Point vortices on the hyperboloid

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POINT VORTICES ON THE HYPERBOLOID

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In Hamiltonian systems with symmetry, many previous studies have centred their attention on compact symmetry groups, but relatively little is known about the effects of noncompact groups. This thesis investigates the properties of the system of N point vortices on the hyperbolic plane $\mathcal{H}_2$, which has noncompact symmetry $SL\,(2,\mathbb{R})$.

The Poisson Hamiltonian structure of this dynamical system is presented and the relative equilibria conditions are found. We also describe the trajectories of relative equilibria with momentum value not equal to zero. Finally, stability criteria are found for a number of cases, focusing on $N = 2, 3$. These results are placed in context with the study of point vortices on the sphere, which has compact symmetry.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Chapter 1

Introduction

The system of point vortices is a topic that has been widely studied on the plane and on the sphere. Rangachari Kidambi and Paul K. Newton provide a through historical summary of research into point vortices, up to 1997, in [13]. In this work they study the motion of three point vortices on the sphere, and give the fixed and relative equilibria conditions.

The governing equations of the system for point vortices on the plane, sphere or hyperboloid are Hamiltonian. By providing a symplectic (Poisson) structure to this dynamical system, the study of relative equilibria can be approached using the energy momentum methods. Moreover, George W. Patrick [27] gives a useful result for the stability of relative equilibria on the reduced space of Hamiltonian systems with compact symmetry.

Sergey Pekarsky and Jerrold E. Marsden [29] use these results to find the stability of relative equilibria conditions for three point vortices on the sphere. Here the corresponding symplectic group action is $SO(3)$, which is a compact group. A numerical study of these results is performed in [19].

The case of point vortices on the hyperboloid model $\mathcal{H}_2$ has been treated in [2, 6, 11, 14]. But, to our knowledge it has not been investigated in detail except by Yoshifumi Kimura in [14], and Seungsu Hwang and Sun-Chul Kim in [11]. Kimura formulates the vortex motion on the sphere and on the hyperboloid $\mathcal{H}_2$. The Hamiltonian structure used in this thesis is the one provided in that reference. Hwang and Kim investigate the fixed and relative equilibria of three point vortices on the hyperboloid, and also study the case where vortices collapse.

In this thesis, we recover the relative equilibria results in [11] using the Hamiltonian structure of this dynamical system. We additionally study the case of two point vortices, which is mentioned in [14] for the particular case of a vortex dipole. Our findings in Chapter 5 coincide with the results provided there. Moreover, we provide stability results and a description of the trajectories for two and three point vortices with momentum value not equal to zero.

Much of the material presented in Chapters 2 and 3 is well known. These chapters introduce the background needed to study of the action $SL(2,\mathbb{R})$ on the hyperboloid $\mathcal{H}_2$. It is shown that for any $\mu \neq 0$ in $\mathfrak{sl}(2,\mathbb{R})^*$ we associate a conic (ellipse, hyperbola or parabola) related to its isotropy subgroup under the coadjoint action. We call this the *type* of μ: μ is elliptic, hyperbolic or parabolic when G_μ is isomorphic to an elliptic, hyperbolic or parabolic transformation, respectively.

In Chapter 4, we set up the Poisson Hamiltonian structure of the N point vortices on the hyperboloid $\mathcal{H}_2$.

The concept of a relative equilibrium is introduced in Chapter 5, and the case $N = 2$ is studied. It is found that every two point vortex configuration on the hyperboloid is a relative equilibrium. In Section 5.4 a list of some existing results of G and G_μ stability of relative equilibria is presented. These results require the symmetry group action to be free and proper, therefore collision of vortices must be discarded. However for action of $SO(3)$ to be free for this dynamical system on the sphere requires $N \geqslant 3$. In this sense, the system of point vortices on the hyperboloid is more interesting than on the sphere, as it allow us to use these results to derive the stability conditions from $N \geqslant 2$. In Theorem 5.4.8 we state that any two point configuration is $SL(2,\mathbb{R})$-stable, and in Corollary 5.4.9 we show that the stronger result of $SL(2,\mathbb{R})_\mu$ stability is obtained if the isotropy momentum value is elliptic.

Chapter 6 treats the case of $N = 3$. A classification of relative equilibria is given in Theorem 6.1 where we find that any relative equilibrium is either an equilateral configuration, or all three vortices lie on a common geodesic. However, in contrast with the sphere, a relative equilibrium on the hyperboloid does not satisfy both of these.

The stability conditions for the relative equilibria of three point vortices are studied in Chapter 7. It is important to point out that the stability conditions for three equilateral point vortices on the hyperboloid model are the same as for the system on the sphere and on the plane. Lemma 7.2.3 shows that the momentum value of a geodesic relative equilibrium X_e is either zero or elliptic.

In Theorem 7.2.3, the $SL\left(2,\mathbb{R}\right)_{\mu}$-stability conditions for an isosceles geodesic configuration with non-zero momentum value μ are given. Finally, in Figure 7.3, some of the stable regions of a geodesic relative equilibrium with three different lengths are shown.

By the time of the final submission of this thesis, Seungsu Hwang and Sun-Chul Kim had the reference [12] published. In this paper they present the relative equilibria conditions for rings of vortices on the hyperboloid, and conclude that any two point vortex configuration is a relative equilibrium. They also comment what the possible trajectory would be for any two point vortex configuration, they particularly discuss the case of the vortex dipole. In contrast with [11], the Hamiltonian used in [12] coincides with the our choice of Hamiltonian. Additionally, our results in relative equilibria agree with theirs, and the description of the trajectory of a vortex dipole is given here in Chapter 5.

Chapter 2

Hyperbolic geometry

Hyperbolic geometry was born as a tool to prove that the fifth Euclidean postulate can not be deduced from the other four. Several models of hyperbolic geometry have been constructed and widely studied since then. The *hyperboloid model*, denoted by $\mathcal{H}_2$, is a model of hyperbolic geometry and is the manifold where our dynamical system would be acting, hence it is essential to describe it.

Every other model is isometrically equivalent to this one and to each other [7], in sections 2.1 and 2.2 we define the relation between $\mathcal{H}_2$ and two additional models. All of the material presented in this chapter is very standard and can be found in the literature. In [7], the hyperboloid and three additional models are described, this paper together with [32] are useful references to follow the contents of this chapter.

2.1 Hyperbolic models

The *hyperboloid model*, also referred in the literature as the *Minkowski model*, is the representation of the hyperbolic plane by the upper sheet of the 2-sheeted hyperboloid in $\mathbb{R}^3$

$$\mathcal{H}_2 = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 - x^2 - y^2 = 1, z > 0\}.$$

The Riemannian metric of this model is

$$ds^2_{\mathcal{H}_2} = dx^2 + dy^2 - dz^2. \tag{2.1}$$

The induced inner product that defines this metric is defined in Section 2.3 together with other characteristics of this hyperbolic model.

Another commonly studied hyperbolic model is the *Poincaré disk model*

$$\mathbb{D} = \left\{(x, y, 0) \mid x^2 + y^2 < 1\right\},$$

with the metric $ds^2_{\mathbb{D}} = 4\frac{dx^2+dy^2}{(1-x^2-y^2)^2}$.

Finally we also relate relate to $\mathcal{H}_2$ the *Klein model* K which is given by

$$K = \left\{(x, y, 1) \mid x^2 + y^2 < 1\right\},$$

with metric $ds^2_K = \frac{dx^2+dy^2}{(1-x^2-y^2)} + \frac{(xdx+ydy)^2}{(1-x^2-y^2)^2}$.

2.1.1 Stereographic projection

As mentioned before, every model is isometrically equivalent to each other. In particular, the connection between the hyperboloid $\mathcal{H}_2$, with the Poincaré disk $\mathbb{D}$ and Klein model K is through stereographic projection. This projection is shown in Figure 2.1, it is easy to visualise what are the corresponding points from one model to another. A point X in the hyperboloid model $\mathcal{H}_2$ is projected to a point X_K

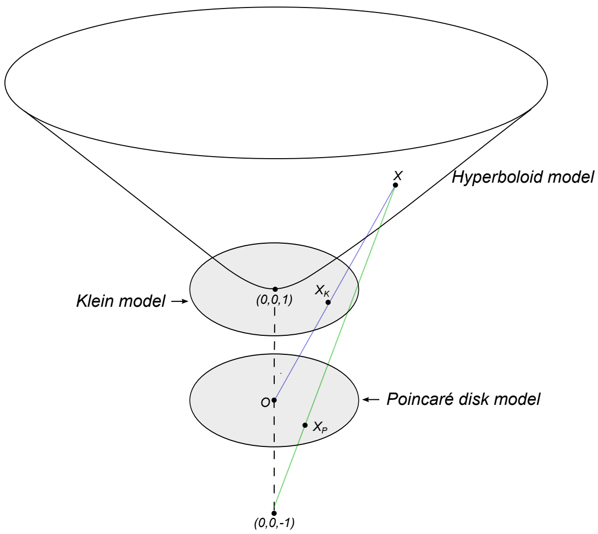


Figure 2.1: Stereographic projection between the hyperboloid model $\mathcal{H}_2$, Klein model K and Poincaré disk model $\mathbb{D}$

in the Klein model K, as the intersection of the plane $z = 1$ with the line passing through X and the origin O. Equivalently, a point X_P in the Poincaré disk model $\mathbb{D}$ corresponds to the intersection of the line through X and $(0,0,-1)$ with the plane $z=0$.

Stereographic projection preserves angles between their corresponding points, this is the reason why we have chosen to additionally describe these models, which are equivalent to the hyperboloid by this kind of projection. While working on numerical integrations for three point vortices on the hyperboloid, we found it easier to carry out the calculations in the projected points in the Poincaré disk $\mathbb{D}$ than in $\mathcal{H}_2$. The projectivization map from $\mathcal{H}_2$ to the Poincaré disk model $\mathbb{D}$ is given by

$$\begin{aligned} \varphi : \mathcal{H}_2 &\to \mathbb{D}, \\ \varphi \begin{pmatrix} x \\ y \\ z \end{pmatrix} &\mapsto \frac{1}{z+1} \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned} \tag{2.2}$$

Its inverse is given by the formula

$$\varphi^{-1} \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \frac{1}{1-r^2} \begin{pmatrix} 2u \\ 2v \\ 1+r^2 \end{pmatrix}, \tag{2.3}$$

where $r^2 = u^2 + v^2$ for $(u,v) \in \mathbb{D}$.

The *null-cone* $\mathfrak{C}$ is defined as the set of vectors $x \in \mathbb{R}^3$ with $\|\overrightarrow{x}\|^2 = \langle x, x\rangle_{\mathcal{H}_2} = 0$. Via the projectivization map (2.2), $\mathfrak{C}$ corresponds to the unit circle in $\mathbb{R}^2$.

Since the pullback of the Riemannian metric of the Poincaré disk $ds^2_{\mathbb{D}}$ is the metric $ds^2_{\mathcal{H}_2}$ for the hyperboloid, that is $\varphi^*\left(ds^2_{\mathbb{D}}\right) = ds^2_{\mathcal{H}_2}$ these projection maps are *isometries.*

2.1.2 Symplectic structure

From [3] a *symplectic structure* on a *2n* dimensional manifold $\mathcal{M}$ is a closed non-degenerate differential 2-form ω^2 on $\mathcal{M}$. The pair, $(\mathcal{M}, \omega^2)$ is called a *symplectic manifold*. A symplectic form on the hyperboloid $\mathcal{H}_2$ is also a symplectic form on the Poincaré disk. For instance

$$\omega_{\mathcal{H}_{\mathbb{D}}} = 2\frac{dx \wedge dy}{(1 - x^2 - y^2)^2}, \tag{2.4}$$

gives a symplectic structure to both the $\mathbb{D}_2$ and to the hyperboloid model via the pullback $\varphi^* (\omega_{\mathbb{D}}) = \omega_{\mathcal{H}_2}$.

In Section 4.2 we will define the *Kostant-Kirillov-Souriau* form that acts on $\mathcal{H}_2$, and hence can be constructed for any of this hyperbolic models. The KKS form gives a symplectic structure to the hyperboloid model $\mathcal{H}_2$, so it allow us to approach the study the dynamics of a Hamiltonian system on the hyperboloid as the dynamics of a symplectic manifold $(\omega_{\mathcal{H}_2}, \mathcal{H}_2)$. The symplectic form (2.4) is $SL(2, \mathbb{R})$ invariant on the hyperboloid $\mathcal{H}_2$ as stated in [23], in this paper it is also shown that the momentum map could be chosen so $(\omega_{\mathcal{H}_2}, \mathcal{H}_2)$ coincides with the KKS symplectic form (4.10).

2.2 Geodesics

The Poincaré disk and Klein model are very similar, they both share the same domain, but not the same hyperbolic lines or *geodesics*. In the Klein model geodesics are straight line segments, while in the Poincaré disk these lines correspond to diameters and circles orthogonal to the boundary.

We can then apply stereographic projection from the Klein model K or Poincaré disk $\mathbb{D}$ to the hyperboloid $\mathcal{H}_2$, as it can be seen in Figure 2.2. The geodesics for hyperboloid model are the intersection of the hyperboloid $\mathcal{H}_2$ and any plane P in $\mathbb{R}^3$ passing through the origin. A proof of this can be found in Section 9 of [7] or in Seungsu Hwang and Sun-Chul Kim's Lemma 1 of [11].

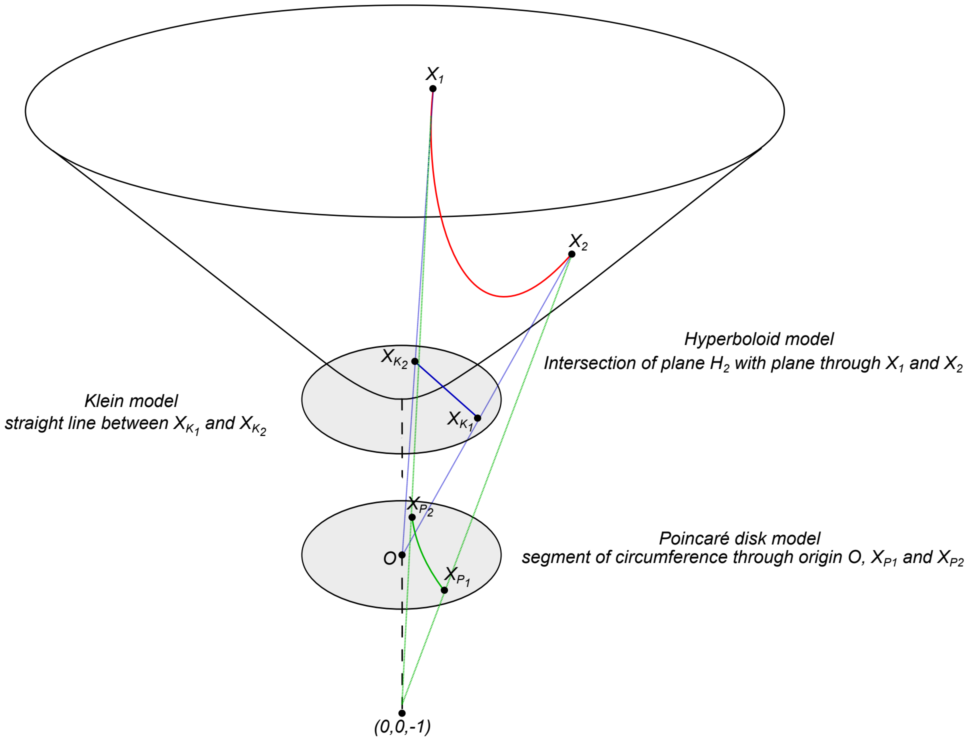


Figure 2.2: Geodesics in the hyperboloid model $\mathcal{H}_2$, Klein model K and Poincaré disk model $\mathbb{D}$

2.3 Hyperboloid model

The *hyperbolic inner product* induced in $\mathbb{R}^3$ that defines $ds^2_{\mathcal{H}_2}$ is

$$\langle X_1, X_2 \rangle_{\mathcal{H}_2} = x_1x_2 + y_1y_2 - z_1z_2, \tag{2.5}$$

for $X_1 = (x_1, y_1, z_1)$ and $X_2 = (x_2, y_2, z_2)$.

We can then rewrite our definition of the hyperboloid as

$$\mathcal{H}_2 = \left\{ X = (x, y, z) \in \mathbb{R}^3 \mid \langle X, X \rangle_{\mathcal{H}_2} = -1, z > 0 \right\}.$$

The *hyperbolic cross product* for this hyperbolic model is defined by

$$(x_1, y_1, z_1) \times_{\mathcal{H}_2} (x_2, y_2, z_2) = (y_1z_2 - z_1y_2, z_1x_2 - x_1z_2, -x_1y_2 + y_1x_2). \tag{2.6}$$

Note that the *volume* of three vectors by means of this hyperbolic model is the same obtained by Euclidean geometry,

$$V = X_1 \times (X_2 \cdot X_3) = X_1 \times_{\mathcal{H}_2} \langle X_2, X_2 \rangle_{\mathcal{H}_2} \tag{2.7}$$

By analogy with Euclidean geometry, we define the *hyperbolic distance* between X_1 and $X_2 \in \mathcal{H}_2$ as the hyperbolic length of the hyperbolic line connecting these two points. We understand an hyperbolic line as the unique path in a geodesic joining these two points. The following lemma is a well known result that relates the inner product with this hyperbolic distance.

Lemma 2.3.1 ([7]). *Let $d(X_1, X_2)$ be the hyperbolic distance between X_1 and $X_2 \in \mathcal{H}_2$, then $\langle X_1, X_2 \rangle_{\mathcal{H}_2} = -\cosh(d(X_1, X_2))$.*

The proof can be found in [7].

Chapter 3

$SL(2, \mathbb{R})$

The *special linear group* $SL(2, \mathbb{R})$ is of obvious interest as the dynamical system of point vortices on the hyperboloid is ruled by the action of this Lie group. We derive important results that will help us to understand the role of this symmetry group in this dynamical system.

In this chapter we firstly present some basic definitions and results for Lie groups and Lie algebras, this material is mainly taken from [4], [10] and [20] . We present every definition for general groups together with its equivalent for matrix groups. Throughout the whole chapter we focus on introducing results for $SL(2, \mathbb{R})$ either as an example or as a remark.

The second section introduces results for the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ and the dual space of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})^*$. Theorem 3.2.1 in Section 3.2.3 relates the geometry of the coadjoint orbits with Möbius transformations, this theorem help us to classify the type of $\mu \in \mathfrak{sl}(2, \mathbb{R})^*$. Although not used now, from the information presented here Proposition 3.2.7 defines which type of $\mu \in \mathfrak{sl}(2, \mathbb{R})^*$ is split, while Proposition 3.2.9 shows that any $\mu \neq 0$ is regular.

3.1 Lie groups and Lie algebras

We denote $\mathcal{M}(n, \mathbb{R})$ as the set of $n \times n$ matrices with real entries. Recall that any $M \subset \mathcal{M}(n, \mathbb{R})$ with matrix multiplication as the group operation is a *matrix group*.

Definition 3.1.1. A *Lie group* is a smooth manifold that is also a group, with the property that the operations of group multiplication, $(g,h) \mapsto g \cdot h$, and inversion, $g \mapsto g^{-1}$, are smooth. A *matrix Lie group* is a matrix group that is also a submanifold of $\mathcal{M}(n,\mathbb{R})$.

Example 3.1.2. The *general linear group* $GL(n,\mathbb{R})$ is an n^2 dimensional submanifold of $\mathcal{M}(n,\mathbb{R})$ consisting of all $n \times n$ invertible matrices. Example 7.18 in [4] proves that this is a matrix Lie group.

Moreover, by definition every matrix Lie group is a manifold, and the operations of matrix multiplication and matrix inversion are smooth. Therefore every matrix Lie group is a Lie group, however the converse is not always true.

Definition 3.1.3. Let G be a Lie group. A closed subgroup $H \leq G$ that is also a submanifold is called a *Lie subgroup of G*.
It is then automatic that the restrictions to H of the multiplication and inverse maps on G are smooth, hence H is also a Lie group.

Example 3.1.4. As shown in Example 7.19 of [4], the *special linear group*

$$SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \,|\det A = 1\}$$

is a Lie subgroup of $GL(n,\mathbb{R})$. Consequently $SL(n,\mathbb{R})$ is a matrix Lie group too.

Definition 3.1.5. A (real) *Lie algebra* is a (real) vector space $\mathcal{A}$ together with a bilinear operation $(v,w) \in \mathcal{A} \times \mathcal{A} \to [v,w] \in \mathcal{A}$, called the *Lie bracket*, such that,

1. $[v,w] = -[w,v]$ for all $v,\ w \in \mathcal{A}$ (skew-symmetry),

2. $[[v,w],u] + [[u,v],w] + [[w,u],v] = 0$ for all $u,\ v,\ w \in \mathcal{A}$ (Jacobi identity).

Definition 3.1.6. The *matrix commutator* of any pair of $n \times n$ matrices A and B is defined as $[A,B] := AB - BA$. A *matrix Lie algebra* is a vector subspace of $\mathcal{M}(n,\mathbb{R})$ for some n with the usual operations of matrix addition and scalar multiplication, that is also closed under the matrix commutator $[\cdot,\cdot]$.

It is not hard to show that for any A, B and $C \in \mathcal{M}(n,\mathbb{R})$ the matrix commutator satisfies the skew-symmetry and Jacobi identity conditions of Definition 3.1.5 . In view of this it is evident that every matrix Lie algebra is a Lie algebra given the Lie bracket as the matrix commutator.

The following proposition is a very useful and well known result, which firstly relies in the fact that the *tangent space at the identity* $T_I G$ of a Lie group G is a vector subspace of $\mathcal{M}(n,\mathbb{R})$, and secondly that the matrix commutator is closed in $T_I G$.

Proposition 3.1.7 ([10])**.** *For any matrix Lie group G, the tangent space at the identity $T_I G$ is a matrix Lie algebra.*

Given a Lie group G we will denote $\mathfrak{g} = T_I G$ as the Lie algebra of G.

Example 3.1.8. The Lie algebra of $GL(n,\mathbb{R})$ is simply

$$\mathfrak{gl}(n,\mathbb{R}) = \{A \in \mathcal{M}(n,\mathbb{R})\}.$$

To illustrate Proposition 3.1.7, in Example 3.1.10 we calculate $\mathfrak{sl}(n,\mathbb{R})$, the Lie algebra of $SL(n,\mathbb{R})$. Before this we introduce the *Implicit Function Theorem for manifolds*, which will be of use not only for the calculations of $\mathfrak{sl}(n,\mathbb{R})$ but for later calculations in Chapter 6.

Theorem 3.1.9 (Implicit Function Theorem for manifolds [4])**.** *Let $f : \mathcal{M} \longrightarrow \mathcal{M}'$ be a smooth map between smooth manifolds of dimensions n and n'. Suppose that for some $q \in \mathcal{M}'$, $df_p := T_p\mathcal{M} \longrightarrow T_{f(p)}\mathcal{M}'$ is surjective for every $p \in \mathcal{N} = f^{-1}q$. Then $\mathcal{N} \subseteq \mathcal{M}$ is a submanifold of dimension $n - n'$ and the tangent space at $p \in \mathcal{N}$ is given by $T_p\mathcal{N} = \ker df_p$.*

Example 3.1.10. $\mathfrak{sl}(n,\mathbb{R})$ is the space of traceless matrices.

