in the first set with $(1/4, 1/2)$ in the second, and with identification of $(q, 0)$ with $(q, 1)$ for all $q \geq 1/4$.

Let E^+ and E^- be the two invariant subspaces of the reducible representation $T(1/4, 1/2)$. We have the following structure theorem.

Theorem 2.2. *Let $\mathfrak{A}$ denote the reduced C^*-algebra $C_r^*(G)$. The Fourier transform $f \mapsto \widehat{f}$ induces an isomorphism of $\mathfrak{A}$ onto the C^*-algebra*

$$\{F \in C_0(\mathcal{Z}, \mathfrak{K}) : F(q)E^+ \subset E^+, F(q)E^- \subset E^- \text{ if } q \leq 1/4\}.$$

where $\mathfrak{K}$ is the C^*-algebra of compact operators on the standard Hilbert space.

Proof. The reduced C^*-algebra is a quotient of the full C^*-algebra $C^*(G)$:

$$1 \to \mathfrak{I} \to C^*(G) \to C_r^*(G) \to 1.$$

The complementary series makes no contribution. The C^*-algebra $\mathfrak{A}$ is a quotient of the full C^*-algebra in [KM] and the primitive ideal space of $\mathfrak{A}$ contains every point in the support of Plancherel measure on $\widehat{G}$ (the unitary dual of G), by Theorem (2.1). $\square$

Note that the Jacobson topology on the primitive ideal spectrum of $\mathfrak{A}$ is exactly right: it has a double point at $q = 1/4 \in \mathcal{Z}$, where the unitary representation $T(1/4, 1/2)$ is reducible. The two subspaces E^+, E^- are invariant subspaces of the representation $T(1/4, 1/2)$ by Eqn.(2.4) in [KM] and in fact the representation $T(1/4, 1/2)$ is the direct sum of two representations in the discrete series:

$$T(1/4, 1/2) = D(1/2, +) \oplus D(1/2, -)$$

see Eqn.(2.4) in [KM, p.40].

3. The K-cycle

We recall the parameter space $\mathcal{Z}$ from §1. The unitary representation theory of G presents us with a continuous field of Hilbert G-modules over $\mathcal{Z}$:

$$\{V_q : q \leq 1/4\}$$
$$\{V_{q,\tau} : q \geq 1/4, 0 \leq \tau \leq 1\}$$

subject to the conditions

$$V_{1/4} = V_{1/4,1/2}, \quad V_{q,0} = V_{q,1} \qquad \forall q \geq 1/4/$$

The Casimir operator on the G-modules V_q and $V_{q,\tau}$ is precisely the multiplication by the parameter q. For $q \leq 1/4$ we also introduce an additional parameter $\ell \geq 1/2$ defined by the equation $q = \ell(1-\ell)$.

Let $\mathfrak{g}$ denote the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$, and let $C(\mathfrak{g})$ denote the Clifford algebra of $\mathfrak{g}$ with respect to the negative definite quadratic form on $\mathfrak{g}$. Let X_0, X_1, X_2 denote an orthonormal basis in $\mathfrak{g}$. Note that the notation in [P, (1.1)] is $l_k = X_k$.

Following the algebraic approach in [HP, Def. 3.1.2] the *Dirac operator* is the element of the algebra $U(\mathfrak{g}) \otimes C(\mathfrak{g})$ given by

$$D = X_0 \otimes c(X_0) + X_1 \otimes c(X_1) + X_2 \otimes c(X_2)$$

where $c(X_k)$ denotes Clifford multiplication by X_k.

Let

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

and set

$$c(X_k) = i\sigma_k, \qquad k = 0, 1, 2$$

Then we have

$$c(X_k)^2 = -1$$

for all $k = 0, 1, 2$.

We have

$$D = i(X_0\sigma_0 + X_1\sigma_1 + X_2\sigma_2) \tag{3}$$
$$= i\begin{pmatrix} X_0 & X_1 + iX_2 \\ X_1 - iX_2 & -X_0 \end{pmatrix} \tag{4}$$

