

Depth and the local Langlands correspondence

Aubert, Anne-Marie and Baum, Paul and Plymen,
Roger and Solleveld, Maarten

2013

MIMS EPrint: **2013.55**

Manchester Institute for Mathematical Sciences
School of Mathematics

The University of Manchester

Reports available from: <http://eprints.maths.manchester.ac.uk/>

And by contacting: The MIMS Secretary
School of Mathematics
The University of Manchester
Manchester, M13 9PL, UK

ISSN 1749-9097

DEPTH AND THE LOCAL LANGLANDS CORRESPONDENCE

ANNE-MARIE AUBERT, PAUL BAUM, ROGER PLYMEN, AND MAARTEN SOLLEVELD

ABSTRACT. Let G be an inner form of a general linear group or a special linear group over a non-archimedean local field. We prove that the local Langlands correspondence for G preserves depths.

CONTENTS

1. Introduction	1
2. The local Langlands correspondence for inner forms of $\mathrm{GL}_n(F)$	3
2.1. The statement of the correspondence	3
2.2. The Jacquet–Langlands correspondence	4
2.3. Depth for Langlands parameters	6
2.4. The depth of representations of $\mathrm{GL}_m(D)$	8
2.5. Conductors of representations of $\mathrm{GL}_m(D)$	12
2.6. Depth preservation	14
3. The local Langlands correspondence for inner forms of $\mathrm{SL}_n(F)$	16
3.1. The statement of the correspondence	16
3.2. Depth preservation	17
References	18

1. INTRODUCTION

Let F be a non-archimedean local field and let G be a connected reductive group over F . Let $\Phi(G)$ denote the collection of equivalence classes of Langlands parameters for G , and $\mathrm{Irr}(G)$ the set of (isomorphism classes of) irreducible smooth G -representations. A central role in the representation theory of such groups is played by the local Langlands correspondence (LLC). It is supposed to be a map

$$\mathrm{Irr}(G) \rightarrow \Phi(G)$$

that enjoys several naturality properties [Bor, Vog]. The LLC should preserve interesting arithmetic information, like local L-functions and ϵ -factors. A lesser-known invariant that makes sense on both sides of the LLC is *depth*.

The depth of a Langlands parameter ϕ is easy to define. For $r \in \mathbb{R}_{\geq 0}$ let $\mathrm{Gal}(F_s/F)^r$ be the r -th ramification subgroup of the absolute Galois group of F .

Date: November 6, 2013.

2010 Mathematics Subject Classification. 20G05, 22E50.

Key words and phrases. representation theory, p-adic group, local Langlands program, division algebra.

The second author was partially supported by NSF grant DMS-1200475.

Then the depth of ϕ is the smallest number $d(\phi) \geq 0$ such that ϕ is trivial on $\text{Gal}(F_s/F)^r$ for all $r > d(\phi)$.

The depth $d(\pi)$ of an irreducible G -representation π was defined by Moy and Prasad [MoPr1, MoPr2], in terms of filtrations $P_{x,r}$ ($r \in \mathbb{R}_{\geq 0}$) of the parahoric subgroups $P_x \subset G$. On the basis of several examples (see below) it is reasonable to expect that for every Langlands parameter $\phi \in \Phi(G)$ with L-packet $\Pi_\phi(G) \subset \text{Irr}(G)$ one has

$$(1) \quad d(\phi) = d(\pi) \quad \text{for all } \pi \in \Pi_\phi(G).$$

This relation would be useful for several reasons. Firstly, it allows one to employ some counting arguments in the local Langlands correspondence, because (up to unramified twists) there are only finitely many irreducible representations and Langlands parameters whose depth is at most a specified upper bound.

Secondly, it would be a step towards a more explicit LLC. One can try to determine the groups $P_{x,r}/P_{x,r+\epsilon}$ ($\epsilon > 0$ small) and their representations explicitly, and to match them with representations of $\text{Gal}(F_s/F)/\text{Gal}(F_s/F)^{r+\epsilon}$.

Thirdly, one can use (1) as a working hypothesis when trying to establish a local Langlands correspondence, to determine whether or not two irreducible representations stand a chance of belonging to the same L-packet.

The most basic case of depth preservation concerns Langlands parameters $\phi \in \Phi(G)$ that are trivial on both the inertia group \mathbf{I}_F and on $\text{SL}_2(\mathbb{C})$. These can be regarded as Langlands parameters of negative depth. Such a ϕ is only relevant for G if G is quasi-split and splits over an unramified extension of F . In that case one can say that an irreducible G -representation has negative depth if it possesses a nonzero vector fixed by a hyperspecial compact subgroup. The Satake isomorphism shows how to each such representation one can associate (in a natural way) a Langlands parameter of the above kind.

The G -representations of depth zero have been subjected to ample study, see for example [GSZ, Mor, DBRe, Mœ]. According to Moy–Prasad, an irreducible representation has depth zero if and only if it has nonzero vectors fixed by the pro-unipotent radical of some parahoric subgroup of G . This includes Iwahori-spherical representations and Lusztig’s unipotent representations [Lus1, Lus2]. A Langlands parameter has depth zero if and only if it is trivial on the wild inertia subgroup of the absolute Galois group of F . For depth zero the equality (1) is conjectured, and proven in certain cases, in [DBRe]. It fits very well with the aforementioned work of Lusztig.

In positive depth there is the result of Yu [Yu2, §7.10], who proved (1) for unramified tori. For $\text{GL}_n(F)$, (1) was claimed in [Yu1, §2.3.6] and proved in [ABPS2, Proposition 4.5]. For $\text{GSp}_4(F)$, (1) is proved in [Gan, § 10]. We refer to [GrRe, Ree, ReYu] for some interesting examples of positive depth Langlands parameters and supercuspidal representations. Most of these examples satisfy (1), but in the introduction of [ReYu] some particular cases are mentioned in which (1) does not hold. So it remains to be seen in which generality the local Langlands correspondence will preserve depths.

In this paper we will prove that the local Langlands correspondence preserves depth for two classes of groups: the inner forms of $\text{GL}_n(F)$ and the inner forms of $\text{SL}_n(F)$. In a few non-split cases, this was done before in [LaRa].

Let D be a division algebra with centre F , of dimension d^2 over F . Then $G = \mathrm{GL}_m(D)$ is an inner form of $\mathrm{GL}_n(F)$ with $n = dm$. There is a reduced norm map $\mathrm{Nrd} : \mathrm{GL}_m(D) \rightarrow F^\times$ and the derived group $G_{\mathrm{der}} := \ker(\mathrm{Nrd} : G \rightarrow F^\times)$ is an inner form of $\mathrm{SL}_n(F)$. Every inner form of $\mathrm{GL}_n(F)$ or $\mathrm{SL}_n(F)$ is isomorphic to one of this kind.

The main steps in the proof of our depth-preservation theorem are:

- With the Langlands classification one reduces the problem to essentially square-integrable representations and elliptic Langlands parameters.
- Express the depth in terms of ϵ -factors and conductors. This is a technical step which involves detailed knowledge of the representation theory of G . Here it is convenient to use an alternative but equivalent version of depth, the normalized level of an irreducible G -representation.
- Show that the Jacquet–Langlands correspondence for $G = \mathrm{GL}_m(D)$ preserves ϵ -factors. Since the LLC for $\mathrm{GL}_m(D)$ is defined as a composition of the Jacquet–Langlands correspondence with the LLC for $\mathrm{GL}_n(F)$ and the latter is known to preserve ϵ -factors, this proves depth-preservation for inner forms of $\mathrm{GL}_n(F)$.
- Relate the depth for G_{der} to depth for G . The relations for irreducible representations and for Langlands parameters are analogous. As the LLC for G_{der} is derived from that for G , this settles depth-preservation for inner forms of $\mathrm{SL}_n(F)$.

This paper develops results presented by the second author in a lecture at the 2013 Arbeitstagung.

The authors wish to thank Vincent Sécherre for some helpful explanations on the construction of simple types for $\mathrm{GL}_m(D)$.

2. THE LOCAL LANGLANDS CORRESPONDENCE FOR INNER FORMS OF $\mathrm{GL}_n(F)$

2.1. The statement of the correspondence.

The local Langlands correspondence for $\mathrm{GL}_n(F)$ was established in the important papers [LRS, HaTa, Hen2, Zel]. Together with the Jacquet–Langlands correspondence this provides the LLC for inner forms $G = \mathrm{GL}_m(D)$ of $\mathrm{GL}_n(F)$, see [HiSa, ABPS1]. For these groups every L-packet is a singleton and the LLC is a canonical bijective map

$$(2) \quad \mathrm{rec}_{D,m} : \mathrm{Irr}(\mathrm{GL}_m(D)) \rightarrow \Phi(\mathrm{GL}_m(D)).$$

A remarkable aspect of Langlands’ conjectures [Vog] is that it is better to consider not just one reductive group at a time, but all inner forms of a given group simultaneously. Inner forms share the same Langlands dual group, so in (2) the right hand side is the same for all inner forms G of the given group. Then one can turn (2) into a bijection by defining a suitable equivalence relation on the set of inner forms and taking the corresponding union of the sets $\mathrm{Irr}(G)$ on the left hand side (see Theorem 2.1 below).

We define the equivalence classes of such inner forms to be in bijection with the isomorphism classes of central simple F -algebras of dimension n^2 via $M_m(D) \mapsto \mathrm{GL}_m(D)$, respectively $M_m(D) \mapsto \mathrm{GL}_m(D)_{\mathrm{der}}$.

As Langlands dual group of G we take $\mathrm{GL}_n(\mathbb{C})$. To deal with inner forms it is advantageous to consider the conjugation action of $\mathrm{SL}_n(\mathbb{C})$ on these two groups. It

induces a natural action of $\mathrm{SL}_n(\mathbb{C})$ on the collection of Langlands parameters for $\mathrm{GL}_n(F)$. For any such parameter ϕ we can define

$$(3) \quad C(\phi) = Z_{\mathrm{SL}_n(\mathbb{C})}(\mathrm{im} \phi), \quad \text{and} \quad \mathcal{S}_\phi = C(\phi)/C(\phi)^\circ.$$

Notice that the centralizers are taken in $\mathrm{SL}_n(\mathbb{C})$ and not in the Langlands dual group.