Proposition 2.9.2 in [10] proves that

$$\begin{aligned} f : GL(n,\mathbb{R}) &\longrightarrow \mathbb{R} \\ A &\longrightarrow \det A \end{aligned}$$

is a submersion, hence its differential is everywhere surjective. Therefore, by Theorem 3.1.9, $SL(n,\mathbb{R}) = \det^{-1}(1)$ is a submanifold of $\mathcal{M}(n,\mathbb{R})$ of dimension n^2-1 and

$$\begin{aligned}\mathfrak{sl}(n,\mathbb{R}) &= T_I SL(n,\mathbb{R}) \\ &= \ker df(I),\end{aligned}$$

where f is the determinant map. That implies

$$\mathfrak{sl}(n,\mathbb{R}) = \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid (d\det(I))(A) = 0\}.$$

Rewriting A as $\frac{d}{dt}\big|_{t=0} I + tA$

$$(d\det(I))(A) = \frac{d}{dt}\Big|_{t=0} \det(I+tA).$$

Given $B \in \mathcal{M}(n,\mathbb{R})$ Jacobi's formula states that $\frac{d}{dt}\det B = \mathrm{tr}\ (\mathrm{adj}(B)\frac{dB}{dt})$, where $\mathrm{tr}\ B$ and $\mathrm{adj}B$ represent the trace and adjugate of B respectively. Thus

$$\begin{aligned}(T_I \det)(A) &= \mathrm{tr}\ \left(\mathrm{adj}\,(I+tA)\,\frac{d\,(I+tA)}{dt}\right)\Big|_{t=0} \\ &= \mathrm{tr}\ (\mathrm{adj}I \cdot A).\end{aligned}$$

Moreover, for B an invertible matrix, $B\ \mathrm{adj}B = \det B\ I$, i.e. $\mathrm{adj}B = \det B\ B^{-1}$, thus $\mathrm{adj}I = I$ and following on from this we get

$$(T_I \det)(A) = \mathrm{tr}\ A.$$

Consequently,

$$\mathfrak{sl}(n,\mathbb{R}) = \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid \mathrm{tr}\ A = 0\}.$$

A basis for $\mathfrak{g}=\mathfrak{sl}\left(2,\mathbb{R}\right)$ is given by

$$\mathcal{B}=\left\{e_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, e_2=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, e_3=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right\}. \tag{3.1}$$

A subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is defined as a *Lie subalgebra* of $\mathfrak{g}$ if it is closed under the Lie bracket in $\mathfrak{g}$.

Proposition 3.1.11 ([20])**.** *Let H be a Lie subgroup of G. Then the Lie algebra of H, denoted by $\mathfrak{h}$, is a Lie subalgebra of $\mathfrak{g}$. Moreover,*

$$\mathfrak{h}=\left\{\xi\in\mathfrak{g}|\exp t\xi\in H \text{ for all } t\in\mathbb{R}\right\}.$$

From the first statement we can conclude that $\mathfrak{sl}\left(n,\mathbb{R}\right)$ is a Lie subalgebra of $\mathfrak{gl}\left(n,\mathbb{R}\right)$.

Definition 3.1.12. The *dual space* of any real vector space V, denoted V^*, is the set of linear maps from V to $\mathbb{R}$, which is itself a real vector space, with the usual operations of addition and scalar multiplication of maps. Given $\mathcal{B}=\{e_1,e_2,..,e_n)$ a basis of V, then the associated dual basis for V^* denoted by $\mathcal{B}'=\{e^1,e^2,...,e^n\}$ is defined by

$$\langle e^i,e_j\rangle:=\delta_{ij},$$

with δ_{ij} the Kronecker delta, and $\langle\cdot,\cdot\rangle:V^*\times V\to\mathbb{R}$ the natural pairing that identifies V with V^*.

When working with the action of a matrix Lie group G in a Euclidean metric space, the trace pairing that identifies $A\in\mathfrak{g}$ with $B\in\mathfrak{g}^*$ is usually defined by $\langle A,B\rangle=\operatorname{tr}\ (AB)$. In contrast, the natural pairing that corresponds to the hyperbolic inner product (2.5) in $\mathcal{H}_2$, and therefore defines the hyperbolic metric $ds^2_{\mathcal{H}_2}$

(2.1), is given by the trace pairing

$$\langle A, B\rangle := \frac{1}{2}\mathrm{tr}\ (AB)\,. \tag{3.2}$$

Therefore, when analysing the action of $SL(2,\mathbb{R})$ in the hyperbolic model $\mathcal{H}_2$, the pairing (3.2) identifies $\mathfrak{sl}(2,\mathbb{R})$ with $\mathfrak{sl}(2,\mathbb{R})^*$.

Example 3.1.13.

$$\mathcal{B}' = \left\{ e^1 = e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e^2 = e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e^3 = -e_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \tag{3.3}$$

is dual to the basis (3.1) of $\mathfrak{sl}(2,\mathbb{R})$. Hence, (3.3) is a basis of $\mathfrak{sl}(2,\mathbb{R})^*$ indeed.

Remark 3.1.14. $\mathfrak{sl}(2,\mathbb{R}) \cong \mathbb{R}^3$ *and* $\mathfrak{sl}(2,\mathbb{R})^* \cong \mathbb{R}^3$.

Any vector $\check{X} = (x, y, z)$ in $\mathbb{R}^3$ can be identified with a 2×2 traceless matrix X by the map

$$\check{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \leftrightarrow X = \begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix}. \tag{3.4}$$

This map is a vector isomorphism and can be used to analyse the action of $SL(2,\mathbb{R})$ on $\check{X} \in \mathcal{H}_2 \subset \mathbb{R}^3$. By looking at the basis of $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(2,\mathbb{R})^*$ is clear that we can associate $\xi \in \mathfrak{s}l(2,\mathbb{R})$ with a vector in $\mathbb{R}^3$ by (3.4). We can also use this identification map to relate $\mu \in \mathfrak{sl}(2,\mathbb{R})^*$ with $\mathbb{R}^3$.

Furthermore, let ξ and $\eta \in \mathfrak{g}$ with $\check{\xi}$ and $\check{\eta} \in \mathbb{R}^3$ their associated vectors, then it is a straightforward calculation to show

$$[\xi, \eta]\check{} = -2\left(\check{\xi} \times_{\mathcal{H}_2} \check{\eta}\right). \tag{3.5}$$

Hence we identify $\mathfrak{sl}(2,\mathbb{R})$ with $\mathbb{R}^3$ using the hyperbolic cross product as a Lie bracket, (3.4) becomes a Lie algebra isomorphism indeed. The exact same conclusion follows from noticing that any vector $\check{X}$ is related to a element $\mu \in \mathfrak{sl}(2,\mathbb{R})^*$ and that (3.5) is also satisfied for elements of $\mathfrak{sl}(2,\mathbb{R})^*$.

3.1.1 Action of Lie groups

We follow the same approach of previous section, introducing first definitions for a general Lie group together with the corresponding definition for a matrix Lie group, which is given either as another definition or as an example.

Definition 3.1.15. A (smooth) *left action* of a Lie group G on manifold $\mathcal{M}$ is a smooth mapping

$$\begin{aligned} \Phi : G \times \mathcal{M} &\rightarrow \mathcal{M} \\ (g, x) &\rightarrow g \cdot x, \end{aligned}$$

such that

1. $\Phi(e, x) = x$ for all $x \in \mathcal{M}$ and e the identity element of G.

2. $\Phi(g, \Phi(h, x)) = \Phi(gh, x)$ for all g, $h \in G$ and $x \in \mathcal{M}$, and

3. For every $g \in G$, the map $\Phi_g : \mathcal{M} \rightarrow \mathcal{M}$, defined by

$$\Phi_g(x) := \Phi(g, x),$$

is a diffeomorphism.

Definition 3.1.16. The *left action of a matrix Lie group* $G \subset GL(n,\mathbb{R})$ on $\mathbb{R}^n$ is given by $\Phi(A, \mathbf{v}) = A\mathbf{v}$ (left matrix multiplication). An analogous definition is given for *right action* using right matrix multiplication.

The identification map (3.4) allow us to analyse the action of $G \subset \mathcal{M}(2,\mathbb{R})$ on $\mathbb{R}^3$, for instance the action of $SL(2,\mathbb{R})$ in $\mathcal{H}_2 \subset \mathbb{R}^3$ is calculated by multiplication of 2×2 matrices.

Definition 3.1.17. The *group orbit of* G through $x \in \mathcal{M}$ is defined as

$$\mathrm{Orb}(x) = \{g \cdot x | g \in G\},$$

and the *isotropy group (stabiliser group)* of x as

$$G_x = \{g \in G | g \cdot x = x\}. \tag{3.6}$$

An action is said to be *transitive* if for every x, $y \in \mathcal{M}$, there exists $g \in G$ such that $g \cdot x = y$. If the action of G in $\mathcal{M}$ has no isotropy groups, that is if for all x such that $g \cdot x = x$ implies $g = e$, then the action is called *free*. If the map of the action is proper, the action is said to be *proper*. Hence, the action is proper if for every pair of sequences $\{x_n\}$ and $\{g_n x_n\}$ convergent in $\mathcal{M}$, the sequence $\{g_n\}$ has a convergent subsequence in G.

Lemma 3.1.18. *The action of $SL(2,\mathbb{R})$ in $\mathcal{H}_2$ is transitive and proper.*

Proof. Let $\check{X}_1 = (x_1, y_1, z_1)$ and $\check{X}_2 = (x_2, y_2, z_2) \in \mathcal{H}_2$ with hyperbolic inner product $k = x_1x_2 + y_1y_2 - z_1z_2$. Then

$$g := \begin{pmatrix} y_2z_1 - z_2y_1 - k & x_1z_2 - z_1x_2 + x_1y_2 - y_1x_2 \\ x_1z_2 - z_1x_2 - (x_1y_2 - y_1x_2) & -(y_2z_1 - z_2y_1) - k \end{pmatrix} \in SL(2,\mathbb{R}), \tag{3.7}$$

satisfies $g \cdot X_1 = X_2$.

In Example 2.3.5 in [26] is shown that the action of Lie groups on themselves is proper. We now show that

$$\begin{aligned} F : G \times G/K &\rightarrow G/K \times G/K \\ (g, hk) &\rightarrow (ghK, hK) \end{aligned}$$

is proper if K is compact.

Let (g_i, h_ik) be a sequence such that $g_ih_iK \rightarrow qK$ and $h_iK \rightarrow hK$. Since the identity is a continuous map $Kh_i^{-1} \rightarrow Kh^{-1}$ must hold, thus

$$\lim_{i\rightarrow\infty} g_ih_iKh_i^{-1} = qKh^{-1}.$$

Let a, $b \in K$, since K is compact the distance between a and b is given by $d(aK, bK) = \min_{k\in K} d(a, bk)$. Therefore,

$$d\left(g_i, qKh^{-1}\right) := \min_{k\in K} d\left(g_i, qkh^{-1}\right) \to 0.$$

Given that K is compact there exists k_0 such that the subsequence $k_{i_j} \to k_0$. Hence,

$$d\left(g_i, qk_0h^{-1}\right) \leqslant d\left(g_i, qk_{i_j}h^{-1}\right) + d\left(qk_{i_j}h^{-1}, qk_0h^{-1}\right),$$

so $g_{i_j} \to qk_0h$ and the map F is proper.

Since $\mathcal{H}_2 \cong SL(2,\mathbb{R})/SO(2)$ the map

$$\begin{aligned} F : SL(2,\mathbb{R}) \times \mathcal{H}_2 &\to \mathcal{H}_2 \times \mathcal{H}_2 \\ (g, m) &\to (g \cdot m, m) \end{aligned}$$

is proper. □

Definition 3.1.19. The action of G on itself by *left multiplication* is the left action defined by

$$\begin{aligned} G \times G &\to G, \\ (g, h) &\to L_g(h) := g \cdot h. \end{aligned}$$

Analogously, the *right multiplication* action is the right action defined by

$$\begin{aligned} G \times G &\to G, \\ (g, h) &\to R_g(h) := h \cdot g. \end{aligned}$$

The action of G on itself by *inner automorphism* is

$$\begin{aligned} G \times G &\to G, \\ (g, h) &\to I_g(h) := (L_g \circ R_{g^{-1}})(h) = ghg^{-1}. \end{aligned}$$

Definition 3.1.20. The *adjoint action of* G on $\mathfrak{g}$ is

$$\begin{aligned} G\times\mathfrak{g} &\to \mathfrak{g}, \\ (g,\xi) &\to Ad_g\xi := T_eI_g(\xi). \end{aligned}$$

Example 3.1.21 ([10])**.** *Adjoint action for matrix Lie groups*

Let $R\in G$ and $B(t)$ a path with $B(0)=I$ and $B'(0)=\xi\in\mathfrak{g}$, we define the *adjoint action* of G on $\mathfrak{g}$ as:

$$Ad_R\xi = T_II_R(\xi) = \left.\frac{d}{dt}\right|_{t=0} I_R(B(t)) = \left.\frac{d}{dt}\right|_{t=0} RB(t)R^{-1} = R\xi R^{-1}.$$

Therefore,

$$Ad_R\xi = R\xi R^{-1}. \tag{3.8}$$

Definition 3.1.22. The *coadjoint action* of G on $\mathfrak{g}^*$ is the inverse dual of the adjoint action:

$$\begin{aligned} G\times\mathfrak{g}^* &\to \mathfrak{g}^*, \\ (g,\mu) &\to Ad^*_{g^{-1}}\mu, \end{aligned}$$

where

$$\left\langle Ad^*_{g^{-1}}\mu,\xi\right\rangle = \langle\mu, Ad_{g^{-1}}\xi\rangle$$

for all $\mu\in\mathfrak{g}^*$, $\xi\in\mathfrak{g}$ and $\langle\cdot,\cdot\rangle : \mathfrak{g}^*\times\mathfrak{g}\to\mathbb{R}$ is the natural pairing.

Example 3.1.23. *Coadjoint action of* $SL(2,\mathbb{R})$ *on* $\mathfrak{sl}(2,\mathbb{R})^*$. Let $\xi\in\mathfrak{sl}(2,\mathbb{R})$ and $\mu\in\mathfrak{sl}(2,\mathbb{R})^*$, by definition

$$\begin{aligned} \langle \mathrm{Ad}^*_{R^{-1}}\mu,\xi\rangle &= \langle\mu,\mathrm{Ad}_{R^{-1}}\xi\rangle \\ &= \frac{1}{2}\mathrm{tr}\,\left(\mu R^{-1}\xi R\right) \\ &= \frac{1}{2}\mathrm{tr}\,\left(R\mu R^{-1}\xi\right) \\ &= \langle R\mu R^{-1},\xi\rangle. \end{aligned}$$

Given that $\langle\cdot,\cdot\rangle$ is non degenerate we conclude

$$\mathrm{Ad}^*_{R^{-1}}\mu = R\mu R^{-1}. \tag{3.9}$$

Remark 3.1.24. Consider the adjoint and coadjoint action of $SL(2,\mathbb{R})$ on $\mathfrak{sl}(2,\mathbb{R})^*$ and let $\check{X} \in \mathbb{R}^3$. Suppose the associated vectors to $\check{X}$ in $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(2,\mathbb{R})^*$ are ξ and μ respectively. Then

$$\mathrm{Ad}^*_{R^{-1}}\mu = \mathrm{Ad}_R\xi, \tag{3.10}$$

that is, the adjoint and coadjoint action at $\check{X}$ are equivalent.

Lemma 3.1.25. *Given μ and $\eta \in \mathfrak{sl}(2,\mathbb{R})^*$ (or $\mathfrak{sl}(2,\mathbb{R})$) such that they are both equal to zero or both different from zero. Suppose $\check{\mu} = (\mu_1,\mu_2,\mu_3)$ and $\check{\eta} = (\eta_1,\eta_2,\eta_3)$ satisfy $\mu_3\eta_3 > 0$, then μ and η are in the same coadjoint (or adjoint) orbit if and only if* $\det\mu = \det\eta$.

Proof. If μ and η are in the same coadjoint orbit then there exists $g \in SL(2,\mathbb{R})$ such that

$$\mathrm{Ad}^*_{g^{-1}}\mu = g\mu g^{-1} = \eta,$$

thus

$$\det\mu = \det\eta.$$

Recall that two $n\times n$ matrices A and B are similar if there exists a $n\times n$ invertible matrix P such that $A = P^{-1}BP$, it is obvious that if μ and η are similar then they are in the same coadjoint orbit.

Let μ and η on the null-cone $\mathfrak{C}$ such that they are both different from zero. We first assume that neither μ_1 or η_1 are equal to zero, that is $\mu_2 \neq \pm\mu_3$ and $\eta_2 \neq \pm\eta_3$. Let $\check{X} = (0,1,1)$ then

$$g_\mu := \begin{pmatrix} -\sqrt{\frac{\mu_2+\mu_3}{2}} & 1 \\ \sqrt{\frac{\mu_3-\mu_2}{2}} & -\frac{\mu_1+\sqrt{2(\mu_2+\mu_3)}}{\mu_2+\mu_3} \end{pmatrix} \in SL(2,\mathbb{R}) \tag{3.11}$$

satisfies $\mathrm{Ad}^*_{g_\mu^{-1}}X = \mu$. Similarly there exists g_η such that $\mathrm{Ad}^*_{g_\eta^{-1}}X = \eta$ which implies $\mathrm{Ad}^*_{g_\mu g_\eta^{-1}}\mu = \eta$. Consequently η is in the same coadjoint orbit of μ. The calculations for one or both of μ_1, η_1 equal to zero, that is collinear with $\check{X}$ or $(0,-1,1)$, are also straightforward.

Now assume that $\det \mu = \det \eta \neq 0$, then both matrices share the same eigenvalues and are similar to the same diagonal matrix D. Since similarity is a transitive property, μ and η are similar and therefore in the same coadjoint orbit indeed.

To expand on this, let P_1 and P_2 the matrix with columns as the eigenvectors of μ and η respectively, that is

$$D = P_1^{-1} \mu P_1,$$

and

$$D = P_2^{-1} \eta P_2.$$

Then,

$$\mathrm{Ad}^*_{P_2 P_1^{-1}} \mu = \eta$$

In accordance with Remark 3.1.24 exactly the same conclusions are obtained when working with the adjoint orbit.

□

3.1.2 Infinitesimal generators

For each vector $\xi \in \mathfrak{g}$, the Lie group action defines an infinitesimal generator vector field $\xi_{\mathcal{P}}$ on the manifold $\mathcal{P}$.

Definition 3.1.26. Let ξ be a vector in $\mathfrak{g}$, and consider the one-parameter subgroup $\{\exp(t\xi) : t \in \mathbb{R}\} \subseteq G$. The orbit of an element x with respect to this subgroup is a smooth path $t \to (exp(t\xi))\, x$ in $\mathcal{M}$. The *infinitesimal generator* associated to ξ at $x \in \mathcal{M}$, denoted $\xi_{\mathcal{M}}(x)$, is the tangent (or velocity) vector to this curve at x, that is:

$$\xi_{\mathcal{M}}(x) := \left.\frac{d}{dt}\right|_{t=0} (\exp(t\xi)\, x) \in T_x\mathcal{M}.$$

The smooth vector field

$$\begin{aligned} \xi_{\mathcal{M}} : \mathcal{M} &\to T\mathcal{M}, \\ x &\to \xi_{\mathcal{M}}(x), \end{aligned}$$

is called the *infinitesimal generator vector field associated to* ξ.

The Lie algebra of the isotropy group G_x, $x \in \mathcal{M}$, called the *isotropy Lie algebra at* x and denoted $\mathfrak{g}_x$, by Proposition 3.1.11 is defined as

$$\begin{aligned} \mathfrak{g}_x &= \{\xi \in \mathfrak{g} \mid \exp t\xi \cdot x = x, \forall t \in \mathbb{R}\} \\ &= \{\xi \in \mathfrak{g} \mid \xi_{\mathcal{M}}(x) = 0\}. \end{aligned} \tag{3.12}$$

Proposition 3.1.27 ([20])**.** *The tangent space at* x *to an orbit* $\mathrm{Orb}(x_0)$ *is*

$$T_x\mathrm{Orb}(x_0) = T_x(G \cdot x_0) = \{\xi_{\mathcal{M}}(x_0) \,|\xi \in \mathfrak{g}\},$$

where $\mathrm{Orb}(x_0)$ *is endowed with the manifold structure making* $G/G_{x_0} \to \mathrm{Orb}(x_0)$ *into a diffeomorphism.*

Definition 3.1.28. The infinitesimal generator map

$$\begin{aligned} \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g}, \\ (\xi, \eta) &\to \mathrm{ad}_\xi(\eta) = \xi_{\mathfrak{g}}(\eta) = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp t\xi}\eta \end{aligned}$$

is called the adjoint action of $\mathfrak{g}$ on itself, even though is not a group action, the **adjoint operator** denoted by ad is defined by

$$\mathrm{ad}_\xi\eta = \xi_{\mathfrak{g}}(\eta) \qquad \text{for all } \eta \in \mathfrak{g}.$$

Example 3.1.29 ([10])**.** *The adjoint operator for matrix Lie algebras.*

Given ξ and $\eta \in \mathfrak{g}$

$$\begin{aligned}
\mathrm{ad}_\xi \eta &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp t\xi}\eta \\
&= \left.\frac{d}{dt}\right|_{t=0} (\exp t\xi)\,\eta\,(\exp t\xi)^{-1} \\
&= \xi\eta - \eta\xi = [\xi,\eta]\,.
\end{aligned}$$

This equation is not exclusive to matrix Lie algebras, it is also derived for any type of Lie algebra following the general definition of a Lie algebra.