Let now π be a unitary representation, in the principal series or the discrete series of G, on a Hilbert space E_π. The self-adjoint operators H_0, H_1, H_2, which act on the Hilbert space E_π, are determined by the following equation [P, p.98]:

$$\exp(-itH_k) = \pi(\exp(tX_k)) \qquad \forall t \in \mathbb{R}, k = 0, 1, 2$$

On each of the Hilbert spaces V_q and $V_{q,\tau}$ we therefore have three self-adjoint operators, namely H_0, H_1 and H_2. These form a field of operators on the field of Hilbert spaces E_π. The spectrum of H_0 is discrete with eigenvalues $m = \ell, \ell+1, \ell+2, \ldots$ and $m = -\ell, \ell-1, -\ell-2, \ldots$ in the case that $q < 1/4$ and with eigenvalues $m \in \tau + \mathbb{Z}$ for $q \geq 1/4$. Each eigenvalue has multiplicity 1 and we let f_m be an orthogonal basis of eigenvectors of H_0 so that

$$H_0 f_m = m f_m.$$

Following [P, p.100], we define

$$H_+ = H_1 + iH_2, \qquad H_- = H_1 - iH_2$$

In addition, we have the following equations

$$\begin{aligned} H_+ f_m &= (q + m(m+1))^{1/2} f_{m+1} \\ H_- f_m &= (q + m(m-1))^{1/2} f_{m-1} \end{aligned}$$

which hold for all m when $q \geq 1/4$ and where the first equation holds for all $m \neq -\ell$, the second for all $m \neq \ell$ when $q < 1/4$. The special cases of $H_+ f_{-\ell}$ and $H_- f_\ell$ are both zero.

We now construct a field of self-adjoint operators, which can be viewed as the formal Fourier Transform of the Dirac operator D, see (3). It is an operator of Dirac type and has the following form:

$$\widehat{D} = \begin{pmatrix} H_0 & H_1 + iH_2 \\ H_1 - iH_2 & -H_0 \end{pmatrix} \tag{5}$$

A crucial observation at this point is the emergence of two dimensional invariant subspaces for $\widehat{D}$. Each such subspace E_m is spanned by a pair of vectors $\begin{pmatrix} f_m \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ f_{m-1} \end{pmatrix}$, where m is in the set $\tau + \mathbb{Z}$ for $q \geq 1/4$ and $m = \ell+1, \ell+2, \ldots$ or $-\ell, -\ell-1, \ldots$ for $q < 1/4$. We have the following equations

$$\widehat{D} \begin{pmatrix} f_m \\ 0 \end{pmatrix} = \begin{pmatrix} H_0 & H_+ \\ H_- & -H_0 \end{pmatrix} \begin{pmatrix} f_m \\ 0 \end{pmatrix} = \begin{pmatrix} m f_m \\ (q + m(m-1))^{1/2} f_{m-1} \end{pmatrix}$$

and

$$\widehat{D} \begin{pmatrix} 0 \\ f_{m-1} \end{pmatrix} = \begin{pmatrix} H_0 & H_+ \\ H_- & -H_0 \end{pmatrix} \begin{pmatrix} 0 \\ f_{m-1} \end{pmatrix} = \begin{pmatrix} (q + m(m-1))^{1/2} f_m \\ -(m-1) f_{m-1} \end{pmatrix}.$$

With respect to this basis, the operator $\widehat{D}$ is given by the following symmetric matrix

$$\begin{pmatrix} m & (q + m(m-1))^{1/2} \\ (q + m(m-1))^{1/2} & -(m-1) \end{pmatrix}.$$

This symmetric matrix has the following eigenvalues

$$\lambda = \frac{1}{2} \pm \sqrt{1/4 + q + 2m(m-1)}.$$

In the case that $q \geq 1/4$ the subspaces E_m for $m \in \tau + \mathbb{Z}$ span the whole of $V_{q,\tau}$. However, for $q < 1/4$ there are a further two 1-dimensional subspaces spanned by the vectors

$$\begin{pmatrix} f_\ell \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ f_{-\ell} \end{pmatrix}$$

These subspaces are invariant since $H_-(f_\ell) = 0$ and $H_+(f_{-\ell}) = 0$.

Note that something very special occurs in the limit-of-discrete series when $q = 1/4$ and $\tau = 1/2$. Here, when $m = 1/2$ we have

$$q + m(m-1) = 0$$

and so in this case the operator matrix $\begin{pmatrix} H_0 & H_+ \\ H_- & -H_0 \end{pmatrix}$ restricted to the 2-dimensional subspace spanned by $\begin{pmatrix} f_{1/2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ f_{-1/2} \end{pmatrix}$ is the diagonal matrix