Via the Langlands correspondence the non-trivial irreducible representations of \mathcal{S}_ϕ are associated to irreducible representations of non-split inner forms of $\mathrm{GL}_n(F)$. For example, consider a Langlands parameter ϕ for $\mathrm{GL}_2(F)$ which is elliptic, that is, whose image is not contained in any torus of $\mathrm{GL}_2(\mathbb{C})$. Then $\mathcal{S}_\phi = Z(\mathrm{SL}_2(\mathbb{C})) \cong \{\pm 1\}$. The pair $(\phi, \mathrm{triv}_{\mathcal{S}_\phi})$ parametrizes an essentially square-integrable representation of $\mathrm{GL}_2(F)$ and $(\phi, \mathrm{sgn}_{\mathcal{S}_\phi})$ parametrizes an irreducible representation of the inner form D^\times , where D denotes a noncommutative division algebra of dimension 4 over F .

The enhanced version of the local Langlands correspondence for all inner forms of general linear groups over nonarchimedean local fields says:

Theorem 2.1. [ABPS2, Theorem 1.1]

There is a canonical bijection between:

- pairs (G, π) with $\pi \in \mathrm{Irr}(G)$ and G an inner form of $\mathrm{GL}_n(F)$, considered up to equivalence;
- $\mathrm{GL}_n(\mathbb{C})$ -conjugacy classes of pairs (ϕ, ρ) with $\phi \in \Phi(\mathrm{GL}_n(F))$ and $\rho \in \mathrm{Irr}(\mathcal{S}_\phi)$.

For these Langlands parameters a character of \mathcal{S}_ϕ determines an inner form of $\mathrm{GL}_n(F)$ via the Kottwitz isomorphism [Kot]. In contrast with the usual LLC, our L-packets for inner forms of general linear groups need not be singletons. To be precise, the packet Π_ϕ contains the unique representation $\mathrm{rec}_{D,m}^{-1}(\phi)$ of $G = \mathrm{GL}_m(D)$ if ϕ is relevant for G , and no G -representations otherwise.

2.2. The Jacquet–Langlands correspondence.

A representation π of G is called essentially square-integrable if $\pi|_{G^{\mathrm{der}}}$ is square-integrable and there exists an unramified character χ of G such that $\pi \otimes \chi$ is unitary. We denote the set of (equivalence classes of) irreducible essentially square-integrable G -representations by $\mathrm{Irr}_{\mathrm{ess}L^2}(G)$. There is a natural bijection between $\mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_n(F))$ and $\mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_m(D))$, discovered first for $\mathrm{GL}_2(F)$ by Jacquet and Langlands [JaLa]. The local Langlands correspondence for $\mathrm{GL}_m(D)$ is constructed with the help of the Jacquet–Langlands correspondence. Here we recall some useful properties of the latter correspondence.

Theorem 2.2. *Let $\mathrm{GL}_m(D)$ be an inner form of $\mathrm{GL}_n(F)$. There exists a canonical bijection*

$$\mathrm{JL} : \mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_n(F)) \rightarrow \mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_m(D))$$

with the following properties:

- (a) *There is a canonical identification of the semisimple elliptic conjugacy classes in $\mathrm{GL}_n(F)$ with those in $\mathrm{GL}_m(D)$. Let $g \in \mathrm{GL}_n(F)$ and $g' \in \mathrm{GL}_m(D)$ be semisimple elliptic elements in the same conjugacy class and let θ_π be the character of $\pi \in \mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_n(F))$. Then*

$$(-1)^n \theta_\pi(g) = (-1)^m \theta_{\mathrm{JL}(\pi)}(g').$$

(b) JL preserves twists with characters of F^\times :

$$\mathrm{JL}(\pi \otimes \chi \circ \det) = \mathrm{JL}(\pi) \otimes \chi \circ \mathrm{Nrd}.$$

(c) JL respects contragredients: $\mathrm{JL}(\pi^\vee) = \mathrm{JL}(\pi)^\vee$.

(d) Let P' be a standard parabolic subgroup of $\mathrm{GL}_m(D)$, with Levi factor $M' = \prod_i \mathrm{GL}_{m_i}(D)$. Let P be the corresponding standard parabolic subgroup of $\mathrm{GL}_n(F)$, with Levi factor $M = \prod_i \mathrm{GL}_{dm_i}(F)$. Then the Jacquet modules $r_P^{\mathrm{GL}_n(F)}(\pi)$ and $r_{P'}^{\mathrm{GL}_m(D)}(\mathrm{JL}(\pi))$ are either both zero or both irreducible and essentially square-integrable. In the latter case

$$\mathrm{JL}(r_P^{\mathrm{GL}_n(F)}(\pi)) = r_{P'}^{\mathrm{GL}_m(D)}(\mathrm{JL}(\pi)).$$

In other words, JL and its version for M and M' respect Jacquet restriction.

(e) JL preserves supercuspidality.

(f) $\mathrm{JL}(\mathrm{St}_{\mathrm{GL}_n(F)}) = \mathrm{St}_{\mathrm{GL}_m(D)}$, where St_G denotes the Steinberg representation of G .

(g) JL preserves γ -factors:

$$\gamma(s, \mathrm{JL}(\pi), \psi) = \gamma(s, \pi, \psi) \quad \text{for any nontrivial character } \psi \text{ of } F.$$

(h) JL preserves L-functions: $L(s, \mathrm{JL}(\pi)) = L(s, \pi)$.

(i) JL preserves ϵ -factors: $\epsilon(s, \mathrm{JL}(\pi), \psi) = \epsilon(s, \pi, \psi)$.

Proof. The correspondence, which is in fact characterized by property (a), is proven over p -adic fields in [DKV] and over local fields of positive characteristic in [Bad]. Properties (b) and (c) are obvious in view of (a). The same goes for property (f) in the case $m = 1$, because then $\mathrm{St}_{\mathrm{GL}_m(D)}$ is just the trivial representation of D^\times . For (d) see [Bad, §5], in particular Proposition B. Obviously (d) implies (e). Property (f) for $m > 1$ follows from (f) for $m = 1$ and property (d). Property (g) was proven over local function fields in [Bad, p. 741], with an argument that also works over p -adic fields.

Properties (h) and (i) were claimed in [DKV], with the difference that the ϵ -factors of π and $\mathrm{JL}(\pi)$ are said to agree only up to a sign $(-1)^{n+m}$. This sign is due to a convention that does not agree with [GoJa], which we use for the definition of ϵ -factors. Unfortunately the argument for (h) and (i) given in [DKV, §B.1] is incorrect. Instead, we will establish (h) by direct calculation.

Let ν_D denote the unramified character $g' \mapsto \|\mathrm{Nrd} g'\|_F$ of $\mathrm{GL}_m(D)$. Consider any $\pi' \in \mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_m(D))$. By [DKV, §B.2] or [Tad, §2] there exist:

- integers a, b, s_σ such that $ab = m$ and s_σ divides ad ;
- an irreducible supercuspidal representation σ of $\mathrm{GL}_a(D)$,

such that π' is a constituent of the parabolically induced representation

$$(4) \quad \Pi' := I_{\mathrm{GL}_a(D)^b}^{\mathrm{GL}_m(D)}(\nu_D^{s_\sigma \frac{1-b}{2}} \sigma \otimes \nu_D^{s_\sigma \frac{3-b}{2}} \sigma \otimes \cdots \otimes \nu_D^{s_\sigma \frac{b-1}{2}} \sigma).$$

By [Jac, Proposition 2.3] the L-function of (4) is the product of L-functions of the inducing representations:

$$(5) \quad L(s, \Pi') = \prod_{k=1}^b L(s, \nu_D^{s_\sigma(k-(1+b)/2)} \sigma) = \prod_{k=1}^b L(s + s_\sigma(k - (1+b)/2), \sigma).$$

By definition $L(s, \pi')^{-1}$ is a monic polynomial in q^{-s} , and by [Jac, 2.7.4] it is a factor of the monic polynomial $L(s, \Pi')^{-1}$. Now there are two cases to be distinguished, depending on whether σ is an unramified representation of D^\times or not.

Case 1: $a = 1, b = m$ and σ is unramified.

There exists an unramified character χ of F^\times such that $\sigma = \chi \circ \text{Nrd}$. By [DKV, §B.2] or [Tad, §2] (4) only has an essentially square-integrable subquotient if $s_\sigma = d$. Then $\pi' \cong \text{St}_{\text{GL}_m(D)} \otimes \chi \circ \text{Nrd}$, so $\text{JL}^{-1}(\pi') \cong \text{St}_{\text{GL}_n(F)} \otimes \chi \circ \det$. With property (f) this enables us to compute the γ -factor. Let ω_F be a uniformizer of F , \mathfrak{o}_F the ring of integers and \mathfrak{p}_F its maximal ideal. Assume that ψ is trivial on \mathfrak{p}_F but not on \mathfrak{o}_F . Then

$$(6) \quad \gamma(s, \pi', \psi) = \gamma(s, \text{St}_{\text{GL}_n(F)} \otimes \chi \circ \det, \psi) = (-1)^n q^{n/2} \frac{1 - q^{-s+(1-n)/2} \chi(\omega_F)}{1 - q^{-s+(1+n)/2} \chi(\omega_F)}.$$

By [GoJa, Proposition 4.4], (5) becomes

$$(7) \quad \prod_{k=1}^m L(s + d(k - (1+m)/2), \chi \circ \text{Nrd}) = \prod_{k=1}^m L(s + d(k - (1+m)/2) + (d-1)/2, \chi).$$

Now we apply [GoJa, Theorem 7.11.4] (it is stated only for $\text{GL}_n(F)$, but the proof with zeros and poles of L-functions goes through because we know $\gamma(s, \pi', \psi)$) and we find that for the L-function of π' we need only the factor $k = m$ of (7):

$$L(s, \pi', \psi) = L(s + (n-1)/2, \chi) = (1 - q^{-s+(1-n)/2} \chi(\omega_F))^{-1}.$$

In particular the whole calculations works with $d = 1$, so

$$(8) \quad L(s, \text{St}_{\text{GL}_m(D)} \otimes \chi \circ \text{Nrd}) = L(s, \text{St}_{\text{GL}_n(F)} \otimes \chi \circ \det) = L(s + (n-1)/2, \chi).$$

Case 2: all other σ .