Definition 3.1.30. The **coadjoint operator** is the map

$$\begin{aligned}
\mathrm{ad}^* : \mathfrak{g}\times\mathfrak{g}^* &\to \mathfrak{g}^*, \\
(\xi,\mu) &\to \mathrm{ad}^*_\xi(\mu)\,,
\end{aligned}$$

such that for all $\xi \in \mathfrak{g}$, the map $\mathrm{ad}^*_\xi : \mathfrak{g}^* \to \mathfrak{g}^*$ is the dual of the adjoint operator, that is

$$\langle \mathrm{ad}^*_\xi \mu, \eta\rangle = \langle \mu, \mathrm{ad}_\xi \eta\rangle \qquad \text{for all } \eta \in \mathfrak{g}.$$

Remark 3.1.31 ([10]). Let ξ, $\eta \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$, then from the definition of the infinitesimal generator

$$\begin{aligned}
\langle \xi_{\mathfrak{g}^*}(\mu), \eta\rangle &= \left\langle \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}^*_{\exp(t\xi)}\mu, \eta \right\rangle \\
&= \left.\frac{d}{dt}\right|_{t=0} \langle \mathrm{Ad}^*_{\exp(t\xi)}\mu, \eta\rangle \\
&= \left\langle \mu, \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(-t\xi)}\eta \right\rangle \\
&= \langle \mu, -\mathrm{ad}_\xi \eta\rangle.
\end{aligned}$$

Thus, the infinitesimal generator of $\mathfrak{g}^*$ is related to the coadjoint operator by

$$\mathrm{ad}^*_\xi(\mu) = -\xi_{\mathfrak{g}^*}(\mu)\,. \tag{3.13}$$

In connection with the definition of (3.12), the isotropy Lie algebra of $\mu \in \mathfrak{g}^*$ is given by

$$\mathfrak{g}_\mu = \left\{\xi \in \mathfrak{g} | \mathrm{ad}^*_\xi \mu = 0\right\}. \tag{3.14}$$

Example 3.1.32. *Coadjoint operator of $SL(2,\mathbb{R})$ on $\mathcal{H}_2$.* Let G be a matrix Lie group, for all ξ, $\eta \in \mathfrak{g}$ and for every $\mu \in \mathfrak{g}^*$

$$\begin{aligned}
\langle \mathrm{ad}^*_\xi \mu, \eta \rangle &= \langle \mu, \mathrm{ad}_\xi \eta \rangle \\
&= \langle \mu, [\xi, \eta] \rangle \\
&= \langle \mu, \xi\eta - \eta\xi \rangle \\
&= \langle \mu, \xi\eta \rangle - \langle \mu, \eta\xi \rangle.
\end{aligned}$$

In our case by the trace pairing (3.2) we get,

$$\begin{aligned}
\langle \mathrm{ad}^*_\xi \mu, \eta \rangle &= \frac{1}{2}\mathrm{tr}\ (\mu\xi\eta) - \frac{1}{2}\mathrm{tr}\ (\mu\eta\xi) \\
&= \frac{1}{2}\mathrm{tr}\ (\mu\xi\eta) - \frac{1}{2}\mathrm{tr}\ (\xi\mu\eta) \\
&= \frac{1}{2}\mathrm{tr}\ (\mu\xi\eta - \xi\mu\eta) \\
&= \frac{1}{2}\mathrm{tr}\ ([\mu, \xi]\, \eta) \\
&= \langle [\mu, \xi], \eta \rangle.
\end{aligned}$$

Hence,

$$\mathrm{ad}^*_\xi \mu = [\mu, \xi] = -[\xi, \mu].$$

Remark 3.1.33. For all ξ, $\eta \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$ we have

$$\begin{aligned}
\mathrm{ad}_\xi \eta &= [\xi, \eta], \\
\mathrm{ad}^*_\xi \mu &= -[\xi, \mu].
\end{aligned}$$

Hence, by the equation (3.5), the hyperbolic cross product $\times_{\mathcal{H}_2}$ is in direct correspondence with the infinitesimal generators of $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(2,\mathbb{R})^*$ by

$$\begin{aligned} \xi_{\mathfrak{g}}\check{\eta} &= -2\left(\check{\xi} \times_{\mathcal{H}_2} \check{\eta}\right), \\ \xi_{\mathfrak{g}^*}\check{\mu} &= -2\left(\check{\xi} \times_{\mathcal{H}_2} \check{\mu}\right). \end{aligned}$$

As mentioned in Remark 3.1.14, $\mathfrak{sl}(2,\mathbb{R}) \cong \mathbb{R}^3$ as well as $\mathfrak{sl}(2,\mathbb{R})^* \cong \mathbb{R}^3$, thus by (3.13)

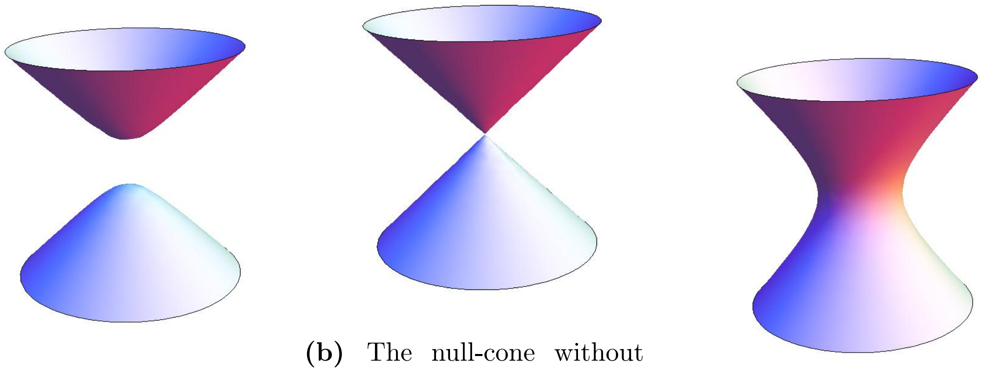
$$\begin{aligned} \mathrm{ad}^*_{\xi}(\mu) &= 2\left(\check{\xi} \times_{\mathcal{H}_2} \check{\mu}\right) \\ &= \mathrm{ad}_{\xi}(\eta) \end{aligned} \tag{3.15}$$

with η and μ the associated elements to $\check{X} \in \mathbb{R}^3$ in $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(2,\mathbb{R})^*$ respectively. This equality was expected as the coadjoint and adjoint orbits for $SL(2,\mathbb{R})$ in the hyperbolic model are equivalent (Remark 3.1.24). Equation (3.15) is the expression of the tangent level of the action of $SL(2,\mathbb{R})$ on $\mathcal{H}_2$, and $\xi_{\mathcal{H}_2}(x) \in T_x\mathrm{Orb}(x)$ is tangent to $\mathrm{Orb}(x)$ the orbit of x.

3.2 Coadjoint geometry

In this section we use some of the results and definitions of the previous section to define some important characteristics of the action of $SL(2\mathbb{R}^3)$ in $\mathfrak{sl}(2,\mathbb{R}) \cong \mathbb{R}^3$ and $\mathfrak{sl}(2,\mathbb{R})^* \cong \mathbb{R}^3$. Recall the *null-cone*, denoted by $\mathfrak{C}$, is defined as the set of all vectors $\check{\xi} \in \mathbb{R}^3$ with determinant $\det \xi = -\left\langle \check{\xi}, \check{\xi} \right\rangle_{\mathcal{H}_2} = 0$, see Figure 3.1(b).

To begin with this section, is important to point out that given $\check{\xi} \neq 0 \in \mathbb{R}^3$, the sign of determinant $\det \xi$ assigns a type of surface as shown in Figure 3.1. If $\check{\xi}$ is inside, outside or on the null-cone $\mathfrak{C}$, then $\det \xi$ is greater, less than or equal to zero respectively. We later show that these surfaces are actually the coadjoint orbits.



(a) A sheet of the two-sheet hyperboloid is the coadjoint orbit for $\check{\xi}$ inside of $\mathfrak{C}$, that is $\det \xi > 0$.

(b) The null-cone without the origin is the coadjoint orbit for $\check{\xi} \neq 0$ such that $\det \xi = 0$. The origin itself is the coadjoint orbit of $\check{\xi} = 0$.

(c) The coadjoint orbit for $\check{\xi}$ outside of $\mathfrak{C}$, that is $\det \xi < 0$, is a one sheet hyperboloid.

Figure 3.1: Coadjoint orbits of the action of $SL(2,\mathbb{R})$ in $\mathbb{R}^3$.

3.2.1 Isotropy subgroups

Recall from Definition 3.1.17 that isotropy subgroups are the group elements that fix a point of the manifold. The isotropy subgroups for any of the other type of actions described before are defined similarly. For instance, the isotropy subgroup for the adjoint action of a matrix Lie group G on $\mathfrak{g}$ is

$$G_\xi = \left\{A \in G \mid A\xi A^{-1} = \xi\right\},$$

with $\xi \in \mathfrak{g}$. Consider $g' \in G_{h\cdot x}$, then $g' \cdot (h \cdot x){=}h \cdot x$, this implies $(h^{-1}g'h) \cdot x{=}x$, therefore if $g{=}h^{-1}g'h \in G_x$ then $g'{=}hgh^{-1} \in hG_xh^{-1}$. Thus $G_{h\cdot x} \subset hG_xh^{-1}$, with the converse inclusion obtained in a similar way we get the following relationship between the two isotropy groups

$$G_{h\cdot x} = hG_xh^{-1}. \tag{3.16}$$

The next theorem classifies the isotropy groups of the adjoint action of $SL(2,\mathbb{R})$ on $\mathfrak{sl}(2,\mathbb{R})$ by the value of the $\det \xi = -\left\langle \check{\xi}, \check{\xi}\right\rangle_{\mathcal{H}_2}$.

Theorem 3.2.1. *Let $G = SL(2,\mathbb{R})$ and $\xi \in \mathfrak{g}$ with G_ξ its isotropy subgroup by the adjoint action of G on $\mathfrak{g} \cong \mathbb{R}^3$. If*

1. *$\check{\xi}$ is inside $\mathfrak{C}$, that is pointing into the null-cone $\mathfrak{C}$, then $G_\xi \cong SO(2,\mathbb{R})$,*

2. $\check{\xi}$ is on $\mathfrak{C}$ we have two posibilities:

a) $G_\xi = SL(2,\mathbb{R})$ when $\check{\xi}=0$ or

b) $G_\xi \cong \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, t \in \mathbb{R} \right\}$,

3. $\check{\xi}$ is outside $\mathfrak{C}$, $G_\xi \cong \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, t \in \mathbb{R}^+ \right\}$,

where $\cong$ means conjugate subgroups of $SL(2,\mathbb{R})$.

Proof. All matrices in $SL(2,\mathbb{R})$ satisfy $A0A^{-1}=0$ where 0 is the 2×2 zero matrix, hence $G_\xi = SL(2,\mathbb{R})$ for $\check{\xi}=0$.

For the remaining cases, that is when $\xi \neq 0$, the proof consists in showing that $G_\xi \cong G_{X_1}$ for a given $\check{X}_1$ with the same sign of determinant. We present the calculations for G_{X_1} only for $\check{\xi} = (\xi_1,\xi_2,\xi_3)$ inside $\mathfrak{C}$. The proof for the other isotropy subgroups is obtained in a similar manner from the explicitly given $\check{X}_1$ in this proof for each particular case.

Note that the boundary for vectors with positive determinant is the cone $\mathfrak{C}$ where the determinant is zero. Since $\mathfrak{C}$ is asymptotic to $\mathcal{H}_2$, the line through a vector $\check{\xi}$ inside the cone $\mathfrak{C}$ intersects $\mathcal{H}_2$ at some point $\check{\xi}'$. As a result, there always exists $k\neq 0$ such that $\check{\xi}' = k\check{\xi} \in \mathcal{H}_2$, so $G_\xi = G_{\xi'}$ is trivially obtained.

Let $\check{X}_1 = (0,0,1) \in \mathcal{H}_2$, from Lemma 3.1.25 there exists $g \in SL(2,\mathbb{R})$ such that $g\cdot X_1 = \mathrm{Ad}_g X_1 = \xi'$, which by (3.16) implies $gG_{X_1}g^{-1} = G_{\xi'}$. Therefore, $G_{X_1} \cong G_{\xi'} = G_\xi$.

We now proceed to calculate G_{X_1}. The representation (3.4) of $\check{X}_1$ by a 2×2 matrix is

$$X_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Take $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,R)$, then

$$AX_1A^{-1} = \begin{pmatrix} -(bd+ac) & a^2+b^2 \\ -(c^2+d^2) & bd+ac \end{pmatrix}.$$

Solving $AX_1A^{-1} = X_1$ we get the set of equations

$$\begin{aligned} bd + ac &= 0, \\ a^2 + b^2 &= 1, \\ c^2 + d^2 &= 1. \end{aligned}$$

Since $A \in SL(2,\mathbb{R})$, if $d \neq 0$ we have $a = \frac{1+bc}{d}$. Substituting a in the first condition above we get $b = -c$, and replacing this in any of the other conditions we get $a = d$. Thus $G_{X_1} = \left\{ \begin{pmatrix} a & -c \\ c & a \end{pmatrix} \mid a^2 + c^2 = 1 \right\}$, without loss of generalty we make $a = \cos\theta$ and $c = \sin\theta$ and obtain

$$G_\xi \cong \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\}. \tag{3.17}$$

This is interpreted as a rotation by θ. The case $d = 0$ is included in (3.17) by taking $\theta = \frac{\pi}{2}$. Therefore, $G_\xi \cong SO(2,\mathbb{R})$ for all $\check{\xi}$ inside the cone $\mathfrak{C}$.

For the case of $\xi \neq 0$ on the cone $\mathfrak{C}$ consider $\check{X}_1 = (0,1,1)$, by Lemma 3.1.25 there exists $g \in G$ such that $g \cdot X_1 = \xi$. Thus we have the congruence relationship $G_{X_1} \cong G_\xi$, and calculations lead to G_{X_1} a parabolic Möbius transformation as stated in the theorem.

Finally, for any $\xi \in \mathfrak{g}$ such that $\det \xi < 0$ there exists a constant $k \neq 0$ such that $\det k\xi = -1$. Again, by Lemma 3.1.25, there exists $g \in G$ such that $g \cdot X_1 = k\xi$ with $\check{X}_1 = (1,0,0)$, resulting in $G_{X_1} \cong G_\xi$. $\square$

3.2.2 Coadjoint orbits

The trace pairing that identifies $\mathfrak{sl}(2,\mathbb{R})$ with $\mathfrak{sl}(2,\mathbb{R})^*$ given by (3.2) is the *Killing form* of $\mathfrak{sl}(2,\mathbb{R})$, which is non degenerate thus, by the definition given in [8], $\mathfrak{sl}(2,\mathbb{R})$ is semisimple. A known fact is that as a consequence of this non-degenerancy for semisimple Lie algebras the adjoint and coadjoint action are equivalent. Here, we have pointed out this result in Remark 3.1.24 for the action of $SL(2,\mathbb{R})$ on $\mathbb{R}^3$.

With the isotropy groups already calculated in Theorem 3.2.1, Lemma 3.1.25 allow us to affirm that the orbits for the coadjoint action of μ in $\mathfrak{sl}(2,\mathbb{R})^*$ are of four types, as pointed out in [22]. The orbits plotted in Figure 3.1 are determined by the determinant of μ, indeed the orbits correspond to the solution of the conditions of the determinant for each isotropy group.

Theorem 3.2.2. *Let $\mu \in \mathfrak{sl}(2,\mathbb{R})^*$. Then the coadjoint orbits are classified as follows:*

1. *If $\det \mu > 0$ then $G_\mu \cong SO(2,\mathbb{R})$, and the coadjoint orbit is one sheet of the hyperboloid of two sheets shown in Figure 3.1(a).*

2. *If $\mu = 0$ then $G_\mu = SL(2,\mathbb{R})$ and the coadjoint orbit is the origin.*

3. *If $\det \mu = 0$ and $\mu \neq 0$ then $G_\mu \cong \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, t \in \mathbb{R} \right\}$. The coadjoint orbit is each sheet $\mathfrak{C}$ with the origin removed.*

4. *If $\det \mu < 0$ then $G_\mu \cong \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, t \in \mathbb{R}^+ \right\}$, and the coadjoint orbit is a one sheeted hyperboloid as shown in Figure 3.1(c).*

3.2.3 Relation between isotropy groups and Möbius transformations

It is easy to see that given $\gamma(z) = \frac{az+b}{cz+d}$ a Möbius transformation and $\lambda \neq 0$, $\theta(z) = \frac{\lambda az+\lambda b}{\lambda cz+\lambda d}$ represent the exact same Möbius transformation of γ. Thus it is always posible to have a Möbius transformation in normalised form taking $\lambda = 1/\sqrt{(ad-bc)}$. And given a normalised Möbius transformation γ, we can construct a map from its 2×2 matrix representation to a 3×3 matrix such that the hyperbolic inner product is preserved ([18]). This lead us to define the map

$$\begin{array}{rcl} \tilde{}: SL(2,\mathbb{R}) & \to & \mathcal{M}(3,\mathbb{R}) \\ B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \to & \tilde{B} = \frac{1}{2}\begin{pmatrix} 2(ad+bc) & -2(ac-bd) & -2(ac+bd) \\ -2(ab-cd) & a^2-b^2-c^2+d^2 & a^2+b^2-c^2-d^2 \\ -2(ab+cd) & a^2-b^2+c^2-d^2 & a^2+b^2+c^2+d^2 \end{pmatrix}, \end{array}$$

whose image preserves the hyperbolic inner product (2.5), and also satisfies

$$\tilde{B}\check{X} = (\mathrm{Ad}^*_{B^{-1}}X)\check{}$$

for any $\check{X} \in \mathbb{R}^3$.

The hyperbolic normal plane to $\check{\mu} \in \mathbb{R}^3$ through X_0 is given by

$$P_{\check{\mu}} = \left\{\check{X} = (x,y,z) \in \mathbb{R}^3 \mid \langle X - X_0, \check{\mu}\rangle_{\mathcal{H}_2} = 0\right\}.$$

Thus the hyperbolic normal plane passing through $\check{\mu}$ itself is defined as

$$P_{\check{\mu}} = \left\{\check{X} \in \mathbb{R}^3 \mid \left\langle\check{X}, \check{\mu}\right\rangle_{\mathcal{H}_2} = \langle\check{\mu}, \check{\mu}\rangle_{\mathcal{H}_2} = \|\check{\mu}\|^2_{\mathcal{H}_2}\right\}.$$

Lemma 3.2.3. *If $B \in SL(2,\mathbb{R})_\mu$ then $P_{\check{\mu}}$ is invariant under the action of $\tilde{B}$.*

Proof. Let $\check{X} \in P_{\check{\mu}}$, then

$$\begin{aligned} \langle\tilde{B}\check{X}, \check{\mu}\rangle_{\mathcal{H}_2} &= \langle\tilde{B}\check{X}, \tilde{B}\check{\mu}\rangle_{\mathcal{H}_2} \\ &= \langle\check{X}, \check{\mu}\rangle_{\mathcal{H}_2} \\ &= \langle\check{\mu}, \check{\mu}\rangle_{\mathcal{H}_2}. \end{aligned}$$

□

In [9] the *generalised orthogonal group* is defined as the set of $(n+k)\times(n+k)$

real matrices A that preserve a symmetric bilinear form $[\cdot,\cdot]_{n,k}$ on $\mathbb{R}^{n+k}$ given by:

$$[x,y]_{n,k} = x_1y_1 + \ldots + x_ny_n - x_{n+1}y_{n+1} - \ldots x_{n+k}y_{n+k},$$

such that $[Ax,Ay]_{n,k} = [x,y]_{n,k}$ for all $x,y \in \mathbb{R}^{n+k}$. This matrix Lie group is denoted by *O(n,k)*.

According to [9], A is in *O(n,k)* if and only if the following conditions are satisfied:

$$\begin{aligned} \left[A^{(l)},A^{(j)}\right]_{n,k} &= 0 & l \neq j, \\ \left[A^{(l)},A^{(l)}\right]_{n,k} &= 1 & 1 \leq l \leq n, \\ \left[A^{(l)},A^{(l)}\right]_{n,k} &= -1 & n+1 \leq l \leq n+k. \end{aligned}$$

where $A^{(i)}$ denotes the i^{th} column vector of A.

From this statement we conclude that the set of hyperbolic rotations respect the x, y and $z-axis$

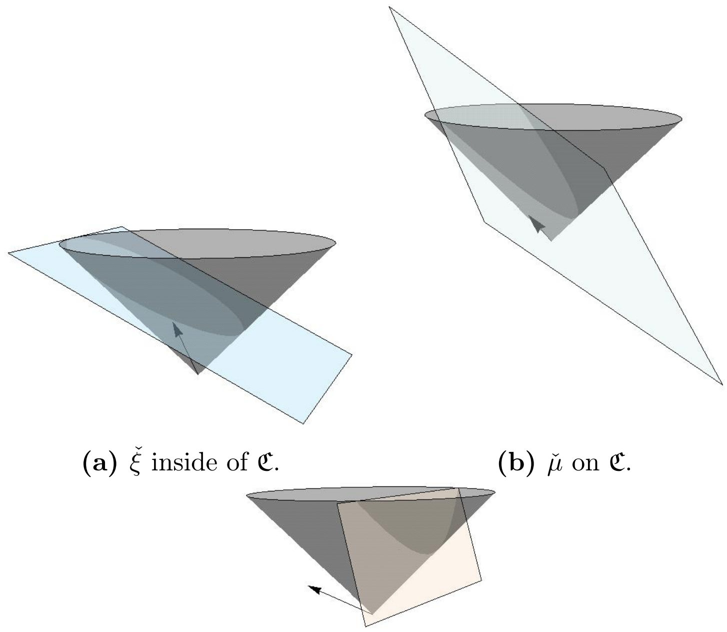
$$R_{\mathcal{H}_2} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{pmatrix}, \begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix}, \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

preserve the Minkowski inner product (2.5) defined for the hyperboloid $\mathcal{H}_2$.

Given $\breve{\mu} \in \mathbb{R}^3$ with $\det\mu > 0$ and Euclidean length $\|\breve{\mu}\|_E = c$, let $\breve{\mu}' = (0,0,c)$. There exists a combination of hyperbolic rotations $A \in R_{\mathcal{H}_2}$ such that $\breve{\mu} = A\breve{\mu}'$ and $P_{\breve{\mu}} = AP_{\breve{\mu}'}$. Moreover, for $\tilde{B} \in G_{\mu'}$ a rotation matrix, points in $P_{\breve{\mu}}$ hyperbolically rotate around $\breve{\mu}$ under the action of $A \circ \tilde{B} = A\tilde{B}$. Evidently as a result of previous statements, the hyperbolic distances are also preserved under this action.

The intersection $P_{\breve{\mu}} \cap \mathcal{H}_2$ is an ellipse, furthermore $A\tilde{B}\left(P_{\breve{\mu}} \cap \mathcal{H}_2\right) \subseteq P_{\breve{\mu}} \cap \mathcal{H}_2$. As a result we conclude that all normal planes to $\breve{\mu}$ when $G_\mu \cong SO(2,\mathbb{R})$ are invariant under $A\tilde{B}$ and intersect $\mathcal{H}_2$ on an ellipse as shown in Figure 3.2(a). This result is very helpful for describing the motion of a point vortices system in $\mathcal{H}_2$ with a momentum value inside $\mathfrak{C}$.

We make a similar analysis for $\breve{\mu}$ outside $\mathfrak{C}$ and $\breve{\mu}$ inside $\mathfrak{C}$, and conclude that the intersection of the normal planes to $\breve{\mu}$ is an hyperbola or a parabola respectively.



(a) $\check{\xi}$ inside of $\mathfrak{C}$. **(b)** $\check{\mu}$ on $\mathfrak{C}$.

(c) $\check{\mu}$ outside of $\mathfrak{C}$.

Figure 3.2: Intersection of the normal planes to $\check{\mu}$ with $\mathcal{H}_2$.

This is shown in Figures 3.2(c) and 3.2(b).

An important conclusion from this is that to any $\mu \neq 0$ in $\mathfrak{sl}(2,\mathbb{R})^*$ we associate a conic (ellipse, hyperbola or parabola) related to its isotropy group. We can call this the *type* of μ: μ is elliptic, hyperbolic or parabolic for G_μ isomorphic to an elliptic, hyperbolic or parabolic transformation, respectively.

3.2.4 Isotropy Lie algebra

The following definition can be found in the book of Brian C. Hall [9].

Definition 3.2.4. If G is a matrix Lie group with Lie algebra $\mathfrak{g}$, then $H \subset G$ is a *connected Lie subgroup* of G if the following conditions are satisfied

1. H is a subgroup of G.

2. Every element of H can be written in the form of $e^{X_1} \cdots e^{X_m}$ with $X_1, ..., X_m \in \mathfrak{h}$.

We now show that every isotropy subgroup of $SL(2,\mathbb{R})$ is connected.

Proposition 3.2.5. *For all $\mu \in sl(2,\mathbb{R})^*$ the isotropy subgroup G_μ is connected.*

Proof. The first condition given in the Definition 3.2.4 are automatically satisfied. We focus then on showing that for any $g \in G = SL(2,\mathbb{R})$ there exists $X \in \mathfrak{sl}(2,\mathbb{R})$ such that $g = e^X$. We calculate $\mathfrak{g}_\mu$ for every type of μ and give $X \in \mathfrak{g}_\mu$ that satisfies condition 3.

1. If μ is elliptic, $\mathfrak{g}_\mu \cong \left\{ \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}, c \in \mathbb{R} \right\}$. Let $X = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \in \mathfrak{g}_\mu$, then $e^X = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Hence, for all $g \in G_\mu$ there exists $X \in G_\mu$ such that $g = e^X$.

2. If $\mu \neq 0$ is parabolic, $\mathfrak{g}_\mu \cong \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, b \in \mathbb{R} \right\}$. Taking $X = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_\mu$ we get $X^2 = 0$. Thus, $e^X = I + X = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and G_μ is connected.