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

For $q < 1/4$, we have the following eigenvector equations

$$\begin{pmatrix} H_0 & H_+ \\ H_- & -H_0 \end{pmatrix} \begin{pmatrix} f_l \\ 0 \end{pmatrix} = l \begin{pmatrix} f_l \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} H_0 & H_+ \\ H_- & -H_0 \end{pmatrix} \begin{pmatrix} 0 \\ f_{-l} \end{pmatrix} = l \begin{pmatrix} 0 \\ f_{-l} \end{pmatrix}.$$

With respect to the basis given by the vectors $\begin{pmatrix} f_l \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ f_{-l} \end{pmatrix}$ our operator matrix has the diagonal form $\begin{pmatrix} l & 0 \\ 0 & l \end{pmatrix}$, so we see that as q approaches $1/4$ and consequently ℓ approaches $1/2$ this matches up with the limit-of-discrete series case of $q = 1/4, \tau = 1/2$, justifying the terminology.

We now consider the restriction of the field of Hilbert spaces to the subspace of $\mathcal{Z}$ where for $q \geq 1/4$ we have $\tau = 1/2$. The q-parameter here identifies the subspace with $\mathbb{R}$. Correspondingly in the Fourier transform picture we are restricting the field of compact-operator valued functions to the line, which induces an isomorphism on K-theory.

Remark 3.1. *The half-line $\{q \in \mathbb{R} : q \geq 1/4\}$ has the following significance in representation theory. The corresponding unitary representations $T(q, 1/2)$ all factor through $\mathrm{SL}_2(\mathbb{R})$ and constitute the odd principal series π_q of $\mathrm{SL}_2(\mathbb{R})$. In particular, the representation $\pi_{1/4}$ is the limit-of-discrete series for $\mathrm{SL}_2(\mathbb{R})$. It is the direct sum of two irreducible representations whose characters θ_+ and θ_- do not vanish on the elliptic set. In this respect, they resemble representations in the discrete series. So the term* limit-of-discrete-series *for $T(1/4, 1/2)$ is surely apt.*

We now attempt to glue together some one-dimensional eigenspaces to form a complex hermitian line bundle L over $\mathbb{R}$. Take $q \geq 1/4$ and consider the subspace $E_{1/2}$. The restriction $\widehat{D}|_{E_{1/2}}$ is given by the matrix

$$\begin{pmatrix} 1/2 & \sqrt{q - 1/4} \\ \sqrt{q - 1/4} & 1/2 \end{pmatrix}$$

from which we readily see that the vector $\begin{pmatrix} f_{1/2} \\ -f_{-1/2} \end{pmatrix}$ is an eigenvector of $\widehat{D}$ with eigenvalue $1/2 - \sqrt{q - 1/4}$. Note that the eigenvalue tends to $1/2$ as $q \to 1/4^+$.

Now for $q < 1/4$ we see that $\begin{pmatrix} f_\ell \\ -f_{-\ell} \end{pmatrix}$ is an eigenvector of $\widehat{D}$ with eigenvalue ℓ tending to $1/2$ as $q \to 1/4^-$.

We can thus define a $\widehat{D}$-invariant complex Hermitian line bundle L as follows.

$$\text{Define } L_q := \begin{cases} \mathbb{C}\begin{pmatrix} f_\ell \\ -f_{-\ell} \end{pmatrix} & \text{for } q \leq 1/4, q = \ell(1-\ell) \\ \\ \mathbb{C}\begin{pmatrix} f_{1/2} \\ -f_{-1/2} \end{pmatrix} & \text{for } q \geq 1/4 \end{cases}$$

On this line-bundle the field of operators $\widehat{D}$ is simply multiplication by the function

$$\omega(q) := \begin{cases} \ell = \frac{1}{2} + \sqrt{1/4 - q} & \text{for } q \leq 1/4 \\ \frac{1}{2} - \sqrt{q - 1/4} & \text{for } q \geq 1/4. \end{cases}$$

In particular we note that $\omega(q)$ tends to $\pm\infty$ as $q \to \mp\infty$. The field of operators induces an operator on the Hilbert module $C_0(\mathbb{R}, L)$ of C_0-sections of the bundle, and this operator is multiplication by ω. This means that the unbounded Kasparov triple $[C_0(\mathbb{R}, L), 1, \omega]$ represents the generator of $KK^1(\mathbb{C}, C_0(\mathbb{R}))$.

Theorem 3.2. *The unbounded Kasparov triple* $[C_0(\mathbb{R}, L), 1, \omega]$ *represents the generator of* $KK^1(\mathbb{C}, C_0(\mathbb{R}))$.