Then [GoJa, Proposition 5.11] says that $L(s, \sigma \otimes \chi) = 1$ for every unramified character χ of $\text{GL}_a(D)$. Hence $L(s, \Pi') = 1$ by (5). We observed above that $L(s, \pi')^{-1}$ is a factor of $L(s, \Pi')$, so $L(s, \pi') = 1$. Because JL is bijective, $\text{JL}^{-1}(\pi')$ is not an unramified twist of the Steinberg representation, so $L(s, \text{JL}^{-1}(\pi')) = 1$ as well. This proves property (h).

In view of the relation

$$(9) \quad \epsilon(s, \pi, \psi) = \gamma(s, \pi, \psi) L(s, \pi) L(1-s, \pi^\vee),$$

(i) follows directly from (c), (g) and (h). \square

We record a particular consequence of equations (6), (8) and (9):

$$(10) \quad \epsilon(s, \text{St}_{\text{GL}_m(D)} \otimes \chi \circ \text{Nrd}, \psi) = (-1)^{n-1} \epsilon(s, \chi, \psi) = (-1)^{n-1} q^{s-1/2} \chi(\omega_F^{-1})$$

for any character ψ of F which is trivial on \mathfrak{p}_F but not on \mathfrak{o}_F .

2.3. Depth for Langlands parameters.

Let F_s be a separable closure of F and let $\text{Gal}(F_s/F)$ be the absolute Galois group of F . We recall some properties of its ramification groups (with respect to the upper numbering), as defined in [Ser, Remark IV.3.1]:

- $\text{Gal}(F_s/F)^{-1} = \text{Gal}(F_s/F)$ and $\text{Gal}(F_s/F)^0 = \mathbf{I}_F$, the inertia group.
- For every $l \in \mathbb{R}_{\geq 0}$, $\text{Gal}(F_s/F)^l$ is a compact subgroup of \mathbf{I}_F . It consists of all $\gamma \in \text{Gal}(F_s/F)$ which, for every finite Galois extension E of F contained in F_s , act trivially on the ring $\mathfrak{o}_E/\mathfrak{p}_E^{i(l,E)}$ (where $i(l,E) \in \mathbb{Z}_{\geq 0}$ can be found with [Ser, §IV.3]).

- $l \in \mathbb{R}_{\geq 0}$ is called a jump of the filtration if

$$\mathrm{Gal}(F_s/F)^{l+} := \bigcap_{l' > l} \mathrm{Gal}(F_s/F)^{l'}$$

does not equal $\mathrm{Gal}(F_s/F)^l$. The set of jumps of the filtration is countably infinite and need not consist of integers.

We define the depth of a Langlands parameter $\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{GL}_n(\mathbb{C})$ as

$$d(\phi) := \inf\{l \geq 0 \mid \mathrm{Gal}(F_s/F)^{l+} \subset \ker \phi\}.$$

We say that $\phi \in \Phi(\mathrm{GL}_n(F))$ is *elliptic* if its image is not contained in any proper Levi subgroup of $\mathrm{GL}_n(\mathbb{C})$.

Let ψ be a nontrivial character of F and let $c(\psi)$ be the largest integer c such that ψ is trivial on \mathfrak{p}_F^{-c} . The ϵ factor of ϕ (and ψ) was defined in [Tat]. It takes the form

$$(11) \quad \epsilon(s, \phi, \psi) = \epsilon(0, \phi, \psi) q^{-s(a(\phi) + nc(\psi))} \quad \text{with } \epsilon(0, \phi, \psi) \in \mathbb{C}^\times.$$

Here $a(\phi) \in \mathbb{Z}_{\geq 0}$ is the Artin conductor of ϕ (called $f(\phi)$ in [Ser, §VI.2]). To study $a(\phi)$ it is convenient to rewrite ϕ in terms of the Weil–Deligne group. For $\gamma \in \mathbf{W}_F$ put

$$\phi_0(\gamma) = \phi(\gamma, 1) \phi\left(1, \begin{pmatrix} \|\gamma\|^{1/2} & 0 \\ 0 & \|\gamma\|^{-1/2} \end{pmatrix}\right),$$

so ϕ_0 is a representation of \mathbf{W}_F which agrees with ϕ on \mathbf{I}_F . Define $N \in \mathfrak{gl}_n(\mathbb{C})$ as the nilpotent element $\log \phi\left(1, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right)$. Then (ϕ_0, N) is the Weil–Deligne representation of $\mathbf{W}_F \times \mathbb{C}$ corresponding to ϕ .

Denote the vector space \mathbb{C}^n endowed with the representation ϕ by V , and write $V_N = \ker(N : V \rightarrow V)$. By definition [Tat, §4.1.6]

$$(12) \quad a(\phi) = a(\phi_0) + \dim(V^{\mathbf{I}_F}/V_N^{\mathbf{I}_F}),$$

$$(13) \quad \epsilon(s, \phi, \psi) = \epsilon(0, \phi, \psi) \det\left(-\mathrm{Frob}|_{V^{\mathbf{I}_F}/V_N^{\mathbf{I}_F}}\right) q^{-s(a(\phi) + nc(\psi))},$$

where Frob denotes a geometric Frobenius element of \mathbf{W}_F .

Lemma 2.3. *For any elliptic $\phi \in \Phi(\mathrm{GL}_n(F))$*

$$(14) \quad d(\phi) := \begin{cases} 0 & \text{if } \mathbf{I}_F \subset \ker(\phi), \\ \frac{a(\phi)}{n} - 1 & \text{otherwise,} \end{cases}$$

Proof. This was proved in [ABPS2, Lemma 4.4] under the additional assumption $\mathrm{SL}_2(\mathbb{C}) \subset \ker \phi$. We will reduce to that special case.

Since ϕ is elliptic, it defines an irreducible n -dimensional representation V of $\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})$. Hence there are irreducible representations (ϕ_1, V_1) of \mathbf{W}_F and (ϕ_2, V_2) of $\mathrm{SL}_2(\mathbb{C})$ such that

$$(15) \quad (\phi, V) = (\phi_1, V_1) \otimes (\phi_2, V_2).$$

In particular $V^{\mathbf{I}_F} = V_1^{\mathbf{I}_F} \otimes V_2$. Suppose first that $V_1^{\mathbf{I}_F} = V_1$. Then $\mathbf{I}_F \subset \ker \phi$, so $d(\phi) = 0$ by definition. Now suppose $V_1^{\mathbf{I}_F} \neq V_1$. As (ϕ_1, V_1) is irreducible and \mathbf{I}_F is normal in \mathbf{W}_F , we must have $V_1^{\mathbf{I}_F} = 0$. Hence $V^{\mathbf{I}_F} = 0$, which by (12) implies $a(\phi) = a(\phi_0)$. By [Ser, Corollary VI.2.1'] $a(\phi_0)$ is additive in V and depends only on

$$\phi_0|_{\mathbf{I}_F} = \phi|_{\mathbf{I}_F} = \phi_1|_{\mathbf{I}_F} \otimes \mathrm{id}_{V_2}.$$

Now it follows from (15) that

$$(16) \quad a(\phi) = a(\phi_1) \dim V_2 = na(\phi_1)/\dim V_1.$$

As $\ker \phi_1 \supset \mathrm{SL}_2(\mathbb{C})$ we may apply [ABPS2, Lemma 4.4], which together with (16) gives

$$d(\phi_1) = \frac{a(\phi_1)}{\dim V_1} - 1 = \frac{a(\phi)}{n} - 1.$$

To conclude, we note that $d(\phi_1) = d(\phi)$ by (15). \square

2.4. The depth of representations of $\mathrm{GL}_m(D)$.

Let $k_D = \mathfrak{o}_D/\mathfrak{p}_D$ be the residual field of D . Let \mathfrak{A} be a hereditary \mathfrak{o}_F -order \mathfrak{A} in $M_m(D)$. The Jacobson radical of \mathfrak{A} will be denoted by \mathfrak{J} . Let $r = e_D(\mathfrak{A})$ and $e = e_F(\mathfrak{A})$ denote the integers defined by $\mathfrak{p}_D\mathfrak{A} = \mathfrak{J}^r$ and $\mathfrak{p}_F\mathfrak{A} = \mathfrak{J}^e$, respectively. We have

$$(17) \quad e_F(\mathfrak{A}) = de_D(\mathfrak{A}).$$

The normalizer in G of \mathfrak{A}^\times will be denoted by

$$\mathfrak{R}(\mathfrak{A}) := \{g \in G : g^{-1}\mathfrak{A}^\times g = \mathfrak{A}^\times\}.$$

Define a sequence of compact open subgroups of $G = \mathrm{GL}_m(D)$ by

$$U^0(\mathfrak{A}) := \mathfrak{A}^\times, \quad \text{and} \quad U^j(\mathfrak{A}) := 1 + \mathfrak{J}^j, \quad j \geq 1.$$

Then \mathfrak{A}^\times is a parahoric subgroup of G and $U^1(\mathfrak{A})$ is its pro-unipotent radical. We define the *normalized level* of an irreducible representation π of G to be

$$(18) \quad d(\pi) := \min \{j/e_F(\mathfrak{A})\},$$

where (j, \mathfrak{A}) ranges over all pairs consisting of an integer $j \geq 0$ and a hereditary \mathfrak{o}_F -order \mathfrak{A} in $M_m(D)$ such that π contains the trivial character of $U^{j+1}(\mathfrak{A})$.