3. If μ is hyperbolic, $\mathfrak{g}_\mu \cong \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, a \in \mathbb{R} \right\}$. Considering
$$X = \begin{pmatrix} \ln t & 0 \\ 0 & \ln t^{-1} \end{pmatrix} \in \mathfrak{g}_\mu,$$
we get $g = e^X$.

4. Finally if $\mu = 0$, $\mathfrak{g}_\mu$ is obviously $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, by the *Iwasawa decomposition* μ must fall in one of the previous cases, therefore is connected.

□

The definitions of *split* and *regular* can be found in various sources, we particularly follow the presentation of [16] and [28].

Definition 3.2.6. Let $\mu \in \mathfrak{g}^*$ and G_μ^0 the identity component of G_μ. We say that μ is *split* if there exists a G_μ^0- invariant complement $\mathfrak{n}_\mu$ to $\mathfrak{g}_\mu$ in $\mathfrak{g}$.

Due to Proposition 3.2.5 for $G = SL(2,\mathbb{R})$, G_μ is connected for all $\mu \in \mathfrak{g}^*$, hence $G_\mu^0 = G_\mu$. Hence is enough to check the existence of a G_μ invariant complement.

Proposition 3.2.7. *Let $G = SL(2,\mathbb{R})$. Then $\mu \in \mathfrak{g}^*$ is not split iff μ is parabolic.*

Proof. Let a, b and $c \in \mathbb{R}$, then any $\xi \in \mathfrak{g}$ is of the form

$$A = \begin{pmatrix} a & b+c \\ b-c & -a \end{pmatrix}.$$

We present the isotropy Lie algebras $\mathfrak{g}_\mu$ for every type of μ and give a G_μ invariant complement $\mathfrak{n}_\mu$ when possible.

1. If μ is elliptic, $\mathfrak{g}_\mu \cong \left\{ \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix} \mid c \in \mathbb{R} \right\}$. Therefore, the natural choice for the complement of $\mathfrak{g}_\mu$ in $\mathfrak{g}$ is $\mathfrak{n}_\mu = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$ which is G_μ invariant, so μ is split.

2. If μ is hyperbolic $\mathfrak{g}_\mu \cong \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \mid a \in \mathbb{R} \right\}$. The complement

$$\mathfrak{n}_\mu = \left\{ \begin{pmatrix} 0 & b+c \\ b-c & 0 \end{pmatrix} \mid b, c \in \mathbb{R} \right\}$$

satisfies to be G_μ invariant, thus μ is split.

3. If μ is parabolic with $\mu \neq 0$, $\mathfrak{g}_\mu \cong \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{R} \right\}$ and the complement of $\mathfrak{g}_\mu$ in $\mathfrak{g}$ is generated by

$$\mathfrak{n}_\mu := \left\{ \begin{pmatrix} a & 0 \\ c & -a \end{pmatrix} \mid a, c \in \mathbb{R} \right\},$$

which is not G_μ invariant so μ is not split.

To illustrate this consider $\check{\xi} = (0,1,1)$, then $\xi \in \mathfrak{g}_\mu$ and $G_\mu = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ with $t \in \mathbb{R}$. The complement is given by $\mathfrak{n}_\mu = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \mid a \in \mathbb{R} \right\}$ and

$\mathrm{Ad}^*_{G_\mu^{-1}} X \neq X$ for any X in $\mathfrak{n}_\mu$.

4. Finally if $\mu = 0$, $\mathfrak{g}_\mu$ is obviously $\mathfrak{g}$ and μ is split.

□

Definition 3.2.8. A point $\mu \in \mathfrak{g}^*$ is **regular** if $\dim \mathfrak{g}_\nu = \dim \mathfrak{g}_\mu$ for every ν in a neighbourhood of μ.

The proof of next proposition follows from the proof of Proposition 3.2.7.

Proposition 3.2.9. *Let $G = SL\,(2,\mathbb{R})$ and $\mu \in \mathfrak{g}^*$. Then every $\mu \neq 0$ is regular.*

Proof. Every coadjoint orbit is 2- dimensional except at the origin. □

Chapter 4

Vortices

The goal of this chapter is to set up the structure of N point vortices in the hyperboloid model $\mathcal{H}_2$ as the dynamics of a Hamiltonian vector field on a symplectic (Poisson) manifold. The first two sections define the basic concepts of *Poisson manifold*, *Hamiltonian vector field* and *momentum map*. Most of the definitions were taken from the book of Jerrold E. Marsden and Tudor S. Ratiu [20], the references [3] and [10] were also used but not extensively.

The concepts of *Casimir functions* and *symplectic leaves* are also defined. Remark 4.2.4 shows that the hyperboloid model is a symplectic leaf (coadjoint orbit) of the action of $SL(2,\mathbb{R})$ on $\mathfrak{sl}(2,\mathbb{R}) \cong \mathbb{R}^3$. We construct the coadjoint orbit symplectic structure related to $\mathcal{H}_2$ mentioned in Chapter 2 by this symmetry group. The last section relates these concepts to the problem of N point vortices on the hyperboloid, and presents the derivation of the momentum map and the differential equations that govern its dynamics.

4.1 Poisson manifolds and Hamiltonian vector field

Definition 4.1.1. A *Poisson bracket* on a finite dimensional manifold $\mathcal{P}$ is a bilinear operation $\{\cdot,\cdot\}$ on $C^\infty(\mathcal{P})$ such that:

- $(C^\infty(\mathcal{P}), \{\cdot,\cdot\})$ is a Lie algebra; and
- $\{FG, H\} = \{F, H\}G + F\{G, H\}$ for all F, G, and $H \in C^\infty(\mathcal{P})$.

A manifold $\mathcal{P}$ endowed with a Poisson bracket on $C^{\infty}(\mathcal{P})$ is called a *Poisson manifold.*

The action of a Lie group G on the Poisson manifold $(\mathcal{P}, \{\cdot,\cdot\})$ is *canonical* if for all F, $G \in C^{\infty}(\mathcal{P})$ and for every $g \in G$

$$\{F \circ \Phi_g, G \circ \Phi_g\} = \{F, G\} \circ \Phi_g,$$

where Φ_g is the action of g in $\mathcal{P}$ (Definition 3.1.15).

Definition 4.1.2. In canonical coordinates $(\mathbf{q} = q_1, q_2, ..., q_N, \mathbf{p} = p_1, p_2, ..., p_N)$ the *canonical Poisson bracket* of two functions $F(\mathbf{q}, \mathbf{p})$ and $G(\mathbf{q}, \mathbf{p})$ is defined by

$$\{F, G\} = \sum_{i=1}^{N} \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right). \tag{4.1}$$

Definition 4.1.3. Let $\mu \in \mathfrak{g}^*$, and F, G two functions defined on $\mathfrak{g}^*$, then $dF(\mu)$ and $dG(\mu) \in (\mathfrak{g}^*)^* \cong \mathfrak{g}$. The *Lie-Poisson bracket*

$$\{F, G\}_{\pm}(\mu) = \pm \langle \mu, [dF(\mu), dG(\mu)] \rangle, \tag{4.2}$$

where $\langle , \rangle$ denotes the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, defines a Poisson structure on $\mathfrak{g}^*$.

Given the basis $\mathcal{B} = \{e_1, e_2, ..., e_r\}$ of $\mathfrak{g}$ and $\mathcal{B}' = \{e^1, e^2, ..., e^r\}$ of $\mathfrak{g}^*$, the *structure constants* C_{ij}^k are defined by

$$[e_i, e_j] = \sum_{k=1}^{r} C_{ij}^k e_k.$$

The Lie-Poisson bracket (4.2) in terms of this coordinate notation is

$$\{F, G\}_{\pm}(\mu) = \pm \sum_{a,b,d=1}^{r} C_{ab}^d \mu_d \frac{\partial F}{\partial \mu_a} \frac{\partial G}{\partial \mu_b}, \tag{4.3}$$

where $\mu = \sum_{a=1}^{r} \mu_a e^a$. The Lie- Poissson bracket induces a Poisson structure in $\mathfrak{g}^*$, hence the pair $\left(\mathfrak{g}^*, \{\cdot,\cdot\}_{\pm}\right)$ is a Poisson manifold.

Proposition 4.1.4 ([10], [20])**.** *Let $\mathcal{P}$ be a Poisson manifold. If $H \in C^{\infty}(\mathcal{P})$, then there is a unique vector field X_H on P such that*

$$X_H[G] = \{G, H\}$$

for all $G \in \mathcal{F}(\mathcal{P})$.

Definition 4.1.5. The vector field X_H is called the *Hamiltonian vector field of H.* We also define the associated *dynamical system* by the differential equation

$$\dot{z} = X_H(z), \tag{4.4}$$

with $z = z(t)$ in the phase space.

Equation (4.4) is valid for any function, in particular for H representing the total energy of a dynamical system with the canonical Poisson bracket (4.1) we obtain

$$X_H(p_i, q_i) = \left(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q_i}\right),$$

which results in *Hamilton's equations*

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}. \end{aligned} \tag{4.5}$$

Proposition 10.7.1 in [20] proves that the equation of motion for the Hamiltonian H with respect to the $\pm$ Lie-Poisson brackets on $\mathfrak{g}^*$ are

$$\frac{d\mu}{dt} = \mp \mathrm{ad}^*_{\frac{\delta H}{\delta \mu}} \mu.$$

Therefore the Hamiltonian vector field of $H : \mathfrak{g}^* \to \mathbb{R}$ is

$$X_H(\mu) = \mp \mathrm{ad}^*_{\frac{\delta H}{\delta \mu}} \mu. \tag{4.6}$$

4.2 Symplectic structure in the coadjoint orbit

Definition 4.2.1. A function $C \in C^\infty(\mathcal{P})$ is called a *Casimir function* of the Poisson structure if

$$\{C, F\} = 0$$

for all $F \in C^\infty(\mathcal{P})$.

Definition 4.2.2. Let $\mathcal{P}$ be a Poisson manifold. We say that z_1, $z_2 \in \mathcal{P}$ are on the same symplectic leaf of $\mathcal{P}$ if there is a piecewise smooth curve in $\mathcal{P}$ joining z_1 and z_2, each segment of which is a trajectory of a locally defined Hamiltonian vector field. This is clearly an equivalence relation, and an equivalence class is called a *symplectic leaf*. The symplectic leaf containing the point z is denoted by $\sum_z$.

An important observation is that, since any Casimir is constant in connected components of $\mathcal{P}$, a Casimir function must be constant on a symplectic leaf $\sum \subset \mathcal{P}$. Furthermore the connected components of the coadjoint orbits are the symplectic leaves of $\mathfrak{g}^*$. Moreover, if $\mathcal{P}$ is finite dimensional then the disjoint union of the symplectic leaves provides a foliation of $\mathcal{P}$, see [20] for a proof of this. Another important result presented there is the next corollary.

Corollary 4.2.3 ([20])**.** *If $C \in C^\infty(\mathfrak{g}^*)$ is* Ad^*- *invariant (constant in orbits), then C is a Casimir function. The converse is also true if all coadjoint orbits are connected.*

Remark 4.2.4 ([20])**.** *Casimir function of $SL(2,\mathbb{R})$ in $\mathbb{R}^3$.* For $G = SL(2,\mathbb{R})$ the determinant function

$$
\begin{aligned}
\det : \mathfrak{g}^* &\rightarrow \mathbb{R} \\
\mu &\rightarrow -\langle \check{\mu}, \check{\mu} \rangle_{\mathcal{H}_2} \\
&= \det \begin{pmatrix} \mu_1 & \mu_2 - \mu_3 \\ \mu_2 + \mu_3 & -\mu_1 \end{pmatrix} \\
&= -\left(\mu_1^2 + \mu_2^2 - \mu_3^2\right),
\end{aligned}
$$

is obviously Ad^*-invariant. Therefore for $\mathfrak{sl}(2,\mathbb{R})^*$ with the Lie-Poisson bracket (4.2) any function of $\mu_1^2 + \mu_2^2 - \mu_3^3$ is a Casimir.

The symplectic leaves are the connected components of the coadjoint orbits described in Theorem 3.2.2 of Chapter 3. Therefore, the hyperboloid $\mathcal{H}_2$ is the symplectic leaf corresponding to the coadjoint orbit of $\mu \in \mathfrak{sl}(2,\mathbb{R})^*$ with determinant 1 and $\mu_3 > 0$. Hence, the Lie- Poisson bracket (4.2) restricted to $\mathcal{H}_2$ induces a Poisson structure in the hyperboloid model. From this point of view we define the Poisson structure for the problem of N point vortices in $\mathcal{H}_2$. Before doing so, we define the coadjoint orbit symplectic structure on $\mathfrak{g}^*$.

Given a Lie group G and $\mathcal{O} \subset \mathfrak{g}^*$ a coadjoint orbit, Theorem 14.3.1 in [20] shows that the *Kostant-Kirillov-Souriau* (KKS) form in $\mathcal{O}$

$$
\omega_{\mathcal{O}}^{\pm}(\mu)\left(\mathrm{ad}_{\xi}^*\mu, \mathrm{ad}_{\eta}^*\mu\right) = \pm\langle \mu, [\xi, \eta] \rangle, \tag{4.7}
$$

defines a symplectic form on $\mathcal{O}$ for all $\mu \in \mathcal{O}$, ξ, $\eta \in \mathfrak{g}$.

Furthermore, the definition of the Lie-Poisson bracket infers a relation with this symplectic form. Theorem 14.4.1 in [20] shows that for any $\mu \in \mathcal{O}$ the symplectic form (4.7) is compatible with the *Lie-Poisson* bracket (4.2) restricted to the coadjoint orbit $\mathcal{O}$ as follows

$$
\{F, G\}_{\pm}|_{\mathcal{O}}(\mu) = \omega_{\mathcal{O}}^{\pm}(\mu)\left(X_{F|_{\mathcal{O}}}, X_{G|_{\mathcal{O}}}\right). \tag{4.8}
$$

In Section 4.1 we derive the Hamiltonian vector field X_F on $\mathfrak{g}^*$ (4.6). Lemma 14.4.2

in [20] proves that using the KKS symplectic form, for any $\mu \in \mathcal{O}$ we have

$$X_{F|_{\mathcal{O}}}(\mu) = \mp \mathrm{ad}^*_{\frac{\delta F}{\delta \mu}} \mu.$$

Hence,

$$\{F, G\}_{\pm} |_{\mathcal{O}}(\mu) = \omega^{\pm}_{\mathcal{O}}(\mu) \left(\mathrm{ad}^*_{\frac{\delta F}{\delta \mu}} \mu, \mathrm{ad}^*_{\frac{\delta G}{\delta \mu}} \mu \right). \tag{4.9}$$

In conclusion, if restricted to a coadjoint orbit, then the Lie-Poisson bracket and the coadjoint orbit symplectic structure KKS are equivalent.

Proposition 4.2.5. *Let $\mu \in \mathcal{O} \subset \mathfrak{sl}(2, \mathbb{R})^*$ and u, $v \in T_\mu \mathcal{H}_2$. Suppose that $\|\breve{\mu}\|^2 = \sum_{i=1}^3 \mu_i^2$ is the Euclidean norm of $\breve{\mu}$ and $\cdot$ denotes the Euclidean product. Then*

$$\omega^{\pm}_{\mathcal{O}}(\mu)(u, v) = \pm \frac{\breve{\mu} \cdot (\breve{u} \times_{\mathcal{H}_2} \breve{v})}{2\|\breve{\mu}\|^2} \tag{4.10}$$

is the KKS symplectic form on the hyperboloid $\mathcal{H}_2$.

Proof. Let ξ, $\eta \in \mathfrak{sl}(2, \mathbb{R})$ such that

$$u = \mathrm{ad}^*_\xi \mu = -[\xi, \mu] \qquad \text{and} \qquad v = \mathrm{ad}^*_\eta \mu = -[\eta, \mu].$$

By Equation (2.7)

$$\begin{aligned} \omega^{\pm}_{\mathcal{O}}(u, v)(\mu) &= \pm \langle \mu, [\xi, \eta] \rangle \\ &= \mp 2 \breve{\mu} \cdot \left(\breve{\xi} \times \breve{\eta} \right). \end{aligned}$$

On other hand, let $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, then for any X, $Y \in \mathbb{R}^3$

$$\begin{aligned} X \times_{\mathcal{H}_2} Y &= M(X \times Y) \\ &= MY \times MX. \end{aligned}$$

Hence,

$$
\begin{aligned}
\check{u} \times_{\mathcal{H}_2} \check{v} &= M\left([\eta, \mu]\check{\ }\right) \times M\left([\xi, \mu]\check{\ }\right) \\
&= 4MM\left(\check{\eta} \times \check{\mu}\right) \times MM\left(\check{\xi} \times \check{\mu}\right) \\
&= 4\left(\check{\eta} \times \check{\mu}\right) \times \left(\check{\xi} \times \check{\mu}\right) \\
&= -4\check{\mu}\left(\check{\mu} \cdot \left(\check{\xi} \times \check{\eta}\right)\right).
\end{aligned}
$$

□

The hyperboloid $\mathcal{H}_2$ together with (4.10) is a symplectic manifold.

4.3 Momentum map

An important characteristic of Hamiltonian systems is that conserved quantities are associated with the symmetry group acting on them. There often exists a momentum map $\mathbf{J} : \mathcal{P} \to \mathfrak{g}^*$ that is conserved by the flow of the Hamiltonian vector. As mentioned in [10] the main idea of defining a momentum map is that if the infinitesimal generator vector field $\xi_{\mathcal{P}}$ is Hamiltonian, then its Hamiltonian function denoted by J_ξ defines a momentum map.

Definition 4.3.1. Given a canonical action of G on a Poisson manifold $\mathcal{P}$, if there exists a map $J : \mathfrak{g} \mapsto C^\infty(\mathcal{P})$ such that for every $\xi \in \mathfrak{g}$ the Hamiltonian vector field satisfies

$$X_{J(\xi)} = \xi_{\mathcal{P}}, \tag{4.11}$$

then the map $\mathbf{J} : \mathcal{P} \to \mathfrak{g}^*$ defined by

$$\langle \mathbf{J}(z), \xi \rangle = J(\xi)(z) \tag{4.12}$$

for all $\xi \in \mathfrak{g}$ and $z \in \mathcal{P}$ is called a *momentum map* for G on $\mathcal{P}$.

Note that the momentum map is not unique, adding a constant to a momentum map of a Hamiltonian system with symmetry is also be a momentum map for the same dynamical system. Another important observation is that the momentum map depends uniquely on the symmetry group and not on the dynamics induced by the Hamiltonian.

Definition 4.3.1 can also be given in terms of the symplectic form ω of symplectic manifold $(\mathcal{M}, \omega)$. The defining equation (4.11) for the momentum map becomes

$$dJ(\xi)(v)(w) = \omega(\xi_{\mathcal{M}}(v), w),$$

where $v \in \mathcal{P}$, $w \in T_v\mathcal{P}$ and $\xi \in \mathfrak{g}$.

The following version of Noether's theorem can be found in [22].

Theorem 4.3.2. Noether's Theorem. *Consider a Hamiltonian action of the Lie group G on the symplectic (or Poisson) $\mathcal{P}$, and let H be an invariant Hamiltonian. Then the flow of the Hamiltonian vector field leaves the momentum functions J_ξ invariant.*

As mentioned before, a coadjoint orbit $\mathcal{O}$ endowed with the Lie-Poisson bracket is a Poisson manifold, thus we can define a momentum map $\mathbf{J}_{\mathcal{O}}$ of the coadjoint action of G on $\mathcal{O}$ with respect to (4.2) following Definition 4.3.1. From equations (4.6) and (3.13) the momentum map should satisfy

$$\mp \mathrm{ad}^*_{\frac{\delta J_{\mathcal{O}}(\xi)}{\delta \mu}} \mu = -\mathrm{ad}^*_\xi \mu \tag{4.13}$$

for every $\mu \in \mathfrak{g}^*$. That implies

$$J_{\mathcal{O}}(\xi)(\mu) = \pm\langle \mu, \xi \rangle,$$

hence $\mathbf{J}_{\mathcal{O}} : \mathcal{O} \to \mathfrak{g}^*$ is defined as

$$\mathbf{J}_{\mathcal{O}}(\mu) = \pm\mu. \tag{4.14}$$

This result is worked out in Example 11.4e) of [20], inspired by this result we present the following proposition.

Proposition 4.3.3. *Let G be a Lie group and let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit with the* KKS *symplectic form $\omega_{\mathcal{O}}$ on $\mathcal{O}$. Then the momentum map $\mathbf{J} : \mathcal{O} \to \mathfrak{g}^*$ of the symplectic manifold $(\mathcal{O}, \Gamma\omega_{\mathcal{O}}, G)$ is given by the homothety*

$$\mu \to \Gamma\mu.$$

Proof. The proof follows from the proofs of Proposition 10.7.1 and Lemma 14.4.2 in [20] where it is proven that the Hamiltonian vector field for $H \in C^\infty(\mathfrak{g}^*)$ must be given by

$$X_H(\mu) = -\frac{1}{\Gamma}\mathrm{ad}^*_{\frac{\delta H}{\delta\mu}}\mu.$$

By definition $J(\xi)$ must satisfy $X_H(\mu) = \xi_{\mathfrak{g}^*}$, that is

$$-\frac{1}{\Gamma}\mathrm{ad}^*_{\frac{\delta J(\xi)}{\delta\mu}}\mu = -\mathrm{ad}^*_{\xi}\mu.$$

Hence,

$$J(\xi)(\mu) = \langle\Gamma\mu, \xi\rangle,$$

which implies

$$\mathbf{J}(\mu) = \Gamma\mu.$$

□

4.4 N point vortices in the hyperboloid model

With the results for symplectic (Poisson) Hamiltonian systems we start the analysis of this dynamical system. We provide the symplectic and Poisson structure of this dynamical system, and present the Hamiltonian that describes the dynamics of this system.

Let $\check{X}_i$ be the vector from the origin in $\mathbb{R}^3$ to the *ith* vortex with non zero vorticity strength Γ_i and let

$$\Delta = \left\{ \check{X} = \left(\check{X}_1, ..., \check{X}_N \right) \in \mathcal{H}_2 \times ... \times \mathcal{H}_2 \mid \text{any two or more } X_i \text{ coincide} \right\}.$$

A candidate for the manifold of this dynamical system consists of N copies of the hyperboloid $\mathcal{H}_2 \times ... \times \mathcal{H}_2$, but we avoid collision of vortices which lead to infinite energy, hence the phase space of this problem is $\mathcal{M} = \mathcal{H}_2 \times ... \times \mathcal{H}_2 \backslash \Delta$.

As previously mentioned the symmetry group acting on this dynamical system is $SL(2, \mathbb{R})$. The *KKS* symplectic form calculated in Remark 4.2.5 induces a symplectic structure in each coadjoint orbit of $\mathfrak{sl}(2, \mathbb{R})^*$, in particular in each copy of $\mathcal{H}_2$. In Remark 4.2.4 we also concluded that the Lie-Poisson bracket (4.2) restricted to $\mathcal{H}_2$ induces a Poisson structure in $\mathcal{H}_2$. Therefore we define the Poisson structure of $\mathcal{M}$ as

$$\{\cdot, \cdot\}_{\mathcal{M}} = \sum_{i=1}^{N} \frac{1}{\Gamma_i} \{\cdot, \cdot\}_i \,, \tag{4.15}$$

where $\{\cdot, \cdot\}_i$ is the Lie-Poisson structure restricted to the *ith* copy of $\mathcal{H}_2$. In a similar way the symplectic form is given by

$$\omega_{\mathcal{M}} (\cdot, \cdot) = \sum_{i=1}^{N} \Gamma_i \omega_{\mathcal{H}_2} (\cdot, \cdot)_i \,, \tag{4.16}$$

where $\omega_{\mathcal{H}_2} (\cdot, \cdot)_i$ is defined in (4.10) for the ith copy of $\mathcal{H}_2$ that contains $\check{X}_i$. With a coadjoint orbit symplectic structure like this, the momentum map for the system of N particles is deduced in the following lemma.