Similarly we have a $\widehat{D}$-invariant line bundle M with

$$M_q := \begin{cases} \mathbb{C}\begin{pmatrix} f_\ell \\ f_{-\ell} \end{pmatrix} & \text{for } q \leq 1/4, q = \ell(1-\ell) \\ \\ \mathbb{C}\begin{pmatrix} f_{1/2} \\ f_{-1/2} \end{pmatrix} & \text{for } q \geq 1/4 \end{cases}$$

on which the operator $\widehat{D}$ is multiplication by the function

$$\varepsilon(q) := \begin{cases} \ell = \frac{1}{2} + \sqrt{1/4 - q} & \text{for } q \leq 1/4 \\ \frac{1}{2} + \sqrt{q - 1/4} & \text{for } q \geq 1/4. \end{cases}$$

In this case we see that $\varepsilon(q) \to +\infty$ as $q \to \pm\infty$ from which we see that the corresponding Kasparov triple $[C_0(\mathbb{R}, M), 1, \varepsilon]$ in $KK^1(\mathbb{C}, C_0(\mathbb{R}))$ is homotopic to a degenerate element and so no contribution is made to KK^1.

We now examine how the remaining 2-dimensional subspaces E_m match up at $q = 1/4$. For $q \geq 1/4$ we have $m = 1/2 + k$ where $k \in \mathbb{Z}$ and we exclude the case $k = 0$ which we have already considered. Now for each $k > 0$ we take the 2-dimensional bundle $N^{(k)}$ whose fibres are $E_{1/2+k}$ for $q \geq 1/4$ and which are $E_{\ell+k}$ for $q \leq 1/4$. These agree at $q = 1/4$ since $\ell = 1/2$ at this point.

For $k < 0$ we take the 2-dimensional bundle $N^{(k)}$ whose fibres are $E_{1/2+k}$ for $q \geq 1/4$ and which are $E_{-\ell+1+k}$ for $q \leq 1/4$. We note that when $\ell = 1/2$ we obtain $E_{-\ell+1+k} = E_{1/2+k}$. Thus for each $k \neq 0$ we can view m as a continuous function of q defined by

$$m(q) := \begin{cases} \ell + k = \frac{1}{2} + \sqrt{1/4 - q} + k & \text{for } q \leq 1/4, k = 1, 2, \ldots \\ -\ell + 1 + k = \frac{1}{2} - \sqrt{1/4 - q} + k & \text{for } q \leq 1/4, k = -1, -2, \ldots \\ \frac{1}{2} + k & \text{for } q \geq 1/4, k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

Recall that the eigenvalues of the restriction of $\widehat{D}$ to E_m are given by

$$\lambda^{\pm} = \frac{1}{2} \pm \sqrt{1/4 + q + 2m(m-1)}.$$

Writing $2m(m-1)$ as $2(m-1/2)^2 - 1/2$ we see that

$$2m(q)(m(q)-1) := \begin{cases} 2(\sqrt{1/4 - q} + k)^2 - 1/2 & \text{for } q \leq 1/4, k = 1, 2, \ldots \\ 2(-\sqrt{1/4 - q} + k)^2 - 1/2 & \text{for } q \leq 1/4, k = -1, -2, \ldots \\ 2k^2 - 1/2 & \text{for } q \geq 1/4, k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

$$= \begin{cases} 2(1/4 - q + k^2 + 2k\sqrt{1/4 - q}) - 1/2 & \text{for } q \leq 1/4, k = 1, 2, \ldots \\ 2(1/4 - q + k^2 - 2k\sqrt{1/4 - q}) - 1/2 & \text{for } q \leq 1/4, k = -1, -2, \ldots \\ 2k^2 - 1/2 & \text{for } q \geq 1/4, k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

$$= \begin{cases} 2(-q + k^2 + 2k\sqrt{1/4 - q}) & \text{for } q \leq 1/4, k = 1, 2, \ldots \\ 2(-q + k^2 - 2k\sqrt{1/4 - q}) & \text{for } q \leq 1/4, k = -1, -2, \ldots \\ 2k^2 - 1/2 & \text{for } q \geq 1/4, k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

$$1/4 + q + 2m(m-1) = \begin{cases} 1/4 - q + 2(k^2 + 2k\sqrt{1/4 - q}) & \text{for } q \leq 1/4, k = 1, 2, \ldots \\ 1/4 - q + 2(k^2 - 2k\sqrt{1/4 - q}) & \text{for } q \leq 1/4, k = -1, -2, \ldots \\ q - 1/4 + 2k^2 & \text{for } q \geq 1/4, k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

In particular we see that the discriminant $1/4 + q + 2m(m-1)$ is always at least 2 so that the eigenvalues must always be distinct, indeed they are respectively $\geq \frac{1}{2} + \sqrt{2}$ and $\leq \frac{1}{2} - \sqrt{2}$. Thus the bundles of positive and negative eigenspaces within $N^{(k)}$ define $\widehat{D}$-invariant line bundles which we denote $N^{(k)\pm}$. Moreover in the positive case the eigenvalues tend to $+\infty$ as $q \to \pm\infty$ and in the negative case eigenvalues tend to $-\infty$ as $q \to \pm\infty$.