Remark 2.4. When π is a representation of $\mathrm{GL}_n(F)$, our notion of normalized level coincides with that of [BuHe, § 12.6]. However when π is a representation of D^\times , the normalized level of π as defined above is not equal to the level $\ell_D(\pi)$ defined in [BuHe, § 54.1]: we have

$$d(\pi) = \frac{1}{d} \ell_D(\pi).$$

This reflects the fact that we have divided by $e_F(\mathfrak{A})$ instead of $e_D(\mathfrak{A})$ in Eqn. (18).

The following proposition will allow to use both results that were written in the setting of the normalized level, as general results on the depth in the sense of Moy and Prasad.

Proposition 2.5. *The normalized level of $\pi \in \mathrm{Irr}(G)$ equals its Moy–Prasad depth.*

Proof. Let us denote the Moy–Prasad depth of (π, V_π) by $d_{\mathrm{MP}}(\pi)$ for the duration of this proof. For any point x of the Bruhat–Tits building $\mathcal{B}(G)$ of G , consider the Moy–Prasad filtrations $P_{x,r}, P_{x,r+}$ ($r \in \mathbb{R}_{\geq 0}$) of the parahoric subgroup $P_x \subset G$ [MoPr1, §2]. We normalize these filtrations by using the valuation on F_s which maps F^\times onto \mathbb{Z} . Then $d_{\mathrm{MP}}(\pi)$ is the minimal $r \in \mathbb{R}_{\geq 0}$ such that $V_\pi^{P_{x,r+}} \neq 0$ for some $x \in \mathcal{B}(G)$, see [MoPr2, §3.4].

Any hereditary \mathfrak{o}_F -order \mathfrak{A} in $M_m(D)$ is associated to a unique facet $\mathcal{F}(\mathfrak{A})$ of $\mathcal{B}(G)$. The filtration $\{U^j(\mathfrak{A}) \mid j \in \mathbb{Z}_{\geq 0}\}$ was compared with the Moy–Prasad groups

for $x \in \mathcal{F}(\mathfrak{A})$ by Broussous and Lemaire. Let $x_{\mathfrak{A}}$ be the barycenter of $\mathcal{F}(\mathfrak{A})$. From [BrLe, Proposition 4.2 and Appendix A] and the definition of $e_F(\mathfrak{A})$ we see that

$$U^j(\mathfrak{A}) = P_{x_{\mathfrak{A}}, j/e_F(\mathfrak{A})} \text{ for all } j \in \mathbb{Z}_{\geq 0}.$$

Hence the definitions of the normalized level and the Moy–Prasad depth are almost equivalent, the only difference being that for $d_{\text{MP}}(\pi)$ we must consider all points of $\mathcal{B}(G)$, whereas for $d(\pi)$ we may only use barycenters of facets of $\mathcal{B}(G)$. Thus it remains to check the following claim: there exists a facet \mathcal{F} of $\mathcal{B}(G)$ with barycenter $x_{\mathcal{F}}$, such that V_{π} has nonzero $P_{x_{\mathcal{F}}, d_{\text{MP}}(\pi)+}$ -invariant vectors.

This is easy to see with the explicit constructions of the groups $P_{x,r}$ at hand, but we prefer not to delve into those details here. In fact, since every chamber of $\mathcal{B}(G)$ intersects every G -orbit, it suffices to consider facets contained in the closure of a fixed "standard" chamber. Then the claim becomes equivalent to saying that $x_{\mathcal{F}}$ is an "optimal point" in the sense of [MoPr1, §6.1]. That is assured by [MoPr1, Remark 6.1], which is applicable because the root system of G is of type A_{m-1} . \square

We fix a uniformizer ϖ_F of \mathfrak{o}_F , and a character $\psi: F \rightarrow \mathbb{C}^{\times}$, which is trivial on \mathfrak{p}_F but not on \mathfrak{o}_F . There is a canonical isomorphism

$$U^j(\mathfrak{A})/U^{j+1}(\mathfrak{A}) \rightarrow \mathfrak{P}^j/\mathfrak{P}^{j+1},$$

given by $x \mapsto x-1$. This leads to an isomorphism from $\mathfrak{P}^{-j}/\mathfrak{P}^{1-j}$ to the Pontrjagin dual of $U^j(\mathfrak{A})/U^{j+1}(\mathfrak{A})$, explicitly given by

$$\beta + \mathfrak{P}^{1-j} \mapsto \psi_{\beta} \quad \beta \in \mathfrak{P}^{-j},$$

with $\psi_{\beta}(1+x) = (\psi \circ \text{tr}_{M_m(D)})(\beta x)$, for $x \in \mathfrak{P}^{j+1}$.

If ρ is an irreducible smooth representation of $\mathfrak{K}(\mathfrak{A})$, then ρ is finite-dimensional. Let $\ell(\rho)$ denote the least integer $\ell \geq -1$ such that $U(\mathfrak{A})^{\ell+1} \subset \ker(\rho)$ (see [BuFr2, § 2.2]). We will call $\ell(\rho)$ the *level* of ρ .

In the case when $\ell(\rho) \geq 1$, consider the restriction of ρ to $U^{\ell(\rho)}(\mathfrak{A})$, it is a representation of the finite abelian group $U^{\ell(\rho)}(\mathfrak{A})/U^{\ell(\rho)+1}(\mathfrak{A})$, so it splits as a sum of characters

$$(19) \quad \rho|_{U^{\ell(\rho)}(\mathfrak{A})} = \bigoplus \psi_{\beta},$$

for various cosets $\beta + \mathfrak{P}^{1-\ell(\rho)}$, with $\beta \in \mathfrak{P}^{-\ell(\rho)}$.

The representation ρ is called *non-degenerate* in the terminology of [BuFr2, § 2], if either $\ell(\rho) \leq 0$, or $\ell(\rho) \geq 1$ and, for some (equivalently, for all) ψ_{β} occurring in (19), we have $(\beta + \mathfrak{P}^{1-\ell(\rho)}) \cap \mathfrak{K}(\mathfrak{A}) \neq \emptyset$.

The next lemma answers the question raised in [BuFr2, § 3.4.(A)] positively.

Lemma 2.6. *Let π be a supercuspidal irreducible representation of G . Then there exists a principal order \mathfrak{A} in $M_m(D)$ such that π contains a non-degenerate representation of $\mathfrak{K}(\mathfrak{A})$.*

Proof. In the case when $m = 1$, we have $G = D^{\times}$, we can take $\mathfrak{A} = \mathfrak{o}_D$. Then $\mathfrak{K}(\mathfrak{A}) = D^{\times}$, and any representation of D^{\times} is non-degenerate.

From now on, we will assume that $m > 1$. It follows from [SéSt1, Corollaire 5.22] that there exists a maximal simple type (J, λ) in G , and an extension Λ of λ to the normalizer $\bar{J} = N_G(\lambda)$ of λ , such that

$$(20) \quad \pi = \text{c-Ind}_{\bar{J}}^G \Lambda.$$

Since (J, λ) is a simple type for G , there exists a *simple stratum* $[\mathfrak{A}, j, 0, \beta]$, such that, either $j = 0$ and $\beta = 0$ (in this case (J, λ) is said to have *level zero*), or $J = J(\mathfrak{A}, \beta)$ see [Séc, § 2.3] (in this case (J, λ) is said to have *positive level*). Here \mathfrak{A} is a hereditary \mathfrak{o}_F -order, j is an integer such that $j \geq 0$, and β is an element of $M_m(D)$ with \mathfrak{A} -valuation $\nu_{\mathfrak{A}}(\beta) \geq -j$, where $\nu_{\mathfrak{A}}(0) := +\infty$, and

$$\nu_{\mathfrak{A}}(\beta) := \max \{i \in \mathbb{Z} : \beta \in \mathfrak{P}^i\}, \quad \text{if } \beta \neq 0.$$

Let E denote the F -algebra generated by β . A simple stratum satisfies in particular the following conditions: E is a field, E^\times normalizes \mathfrak{A} and $\nu_{\mathfrak{A}}(\beta) = -j$.

Let B denote the commuting algebra of E in $M_m(D)$. Then B is a central simple algebra over E . Hence there exist an integer m' and a division algebra D' of index d' over E such that B is isomorphic to the E -algebra $M_{m'}(D')$. There is also an F -algebra $A(E)$ such that $M_m(D) \simeq M_{m'}(A(E))$, see [Séc, (6)]. In the level zero case, we have $E = F$, $D' = D = A(E)$, and $m' = m$. Let \mathfrak{B} denote the intersection of \mathfrak{A} with B . Then \mathfrak{B} is the \mathfrak{o}_E -order that corresponds, under the isomorphism $B \simeq M_{m'}(D')$, to the standard principal \mathfrak{o}_E -order of $M_{m'}(D')$ defined by the partition (t', \dots, t') (r' -times) of m' , that is, the \mathfrak{o}_E -order of $r' \times r'$ -block matrices

$$\begin{pmatrix} \mathfrak{o}_{D'} & \cdots & \cdots & \cdots & \mathfrak{o}_{D'} \\ \mathfrak{p}_{D'} & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \mathfrak{p}_{D'} & \cdots & \cdots & \mathfrak{p}_{D'} & \mathfrak{o}_{D'} \end{pmatrix},$$

in which each block has size $t' \times t'$. We have $m' = r't'$ and $r' = e_{D'}(\mathfrak{B})$.

The representation λ of J has the form $\lambda = \kappa \otimes \tau$, where κ is certain representation of J (a β -*extension*) in the positive level case, while κ is trivial in the level zero case. On the other side, the representation τ is the inflation of a representation of the group

$$J/J^1 \simeq U(\mathfrak{B})/U^1(\mathfrak{B}) \simeq \mathrm{GL}_{t'}(k_{D'})^{r'},$$

where $J^1 = J \cap U^1(\mathfrak{A})$, of the form $\sigma^{r'}$, where σ is a cuspidal irreducible representation of $\mathrm{GL}_{t'}(k_{D'})$. Here $k_{D'} = \mathfrak{o}_{D'}/\mathfrak{p}_{D'}$.