Lemma 4.4.1. *Let G be a Lie group and let $\mathcal{M} = \mathcal{O}_1 \times ... \times \mathcal{O}_N$ the manifold with symplectic structure $\omega = \sum_{i=1}^{N} \Gamma_i \omega_i$ where ω_i is the* KKS *symplectic form on the ith coadjoint orbit $\mathcal{O}_i$. Let $\mathbf{J}_i (X_i)$ be a momentum map for the symplectic manifold $(\mathcal{O}_i, \Gamma_i \omega_i, G)$. Then the map $\mathbf{J} : \mathcal{M} \to \mathfrak{g}^*$ defined by*

$$\mathbf{J} (X_1, ..., X_N) = \mathbf{J}_1 (X_1) + ... + \mathbf{J}_N (X_N)$$

is a momentum map for the symplectic manifold $(\mathcal{M}, \omega, G)$.

Proof. Let $X_i \in \mathcal{O}_i \subset \mathfrak{g}^*$ and $v_i \in T_{X_i}\mathcal{O}_i$, then there exists $\eta \in \mathfrak{g}$ such that $v_i = \mathrm{ad}^*_\eta (X_i)$. Thus by Proposition 4.3.3 we obtain

$$\begin{aligned} d\mathbf{J}(X_i)(v_i) &= \left.\frac{d}{dt}\right|_{t=0} \mathbf{J}\left(\mathrm{Ad}^*_{\exp(t\eta)} X_i\right) \\ &= \left.\frac{d}{dt}\right|_{t=0} \Gamma_i \mathrm{Ad}^*_{\exp(t\eta)} X_i \\ &= \Gamma_i \mathrm{ad}^*_\eta X_i. \end{aligned}$$

On other hand, given $X_1, ..., X_N \in \mathcal{M}$ and $v = v_1, ..., v_N \in T_{X_1}\mathcal{O}_1 \times ... \times T_{X_N}\mathcal{O}_N$

$$\begin{aligned} dJ(\xi)(X_1, ..., X_N)(v) &= \sum_{i=1}^N \Gamma_i \omega_i \left(\xi_{\mathfrak{g}^*} X_i, v_i\right) \\ &= \sum_{i=1}^N \Gamma_i \omega_i \left(-\mathrm{ad}^*_\xi X_i, \mathrm{ad}^*_\eta X_i\right) \\ &= \sum_{i=1}^N \Gamma_i \langle X_i, [\eta, \xi]\rangle \\ &= \sum_{i=1}^N \Gamma_i \langle X_i, \mathrm{ad}_\eta \xi\rangle \\ &= \sum_{i=1}^N \langle \Gamma_i \mathrm{ad}^*_\eta X_i, \xi\rangle \\ &= \sum_{i=1}^N \langle d\mathbf{J}(X_i)(v_i), \xi\rangle. \end{aligned}$$

Hence,

$$J(\xi)(X_1, ..., X_N) = \langle \sum_{i=1}^N \mathbf{J}(X_i), \xi\rangle.$$

□

As expected the same result is derived from Equation (4.13) in terms of the Poisson structure (4.15). Let $G = SL(2, \mathbb{R})$ acting in $\mathcal{M}$ with symplectic form (4.16), let $X \in \mathcal{M}$ denote the set $X = (X_1, ..., X_N)$, where $X_i \in \mathfrak{sl}(2, \mathbb{R})^*$ is the corresponding 2×2 matrix to the $\check{X}_i$ vortex. Then the momentum map of this N particle system is

$$\mathbf{J}(X_1, ..., X_N) = \sum_{i=1}^N \Gamma_i X_i. \tag{4.17}$$

J. -M. Souriau in [31] proves that when the symmetry group is semisimple the momentum map of a symplectic manifold is coadjoint equivariant. $SL(2,\mathbb{R})$ is semisimple, hence the momentum map $\mathbf{J}$ satisfies

$$\mathbf{J}\left(g\cdot X\right)=\mathrm{Ad}_{g^{-1}}\mathbf{J}\left(X\right)$$

for all $g\in SL\left(2,\mathbb{R}\right)$, $X\in\mathcal{M}$.

By Noether's theorem (Theorem 4.3.2) the momentum map (4.17) is a conserved quantity under the flow of any Hamiltonian, particularly for the Hamiltonian representing the energy of this dynamical system.

Y. Kimura [14], constructs the Hamiltonian corresponding to the dynamics of N point vortices in the hyperboloid $\mathcal{H}_2$. This Hamiltonian, in terms of the hyperbolic inner product (2.5) is given by

$$H=-\frac{1}{4\pi}\sum\Gamma_i\Gamma_j\ln\frac{\langle\check{X}_i,\check{X}_j\rangle_{\mathcal{H}_2}+1}{\langle\check{X}_i,\check{X}_j\rangle_{\mathcal{H}_2}-1}. \tag{4.18}$$

Note that if all vorticities have the same sign, as two points get closer, i.e. as the distance between them tends to 0 in $\mathcal{H}_2$, so is $H\to\infty$ as expected when a collision occurs. This also means that the Hamiltonian $\mathcal{H}$ has a minimum (of energy) at some point and this should be an equilibrium point indeed.

As pointed out before, as the hyperboloid is a symplectic leaf, as it is connected component of a coadjoint orbit of $\mathfrak{g}^*=\mathfrak{sl}\left(2,\mathbb{R}\right)^*\cong\mathbb{R}^3$, we derive the equations of this dynamical system through the Lie-Poisson bracket.

The structure constants associated to the basis of $\mathfrak{g}=\mathfrak{sl}\left(2,\mathbb{R}\right)$ with the basis (3.1) are $C^1_{23}=-2$ and $C^2_{13}=C^1_{23}=2$. Let $\mu\in\mathfrak{sl}\left(2,\mathbb{R}\right)^*\cong\mathbb{R}^3$ with $\check{\mu}=(\mu_1,\mu_2,\mu_3)$ the associated vector by (3.2). Then the Lie-Poisson bracket (4.2) in $\mathfrak{sl}\left(2,\mathbb{R}\right)^*$ is

$$\begin{aligned}\{F,G\}_\pm\left(\mu_1,\mu_2,\mu_3\right) &= \pm 2\left(\mu_1\left(-\frac{\partial F}{\partial\mu_2}\frac{\partial G}{\partial\mu_3}+\frac{\partial F}{\partial\mu_3}\frac{\partial G}{\partial\mu_2}\right)+\mu_2\left(\frac{\partial F}{\partial\mu_1}\frac{\partial G}{\partial\mu_3}-\frac{\partial F}{\partial\mu_3}\frac{\partial G}{\partial\mu_1}\right)\right.\\ &\left.+\mu_3\left(\frac{\partial F}{\partial\mu_1}\frac{\partial G}{\partial\mu_2}-\frac{\partial F}{\partial\mu_2}\frac{\partial G}{\partial\mu_1}\right)\right).\end{aligned} \tag{4.19}$$

Recall the Poisson structure for $\mathcal{M}$ is given by (4.15), considering only the positive sign we obtain

$$\{F,G\}_{\mathcal{M}}(\mu_1,\mu_2,\mu_3)=\sum_{i=1}^{N}\frac{1}{\Gamma_i}\{F,G\}_i(\mu_1,\mu_2,\mu_3).$$

According to (4.4) we can now easily state the equations of motion for a point vortex $\check{X}_r=(X_r^1,X_r^2,X_r^3)\in\mathcal{H}_2$ as

$$\dot{X}_r^i=\left\{X_r^i,H\right\}\left(X_r^1,X_r^2,X_r^3\right),$$

with $i=1$, 2 or 3. With this notation $\langle\check{X}_r,\check{X}_s\rangle_{\mathcal{H}_2}=X_r^1X_s^1+X_r^2X_s^2-X_r^3X_s^3$ implies

$$\frac{\partial}{\partial X_r^i}\langle\check{X}_r,\check{X}_s\rangle_{\mathcal{H}_2}=\tau_i X_s^i,$$

where

$$\tau_i=\begin{cases}1 & \text{if } i=1,2,\\ -1 & \text{if } i=3.\end{cases}$$

Following on from this,

$$\frac{\partial\mathcal{H}}{\partial X_r^i}=\tau_i\left(\frac{\Gamma_r}{2\pi}\sum_{p\neq r}\Gamma_p\frac{X_p^i}{\langle X_r,X_p\rangle_{\mathcal{H}_2}^2-1}\right), \tag{4.20}$$

leads to the system of differential equations that describe this system

$$\dot{\check{X}}_r=\frac{1}{\pi}\sum_{p\neq r}\Gamma_p\frac{\check{X}_p\times_{\mathcal{H}_2}\check{X}_r}{\left\langle\check{X}_r,\check{X}_p\right\rangle_{\mathcal{H}_2}^2-1}. \tag{4.21}$$

This equation differs from the differential equations derived by Yoshifumi Kimura in [14] by a factor of 2 in the denominator, which is carried by the choice of the basis of the Lie algebra $\mathcal{B}$ (3.1). On other hand the differential equation presented by Seungsu Hwang and Sun-Chul Kim in [11] for this dynamical system differs by an additional factor of $\left\langle\check{X}_r,\check{X}_p\right\rangle_{\mathcal{H}_2}^2+1$ in the denominator.

We can now refer to this dynamical system by the quintuple

$$\left(\mathcal{M}, \omega_{\mathcal{M}}\left(\{\cdot,\cdot\}_{\mathcal{M}}\right), H, SL\left(2,\mathbb{R}\right), \mathbf{J}\right)$$

and use the results of symplectic (Poisson) Hamiltonian systems to derive its relative equilibria and stability conditions.

Chapter 5

Two vortices

In this chapter we consider the system of two vortices $X = (X_1, X_2) \in \mathcal{M}$ with their corresponding nonzero vorticity strength $\Gamma = (\Gamma_1, \Gamma_2)$. The Poisson Hamiltonian structure of this dynamical system $(\mathcal{M}, \omega(\cdot,\cdot)_{\mathcal{M}}, H, SL(2,\mathbb{R}), \mathbf{J})$ has been defined in Section 4.4.

From now we drop the bold notation of (4.17) and we denote the momentum map $\mathbf{J}$ simply by J. Therefore the momentum map of this dynamical system is $J(X_1, X_2) = \Gamma_1 X_1 + \Gamma_2 X_2$. Notice that a line passing through the origin O and $\check{X}_i$ intersects the hyperboloid $\mathcal{H}_2$ at exactly one point, $\check{X}_i$ itself, this implies that the vortices can not be parallel, therefore the momentum map $J(X_1, X_2)$ must be always different from 0.

Given the initial parameters of hyperbolic distance $c = d(X_1, X_2)$ and ratio of vorticity $\gamma = \frac{\Gamma_1}{\Gamma_2}$, we find the momentum isotropy subgroup associated to these conditions. Before this, we find the relationship between c and γ in terms of the determinant of the momentum value for these specific parameters. In Section 5.2 the types of trajectories followed by this system are described.

We introduce the concept of relative equilibria following the approach of Montaldi [22], please refer there for more details and proofs. In Section 5.3 we calculate the relative equilibria conditions for the system of N point vortices in the hyperboloid and present the explicit *angular velocity* for $N = 2$.

In the last section of this chapter we introduce the concepts of G and G_μ stability, and show that every configuration of two point vortices on the hyperboloid is both $SL(2,\mathbb{R})$ and leafwise stable. The stronger result of $SL(2,\mathbb{R})_\mu$ stability is obtained for configurations with elliptic momentum value ($\det\mu > 0$).

5.1 Momentum isotropy groups and relation with hyperbolic distance

This section relates the type of momentum value μ to the hyperbolic distance $c = d\left(\check{X}_1, \check{X}_2\right)$ and the ratio of vorticities $\gamma = \frac{\Gamma_1}{\Gamma_2}$. This relation is later used to conclude that G_μ-stability will depend only on c and γ in Theorem 5.4.8.

Recall, that the type of μ is given by the determinant of $\mu = J(X_1, X_2)$, which could be obtained either by the hyperbolic inner product or the map (3.4)

$$\det\mu = -\langle\check{\mu},\check{\mu}\rangle_{\mathcal{H}_2} = (\Gamma_1 z_1 + \Gamma_2 z_2)^2 - (\Gamma_1 x_1 + \Gamma_2 x_2)^2 - (\Gamma_1 y_1 + \Gamma_2 y_2)^2 .$$

Defining $\gamma = \frac{\Gamma_1}{\Gamma_2}$ and $t = \left\langle \check{X}_1, \check{X}_2 \right\rangle_{\mathcal{H}_2}$ we get

$$\begin{aligned} \frac{\det\mu}{\Gamma_2^2} &= -2\gamma t + \gamma^2 \det X_1 + \det X_2 \\ &= -2\gamma t + \gamma^2 + 1. \end{aligned}$$

Solving for γ:

$$\gamma = t \pm \sqrt{t^2 - 1 + \frac{\det\mu}{\Gamma_2^2}}. \tag{5.1}$$

In Lemma 2.3.1 the hyperbolic distance $c = d\left(\check{X}_1, \check{X}_2\right)$ is related to t by by the equation

$$t = -\cosh c.$$

Using this distance condition in (5.1) we get the relation between the ratio of vorticities γ and the hyperbolic distance c between the vortices as the solution curves:

$$\gamma_1\left(c, \det \mu\right) = -\cosh c + \sqrt{\sinh^2 c + \frac{\det \mu}{\Gamma_2^2}},$$
$$\gamma_2\left(c, \det \mu\right) = -\cosh c - \sqrt{\sinh^2 c + \frac{\det \mu}{\Gamma_2^2}}.$$

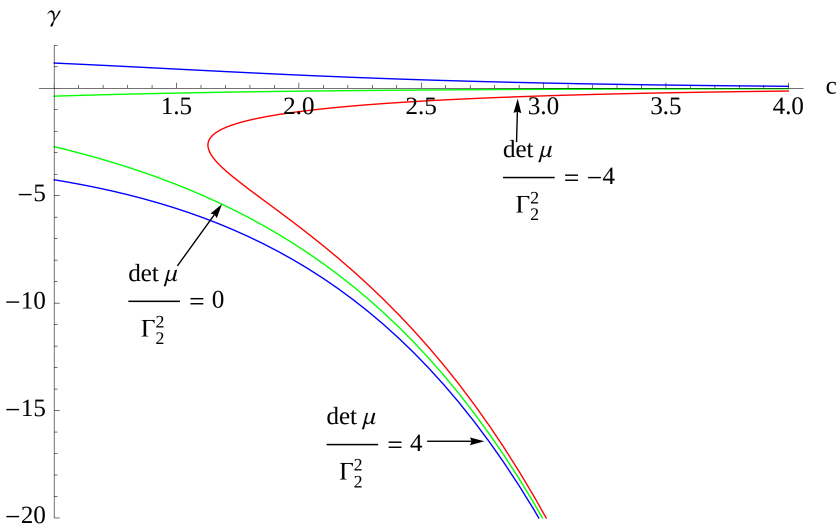


Figure 5.1: Hyperbolic distance c- vorticity ratio γ for different values of $\det \mu$.

In Figure 5.1 we plot the solution curves for specific values of $\det \mu$. If $\det \mu = 0$ the solution curves are $\gamma_1\left(c, 0\right) = -e^{-c}$ and $\gamma_2\left(c, 0\right) = -e^{c}$. Given any $\gamma_i\left(c, \det \mu\right)$, with $i = 1$ or 2, such that $\gamma_2\left(c, 0\right) < \gamma_i\left(c, \det \mu\right) < \gamma_1\left(c, 0\right)$, it is verified that $\det \mu < 0$. That is the area between the solution curves of $\det \mu = 0$ must satisfy the condition $\det \mu < 0$. Similarly, for any $\gamma_i\left(c, \det \mu\right) > \gamma_1\left(c, 0\right)$ or $\gamma_i\left(c, \det \mu\right) < \gamma_2\left(c, 0\right)$, $\det \mu$ must be greater than zero.

Thus we have set the general conditions for the isotropy subgroups of the momentum value with respect to the boundary condition $\det \mu = 0$.

Recall from Theorem 3.2.1 that if $\det \mu > 0$ the isotropy subgroup G_μ is conjugate to the group of rotations $SO\left(2, \mathbb{R}\right)$. An analogous conclusion is set for $\det \mu < 0$ where the isotropy group would be conjugate to the group of hyperbolic Möbius transformations. The value $\gamma = 0$ is obviously excluded as well as $\Gamma_1 = 0$ and $\Gamma_2 = 0$. Since $\mu \neq 0$ for the system of two point vortices, the sign of $\det \mu$ offers a classification of isotropy subgroups as illustrated in Figure 5.2.

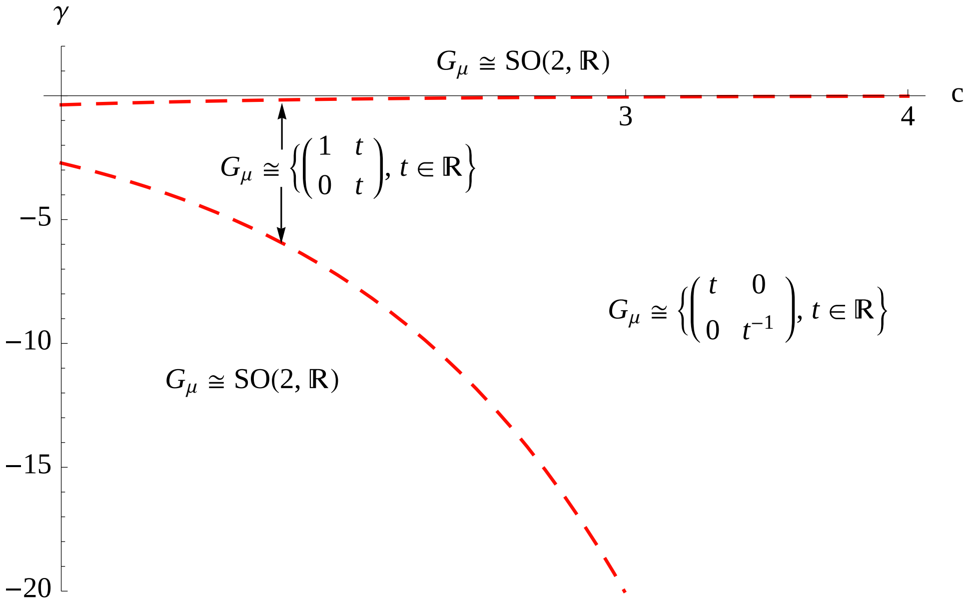


Figure 5.2: Classification of the isotropy subgroups in terms of the hyperbolic distance c and vorticity ratio γ.

For instance the vortex dipole treated in [14], that is $\Gamma_1 = -\Gamma_2$ with $\gamma(c, \det \mu) = -1$, leads to $\gamma_2\,(c, 0) < \gamma(c, \det \mu) < \gamma_1\,(c, 0)$ which implies that

$$G_\mu \cong \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, t \in \mathbb{R} \right\}.$$

Hence the vortices move parallel to each other on hyperbolas as shown in Figure 5.3(c). Meanwhile, for any $\Gamma_1\Gamma_2 > 0$ we obtain $\gamma\,(c, \det \mu) > 0$, that is $\gamma\,(c, \det \mu) > \gamma_1\,(c, 0)$. This shows that for $\Gamma_1\Gamma_2 > 0$ the momentum isotropy subgroup is conjugate to the group of rotations $SO\,(2, \mathbb{R})$ indeed.

5.2 Trajectories of the vortices

By the Implicit Function Theorem $\dim J^{-1}\,(\mu) = 1$ for two point vortices. We also know that $G_\mu \cdot X \subset J^{-1}\,(\mu)$ so they must be equal. On other hand, the momentum map is preserved under the flow of the Hamiltonian by Noether's theorem (Theorem 4.3.2). Therefore the symmetry group should act only by the momentum value isotropy subgroups G_μ. Thus the motion of this system must be on this group orbit $G_\mu \cdot X$.

As shown in Section 3.2.3, the set of points in a hyperbolic normal plane $P_{\check{\mu}}$ to $\check{\mu}$ is G_μ invariant. To describe what type of trajectories the vortex $\check{X}_i$ would follow, we consider a hyperbolic normal plane to $\check{\mu}$ passing through $\check{X}_i$. The intersection of this plane with the hyperboloid $\mathcal{H}_2$ defines the trajectory of the vortex.

Consider for example $\check{\mu}$ inside the null-cone $\mathfrak{C}$, then X_1 and X_2 would rotate around $\check{\mu}$ on an ellipse that intersects the hyperboloid as it can be seen in Figure 5.3(a). If $\check{\mu}$ lies outside the null cone $\mathfrak{C}$ the vortices would move on hyperbolas as shown in Figure 5.3(c). Finally, if $\check{\mu}$ is on the null cone $\mathfrak{C}$ the vortices would move on parabolas as it can be seen in Figure 5.3(b).

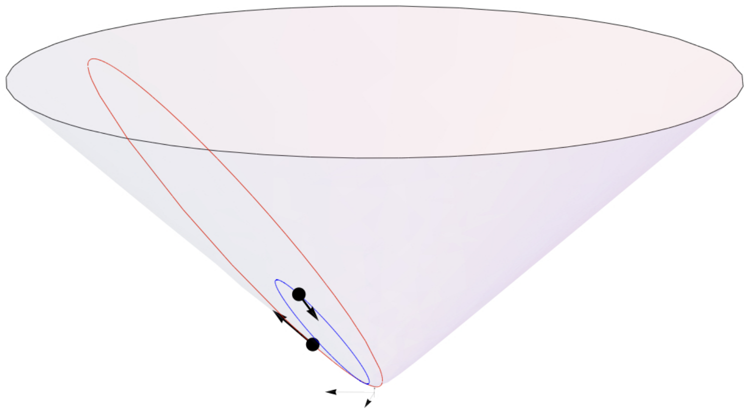
5.3 Relative equilibria

Definition 5.3.1. A *relative equilibrium* is a trajectory $\gamma(t)$ in a manifold $\mathcal{P}$ such that for each $t \in \mathbb{R}$ there is a symmetry transformation $g_t \in G$ for which $\gamma(t) = g_t \cdot \gamma(0)$.

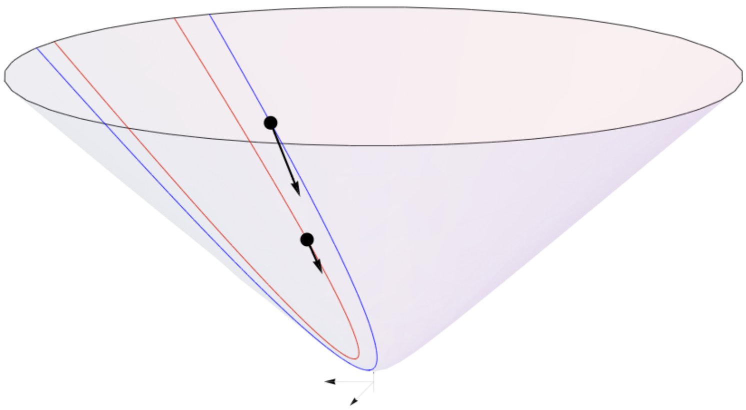
As noted in [22], any trajectory of an invariant group orbit is a relative equilibrium, and if a relative equilibrium $\gamma(t)$ is the trajectory through x, then $g \cdot \gamma(t)$ is the trajectory through $g \cdot x$, hence the group orbit $\gamma(t)$ is invariant. We can then think of a relative equilibrium as a group orbit that is invariant under the dynamics.