This establishes the following result.

Theorem 3.3. *Each individual line bundle thus gives a Kasparov triple* $[C_0(\mathbb{R}, N^{(k)\pm}), 1, \lambda^{\pm}]$ *(where we view the eigenvalue* $\lambda^{\pm}$ *as a function of* q*) which is homotopic to a degenerate element.*

Finally, we have

Theorem 3.4. *The* K*-theory is concentrated in the complex hermitian line bundle* L *constructed above.*

Proof. In this context, $KK^1(\mathbb{C}, C_0(\mathbb{R}))$ is countably additive. □

4. K-theory

Let

$$A = \{z \in \mathbb{C}; |z| \leq 1\}, \qquad B = [1,2]$$

and let

$$\mathcal{Y} = A \cup B.$$

The coordinate change

$$r = (q + 1/4)^{-1}, \qquad s = 2 - 1/2\ell$$

transforms, by a continuous injective map, the locally compact parameter space $\mathcal{Z}$ into the compact parameter space $\mathcal{Y}$.

Lemma 4.1. *The reduced C^*-algebra $\mathfrak{A}$ is strongly Morita equivalent to the C^*-algebra $\mathfrak{B}$ of all 2×2-matrix-valued functions on the compact Hausdorff space Y which are diagonal on B, and vanish at 0 and 2:*

$$\mathfrak{B} := \{F \in C(\mathcal{Y}, M_2(\mathbb{C})) : F(y) \text{ is diagonal on } B, F(0) = 0 = F(2)\}$$

The computation of the K-theory of $\mathfrak{A}$ is done as follows. Define a new C^*-algebra as follows:

$$\mathfrak{C} := \{F \in C(\mathcal{Y}, M_2(\mathbb{C})) : F(y) \text{ is diagonal on } B, F(2) = 0\}$$

The map

$$\mathfrak{C} \to M_2(\mathbb{C}), \qquad f \mapsto f(0)$$

then fits into an exact sequence of C^*-algebras

$$1 \to \mathfrak{B} \to \mathfrak{C} \to M_2(\mathbb{C}) \to 0$$

This leads to the six-term exact sequence

$$\begin{array}{ccccc} K_0(\mathfrak{B}) & \longrightarrow & K_0(\mathfrak{C}) & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(\mathfrak{C}) & \longleftarrow & K_1(\mathfrak{B}) \end{array}$$

Note that $\mathcal{Y} = A \cup B$ is a contractible space. The following homotopy is well-adapted to the C^*-algebra $\mathfrak{C}$. Given $z = x + iy \in \mathcal{Y}$ set h_t as follows:

$$h_t(x + iy) = \begin{cases} x + (1 - 2t)iy & 0 \leq t \leq 1/2 \\ x + (2 - x)(2t - 1) & 1/2 \leq t \leq 1 \end{cases}$$

This is a homotopy equivalence from $\mathcal{Y}$ to the point $\{2\}$. Note that the homotopy h_t with $1/2 \leq t \leq 1$ moves along the interval B towards the point $\{2\}$.

This induces a homotopy equivalence from $\mathfrak{C}$ to the zero C^*-algebra $\mathfrak{O}$:

$$\mathfrak{C} \sim_h \mathfrak{O}$$

i.e. $\mathfrak{C}$ is a contractible C^*-algebra.

This leads to the six-term exact sequence

$$\begin{array}{ccccc} K_0(\mathfrak{B}) & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & 0 & \longleftarrow & K_1(\mathfrak{B}) \end{array}$$

Since K_j is an invariant of strong Morita equivalence, we have the following result.

Theorem 4.2. *Let $\mathfrak{A}$ denote the reduced C^*-algebra of the universal cover of $\mathrm{SL}_2(\mathbb{R})$. Then*

$$K_0(\mathfrak{A}) = 0, \qquad K_1(\mathfrak{A}) = \mathbb{Z}$$

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