Let $\varpi_{D'}$ be a uniformizer of D' such that the conjugation action of the element $\varpi_{D'}$ on $\mathfrak{o}_{D'}$ induces a generator of the Galois group $\mathrm{Gal}(k_{D'}/k_E)$. The group $\mathrm{Gal}(k_{D'}/k_E)$ acts naturally on $\mathrm{GL}_s(k_{D'})$ and on the representation σ . Let l denote the length of the $\mathrm{Gal}(k_{D'}/k_E)$ -orbit of σ .

The group \bar{J} (as in [Séc, § 5.1]) is generated by J and the element

$$h = \begin{pmatrix} 0 & \mathrm{I}_{r'-1} \\ \varpi_{D'}^l & 0 \end{pmatrix},$$

where $\mathrm{I}_{r'-1}$ is the identity matrix in $M_{r'-1}(M_{t'}(A(E)))$, see in [Séc, (17), (20)]. The element h belongs to $\mathfrak{K}(\mathfrak{B})$. Moreover, since π is supercuspidal, the order \mathfrak{B} is maximal (that is, $r' = e_{D'}(\mathfrak{B}) = 1$), hence there is a unique (up to G -conjugation) choice for \mathfrak{A} . Then \mathfrak{A} is principal and *sound*, see [SéSt1, (5.5)]. This means that

$$\mathfrak{K}(\mathfrak{A}) \cap B^\times = \mathfrak{K}(\mathfrak{B}),$$

so h normalizes \mathfrak{A} . It follows that the group \bar{J} is contained in the normalizer $\mathfrak{K}(\mathfrak{A})$ of \mathfrak{A} in G . We set

$$(21) \quad \rho := \text{Ind}_{\bar{J}}^{\mathfrak{K}(\mathfrak{A})}(\Lambda).$$

We will see that ρ is non-degenerate. The level $\ell(\rho)$ of ρ is equal to j and, since κ is a β -extension, by the construction of κ , the restriction of ρ to $U_j(\mathfrak{A})$ contains the character ψ_β of $U_j(\mathfrak{A})$ that corresponds to $\beta + \mathfrak{P}^{1-j}$, that is contained in $\mathfrak{K}(\mathfrak{A})$ as required. \square

Let $[\mathfrak{A}, j, j-1, \beta]$ be a stratum in $M_m(D)$. Let g denote the greatest common divisor of j and e , and set $y_\beta := \varpi_F^{j/g} \beta^{e/g}$. We consider y_β as an element of $M_n(F)$. Its characteristic polynomial belongs to $\mathfrak{o}_F[X]$. Let $\varphi_\beta(X) \in k_F[X]$ denote the reduction modulo \mathfrak{p}_F of the latter. The polynomial $\varphi_\beta(X)$ is also the characteristic polynomial of \bar{y}_β , the reduction modulo \mathfrak{p}_F of y_β . The stratum $[\mathfrak{A}, j, j-1, \beta]$ is called *fundamental* if $\varphi_\beta(X)$ is not a power of X , see [SéSt1, Déf. 3.9] or equivalently [Bro, Def. 1.17].

Lemma 2.7. *Assume that there exists a stratum $[\mathfrak{A}', j', j'-1, 0]$ in $M_m(D)$ such that*

$$\beta \in (\mathfrak{P}')^{1-j'}, \quad \text{and} \quad \frac{j'-1}{e'} < \frac{j}{e},$$

where \mathfrak{P}' denotes the Jacobson radical of \mathfrak{A}' , $e = e_F(\mathfrak{A})$ and $e' = e_F(\mathfrak{A}')$. Then $[\mathfrak{A}, j, j-1, \beta]$ is not fundamental.

Proof. The proof follows the same lines as [BuKu1, (2.6.2)], which provides the analogous result when $D = F$. Then the element β^e lies in $(\mathfrak{P}')^{e(1-j')}$. Hence $\varpi_F^j \beta^e$ lies in $\mathfrak{A} \cap (\mathfrak{P}')^{e+e'j-ej'} \cap \mathfrak{A}$, since $\mathfrak{p}_F \mathfrak{A}' = (\mathfrak{P}')^{e'}$ and $\beta \in \mathfrak{p}_F^{-1} \mathfrak{P}^{-j}$. By assumption, $e'j - ej' \geq 0$, so the element $\varpi_F^j \beta^e$ lies in \mathfrak{P}' . Since any element of the radical of any hereditary \mathfrak{o}_F -order in $M_m(D)$ has characteristic polynomial (in a variable X) congruent to $X^m \pmod{\mathfrak{p}_F}$, we obtain that

$$(\varpi_F^j \beta^e)^m \in \mathfrak{p}_F \mathfrak{A} \subset \mathfrak{P},$$

and hence $\beta^{em} \in \mathfrak{P}^{1-em}$. Then [Bro, Lemma 1.1.8] implies that $[\mathfrak{A}, j, j-1, \beta]$ is not fundamental. \square

Recall that one says that two strata $[\mathfrak{A}, j, j-1, \beta]$ and $[\mathfrak{A}', j', j'-1, \beta']$ *intertwine* if there exists $g \in G$ such that

$$(22) \quad g^{-1}(\beta + \mathfrak{P}^{1-j})g \cap (\beta' + (\mathfrak{P}')^{1-j'}) \neq \emptyset.$$

Proposition 2.8. *If two fundamental strata $[\mathfrak{A}, j, j-1, \beta]$ and $[\mathfrak{A}', j', j'-1, \beta']$ intertwine, then $j/e = j'/e'$.*

Proof. Assume that $[\mathfrak{A}, j, j-1, \beta]$ and $[\mathfrak{A}', j', j'-1, \beta']$ intertwine. Hence there exists an element g in G such that (22) holds. By the definition of a stratum, we have $\beta' + (\mathfrak{P}')^{1-j'} \subset (\mathfrak{P}')^{-j'}$. It follows that $[\mathfrak{A}, j, j-1, \beta]$ and $[\mathfrak{A}', j'+1, j', 0]$ intertwine. Replacing (if necessary) the stratum $[\mathfrak{A}, j, j-1, \beta]$ by a G -conjugate, we may assume that $\beta \in (\mathfrak{P}')^{1-j'}$. If we suppose that $j'/e' < j/e$, by applying Lemma 2.7, we get that $[\mathfrak{A}, j, j-1, \beta]$ is not fundamental. Hence we have $j'/e' \geq j/e$. By interchanging the roles of $[\mathfrak{A}, j, j-1, \beta]$ and $[\mathfrak{A}', j', j'-1, \beta']$, we obtain that $j/e \geq j'/e'$. Hence the equality holds. \square

2.5. Conductors of representations of $\mathrm{GL}_m(D)$.

Let $\epsilon(s, \pi, \psi)$ denote the Godement–Jacquet local constant [GoJa]. It takes the form

$$(23) \quad \epsilon(s, \pi, \psi) = \epsilon(0, \pi, \psi) q^{-f(\pi, \psi)s}, \quad \text{where } \epsilon(0, \pi, \psi) \in \mathbb{C}^\times.$$

Recall that $c(\psi)$ is the largest integer c such that $\mathfrak{p}_F^{-c} \subset \ker \psi$. In the previous section we had $c(\psi) = -1$.

A representation of D^\times is called *unramified* if it is trivial on \mathfrak{o}_D^\times . An unramified representation of D^\times is a character and has depth zero.

Proposition 2.9. *Let π be a supercuspidal irreducible representation of G . We have*

$$(24) \quad f(\pi, \psi) = \begin{cases} n(c(\psi) + 1) - 1 & \text{if } m = 1 \text{ and } \pi \text{ is unramified,} \\ n(d(\pi) + 1 + c(\psi)) & \text{otherwise.} \end{cases}$$

Proof. We suppose first that $m = 1$ (so $d = n$) and π is unramified. The required formula can be read off from (10) if $c(\psi) = -1$. For general ψ , applying [BuFr1, Theorem 3.2.11] and taking in account [BuFr1, (1.2.7), (1.2.8), (1.2.10)], we obtain

$$f(\pi, \psi) = (d(1 - d - dc(\psi))) \cdot \left(-\frac{1}{d}\right) = d + dc(\psi) - 1.$$

Hence the first case of Eqn. (24) holds.

From now on, we will assume that $m \geq 2$ or π is ramified. Then Lemma 2.6, combined with the fact that the Godement–Jacquet L-function $L(s, \pi)$ is 1, shows that π satisfies the conditions of Theorem 3.3.8 of [BuFr2]. We take π and ρ as in (20) and (21), respectively. Recall that $n = md$. By applying the formula of [BuFr2, Theorem 3.3.8 (iv)], we obtain

$$q^{f(\pi, \psi)} = \left[\mathfrak{A} : \mathfrak{p}_F^{c(\psi)+1} \mathfrak{P}^j \right]^{1/n}.$$

On the other hand, the \mathfrak{o}_F -order is G -conjugate to the standard principal \mathfrak{o}_F -order of $M_m(D)$ defined by the partition (t, \dots, t) (r -times) of m , where $m = rt$ and $r = e_D(\mathfrak{A})$. Hence we have $\mathfrak{A}/\mathfrak{P} \simeq (M_t(k_D))^r$. It follows that

$$[\mathfrak{A} : \mathfrak{P}^j] = (q^d)^{t^2 r} = q^{drt^2}.$$

Hence we get

$$f(\pi, \psi) = \frac{drt^2(j + e + ec(\psi))}{n} = n \left(\frac{j}{e} + 1 + c(\psi) \right),$$

since $drt^2 = nt = n^2/e$. By the construction of the type (J, λ) , we have $d(\pi) \leq j/e$. Conversely, let $[\mathfrak{A}', j', j' - 1, \beta']$ be a stratum contained in π . Then if $[\mathfrak{A}, j, j - 1, \beta]$ is such that its normalized level j/e is minimal among the normalized levels of all the strata contained in π , it is necessarily fundamental [Bro, Theorem 1.2.1. (ii)]. Finally, if two fundamental strata are contained in π , then they must intertwine, and thus have the same normalized level by Proposition 2.8. Therefore $e/j = d(\pi)$. \square

Theorem 2.10 below proves the validity of Conjecture 4.3 of [LaRa]. In the case when F has characteristic 0, it is due to Lansky and Raghuram for the groups $\mathrm{GL}_n(F)$ and D^\times , [LaRa, Theorem 3.1], and for certain representations of $\mathrm{GL}_2(D)$, [LaRa, Theorem 4.1]. Our proof is inspired by those of these results.