If we consider the reduced space $\mathcal{P}/G$, a group orbit is just a point, and an invariant group orbit is a point that is invariant under the dynamics. In other words a relative equilibrium in the whole space is just an equilibrium point in the reduced space. Let $\mu = J(x)$ with x a relative equilibrium point. By Noether's theorem the momentum map J is invariant under the dynamics, hence the level sets $J^{-1}(\mu)$ are invariant under the flow of the Hamiltonian vector field, and x is a critical point of $H\Big|_{J^{-1}(\mu)}$. This result is part of the following proposition which is given by many authors, the complete proof can be found in [22].

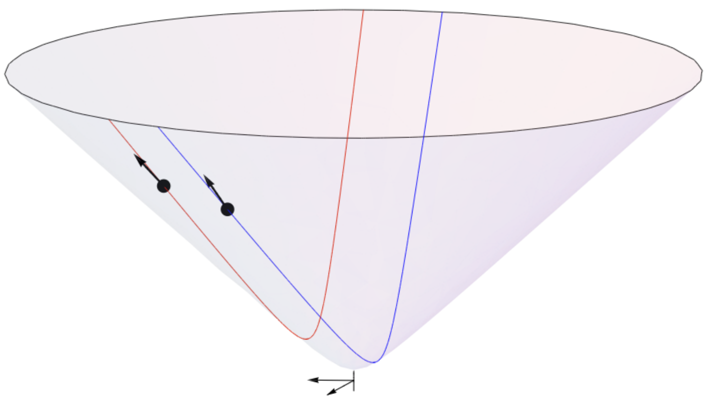
Proposition 5.3.2. *Let J be a momentum map for the G action on $\mathcal{P}$ and let H be a G-invariant Hamiltonian on $\mathcal{P}$. Let $x \in \mathcal{P}$ and let $\mu = J(x)$. Then the following are equivalent:*



(a) $\check{\mu}$ inside $\mathfrak{C}$, with $\Gamma_1 = 1$ and $\Gamma_2 = 3$.



(b) $\check{\mu}$ on $\mathfrak{C}$, with $\Gamma_1 = 1$ and $\Gamma_2 = -\frac{1}{2}$.



(c) $\check{\mu}$ outside $\mathfrak{C}$, with $\Gamma_1 = 1$ and $\Gamma_2 = -1$.

Figure 5.3: Trajectories of the vortices for $\check{\mu}$ inside, outside and on the null-cone $\mathfrak{C}$.

1. *The trajectory $\gamma(t)$ through x is a relative equilibrum,*

2. *The group orbit $G \cdot x$ is invariant under the dynamics,*

3. $\exists \xi \in \mathfrak{g}$ *such that* $\gamma(t) = \exp(t\xi) \cdot x, \forall t \in \mathbb{R}$,

4. $\exists \xi \in \mathfrak{g}$ *such that x is a critical point of* $H_\xi = H - \langle J, \xi \rangle$,

5. *x is a critical point of the restriction of H to the level set of $J^{-1}(\mu)$.*

5.3.1 Relative equilibria of two point vortices on the hyperboloid $\mathcal{H}_2$.

The trajectories of this system are the group orbits of G_μ through $\check{X}$. In Section 5.2 it was shown that the group orbit $G_\mu \cdot X$ coincides with $J^{-1}(\mu)$, so this group orbit is invariant under the dynamics, implying that all trajectories in it are relative equilibria.

Summing up, given X_1 and X_2 in $\mathcal{H}_2$ the trajectories of this vortices are in relative equilibrium and are conics determined by the determinant of the momentum value J as shown in Figure 5.3. Having only the hyperbolic distance c between the vortices and the ratio of vorticities γ we could also know the type of trajectory by looking at Figure 5.2.

Now we would like to know how fast the vortices are moving. Given that J is invariant under the dynamics, the vector velocity of the momentum map is

$$\dot{J} = \Gamma_1 \dot{X}_1 + \Gamma_2 \dot{X}_2 = 0.$$

Velocities are tangent vectors to the hyperboloid, hence $\dot{X}_1$ and $\dot{X}_2$ are parallel and the two vector velocities must be on the same plane. It can also be noted that if Γ_1 and Γ_2 have the same sign then the velocity directions are opposite, if the signs are different then $\dot{X}_1$ and $\dot{X}_2$ have the same direction. If $|\Gamma_1| > |\Gamma_2|$ then X_2 moves slower than X_1. We get the exact same relation by using the Equation (4.21) derived in the previous chapter.

The vector ξ of (3) and (4) of Proposition 5.3.2 is often called the *angular velocity* of the relative equilibrium. We find the general conditions for relative equilibria of the system of N point vortices and the angular velocity ξ for $N = 2$.

The augmented Hamiltonian $H_\xi = H - \langle J, \xi \rangle$ for $\xi \in \mathfrak{sl}(2, \mathbb{R})$ of the system of N point vortices in $\mathcal{H}_2$ is given by

$$H_\xi = -\frac{1}{4\pi} \sum_{r \neq s}^{N} \Gamma_r \Gamma_s \ln \left(\frac{\left\langle \check{X}_r, \check{X}_s \right\rangle_{\mathcal{H}_2} + 1}{\left\langle \check{X}_r, \check{X}_s \right\rangle_{\mathcal{H}_2} - 1} \right) - \sum_{i=1}^{3} \sum_{r=1}^{N} \tau_i \xi_i \Gamma_r X_r^i \tag{5.2}$$

with $\check{X}_r = (X_r^1, X_r^2, X_r^3)$, $\check{X}_s = (X_s^1, X_s^2, X_s^3) \in \mathcal{H}_2$, with strength vorticities Γ_r and Γ_s respectively, and

$$\tau_i = \begin{cases} 1 & \text{if } i = 1, 2, \\ -1 & \text{if } i = 3. \end{cases}$$

It is clear that relative equilibria must be restrained to $\mathcal{H}_2$, this restriction is included with the addition of a Lagrange multiplier to (5.2). For this reason we find the critical points of the *extended Hamiltonian* defined as

$$\mathcal{H}_\xi = -\frac{1}{4\pi} \sum_{r \neq s}^{N} \Gamma_r \Gamma_s \ln \left(\frac{\left\langle \check{X}_r, \check{X}_s \right\rangle_{\mathcal{H}_2} + 1}{\left\langle \check{X}_r, \check{X}_s \right\rangle_{\mathcal{H}_2} - 1} \right) - \sum_{i=1}^{3} \sum_{r=1}^{N} \tau_i \xi_i \Gamma_r X_r^i + \sum_{r=1}^{N} \lambda_r \left(\langle \check{X}_r, \check{X}_r \rangle_{\mathcal{H}_2} + 1 \right). \tag{5.3}$$

With this notation $\langle \check{X}_r, \check{X}_s \rangle_{\mathcal{H}_2} = \sum_{i=1}^{3} \tau_i X_r^i X_s^i$, thus $\frac{\partial}{\partial X_r^i} \langle \check{X}_r, \check{X}_s \rangle_{\mathcal{H}_2} = \tau_i X_s^i$, and

$$\frac{\partial \mathcal{H}_\xi}{\partial X_r^i} = \tau_i \left(\frac{\Gamma_r}{2\pi} \sum_{p \neq r} \Gamma_p \frac{X_p^i}{\left\langle \check{X}_r, \check{X}_p \right\rangle_{\mathcal{H}_2}^2 - 1} - \Gamma_r \xi_i + \lambda_r X_r^i \right) = 0. \tag{5.4}$$

This leads to the general condition for relative equilibria, given as the solutions of the following equation of *angular velocity*

$$\xi_i = \frac{1}{2\pi} \sum_{p \neq r} \Gamma_p \frac{X_p^i}{L_{pr}} + \frac{\lambda_r}{\Gamma_r} X_r^i, \qquad \forall r \in \{1, 2, ...N\} \text{ and } i \in \{1, 2, 3\}, \tag{5.5}$$

where L_{pr} denotes $\langle X_p, X_r \rangle_{\mathcal{H}_2}^2 - 1$. The vector ξ represents the angular velocity, that is $\dot{\check{X}}_i = \check{\xi} \times_{\mathcal{H}_2} \check{X}_i$.

Theorem 5.3.3 (Relative Equilibria of Two Point Vortices on the Hyperboloid). *Let $X = (X_1, X_2) \in \mathcal{M}$ be two point vortices with strength vorticity $\Gamma = (\Gamma_1, \Gamma_2)$. Then every trajectory of this pair of vortices is in relative equilibrium with angular velocity*

$$\xi = \frac{1}{2\pi L} J\left(X_1, X_2\right),$$

where $L = \langle X_1, X_2\rangle^2_{\mathcal{H}_2} - 1 = \cosh^2\left(d\left(X_1, X_2\right)\right) - 1$.

Proof. The group orbits are invariant under the dynamics, hence on relative equilibria, this argument was already given at the start of this section.

The Lagrange multipliers that solve (5.5) are

$$\lambda_1 = \frac{\Gamma_1^2}{2\pi L},$$
$$\lambda_2 = \frac{\Gamma_2^2}{2\pi L}.$$

Therefore, the *angular velocity* ξ is given by (5.5). □

Note that since $L > 0$ the vectors $\check{\xi}$ and $\check{J}$ have the same direction, therefore the vortices rotate hyperbolically around $\check{J}$.

5.4 Stability of relative equilibria

Intuitively, a group orbit nearby a *stable relative equilibrium* remains nearby with the flow of the Hamiltonian vector field X_H. We follow the definition of stability for systems with symmetry given by George W. Patrick in [27].

Definition 5.4.1. Let $(\mathcal{P}, \{\cdot,\cdot\}, J, H, G)$ be a Poisson Hamiltonian system with symmetry, let ϕ_t be the flow of the Hamiltonian vector field X_H and G' a subgroup of G. A relative equilibrium $x_e \in \mathcal{P}$ is called *G'-stable* if for all G'-invariant neighbourhoods V of $G' \cdot x_e$ there exists a G'-invariant neighbourhood U of x_e such that $\phi_t(x) \in V$ for all $x \in U$ and for all t.

A relative equilbrium x_e with momentum value $J(x_e) = \mu$ is called *leafwise stable* ([28]) if it is G_μ stable for the flow ϕ_t restricted to the momentum level set

$J^{-1}(\mu)$. Evidently, a relative equilibrium that is G- stable is also stable in the reduced space $\mathcal{P}/G_\mu$. However, the converse is not always true (see [15, 17, 28] and for examples). On other hand, G_μ stability does imply G-stability as shown by the following proposition.

Proposition 5.4.2. *Let $(\mathcal{P}, \{\cdot,\cdot\}, J, H, G)$ be a Poisson Hamiltonian system with symmetry. Suppose $x_e \in \mathcal{P}$ is a relative equilibrium with $\mu = J(x_e)$. If x_e is G_μ-stable then it is G-stable.*

Proof. Suppose the flow of H in $\mathcal{P}$ is given by $\phi_t : \mathcal{P} \to \mathcal{P}$.

Let V be a G-invariant neighbourhood of $G \cdot x_e$ and obviously it is also G_μ-invariant neighbourhood of $G \cdot x_e$. Therefore, there exists a G_μ invariant $U_1 \subset V$ such that $\phi_t(x) \in V$ for all $x \in U_1$ and for all t. Then $U = G \cdot U_1$ is the required G-invariant subset in V.

□

In the reduced space, a relative equilibria becomes a point and the definition of G-stability coincides with the definition of Lyapunov stability if $J^{-1}(\mathcal{O})/G$ is a symplectic manifold. This holds if the action of the Lie group G is free and proper. Since collinear point vortices on the sphere can be fixed under the action of $SO(2,\mathbb{R})$, the study of G- stability of the system of N point vortices on the sphere is restricted to $N \geqslant 3$. However, this limitation does not appear when studying this dynamical system in the hyperboloid. Hence by avoiding collisions the action of $SL(2,\mathbb{R})$ on $\mathcal{M} = \mathcal{H}_2 \times ... \times \mathcal{H}_2 \backslash \Delta$ is always free for $N \geqslant 2$. The next version of the Dirichlet criterion for stability can be found in [24].

Theorem 5.4.3 (Dirichlet's theorem)**.** *Let X be a dynamical system on the manifold $\mathcal{P}$, and let $x \in \mathcal{P}$ be an equilibrium. Let $C \in C^\infty(M)$ be a conserved quantity of X that is, $C \circ F_t = C$ for all time t, where F_t is the flow associated to X. If C is such that $dC(x) = 0$ and the quadratic form $d^2C(x)$ is definite, then the equilibrium x is stable*

Energy methods have adapted Dirichlet's theorem to the study of stability of relative equilibria of symmetric Hamiltonian systems, testing for definiteness of the second variation of an augmented Hamiltonian (energy function) at x_e. Since a relative equilibrium x_e is a critical point of $H\Big|_{J^{-1}(\mu)}$, by Dirichlet's theorem if the quadratic form $d^2H\Big|_{J^{-1}(\mu)}$ is definite at x_e, then x_e is stable in the reduced space $\mathcal{P}/G_\mu$. Despite this being an effective technique for determining leafwise stability is not useful for determining stability in the whole $\mathcal{P}$.

The *Energy-Casimir method* consists in finding a Casimir C such that x_e is a critical point of the Hamiltonian $H + C$, the energy function for this method becomes $H + C$ and stability is determined by definiteness of $d^2\left(H + C\right)(x_e)$. This method developed by Arnold does not have the limitation of working on the reduced space, but carries the condition of finding a Casimir which can not always be found. This motivated the construction of *Energy-momentum method* which can be used in Poisson systems where there are not Casimir functions for every point.

The augmented Hamiltonian for the *Energy-momentum method* is given by $H_\xi = H - \langle J, \xi\rangle$. This method consists in testing definiteness of d^2H_ξ on a normal space to the action of $\mathfrak{g}_\mu$ at x_e, if this resulted to be definite x_e was initially called *formally stable*. Is not until 1992 that George W. Patrick relates formal stability with the concept of G_μ stability defined above in the following theorem for regular points (Definition 3.2.6).

Theorem 5.4.4 ([27]). *Let $(\mathcal{P}, \omega, H, G, J)$ be a symplectic Hamiltonian system with symmetry. Suppose x_e is a regular relative equilibrium, the action of G_μ on $\mathcal{P}$ is proper, and $\mathfrak{g}$ admits an inner product invariant under the adjoint action of G_μ. Then $dH_\xi(x_e) = 0$, and x_e is G_μ stable if it is formally stable; that is if $d^2H_\xi(x_e)$ is positive or negative definite on some(and hence any) complement to $\mathfrak{g}_\mu \cdot x_e$ in $T_{x_e}J^{-1}(\mu)$.*

We provide stability results obtained from the energy momentum method. But before doing so, in order to find a symplectic normal space for the use of this method, the decomposition of the tangent space $T_{x_e}\mathcal{P}$ is presented .

5.4.1 Witt decomposition

Regardless of the fact that stability is determined only on the reduced space, computing the definiteness of the Hessian d^2H_ξ on a symplectic normal space becomes much simpler. In this section we present the *Witt decomposition* (following the presentation of [5, 30]) which is a decomposition of $T_x\mathcal{P}$ for any $x \in \mathcal{P}$.

We consider that the action of G on the symplectic manifold $(\mathcal{P}, \omega, J, H, G)$ is free and proper, and the momentum map J is coadjoint equivariant. Let $J(x) = \mu$ and $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{n}$ a splitting of $\mathfrak{g}$. Let

$$
\begin{aligned}
T_0 &= \mathfrak{g}_\mu \cdot x, \\
T_1 &= \mathfrak{n} \cdot x.
\end{aligned}
$$

Then

$$
\begin{aligned}
T_x (G \cdot x) &= \mathfrak{g} \cdot x \\
&= T_0 \oplus T_1.
\end{aligned}
$$

By the Implicit Function Theorem 3.1.9 we have $\ker dJ(x) = T_x J^{-1}(\mu)$, hence

$$
T_0 = T_x (G_\mu \cdot x) \subset \ker dJ(x).
$$

The complement N_1 to T_0 in $\ker dJ(x)$, is called the *symplectic slice at x*. Let N be the normal space to the group orbit $G \cdot x$. In the sense of the symplectic form N is the normal space to T, then $N_1 \subset N$. The restriction of the symplectic form on $T_x\mathcal{P}$ to T_1 and N_1 is nondegenerate, in view of this we say that T_1 and N_1 are symplectic. Furthermore, N_1 is a maximal subspace of N which is symplectic, for that reason we also refer to this slice as the *symplectic normal space at x*.

Let N_0 be the complement to N_1 in N, that is, $N = N_0 \oplus N_1$. The tangent space to $\mathcal{P}$ at x is therefore decomposed as

$$T_x\mathcal{P} = T_0 \oplus T_1 \oplus N_0 \oplus N_1.$$

Geometrically, T_1 represents the change of group orbit and N_0 the change from one momentum level set to another.

5.4.2 G-stability and G_μ-stability.

The topological characteristics of μ play an important role for stability results. If the action of G on the manifold $\mathcal{P}$ is free and proper, then a relative equilibrium x_e is G-stable if the Hessian of the reduced Hamiltonian is definite in x_e, and provided $\mu = J(x_e)$ is regular. This result is due to Arnold (Apendix 2 of [3]) and Libermann and Marle ([17]). Furthermore, Montaldi ([21]) shows that the same result is obtained by only requiring $\mathfrak{g}^*/G$ Hausdorff at G_μ, this is always granted if μ is regular, but could also be obtained for non regular points as the following proposition shows.

Proposition 5.4.5 ([28])**.** *Let $(\mathcal{P}, \omega, H, G, J)$ be a symplectic Hamiltonian system with symmetry. Suppose G acts freely and properly in $\mathcal{P}$ and $\mu \in \mathfrak{g}^*$.*

1. *If μ is regular, then $\mathfrak{g}_\mu$ is Abelian and $\mathfrak{g}^*/G$ is Hausdorff at $G \cdot \mu$.*
2. *Let μ be split, then $\mathfrak{g}^*/G$ is Hausdorff at $G \cdot \mu$ if there exists a G^0_μ-invariant inner product on $\mathfrak{g}_\mu$.*

Under the same assumptions as above, George W. Patrick, Mark Roberts and Claudia Wulff gather the results of G- stability mentioned before in the next corollary.

Corollary 5.4.6 ([28])**.** *Let x_e be a relative equilibrium of H with generator ξ_e. Suppose that $\mathfrak{g}^*/G$ is Hausdorff at $\mu = J(x_e)$ and that the Hessian $d^2H_{\xi_e}(x_e)$ is (positive or negative) definite when restricted to any symplectic normal space at x_e. Then x_e is G-stable.*

As mentioned before, G-stability implies leafwise stability but it does not necessarily imply G_μ stability. The existence of a G_μ invariant inner product on $\mathfrak{g}$ in

Patrick's (Theorem 5.4.4) implies that every μ is split. This inspired E. Lerman and S. F. Singer ([16]) to drop the regularity condition for proper group actions and prove G_μ- stability is also obtained for split relative equilibria if there exists a G_μ invariant inner product on $\mathfrak{g}^*$ and $d^2 H_\xi|_{N_1}(X_e)$ is definite. This is also shown by J-P. Ortega and Tudor S. Ratiu in [25]. The next corollary connects these results with G-stable relative equilibrium.

Corollary 5.4.7 ([28])**.** *Let $(\mathcal{P}, \omega, H, G, J)$ be a symplectic Hamiltonian system with symmetry. Suppose G acts freely and properly in $\mathcal{P}$, and x_e is a relative equilibrium with momentum value $\mu \in \mathfrak{g}^*$. If x_e is G-stable and there exists a G_μ^0 invariant inner product on $\mathfrak{g}^*$, then x_e is G_μ^0-stable.*

Under the same assumptions Ortega and Ratiu also prove that if $\dim N_1 = 0$ then x_e is always a G_μ stable relative equilibrium. In Remark 4.6 they treat the case of $SL(2,\mathbb{R})$ to highlight the importance of the existence of a G_μ invariant inner product in $\mathfrak{g}^*$. In that particular example it is shown that $x_e = 0$ is not always G-stable even though $\dim N_1 = 0$. This example is also treated in [28] where is shown that if the angular velocity ξ is pointing out of the cone $\mu = 0$, is leafwise stable but not stable. Nevertheless if ξ is pointing inside the null-cone $\mathfrak{C}$ then $\mu = 0$ is G-stable.

Any $\mu = J(x_e) \neq 0$ in $\mathfrak{sl}(2,\mathbb{R})^*$ is regular by Proposition 3.2.9, therefore Proposition 5.4.5 together with Corollary 5.4.6 permit us to determine $SL(2,\mathbb{R})$-stability by definiteness of $H_\xi|_{N_1}(x_e)$.

Moreover for any type of $\mu \neq 0$ the isotropy Lie algebra $\mathfrak{g}_\mu$ is Abelian so there is an invariant inner product on $\mathfrak{g}_\mu^*$ but not always on $\mathfrak{g}^*$. However if μ is elliptic μ the isotropy subgroup $G_\mu = SO(2,\mathbb{R})_\mu$ is compact (Theorem 3.2.1), and the stronger result of $SL(2,\mathbb{R})_\mu$ is also obtained by Corollary 5.4.7 if μ is $SL(2,\mathbb{R})$- stable. Summing up, only for elliptic μ definiteness of $H_\xi|_{N_1}(x_e)$ implies both $SL(2,\mathbb{R})$ and $SL(2,\mathbb{R})_\mu$ stability.

5.4.3 Stability of relative equilibria of two point vortices.

As mentioned before, the action of $G = SL(2, \mathbb{R})$ on $\mathcal{M}$ is proper (Lemma 3.1.18) for all N, and free for $N \geq 2$.

For the system of two point vortices, any relative equilibrium x_e has momentum value $J(x_e) = \mu \neq 0$, hence μ is regular by Proposition 3.2.9. As mentioned before, by Proposition 5.4.5 and Corollary 5.4.6, $SL(2, \mathbb{R})$- stability is determined upon definiteness of $d^2 H_{\xi_e}(x_e)$ on the symplectic normal space N_1 at x_e. The Witt decomposition of the tangent space which requires

$$N_1 \oplus \mathfrak{g}_\mu \cdot x_e = \ker dJ(x_e).$$

From Theorem 3.2.7 the dimension of $\mathfrak{g}_\mu \cdot x_e$ is equal to one. Since we avoid collisions dJ is always surjective, so by the Implicit Function Theorem 3.1.9 the dimension of $\ker dJ(x_e) = 1$. Therefore the dimension of the symplectic normal space N_1 must be zero, so the assumptions of Corollary 5.4.6 and Corollary 5.4.7 are satisfied and we have proved the following theorem.

Theorem 5.4.8 ($SL(2, \mathbb{R})$**-stability of relative equilibria of two point vortices).** *Let $X_e = (X_1, X_2) \in \mathcal{M}$ be two point vortices with strength vorticity $\Gamma = (\Gamma_1, \Gamma_2)$. Then every trajectory of this vortices is $SL(2, \mathbb{R})$- stable in $\mathcal{M}$.*

The analysis of Figure 5.2 and Corollary 5.4.7 prove the next result.