Theorem 2.10. *The depth $d(\pi)$ and the conductor $f(\pi) := f(\pi, \psi) - nc(\psi)$ of each essentially square-integrable irreducible representation π of $\mathrm{GL}_m(D)$ are linked by the following relation:*

$$(25) \quad d(\pi) = \begin{cases} 0 & \text{if } \pi \text{ is an unramified twist of } \mathrm{St}_{\mathrm{GL}_m(D)}, \\ \frac{f(\pi) - n}{n} & \text{otherwise.} \end{cases},$$

In particular

$$(26) \quad d(\pi) = \max \left\{ \frac{f(\pi) - n}{n}, 0 \right\}.$$

Proof. Let $\pi \in \mathrm{Irr}_{\mathrm{ess}L^2}(\mathrm{GL}_m(D))$. We use the same notation as for π' in the proof of Theorem 2.2.h, so π is consistent of

$$I_{\mathrm{GL}_a(D)^b}^{\mathrm{GL}_m(D)}(\nu_D^{s_\sigma \frac{(1-b)}{2}} \sigma \otimes \nu_D^{s_\sigma \frac{(3-b)}{2}} \sigma \otimes \cdots \otimes \nu_D^{s_\sigma \frac{(b-1)}{2}} \sigma),$$

where $\sigma \in \mathrm{Irr}(\mathrm{GL}_a(D))$ is supercuspidal. Since the depth is preserved by parabolic induction [MoPr2, Theorem 5.2], we get

$$d(\pi) = d(\nu_D^{s_\sigma \frac{(1-b)}{2}} \sigma \otimes \nu_D^{s_\sigma \frac{(3-b)}{2}} \sigma \otimes \cdots \otimes \nu_D^{s_\sigma \frac{(b-1)}{2}} \sigma).$$

It follows that

$$(27) \quad d(\pi) = d(\sigma).$$

We will apply Proposition 2.9 to the supercuspidal representation σ of $\mathrm{GL}_a(D)$. In the special case σ is an unramified representation of D^\times (hence $a = 1$ in this case), Eqn. (24) gives

$$f(\sigma, \psi) = d(c(\psi) + 1) - 1,$$

that is, $f(\sigma) = d - 1$. Hence we get

$$\frac{f(\sigma) - d}{d} = -\frac{1}{d}.$$

Then it implies that

$$\max \left\{ \frac{f(\sigma) - d}{d}, 0 \right\} = \max \left\{ -\frac{1}{d}, 0 \right\} = 0 = d(\sigma),$$

in other words, Eqn. (26) holds for the unramified representations of D^\times .

In the other cases (that is, $a \neq 1$ or σ is ramified), (24) gives $f(\sigma) = ad(d(\sigma) + 1)$, that is,

$$(28) \quad \frac{f(\sigma)}{ad} = d(\sigma) + 1.$$

Since $d(\sigma) \geq 0$ (by definition of the depth), we obtain that

$$(29) \quad d(\sigma) = \max \left\{ \frac{f(\sigma) - ad}{ad}, 0 \right\}.$$

Hence (25) holds for every supercuspidal irreducible representation of $\mathrm{GL}_a(D)$, with $a \geq 1$ an arbitrary integer.

Recall that s_σ is an integer dividing ad , say $ad = a^*s_\sigma$ with $a^* \in \mathbb{Z}$. The image $\text{JL}^{-1}(\sigma)$ of σ under the Jacquet-Langlands correspondence is equivalent to the Langlands quotient of the parabolically induced representation

$$I_{\text{GL}_{a^*s_\sigma}(F)}^{\text{GL}_{a^*s_\sigma}(F)}(\nu_F^{\frac{(1-s_\sigma)}{2}} \sigma^* \otimes \nu_F^{\frac{(3-s_\sigma)}{2}} \sigma^* \otimes \cdots \otimes \nu_F^{\frac{(s_\sigma-1)}{2}} \sigma^*),$$

where σ^* is a unitary supercuspidal irreducible representation of $\text{GL}_{a^*}(F)$ and $\nu_F(g^*) = |\det(g^*)|_F$.

The representation $\text{JL}^{-1}(\pi)$ is equivalent to a constituent of a parabolically induced representation

$$I_{\text{GL}_{ad}(F)^b}^{\text{GL}_{adb}(F)}(\nu_F^{\frac{(1-b)}{2}} \sigma^* \otimes \nu_F^{\frac{(3-b)}{2}} \sigma^* \otimes \cdots \otimes \nu_F^{\frac{(b-1)}{2}} \sigma^*),$$

where σ^* is a supercuspidal irreducible representation of $\text{GL}_{ad}(F)$. We recall from [Hen, § 2.6] the formula describing the epsilon factor of $\text{JL}^{-1}(\pi)$ in terms of the local factors of σ^* :

$$(30) \quad \epsilon(s, \text{JL}^{-1}(\pi), \psi) = \prod_{i=0}^{b-1} \epsilon(s+i, \sigma^*, \psi) \prod_{j=0}^{b-2} \frac{L(-s-j, \check{\sigma}^*)}{L(s+j, \sigma^*)}.$$

By Theorem 2.2.d the representation

$$\sigma := \text{JL}^{-1}(\sigma^*) \in \text{Irr}(\text{GL}_a(D)), \quad \text{with } ad = a^*,$$

is supercuspidal. Since the Jacquet-Langlands correspondence preserves the ϵ -factors (see Theorem 2.2.i) we have

$$\epsilon(s, \text{JL}^{-1}(\pi), \psi) = \epsilon(s, \pi, \psi) \quad \text{and} \quad \epsilon(s, \sigma^*, \psi) = \epsilon(s, \sigma, \psi).$$

Thus we have obtained the following formula

$$(31) \quad \epsilon(s, \pi, \psi) = \prod_{i=0}^{b-1} \epsilon(s+i, \sigma, \psi) \prod_{j=0}^{b-2} \frac{L(-s-j, \check{\sigma})}{L(s+j, \sigma)}.$$

It follows from (31) that

$$(32) \quad f(\text{St}_{\text{GL}_m(D)} \otimes \chi) = n - 1$$

for any unramified character χ , and

$$(33) \quad f(\pi) = b f(\sigma)$$

in the case when $m \neq 1$ or σ is ramified. Then we get

$$(34) \quad d(\pi) = \max \left\{ \frac{bf(\sigma) - bad}{bad}, 0 \right\} = \max \left\{ \frac{f(\pi) - n}{n}, 0 \right\}. \quad \square$$

2.6. Depth preservation.

Corollary 2.11. *The Jacquet-Langlands correspondence preserves the depth of essentially square-integrable representations of $\text{GL}_m(D)$.*

Proof. Theorem 2.2.i shows in particular that the Jacquet-Langlands correspondence preserves conductors. Now Theorem 2.10 shows that it preserves depths as well. \square

Theorems 2.10 and 2.2 are also the crucial steps to show that the local Langlands correspondence for inner forms of $\mathrm{GL}_m(D)$ preserves depths. With similar considerations we show that it also preserves L-functions, ϵ -factors and γ -factors. We abbreviate these three to "local factors". For the basic properties of the local factors of Langlands parameters we refer to [Tat].

Theorem 2.12. *The local Langlands correspondence for representations of $\mathrm{GL}_m(D)$ preserves L-functions, ϵ -factors, γ -factors and depths. In other words, for every irreducible smooth representation π of $\mathrm{GL}_m(D)$:*

$$\begin{aligned} L(s, \pi) &= L(s, \mathrm{rec}_{D,m}(\pi)), \\ \epsilon(s, \pi, \psi) &= \epsilon(s, \mathrm{rec}_{D,m}(\pi), \psi), \\ \gamma(s, \pi, \psi) &= \gamma(s, \mathrm{rec}_{D,m}(\pi), \psi), \\ d(\pi) &= d(\mathrm{rec}_{D,m}(\pi)). \end{aligned}$$

Proof. It is well-known that the local Langlands correspondence for $\mathrm{GL}_n(F)$ preserves local factors, see the introduction of [HaTa].

Assume first that π is essentially square-integrable. Recall the notations of the ϵ factors of π and of $\phi := \mathrm{rec}_{D,m}(\pi) \in \Phi(\mathrm{GL}_m(D))$ from (11) and (23). By definition

$$\mathrm{rec}_{D,m}(\pi) = \mathrm{rec}_{F,n}(\mathrm{JL}^{-1}(\pi)),$$

so by Theorem 2.2 $\mathrm{rec}_{D,m}$ preserves the ϵ -factors of π :

$$(35) \quad \epsilon(0, \phi, \psi) q^{-s(a(\phi)+nc(\psi))} = \epsilon(s, \phi, \psi) = \epsilon(s, \pi, \psi) = \epsilon(0, \pi, \psi) q^{-sf(\pi, \psi)}.$$

Hence, with the notation from Theorem 2.10:

$$(36) \quad f(\pi) = f(\pi, \psi) - nc(\psi) = a(\phi).$$

The properties of $\mathrm{rec}_{F,n}$ imply that ϕ is elliptic. By combining Lemma 2.3 with Theorem 2.10 and (36), we obtain that $d(\phi) = d(\pi)$ whenever π is essentially square-integrable.

Now let π be any irreducible representation of $\mathrm{GL}_m(D)$. By the Langlands classification, there exist a parabolic subgroup $P \subset \mathrm{GL}_m(D)$ with Levi factor M and an irreducible essentially square-integrable representation ω of M , such that π is a quotient of $I_P^{\mathrm{GL}_m(D)}(\omega)$. Moy and Prasad proved in [MoPr2, Theorem 5.2] that π and ω have the same depth. By [Jac, Theorem 3.4] π and ω have the same L-functions and ϵ -factors and by [Jac, (2.3) and (2.7.3)] they also have the same γ -factors.

On the other hand, M is isomorphic to a product of groups of the form $\mathrm{GL}_{m_i}(D)$, so the local Langlands correspondence for M is simply the product of that for the $\mathrm{GL}_{m_i}(D)$. The Langlands parameters $\mathrm{rec}_{D,m}(\pi)$ and $\mathrm{rec}_M(\omega)$ are related via an inclusion of the complex dual groups $\prod_i \mathrm{GL}_{dm_i}(\mathbb{C}) \rightarrow \mathrm{GL}_n(\mathbb{C})$. Hence these two Langlands parameters also have the same depth and local factors.

As we already proved that the LLC preserves depths for essentially square-integrable representations of $\mathrm{GL}_m(D)$ or M , we can conclude that

$$d(\pi) = d(\omega) = d(\mathrm{rec}_M(\omega)) = d(\mathrm{rec}_{D,m}(\pi)),$$

and similarly for the local factors. □

3. THE LOCAL LANGLANDS CORRESPONDENCE FOR INNER FORMS OF $\mathrm{SL}_n(F)$

3.1. The statement of the correspondence.

Recall that F is a non-archimedean local field and that the equivalence classes of inner forms of $\mathrm{SL}_n(F)$ are in bijection with the isomorphism classes of central simple F -algebras of dimension n^2 , via $M_m(D) \mapsto \mathrm{GL}_m(D)_{\mathrm{der}}$. The local Langlands correspondence for $\mathrm{GL}_m(D)_{\mathrm{der}}$ is implied by that for $\mathrm{GL}_m(D)$, in the following way. Given a Langlands parameter

$$\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{PGL}_n(\mathbb{C})$$

which is relevant for $\mathrm{GL}_m(D)_{\mathrm{der}}$, lift it to a Langlands parameter

$$\bar{\phi} : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{GL}_n(\mathbb{C})$$

for $\mathrm{GL}_m(D)$. Then $\mathrm{rec}_{m,D}^{-1}(\bar{\phi})$ is an irreducible representation of $\mathrm{GL}_m(D)$ which, upon restriction to $\mathrm{GL}_m(D)_{\mathrm{der}}$, decomposes as a finite direct sum of irreducible representations. The packet $\Pi_{\phi}(\mathrm{GL}_m(D)_{\mathrm{der}})$ is defined as the set of irreducible constituents of $\mathrm{rec}_{m,D}^{-1}(\bar{\phi})|_{\mathrm{GL}_m(D)_{\mathrm{der}}}$.

For these groups it is more interesting to consider the enhanced Langlands correspondence, where ϕ is supplemented with an irreducible representation of a finite group. In addition to the groups defined in (3), we write

$$(37) \quad \mathcal{Z}_{\phi} = Z(\mathrm{SL}_n(\mathbb{C}))/Z(\mathrm{SL}_n(\mathbb{C})) \cap C(\phi)^{\circ} \cong Z(\mathrm{SL}_n(\mathbb{C}))C(\phi)^{\circ}/C(\phi)^{\circ}.$$

An enhanced Langlands parameter is a pair (ϕ, ρ) with $\rho \in \mathrm{Irr}(\mathcal{S}_{\phi})$. The groups in (3), (37) are related to the more usual component group

$$S_{\phi} := Z_{\mathrm{PGL}_n(\mathbb{C})}(\mathrm{im} \phi)/Z_{\mathrm{PGL}_n(\mathbb{C})}(\mathrm{im} \phi)^{\circ}$$

by the short exact sequence

$$1 \rightarrow \mathcal{Z}_{\phi} \rightarrow \mathcal{S}_{\phi} \rightarrow S_{\phi} \rightarrow 1.$$

Hence \mathcal{S}_{ϕ} has more irreducible representations than S_{ϕ} . Via the enhanced Langlands correspondence the additional ones are associated to irreducible representations of non-split inner forms of $\mathrm{SL}_n(F)$. The following result is due to Hiraga and Saito [HiSa] for generic representations of $\mathrm{GL}_m(D)$ when $\mathrm{char} F = 0$.

Theorem 3.1. [ABPS2, Theorem 1.2]

There exists a bijective correspondence between:

- pairs $(\mathrm{GL}_m(D)_{\mathrm{der}}, \pi)$ with $\pi \in \mathrm{Irr}(\mathrm{GL}_m(D)_{\mathrm{der}})$ and $\mathrm{GL}_m(D)_{\mathrm{der}}$ an inner form of $\mathrm{SL}_n(F)$, considered up to equivalence;
- $\mathrm{SL}_n(\mathbb{C})$ -conjugacy classes of pairs (ϕ, ρ) with $\phi \in \Phi(\mathrm{SL}_n(F))$ and $\rho \in \mathrm{Irr}(\mathcal{S}_{\phi})$.

Here the group $\mathrm{GL}_m(D)_{\mathrm{der}}$ determines $\rho|_{\mathcal{Z}_{\phi}}$ and conversely. The correspondence satisfies the desired properties from [Bor, §10.3], with respect to restriction from inner forms of $\mathrm{GL}_n(F)$, temperedness and essential square-integrability of representations.

We remark that the above bijection need not be canonical if $\Pi_{\phi}(\mathrm{GL}_m(D)_{\mathrm{der}})$ has more than one element.

3.2. Depth preservation.

The depth of a Langlands parameter $\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{PGL}_n(\mathbb{C})$ for an inner form of $\mathrm{SL}_n(F)$ is defined as in Section 2.3:

$$d(\phi) = \inf\{l \in \mathbb{R}_{\geq 0} \mid \mathrm{Gal}(F_s/F)^{l+} \subset \ker \phi\}.$$

Lemma 3.2. *$d(\phi)$ is the minimum, over all lifts $\bar{\phi} \in \Phi(\mathrm{GL}_n(F))$ of ϕ , of the numbers $d(\bar{\phi})$.*

Proof. For every lift $\bar{\phi} \in \Phi(\mathrm{GL}_n(F))$ of ϕ we have $\ker \bar{\phi} \subset \ker \phi$, so $d(\bar{\phi}) \geq d(\phi)$. Since the possible depths are bounded below, we can choose a $\bar{\phi}$ whose depth is minimal among the lifts of ϕ . Suppose that, in contrast to what we want to show, $d(\bar{\phi}) > d(\phi)$.

We temporarily abbreviate $\mathrm{Gal}(F_s/F)^l$ to Γ^l . There is an $l > d(\phi)$ such that $\bar{\phi}$ is nontrivial on Γ^l . Moreover the restriction of $\bar{\phi}$ to Γ^l is a character, because $\bar{\phi}$ lifts ϕ . As $\bar{\phi}|_{\mathbf{W}_F}$ is a smooth group homomorphism, its kernel is an open subgroup of \mathbf{W}_F . Let $r \geq l$ be the minimal number such that $\Gamma^{r+} \subset \ker \bar{\phi}$. Then the restriction of $\bar{\phi}$ to Γ^r can be considered as a character χ of Γ^r/Γ^{r+} . By [Ser, Theorem IV.2.7] there exists an $i \in \mathbb{Z}_{\geq 0}$ such that the Artin reciprocity map yields an injection

$$(38) \quad \theta_i : \Gamma^r/\Gamma^{r+} \rightarrow 1 + \mathfrak{p}_F^i/1 + \mathfrak{p}_F^{i+1}.$$

As $F^\times/1 + \mathfrak{p}_F^{i+1}$ is abelian and \mathbb{C}^\times is a divisible group, we can extend the character $\chi \circ \theta_i^{-1}$ of $\mathrm{im}(\theta_i)$ to a character of $F^\times/1 + \mathfrak{p}_F^{i+1}$. This can be regarded as a smooth character of F^\times , which by local class field theory yields a character $\bar{\chi}$ of \mathbf{W}_F that extends χ . Then $\bar{\phi} \otimes (\bar{\chi}^{-1} \otimes 1_{\mathrm{SL}_2(\mathbb{C})})$ is another lift of ϕ , whose kernel does contain Γ^{r+} . This contradicts the above minimality properties of $\bar{\phi}$ and r . \square

For the depth of an irreducible representation of $\mathrm{GL}_m(D)_{\mathrm{der}}$ there are two candidates. Besides the Moy–Prasad depth one can define the normalized level, just as in (18). This was done for representations of $\mathrm{SL}_n(F)$ in [BuKu2]. However, Proposition 2.5 quickly reveals that these two notions agree:

Corollary 3.3. *The Moy–Prasad depth of an irreducible representation of $\mathrm{GL}_m(D)_{\mathrm{der}}$ equals its normalized level.*

Proof. By definition $\mathrm{GL}_m(D)$ and $\mathrm{GL}_m(D)_{\mathrm{der}}$ have the same Bruhat–Tits building. The descriptions of the two kinds of depth given in the proof of Proposition 2.5 still apply, we only have to replace the relevant compact subgroups by their intersections with $\mathrm{GL}_m(D)_{\mathrm{der}}$. Since this is the same change on both sides, the entire proof of Proposition 2.5 carries over to the present setting. \square

Finally, we use our results for the inner forms of $\mathrm{GL}_n(F)$ to show that the local Langlands correspondence for inner forms of $\mathrm{SL}_n(F)$ preserves depths.