Corollary 5.4.9 ($SL(2, \mathbb{R})_\mu$**-stability of relative equilibria of two point vortices).** *Let $X_e = (X_1, X_2) \in \mathcal{M}$ be two point vortices with strength vorticity $\Gamma = (\Gamma_1, \Gamma_2)$ and hyperbolic distance $c = d\left(\check{X}_1, \check{X}_2\right)$. Suppose the momentum value is given by $\mu = J(X_1, X_2) \in \mathfrak{sl}(2, \mathbb{R})^*$. If either $\frac{\Gamma_1}{\Gamma_2} > -e^{-c}$ or $\frac{\Gamma_1}{\Gamma_2} < -e^{c}$ then X_e is $SL(2, \mathbb{R})_\mu$- stable in $\mathcal{M}$. Otherwise is only leafwise stable.*

□

Chapter 6

Relative equilibria of three vortices

In this chapter we study the case of $N = 3$. Using the geometric approach of symmetric Hamiltonian systems, we classify the relative equilibria for a 3-vortex system. In Remark 6.1.2, we point out how this classification can also be derived from results previously obtained by Seungsu Hwang and Sun-Chul Kim in [11]. The relative equilibria conditions in terms of the strength of vorticities Γ_i are given in sections 6.2 and 6.3.

6.1 Relative Equilibria

The next theorem demonstrates the existence and classification of relative equilibria for $N = 3$.

Theorem 6.1.1 (Relative equilibria of three point vortices on the hyperboloid). *Let $X = (X_1, X_2, X_3) \in \mathcal{M}$ be three point vortices in the hyperboloid $\mathcal{H}_2$, with strength vorticities $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$. Then relative equilibria is one of the following cases:*

1) Equilateral configurations,

2) Geodesic configuration with two equal lengths,

3) Geodesic configuration with three different lengths.

Proof. Any three point vortices configuration can be obtained by hyperbolic rotations to the set

$$\begin{aligned} \check{X}_1 &= (x_1, y_1, z_1)\,, \\ \check{X}_2 &= (0, 0, 1)\,, \\ \check{X}_3 &= (x_3, y_3, z_3)\,. \end{aligned}$$

The dynamics remain invariant under this type of transformation, thus the relative equilibria conditions of this set are the same for any other set of vortices.

From (5.5), relative equilibria conditions for this set imply that vortices should satisfy one of the following possibilities:

1) $z_1 = z_3$ and $\langle \check{X}_1, \check{X}_3 \rangle_{\mathcal{H}_2} = \pm z_3$,

2) $z_1 = z_3$, $\Gamma_1 x_1 = -\Gamma_3 x_3$ and $\Gamma_1 y_1 = -\Gamma_3 y_3$,

3) $y_1 = y_3 = 0$.

In the first case, setting $z_1 = z_3$ enables us to study $\langle \check{X}_1, \check{X}_3 \rangle$ on the $x - y$ plane, thus there exists $r > 0$ such that $x_3^2 + y_3^2 = r^2$ and $x_1^2 + y_1^2 = r^2$ are satisfied. Since $\check{X}_3 \in \mathcal{H}_2$ we have $z_3 = \sqrt{r^2 + 1}$.

Provided $\langle \check{X}_1, \check{X}_3 \rangle = \pm z_3$, and θ the angle between $\check{X}_1$ and $\check{X}_3$ in the $x - y$ plane, we derive the equation

$$\pm \left(r^2 \cos\theta - r^2 - 1 \right) = \sqrt{r^2 + 1}.$$

Since $r^2 \cos\theta - r^2 - 1 < 0$ and $\sqrt{r^2 + 1} > 0$, we conclude that $-\left(r^2 \cos\theta - r^2 - 1\right) = \sqrt{r^2 + 1}$ must hold. That is, $\langle \check{X}_1, \check{X}_3 \rangle = -z_3$. Note that $\langle \check{X}_1, \check{X}_2 \rangle = \langle \check{X}_2, \check{X}_3 \rangle = -z_3$, hence all hyperbolic distances are equal for case 1).

For the second case, let $a = \frac{\Gamma_3}{\Gamma_1}$ then $x_1 = a x_3$ and $y_1 = a y_3$. Thus,

$$\begin{aligned} x_1^2 + y_1^2 &= z_1^2 - 1 \\ &= z_3^2 - 1, \end{aligned}$$

leads to $a^2\left(x_3^2+y_3^2\right)=z_3^2-1$, which implies $a=\pm 1$ and $\Gamma_1=\pm\Gamma_3$. Note that $\Gamma_1\neq-\Gamma_3$ otherwise $X_1=X_3$, and therefore the only possible solution is $\Gamma_3=\Gamma_1$. In conclusion, $x_1=-x_3$ and $y_1=-y_3$, which means that the vortices lie on a geodesic with $\check{X}_1$ and $\check{X}_3$ symmetrically opposite with the same hyperbolic distance to $\check{X}_2$.

Finally, if $y_1=y_3=0$, the vortices lie on the $x-z$ plane which is indeed a geodesic. Following the calculations of (5.5), we are required to dismiss $x_1=-x_3$ and $x_1=\pm 2x_3\sqrt{1+x_3}$. These excluded conditions represent configurations where $\langle\check{X}_1,\check{X}_2\rangle=\langle\check{X}_2,\check{X}_3\rangle$ and $\langle\check{X}_1,\check{X}_3\rangle=\langle\check{X}_2,\check{X}_3\rangle$, that are included in the second case. Equilateral configurations are obviously discarded. To sum up, this case contains only vortices in a geodesic with different hyperbolic distances between them. The additional condition on the values of Γ is derived in Theorem 6.3. □

Remark 6.1.2. Although not mentioned explicitly by Hwang and Kim, the classification of relative equilibria in the previous theorem can also be deduced from formulae (17) – (19) in [11]. Recall, from Section 2.2, that a geodesic on the hyperboloid model is the intersecting curve of a plane through the origin with the hyperboloid $\mathcal{H}_2$. As was also pointed out in [11], contrary to the system on a sphere, it is not possible to have an equilateral configuration in a geodesic of the hyperboloid. Therefore, for an equilateral configuration

$$V=\check{X}_1\cdot_{\mathcal{H}_2}\left(\check{X}_2\times_{\mathcal{H}_2}\check{X}_3\right)\neq 0.$$

This leads us to the conclusion that a relative equilibrium is either an equilateral configuration or on a geodesic, but not both.

Moreover, formulae (17) – (19) in [11] coincide with the results of Rangachari Kidambi and Paul K. Newton [13] for the study of three point vortices on a sphere, but in that case an equilateral configuration can be on a geodesic (great circle).

6.2 Equilateral relative equilibria

Theorem 6.2.1 (Relative equilibria for equilateral configurations). *Every equilateral configuration X_e of point vortices (X_1, X_2, X_3) in $\mathcal{M}$ with strength vorticity $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$ is a relative equilibrium. The angular velocity of X_e is given by*

$$\xi = \frac{1}{2\pi L} J\left(X_1, X_2, X_3\right), \tag{6.1}$$

where $L = \langle \check{X}_i, \check{X}_j \rangle^2_{\mathcal{H}_2} - 1$ for all $i,\ j \in \{1, 2, 3\}$.

Proof. From (5.5) the angular velocity is given by

$$\xi_i = \frac{1}{2\pi L} \sum_{p \neq r}^{3} \Gamma_p X_p^i + \frac{\lambda_r}{\Gamma_r} X_r^i,$$

with $\check{X}_r = (X_r^1, X_r^2, X_r^3)$ and $\check{X}_p = \left(X_p^1, X_p^2, X_p^3\right)$. Therefore the equalities to solve are

$$\frac{2\pi L \lambda_1}{\Gamma_1} X_1^i + \Gamma_2 X_2^i + \Gamma_3 X_3^i = \Gamma_1 X_1^i + \frac{2\pi L \lambda_2}{\Gamma_2} X_2^i + \Gamma_3 X_3^i = \Gamma_1 X_1^i + \Gamma_2 X_2^i + \frac{2\pi L \lambda_3}{\Gamma_3} X_3^i,$$

for $i \in \{1, 2, 3\}$. As mentioned before, a line passing through the origin O and a point $\check{X}_i$ on $\mathcal{H}_2$ intersects the hyperboloid at exactly one point, $\check{X}_i$ itself, so the vortices can not be parallel. Consequently

$$\lambda_i = \frac{\Gamma_i^2}{2\pi L} \quad \forall i \in \{1, 2, 3\},$$

which leads to

$$\begin{aligned} \xi &= \frac{1}{2\pi L} \left(\Gamma_1 X_1 + \Gamma_2 X_2 + \Gamma_3 X_3\right) \\ &= \frac{1}{2\pi L} J\left(X_1, X_2, X_3\right). \end{aligned}$$

□

The exact same angular velocity (6.1) is obtained for equilateral configurations of point vortices on the sphere by Rangachari Kidambi and Paul K. Newton in [13] and Sergey Pekarsky and Jerrold E. Marsden in [29].

Remark 6.2.2. As previously mentioned in Remark 6.1.2, an equilateral relative equilibrium $X_e = (X_1, X_2, X_3)$ is not on a geodesic, thus $\check{X}_1$, $\check{X}_2$ and $\check{X}_3$ are linearly independent in $\mathbb{R}^3$. As $\Gamma_i \neq 0$ for all $i \in \{1,2,3\}$,

$$\mu = J\left(X_e\right) = \Gamma_1 X_1 + \Gamma_2 X_2 + \Gamma_3 X_3 \neq 0.$$

Therefore the trajectory is one of the conics described in Section 3.2.3, where we classified this motion by the value of $\det \mu$. Also, since $\mu \neq 0$, the angular velocity ξ is always different from zero implying that X_e is never an equilibrium point. Moreover,

$$\dot{\check{X}}_i = \frac{1}{2\pi L} \check{\mu} \times_{\mathcal{H}_2} \check{X}_i$$

is satisfied for all $i \in \{1,2,3\}$, thus the vortices "rotate hyperbolically" around the momentum value $\check{\mu}$.

6.3 Geodesic relative equilibria

Let $\check{X}_e$ be given by

$$\begin{aligned} \check{X}_1 &= \left(x_1, 0, \sqrt{1+x_1^2}\right), \\ \check{X}_2 &= \left(0,0,1\right), \\ \check{X}_3 &= \left(-x_3, 0, \sqrt{1+x_3^2}\right), \end{aligned} \tag{6.2}$$

with $x_1,\ x_3 > 0$. Any other set of vortices $\check{X}_e' = \left(\check{X}_1', \check{X}_2', \check{X}_3'\right)$ on a geodesic, with $\check{X}_2'$ between $\check{X}_1'$ and $\check{X}_3$, can be matched with (6.2) by hyperbolic rotations. Since the hyperbolic inner product remains invariant under hyperbolic rotations, the dynamics between the vortices are the same. Therefore, the relative equilibria conditions derived for this set are the same for any other geodesic configuration. Having noted this, whenever studying the relative equilibria, or stability of geodesic configurations, we will make use of this set.

Theorem 6.3.1 (Relative equilibria of geodesic configurations). *Let $X_e = (X_1, X_2, X_3) \in \mathcal{M}$ be a configuration of point vortices on a geodesic of the hyperboloid, with strength of vorticity $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$. Suppose $\check{X}_2$ is between $\check{X}_1$ and $\check{X}_3$, and that $L_{ij} = \langle \check{X}_i, \check{X}_j \rangle^2_{\mathcal{H}_2} - 1$ with $i \in \{1,2,3\}$. Then X_e is a relative equilibrium point if*

$$\sqrt{L_{23}}\,(L_{13} - L_{12})\,\Gamma_1 + \sqrt{L_{13}}\,(L_{23} - L_{12})\,\Gamma_2 + \sqrt{L_{12}}\,(L_{23} - L_{13})\,\Gamma_3 = 0 \qquad (6.3)$$

Proof. We calculate the relative equilibrium conditions of (6.2) in terms of L_{ij}. In this setting $\sqrt{L_{12}} = x_1$, $\sqrt{L_{23}} = x_3$ and

$$\sqrt{L_{13}} = \sqrt{L_{12}}\sqrt{1 + L_{23}} + \sqrt{L_{23}}\sqrt{1 + L_{12}}. \qquad (6.4)$$

The angular velocity $\xi = (\xi_1, \xi_2, \xi_3)$ in (5.5) gives the relative equilibrium condition. Let $r \in \{1,2,3\}$ and denote by ξ_i^r the expression of (5.5) for the i-th component of ξ. It is clear that $\xi_i^1 = \xi_i^2 = \xi_i^3$ for all $i \in \{1,2,3\}$. Solving for λ_1 in $\xi_1^1 = \xi_1^2$ and for λ_3 in $\xi_1^2 - \xi_1^3 = 0$ we obtain

$$\begin{aligned}
\lambda_1 &= \frac{1}{2\pi}\frac{\Gamma_3\,(L_{23} - L_{13})\,\sqrt{L_{12}} + \Gamma_1\sqrt{L_{23}}L_{13}}{L_{12}\sqrt{L_{23}}L_{13}}\Gamma_1, \\
\lambda_3 &= \frac{1}{2\pi}\frac{\Gamma_1\,(L_{12} + L_{13})\,\sqrt{L_{23}} + \Gamma_3 L_{13}\sqrt{L_{12}}}{\sqrt{L_{12}}L_{23}L_{13}}\Gamma_3.
\end{aligned}$$

Solving $\xi_3^1 - \xi_3^2 = 0$ implies that

$$\begin{aligned}
\lambda_2 &= \frac{1}{2\pi}\frac{\Gamma_3\,(L_{23} - L_{13})\,\sqrt{L_{12}}\left(\sqrt{L_{12}}\sqrt{1 + L_{23}} + \sqrt{L_{23}}\sqrt{1 + L_{12}}\right) + \Gamma_2 L_{23} L_{13}}{L_{12}L_{23}L_{13}}\Gamma_2 \\
&= \frac{1}{2\pi}\frac{\Gamma_3\,(L_{23} - L_{13})\,\sqrt{L_{12}} + \Gamma_2 L_{23}\sqrt{L_{13}}}{L_{23}L_{12}\sqrt{L_{13}}}\Gamma_2,
\end{aligned}$$

must hold. Using the notation of (5.5) we note that $X_r^2 = 0$ for all r, thus ξ_2^r is always zero. Therefore, the only equation left to be satisfied is

$$\begin{aligned}
\xi_3^1 - \xi_3^3 &= \frac{1}{2\pi}\left(\sqrt{1+L_{12}}\left(\frac{1}{L_{12}} - \frac{1}{L_{13}}\right) + \frac{(L_{13}-L_{12})\sqrt{1+L_{23}}}{\sqrt{L_{12}}\sqrt{L_{23}}L_{13}}\right)\Gamma_1 \\
&+ \left(\sqrt{1+L_{23}}\left(\frac{1}{L_{13}} - \frac{1}{L_{23}}\right) - \frac{\sqrt{1+L_{23}}\,(L_{23}-L_{13})\sqrt{1+L_{12}}}{{L_{23}}^{3/2}\sqrt{L_{12}}L_{13}}\right)\Gamma_3 \\
&+ \left(\frac{1}{L_{12}} - \frac{1}{L_{23}}\right)\Gamma_2\Big) = 0,
\end{aligned}$$

which requires

$$\begin{aligned}
\frac{\sqrt{L_{12}}\sqrt{1+L_{23}} + \sqrt{L_{23}}\sqrt{1+L_{12}}}{\sqrt{L_{12}}\sqrt{L_{23}}\sqrt{L_{13}}}\left(\frac{(L_{13}-L_{12})\,\Gamma_1}{\sqrt{L_{12}}} + \frac{(L_{23}-L_{13})\,\Gamma_3}{\sqrt{L_{23}}}\right) & \\
+\left(\frac{1}{L_{12}} - \frac{1}{L_{23}}\right)\Gamma_2 &= 0,
\end{aligned}$$

that is

$$\frac{1}{\sqrt{L_{12}}\sqrt{L_{23}}}\left(\frac{(L_{13}-L_{12})\,\Gamma_1}{\sqrt{L_{12}}} + \frac{(L_{23}-L_{13})\,\Gamma_3}{\sqrt{L_{23}}}\right) + \frac{(L_{23}-L_{12})\,\Gamma_2}{L_{12}L_{23}} = 0.$$

□

Corollary 6.3.2. *Any geodesic configuration with momentum value $\mu = J(X_e) = 0$ is a relative equilibrium.*

Proof. The coadjoint isotropy subgroup of $\mu = 0$ is the whole group $SL(2,\mathbb{R})$, thus $\dim SL(2,\mathbb{R})_\mu \cdot X_e = 3$, and by the Implicit Function Theorem $\dim J^{-1}(0) = 3$. Since $SL(2,\mathbb{R})_{\mu=0} \cdot X_e \subset J^{-1}(0)$ they must be equal. Thus, the group orbit of $\mu = 0$ is invariant under the dynamics and X_e is a relative equilibrium.

Another way of proving this is by considering the momentum value μ of (6.2). Recall, $x_1 = \sqrt{L_{12}}$ and $x_3 = \sqrt{L_{23}}$. If X_e is an isosceles geodesic relative equilibrium then $\Gamma_3 = \Gamma_1$, $L_{12} = L_{23}$ and $\mu = 0$ for $\Gamma_1 = -\frac{\Gamma_2}{2\sqrt{1+L_{23}}}$. Therefore

$$\begin{aligned}
\sqrt{L_{23}}\,(L_{13}-L_{12})\,\Gamma_1 + \sqrt{L_{13}}\,(L_{23}-L_{12})\,\Gamma_2 + \sqrt{L_{12}}\,(L_{23}-L_{13})\,\Gamma_3 &= \\
\left(\sqrt{L_{23}}\,(L_{13}-L_{23}) + \sqrt{L_{23}}\,(L_{23}-L_{13})\right)\Gamma_1 &= 0,
\end{aligned}$$

so X_e is a relative equilibrium. On other hand if X_e is not an isosceles geodesic relative equilibrium, then $\mu = 0$ for $\Gamma_1 = \frac{\Gamma_3\sqrt{L_{23}}}{\sqrt{L_{12}}}$ and $\Gamma_2 = -\frac{\Gamma_3\sqrt{L_{13}}}{\sqrt{L_{12}}}$. Thus

$$\begin{aligned}\sqrt{L_{23}}\left(L_{13}-L_{12}\right)\Gamma_1+\sqrt{L_{13}}\left(L_{23}-L_{12}\right)\Gamma_2+\sqrt{L_{12}}\left(L_{23}-L_{13}\right)\Gamma_3 &= \\ L_{23}\left(L_{13}-L_{12}\right)\Gamma_3-L_{13}\left(L_{23}-L_{12}\right)\Gamma_3+L_{12}\left(L_{23}-L_{13}\right)\Gamma_3 &= 0.\end{aligned}$$

□

Remark 6.3.3. Given any geodesic configuration X_e, there exists Γ_1, Γ_2 and Γ_3 such that X_e is a relative equilibrium. As shown in Figure 6.1, excluding $\Gamma_i = 0$, there exists a "plane" of relative equilibria in the coordinate space of Γ_1, Γ_2 and Γ_3, and this plane contains a line representing those Γ for which $\mu = J\left(X_e\right) = 0$.

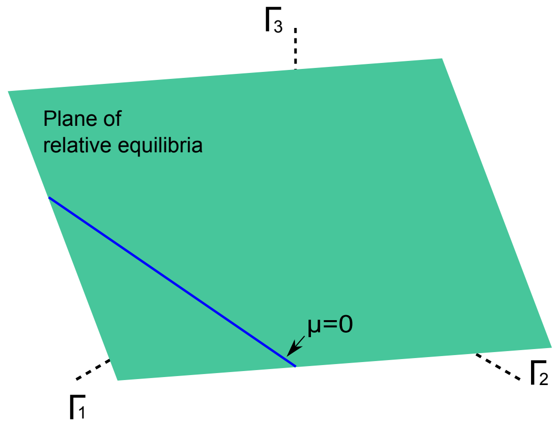


Figure 6.1: Graph of the relative equilibria of geodesic configurations for fixed hyperbolic distances.

As an equilateral configuration can not be on a geodesic, the only two options for three vortices on a geodesic are an isosceles or a configuration with three different lengths. The next two propositions give explicit conditions for the strength of vorticities of a relative equilibrium point.

Proposition 6.3.4 (Relative equilibria of an isosceles geodesic configuration). *Let $X_e = (X_1, X_2, X_3) \in \mathcal{M}$ be a configuration of point vortices on a geodesic of the hyperboloid with strength of vorticity $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$. Suppose that $\check{X}_2$ is between $\check{X}_1$ and $\check{X}_3$, then X_e is a isosceles geodesic relative equilibrium point if and only if $\Gamma_1 = \Gamma_3$.*

Proof. Let X_e be given by the set (6.2) with $x_1 = x_3$. Straightforward calculations show that for Equation (6.3) to be satisfied $\Gamma_1 = \Gamma_3$ must hold. Conversely, substituting L_{13} in terms of L_{12} and L_{23} in (6.3) with $\Gamma_1 = \Gamma_3$ leads to $L_{12} = L_{23}$. □

If the configuration is not isosceles, the next proposition follows immediately from Theorem 6.3.1.

Proposition 6.3.5. *Let $X_e = (X_1, X_2, X_3) \in \mathcal{M}$ be a configuration of point vortices on a geodesic of the hyperboloid with strength of vorticity $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$. Suppose that $\langle \check{X}_1, \check{X}_2 \rangle_{\mathcal{H}_2} \neq \langle \check{X}_2, \check{X}_3 \rangle_{\mathcal{H}_2}$ with $\check{X}_2$ between $\check{X}_1$ and $\check{X}_3$. Then X_e is a relative equilibrium point if*

$$\Gamma_2 = \frac{\Gamma_3 \left(-L_{23} + L_{13}\right) \sqrt{L_{12}} + \Gamma_1 \left(-L_{13} + L_{12}\right) \sqrt{L_{23}}}{\sqrt{L_{13}} \left(L_{23} - L_{12}\right)},$$

and Γ_1, Γ_3 are such that $\Gamma_2 \neq 0$. □

Chapter 7

Stability for three vortices

In this chapter we present a symplectic normal space for N point vortices in Theorem 7.1.1, and in Propositions 7.1.2 and 7.1.3, we provide a symplectic slice for the equilateral and geodesic three point vortex configurations. Finally, using the symplectic slice, in Section 7.2 we present $SL(2,\mathbb{R})$ and $SL(2,\mathbb{R})_{\mu}$-stability results for relative equilibria.

7.1 Symplectic normal space.

By dimension count, the symplectic normal space N_1 to a relative equilibrium point X_e is of dimension two, so the stability results are not as trivial as for $N = 2$.

From Equation (5.4) the Hessian of the augmented Hamiltonian for N point vortices in $\mathcal{H}_2$ is given by

$$\frac{\partial^2 \mathcal{H}_\xi}{\partial X_r^i \partial X_s^j}(X_e) = \tau_i \begin{cases} -\frac{\Gamma_r}{\pi}\tau_j \sum\limits_{p\neq r}\Gamma_p \frac{\langle X_r, X_p\rangle_{\mathcal{H}_2} X_p^j X_p^i}{L_{rp}^2} + \lambda_r \delta^{ij} & \text{if } r = s, \\ \frac{\Gamma_r\Gamma_s}{2\pi L_{rs}}\left(\delta^{ij} - 2\tau_j \frac{\langle X_r, X_s\rangle_{\mathcal{H}_2} X_r^j X_s^i}{L_{rs}}\right) & \text{if } r \neq s, \end{cases} \tag{7.1}$$

$$\delta^{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The next step is now to find a symplectic normal space N_1 at X_e and test definiteness of (7.1) on N_1 at X_e.

S. Pekarsky and J. E. Marsden introduce a symplectic slice for three point vortices on the sphere in [29]. Following their presentation we have constructed a symplectic slice for the system of N point vortices in the hyperboloid.