Theorem 3.4. *Let ϕ be a Langlands parameter for $\mathrm{SL}_n(F)$ which is relevant for $\mathrm{GL}_m(D)_{\mathrm{der}}$. For every $\pi \in \Pi_\phi(\mathrm{GL}_m(D)_{\mathrm{der}})$ there is an equality of depths:*

$$d(\pi) = d(\phi).$$

Proof. Choose $\bar{\phi}$ as in Lemma 3.2 and write $\bar{\pi} = \mathrm{rec}_{D,m}^{-1}(\bar{\phi})$. In view of Theorem 2.12 we have

$$d(\bar{\pi}) = d(\bar{\phi}) = d(\phi).$$

Theorem 2.12 and the compatibility of the local Langlands correspondence with character twists imply that

$$d(\bar{\pi} \otimes \chi) \geq d(\bar{\pi}) \text{ for every character } \chi \text{ of } \mathrm{GL}_m(D).$$

By definition π is an irreducible constituent of the restriction of $\bar{\pi}$ to $\mathrm{GL}_m(D)_{\mathrm{der}}$. In this situation [BuKu2, Lemma 1.12] says that $d(\pi) = d(\bar{\pi})$. We remark that, although [BuKu2] discusses only $\mathrm{SL}_n(F)$, the proof of this result is also valid for its inner forms. \square

REFERENCES

- [ABPS1] A.-M. Aubert, P. Baum, R.J. Plymen, M. Solleveld, “Geometric structure and the local Langlands conjecture”, arXiv:1211.0180.
- [ABPS2] A.-M. Aubert, P.F. Baum, R.J. Plymen, M. Solleveld, “The local Langlands correspondence for inner forms of SL_n ”, preprint, arXiv:1305.2638, 2013.
- [Bad] A. I. Badulescu, “Correspondance de Jacquet-Langlands pour les corps locaux de caractéristique non nulle”, Ann. Sci. Éc. Norm. Sup. (4) **35** (2002), 695–747.
- [Bor] A. Borel, “Automorphic L-functions”, Proc. Symp. Pure Math **33.2** (1979), 27–61.
- [Bro] P. Broussous, “Minimal strata for $\mathrm{GL}_m(D)$ ”, J. reine angew. Math. **514** (1999), 199–236.
- [BrLe] P. Broussous, B. Lemaire, “Building of $\mathrm{GL}(m, D)$ and centralizers”, Transformations Groups **7**, No. 1 (2002),
- [BuFr1] C.J. Bushnell, A. Frölich, “Gauss sums and p -adic division algebras”, Lecture Notes in Math. **987**, Springer-Verlag, 1983.
- [BuFr2] C.J. Bushnell, A. Frölich, “Non-abelian congruence Gauss sums and p -adic simple algebras”, Proc. London Math. Soc. (3) **50** (1985), 207–264.
- [BuHe] C.J. Bushnell, G. Henniart, *The local Langlands conjecture for $\mathrm{GL}(2)$* , Grundlehren der mathematischen Wissenschaften, **335**, Springer-Verlag, 2006.
- [BuKu1] C.J. Bushnell, P.C. Kutzko, *The admissible dual of $\mathrm{GL}(n)$ via compact open subgroups*, Ann. Math. Study **129**, Princeton Univ. Press 1993.
- [BuKu2] C.J. Bushnell, P.C. Kutzko, “The admissible dual of $\mathrm{SL}(N)$ I”. Ann. scient. Éc. Norm. Sup. (4) **26** (1993), 261–280
- [DBRe] S. DeBacker, M. Reeder, “Depth zero supercuspidal L-packets and their stability”, Ann. of Math. **169.3** (2009), 795–901.
- [DKV] P. Deligne, D. Kazhdan, M.-F. Vigneras, “Représentations des algèbres centrales simples p -adiques”, pp. 33–117 in: *Représentations des groupes réductifs sur un corps local*, Travaux en cours, Hermann, 1984.
- [Gan] R. Ganapathy, “The local Langlands correspondence for GSp_4 over local function fields”, preprint, arXiv:1305.6088, 2013.
- [GoJa] R. Godement, H. Jacquet, *Zeta functions of simple algebras*, Lecture Notes in Mathematics **260**, Springer-Verlag, 1972.
- [GSZ] M. Grabitz, A.J. Silberger, W. Zink, “Level zero types and Hecke algebras for local central simple algebras”, J. Number Theory **91** (2001), 92–125.
- [GrRe] B.H. Gross, M. Reeder, “Arithmetic invariants of discrete Langlands parameters”, Duke Math. J. **154** (2010), 431–508.
- [HaTa] M. Harris, R. Taylor, “The geometry and cohomology of some simple Shimura varieties”, Annals of Math. Studies **151**, Princeton University Press, Princeton NJ, 2001.
- [Hen] G. Henniart, “On the local Langlands conjecture for $\mathrm{GL}(n)$: The cyclic case”, Ann. of Math. **123** (1986), 143–203.
- [Hen2] G. Henniart, “Une preuve simple de conjectures de Langlands pour $\mathrm{GL}(n)$ sur un corps p -adique”, Inv. Math. **139** (2000), 439–455.
- [HiSa] K. Hiraga, H. Saito, “On L-packets for inner forms of SL_n ”, Mem. Amer. Math. Soc. 1013, Vol. **215** (2012).
- [Jac] H. Jacquet, “Principal L-functions of the linear group”, Proc. Symp. Pure Math **33.2** (1979), 63–86.
- [JaLa] H. Jacquet, R. Langlands, *Automorphic forms on $\mathrm{GL}(2)$* , Lecture Notes in Mathematics **114**, Springer-Verlag, 1970

- [Kot] R.E. Kottwitz, “Stable trace formula: cuspidal tempered terms”, *Duke Math. J.* **51.3** (1984), 611–650.
- [LaRa] J. Lansky, A. Raghuram, “On the correspondence of representations between $GL(n)$ and division algebras”, *Proc. Amer. Math. Soc.* **131.5** (2002), 1641–1648.
- [LRS] G. Laumon, M. Rapoport, U. Stuhler, “ \mathcal{D} -elliptic sheaves and the Langlands correspondence”, *Invent. Math.* **113** (1993), 217–238.
- [Lus1] G. Lusztig, “Classification of unipotent representations of simple p -adic groups”, *Int. Math. Res. Notices* **11** (1995), 517–589.
- [Lus2] G. Lusztig, “Classification of unipotent representations of simple p -adic groups II”, *Represent. Theory* **6** (2002), 243–289.
- [Mœ] C. Mœglin, “Stabilité en niveau 0, pour les groupes orthogonaux impairs p -adiques”, *Documenta Math.* **9** (2004), 527–564.
- [Mor] L. Morris, “Tamely ramified intertwining algebras”, *Invent. Math.* **114.1** (1993), 1–54.
- [MoPr1] A. Moy, G. Prasad, “Unrefined minimal K -types for p -adic groups”, *Inv. Math.* **116** (1994), 393–408.
- [MoPr2] A. Moy, G. Prasad, “Jacquet functors and unrefined minimal K -types”, *Comment. Math. Helvetici* **71** (1996), 98–121.
- [Ree] M. Reeder, “Supercuspidal L-packets of positive depth and twisted Coxeter elements”, *J. reine angew. Math.* **620** (2008), 1–33.
- [ReYu] M. Reeder, J.-K. Yu, “Epipelagic representations and invariant theory”, *J. Amer. Math. Soc.* (2013), DOI: <http://dx.doi.org/10.1090/S0894-0347-2013-00780-8>
- [Séc] V. Sécherre, “Représentations lisses de $GL_m(D)$ III: types simples”, *Ann. Scient. Éc. Norm. Sup.* **38** (2005), 951–977.
- [SéSt1] V. Sécherre, S. Stevens, “Représentations lisses de $GL_m(D)$ IV: représentations supercuspidales”, *J. Inst. Math. Jussieu* **7.3** (2008), 527–574.
- [Ser] J.-P. Serre, *Corps locaux*. Hermann, Paris, 1962.
- [Tad] M. Tadić, “Induced representations of $GL(n, A)$ for p -adic division algebras A ”, *J. reine angew. Math.* **405** (1990), 48–77.
- [Tat] J. Tate, “Number theoretic background”, *Proc. Symp. Pure Math* **33.2** (1979), 3–26.
- [Vog] D. Vogan, “The local Langlands conjecture”, pp. 305–379 in: *Representation theory of groups and algebras*, *Contemp. Math.* **145**, American Mathematical Society, Providence RI, 1993.
- [Yu1] J.-K. Yu, “Bruhat-Tits theory and buildings”, pp. 53–77 in: *Ottawa lectures on admissible representations of reductive p -adic groups*, Fields Institute Monographs, American Mathematical Society, 2009
- [Yu2] J.-K. Yu, “On the local Langlands correspondence for tori”, pp. 177–183 in: *Ottawa lectures on admissible representations of reductive p -adic groups*, Fields Institute Monographs, American Mathematical Society, 2009
- [Zel] A.V. Zelevinsky, “Induced representations of reductive p -adic groups II. On irreducible representations of $GL(n)$ ”, *Ann. Sci. École Norm. Sup.* (4) **13.2** (1980), 165–210.

INSTITUT DE MATHÉMATIQUES DE JUSSIEU – PARIS RIVE GAUCHE, U.M.R. 7586 DU C.N.R.S.,
 U.P.M.C., 4 PLACE JUSSIEU 75005 PARIS, FRANCE
E-mail address: aubert@math.jussieu.fr

MATHEMATICS DEPARTMENT, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802,
 USA
E-mail address: baum@math.psu.edu

SCHOOL OF MATHEMATICS, SOUTHAMPTON UNIVERSITY, SOUTHAMPTON SO17 1BJ, ENGLAND
 and SCHOOL OF MATHEMATICS, MANCHESTER UNIVERSITY, MANCHESTER M13 9PL, ENGLAND
E-mail address: r.j.plymen@soton.ac.uk plymen@manchester.ac.uk

RADBOD UNIVERSITEIT NIJMEGEN, HEYENDAALSEWEG 135, 6525AJ NIJMEGEN, THE NETHERLANDS
E-mail address: m.solleveld@science.ru.nl