Theorem 7.1.1 (**Symplectic slice for N point vortices on the hyperboloid**). *Let $X_e = (X_1, ..., X_N) \in \mathcal{M}$ be a set of point vortices in relative equilibria with vorticities $\Gamma = (\Gamma_1, ..., \Gamma_N)$, where Γ_i is the vorticity corresponding to X_i for $i \in \{1, 2, ..., N\}$. Suppose that D_1 and D_2 are two independent vectors in $\mathbb{R}^3$, such that the plane through them does not contain any of the vortices. Then $N_1 = \langle \eta, \zeta \rangle$ is a symplectic normal space at X_e, where*

$$\left.\begin{aligned} \eta = & \left(a_1 D_1 \times_{\mathcal{H}_2} \check{X}_1, ..., a_N D_1 \times_{\mathcal{H}_2} \check{X}_N\right), \\ \zeta = & \left(b_1 D_2 \times_{\mathcal{H}_2} \check{X}_1, ..., b_N D_2 \times_{\mathcal{H}_2} \check{X}_N\right), \end{aligned}\right\} \in T_{X_e}\mathcal{M}$$

with $a = (a_1, ..., a_N)$, $b = (b_1, ..., b_N) \in \mathbb{R}^N$ defined by

$$\begin{aligned} \sum_i \Gamma_i a_i D_1 \times_{\mathcal{H}_2} \check{X}_i &= 0, \\ \sum_i \Gamma_i b_i D_2 \times_{\mathcal{H}_2} \check{X}_i &= 0. \end{aligned}$$

Proof. Given $\xi = (\xi_1, ..., \xi_N) \in T_{X_e}\mathcal{M}$, every component $\xi_i \in T_{X_i}\mathcal{H}_2$, and

$$DJ\left(X_e\right) \cdot \xi = \sum_i \Gamma_i \xi_i.$$

Hence, by the Implicit Function Theorem we have

$$\begin{aligned} \ker DJ\left(X_e\right) &= \{\xi \in T_{X_e}\mathcal{M} | \sum_i \Gamma_i \xi_i = 0\} \\ &= T_{X_e} J^{-1}\left(\mu\right), \end{aligned}$$

where $\mu = J\left(X_e\right)$ and $\dim\left(\ker dJ\left(X_e\right)\right) = 2N - 3$.

Since $\dim\left(\mathfrak{g}_\mu \cdot X_e\right) = 1$, a symplectic normal space N_1 at X_e would have to be of dimension $2N - 4$. Consider D_1 and D_2 two independent vectors such that the plane through them has no vortices. The vectors

$$\left.\begin{aligned} \eta = \quad & \left(a_1 D_1 \times_{\mathcal{H}_2} \check{X}_1, ..., a_N D_1 \times_{\mathcal{H}_2} \check{X}_N\right), \\ \zeta = \quad & \left(b_1 D_2 \times_{\mathcal{H}_2} \check{X}_1, ..., b_N D_2 \times_{\mathcal{H}_2} \check{X}_N\right), \end{aligned}\right\} \quad \in T_{X_e}\mathcal{M}$$

are hyperbolically perpendicular to D_1, D_2. Moreover, η and ζ are also hyperbolically perpendicular to $\check{X}_e$, thus the group orbit $\mathfrak{g}_\mu \cdot X_e$ is also perpendicular to $N_1 = \langle \eta, \zeta \rangle$.

N_1 appears as a candidate for a symplectic normal space at X_e. By the restricting N_1 to

$$\begin{aligned} \sum_i \Gamma_i a_i D_1 \times_{\mathcal{H}_2} \check{X}_i &= 0, \\ \sum_i \Gamma_i b_i D_2 \times_{\mathcal{H}_2} \check{X}_i &= 0, \end{aligned}$$

we guarantee that $N_1 \subset \ker dJ\,(X_e)$. As there are $N-2$ independent solutions for η_i, and $N-2$ independent solutions for ζ_i, the dimension of N_1 is $2N-4$ as required. $\square$

Proposition 7.1.2 (Symplectic slice for equilateral configurations of three point vortices). *Given three point vortices $X_e = (X_1, X_2, X_3) \in \mathcal{M}$ such that the hyperbolic distance between them is the same, a sympletic slice to X_e is generated by*

$$\eta := \begin{pmatrix} \frac{1}{\Gamma_1}\left(D_1 \times_{\mathcal{H}_2} \check{X}_1\right) \\ \frac{1}{\Gamma_2}\left(D_1 \times_{\mathcal{H}_2} \check{X}_2\right) \\ (0,0,0) \end{pmatrix} \quad \textit{and} \quad \zeta := \begin{pmatrix} (0,0,0) \\ \frac{1}{\Gamma_2}\left(D_2 \times_{\mathcal{H}_2} \check{X}_2\right) \\ \frac{1}{\Gamma_3}\left(D_2 \times_{\mathcal{H}_2} \check{X}_2\right) \end{pmatrix},$$

with $D_1 = \check{X}_1 + \check{X}_2$ and $D_2 = \check{X}_2 + \check{X}_3$.

Proof. Let $\hat{n} = \check{X}_1 \times \check{X}_2 + \check{X}_1 \times \check{X}_3 + \check{X}_2 \times \check{X}_3$. If $D_1 = \check{X}_1 + \check{X}_2$ and $D_2 = \check{X}_2 + \check{X}_3$, the vector $\hat{n} = D_1 \times D_2$ is clearly normal to the plane through D_1 and D_2.

Recall that the volume V obtained in a space with Euclidean geometry is the same as the one obtained with hyperbolic geometry, that is, $V = \check{X}_1 \cdot_{\mathcal{H}_2} \left(\check{X}_2 \times_{\mathcal{H}_2} \check{X}_3\right) = \check{X}_1 \cdot \left(\check{X}_2 \times \check{X}_3\right)$. Since equilateral configurations do not lie on a geodesic (Remark

6.1.2) $V \neq 0$. It is a straightforward calculation to show that $\hat{n}$ satisfies

$$\begin{aligned} \hat{n} \cdot \check{X}_1 = & \quad \hat{n} \cdot \check{X}_3 \\ = & \quad V \neq 0, \end{aligned}$$

and

$$\hat{n} \cdot \check{X}_2 = -V \neq 0.$$

This means that every vortex shares a component with $\hat{n}$. Hence none of the vortices is contained in the plane generated by D_1 and D_2.

From Theorem 7.1.1 we now define the symplectic normal space at X_e as the space generated by

$$\left.\begin{aligned} \eta = & \quad \left(a_1 D_1 \times_{\mathcal{H}_2} \check{X}_1, ..., a_N D_1 \times_{\mathcal{H}_2} \check{X}_N\right) \\ \zeta = & \quad \left(b_1 D_2 \times_{\mathcal{H}_2} \check{X}_1, ..., b_N D_2 \times_{\mathcal{H}_2} \check{X}_N\right) \end{aligned}\right\} \qquad \in T_{X_e}\mathcal{M},$$

restricted to

$$\begin{aligned} \sum_i \Gamma_i a_i D_1 \times_{\mathcal{H}_2} \check{X}_i &= 0, \\ \sum_i \Gamma_i b_i D_2 \times_{\mathcal{H}_2} \check{X}_i &= 0. \end{aligned}$$

Expanding this system of equations we get

$$\begin{aligned} \Gamma_1 a_1 \check{X}_2 \times_{\mathcal{H}_2} \check{X}_1 + \Gamma_2 a_2 \check{X}_1 \times_{\mathcal{H}_2} \check{X}_2 + \Gamma_3 a_3 \left(\check{X}_1 \times_{\mathcal{H}_2} \check{X}_3 + \check{X}_2 \times_{\mathcal{H}_2} \check{X}_3\right) &= 0, \\ \Gamma_1 b_1 \left(\check{X}_2 \times_{\mathcal{H}_2} \check{X}_1 + \check{X}_3 \times_{\mathcal{H}_2} \check{X}_1\right) + \Gamma_2 a_2 \check{X}_3 \times_{\mathcal{H}_2} \check{X}_2 + \Gamma_3 a_3 \check{X}_2 \times_{\mathcal{H}_2} \check{X}_3 &= 0. \end{aligned}$$

A set of independent solutions to this is $a_1 = \frac{1}{\Gamma_1}$, $a_2 = \frac{1}{\Gamma_2}$, $a_3 = 0$, $b_1 = 0$, $b_2 = \frac{1}{\Gamma_2}$ and $b_3 = \frac{1}{\Gamma_3}$. $\square$

As mentioned in Section 6.3, the dynamics of a set of vortices $\check{X}_1$, $\check{X}_2$, $\check{X}_3$ lying on a geodesic such that $a = \langle \check{X}_1, \check{X}_2 \rangle_{\mathcal{H}_2} = \langle \check{X}_2, \check{X}_3 \rangle_{\mathcal{H}_2}$ are equivalent to those of (6.2) with $x_1 = x_3$. Therefore the stability results obtained from the dynamics of that set are the same for any other geodesic configuration with two equal lengths. The following proposition presents a symplectic normal space for this particular set of vortices, which would be used in Theorem 7.2.4 to obtain some stability results for this type of configuration.

Proposition 7.1.3 (Symplectic slice for an isosceles geodesic configuration). *Let $X_e \in \mathcal{M}$ be the set (6.2) with $x_1 = x_3$ and strength of vorticity $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3 = \Gamma_1)$, a symplectic normal space N_1 to X_e is generated by*

$$\eta := \begin{pmatrix} \frac{1}{\Gamma_1}\left(D_1 \times_{\mathcal{H}_2} \check{X}_1\right) \\ \frac{1}{\Gamma_2}\left(D_1 \times_{\mathcal{H}_2} \check{X}_2\right) \\ (0,0,0) \end{pmatrix} \qquad \text{and} \qquad \zeta := \begin{pmatrix} \frac{1}{\Gamma_1}\left(D_2 \times_{\mathcal{H}_2} \check{X}_1\right) \\ \frac{2k}{\Gamma_2}\left(D_2 \times_{\mathcal{H}_2} \check{X}_2\right) \\ \frac{1}{\Gamma_3}\left(D_2 \times_{\mathcal{H}_2} \check{X}_3\right) \end{pmatrix},$$

where $D_1 = \check{X}_1 + \check{X}_2$, $D_2 = (0,1,0)$ and $k = -\sqrt{x_1^2+1}$.

Proof. Given $\Gamma_1 = \Gamma_3$ this set is a relative equilibrium point by Theorem 6.3.4. The construction of N_1 follows from Theorem 7.1.1. □

Proposition 7.1.4 (Symplectic slice for a geodesic configuration with three different lengths). *Let $X_e \in \mathcal{M}$ be the set (6.2) with strength of vorticity $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$ such that X_e is a relative equilibrium point. Then a symplectic normal space N_1 to X_e is generated by*

$$\eta := \begin{pmatrix} \frac{1}{\Gamma_1}\left(D_1 \times_{\mathcal{H}_2} \check{X}_1\right) \\ \frac{1}{\Gamma_2}\left(D_1 \times_{\mathcal{H}_2} \check{X}_2\right) \\ (0,0,0) \end{pmatrix} \qquad \text{and} \qquad \zeta := \begin{pmatrix} \frac{a x_3}{\Gamma_1}\left(D_2 \times_{\mathcal{H}_2} \check{X}_1\right) \\ \frac{x_1^2 - x_3^2}{\Gamma_2}\left(D_2 \times_{\mathcal{H}_2} \check{X}_2\right) \\ \frac{a x_1}{\Gamma_3}\left(D_2 \times_{\mathcal{H}_2} \check{X}_3\right) \end{pmatrix},$$

where $D_1 = \check{X}_1 + \check{X}_2$, $D_2 = (0,1,0)$ and $a = x_3\sqrt{1+x_1{}^2} - x_1\sqrt{1+x_3{}^2}$. □

7.2 $SL(2,\mathbb{R})$ and $SL(2,\mathbb{R})_\mu$ stability

Here we find G and G_μ stability conditions based on the results of the energy methods introduced before in Chapter 5. Recall from Section 5.4 that a relative equilibrium X_e is G'-stable if, under the dynamics, the G' group orbits of nearby points remain close to $G' \cdot X_e$.

7.2.1 Equilateral relative equilibria

Remark 6.2.2 shows that $J(X_e) \neq 0$ for equilateral configurations of point vortices on the hyperboloid. Thus, by Proposition 3.2.9, every $\mu = J(X_e)$ is regular for this type of relative equilibrium. The results of Proposition 5.4.5 for regular points together with Corollary 5.4.6 and Corollary 5.4.7 lead to the following theorem.

Theorem 7.2.1 ($SL(2,\mathbb{R})$ **and** $SL(2,\mathbb{R})_\mu$ **stability of equilateral configurations).** *An equilateral configuration* $X_e = (X_1, X_2, X_3) \in \mathcal{M}$ *with vorticity strength* $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$ *and momentum value* $\mu = J(X_e)$ *is* $SL(2,\mathbb{R})_\mu$*-stable if*

$$\sum_{i \neq j} \Gamma_i \Gamma_j > 0. \tag{7.2}$$

However, if

$$\sum_{i \neq j} \Gamma_i \Gamma_j < 0, \tag{7.3}$$

then X_e *is* $SL(2,\mathbb{R})$*-unstable.*

Proof. Let $L = \langle \check{X}_i, \check{X}_j \rangle^2_{\mathcal{H}_2} - 1$ for any $i, j \in \{1,2,3\}$, and N_1 the symplectic normal space at X_e derived in Theorem 7.1.2. The Hessian of H_ξ (7.1) restricted to N_1 is given by

$$\left.\frac{\partial^2 \mathcal{H}_\xi}{\partial X_r^i \partial X_s^j}\right|_{N_1} (X_e) = \frac{V^2}{\pi L^2} \begin{pmatrix} -\frac{\Gamma_3}{\Gamma_1} - \frac{\Gamma_3}{\Gamma_2} & 1 \\ 1 & -\frac{\Gamma_1}{\Gamma_2} - \frac{\Gamma_1}{\Gamma_3} \end{pmatrix}.$$

Thus,

$$\begin{aligned} \det \left(\left.\frac{\partial^2 \mathcal{H}_\xi}{\partial X_r^i \partial X_s^j}\right|_{N_1} (X_e) \right) &= \frac{V^2}{\pi L^2} \left(\frac{\Gamma_3\Gamma_2 + \Gamma_1\Gamma_2 + \Gamma_1\Gamma_3}{\Gamma_2^2} \right) \\ &= \frac{V^2}{\pi L^2 \Gamma_2^2} \sum_{i \neq j} \Gamma_i \Gamma_j, \end{aligned}$$

where $V^2 = \left(\check{X}_1 \cdot_{\mathcal{H}_2} \left(\check{X}_2 \times_{\mathcal{H}_2} \check{X}_3\right)\right)^2 \neq 0$. The Hessian of a function of two variables is definite if its determinant is greater than zero, hence $SL(2,\mathbb{R})$ stability is obtained from Corollary 5.4.6.

Since $\mu \neq 0$, μ is either elliptic, parabolic or hyperbolic, with determinant

$$\det \mu = 2k \sum_{i \neq j} \Gamma_i \Gamma_j + \sum_i \Gamma_i^2. \tag{7.4}$$

Hence, for μ parabolic or hyperbolic only (7.3) holds, implying that x_e is $SL\left(2, \mathbb{R}\right)_\mu$ unstable by Proposition 5.4.2. If μ is elliptic, the $SL\left(2, \mathbb{R}\right)_\mu$ stability results follow from Corollary 5.4.7. □

Corollary 7.2.2. *Every hyperbolic and parabolic equilateral relative equilibrium is $SL\left(2, \mathbb{R}\right)$ and $SL\left(2, \mathbb{R}\right)_\mu$ unstable.* □

It is noteworthy that the G-stability conditions for three equilateral vortices on the hyperboloid coincide with those for the system on the sphere [22, 29] and on the plane [1].

On the plane, a relative equilibrium with $\sum_{i \neq j} \Gamma_i \Gamma_j = 0$ is marginally stable [1]. Meanwhile, for the system on the sphere, Marsden, Pekarsky and Shkoller [19] perfomed numerical integrations using Matlab ODE45 package. They observed changes of the stability for $\sum_{i \neq j} \Gamma_i \Gamma_j = 0$. They also conjecture that a Hamiltonian bifurcation occurs, as has also been mentioned in the references [22, 29].

For equilateral configurations on the hyperboloid, we have performed numerical integrations in Maple which suggest that a bifurcation occurs at $\sum_{i \neq j} \Gamma_i \Gamma_j = 0$. This is actually the equation of a cone as shown in Figure 7.1, where the stability depends only on the choice of Γ's. The points with Γ outside this cone, that is the points for which $\sum_{i \neq j} \Gamma_i \Gamma_j < 0$, are $SL\left(2, \mathbb{R}\right)$-unstable. On other hand, any equilateral configuration with Γ inside the cone is a $SL\left(2, \mathbb{R}\right)_\mu$-stable relative equilibrium, and follows the trajectory of an ellipse rotating around its momentum value.

7.2.2 Geodesic relative equilibria

The type of $\mu = J\left(X_e\right)$ is decisive for determining the stability of geodesic relative equilibrium. For instance, if μ is elliptic, that is $\det \mu > 0$, then $SL\left(2, \mathbb{R}\right)_\mu$- stability is automatically obtained for X_e a $SL\left(2, \mathbb{R}\right)$-stable relative equilibrium. Moreover, this type does not change under hyperbolic rotations. Therefore, calculating the

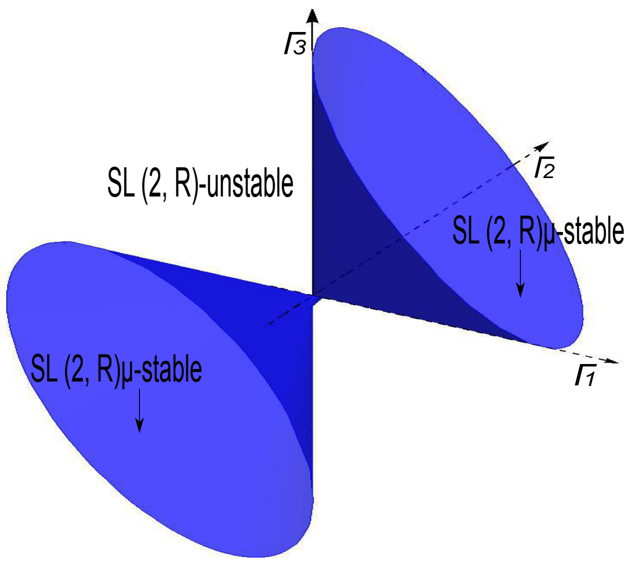


Figure 7.1: Graph of the cone $\sum_{i\neq j}^{3} \Gamma_i \Gamma_j = 0$.

possible types of μ, using (6.2), we can determine what are the type of μ observed for geodesic configurations.

Lemma 7.2.3. *A geodesic relative equilibrium X_e has a momentum value $\mu = J(X_e)$ which is either elliptic or zero.*

Proof. Using the notation of Section 6.3 we consider the set of vortices (6.2) with $x_1 = x_3$. The determinant of the momentum value $\mu = J(X_1, X_2, X_3)$ is

$$\det \mu = \left(2\Gamma_3\sqrt{1+x_1^2} + \Gamma_2\right)^2, \tag{7.5}$$

therefore $\det \mu > 0$ for all $a \neq \frac{\Gamma_2}{2\Gamma_3}$, otherwise $\mu = 0$. Suppose now that $x_1 \neq x_3$ and $\Gamma_1 \neq \Gamma_3$, the determinant of the momentum value is

$$\begin{aligned}\det \mu \;\; = \;\; & 8\frac{(\Gamma_1 x_1 - \Gamma_3 x_3)^2}{k^2}\left(\left(\left(\frac{1}{4}+x_3^2\right)x_1^2 + \frac{1}{4}x_3^2\right)x_3 x_1\sqrt{1+x_3^2}\sqrt{1+x_1^2} + \right. \\ & \left. + \left(\frac{1}{8}+\frac{3}{4}x_3^2 + x_3^4\right)x_1^4 + \left(\frac{3}{4}x_3^4+\frac{3}{8}x_3^2\right)x_1^2 + \frac{1}{8}x_3^4\right),\end{aligned}$$

where $k = (x_1 - x_3)(x_1 + x_3)\left(x_3\sqrt{1+x_1^2} + x_1\sqrt{1+x_3^2}\right)$. Recall that $x_1 > 0$ and $x_3 > 0$, hence μ is elliptic provided $\Gamma_1 \neq \Gamma_3 x_3/x_1$, otherwise $\mu = 0$. □

Formal stability was introduced in Theorem 5.4.4, a relative equilibrium X_e is formally stable if $d^2H_\xi(X_e)$ is definite when restricted to a symplectic slice at X_e. As the coadjoint isotropy subgroup of $\mu = 0$ is the whole $SL(2,\mathbb{R})$, the dimension of $\mathfrak{g}_\mu \cdot X_e = 3$. Thus, by the Implicit Function Theorem 3.1.9, the dimension of a symplectic slice at X_e must be zero. This implies that $\mu = 0$ is formally stable. Furthermore, a configuration with $\mu = 0$ is trivially leafwise stable and, as discussed in Section 5.4.2, G-stable if the angular velocity ξ points into the null-cone $\mathfrak{C}$. Straightforward calculations show that the angular velocity of (6.2) with momentum value $\mu = 0$ is elliptic, hence $\mu = 0$ is G-stable.

Theorem 7.2.4 ($SL(2,\mathbb{R})$ **and** $SL(2,\mathbb{R})_\mu$**-stability of isosceles geodesic relative equilibria).** *Let* $X_e = (X_1, X_2, X_3) \in \mathcal{M}$ *be a configuration of point vortices lying on a geodesic of the hyperboloid with strength of vorticity* $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3 = \Gamma_1)$. *Suppose* $\mu = J(X_e)$ *and* $\langle \check{X}_1, \check{X}_2\rangle_{\mathcal{H}_2} = \langle \check{X}_2, \check{X}_3\rangle_{\mathcal{H}_2} = a \neq \frac{\Gamma_2}{2\Gamma_1}$ *then* μ *is elliptic. Furthermore, let*

$$A(\Gamma_1,\Gamma_2,a) = \frac{1}{\Gamma_1}\Big(512A_1(\Gamma_1,\Gamma_2,a) + A_2(\Gamma_1,\Gamma_2,a)\Big), \tag{7.6}$$

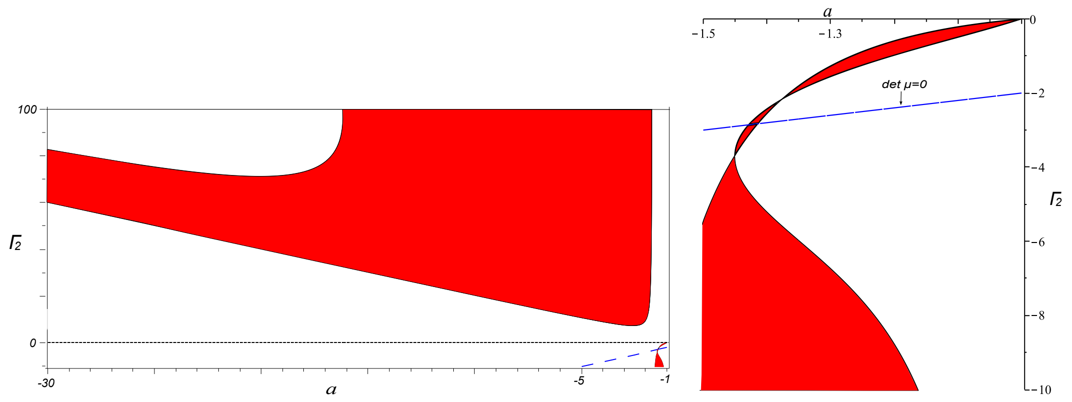
with

$$\begin{aligned} A_1(\Gamma_1,\Gamma_2,a) \;=\;& \Gamma_1^2 a^9 - 2\Gamma_1 a^8\Big(\Gamma_1 - \frac{\Gamma_2}{4}\Big) + a^7\Big(-\frac{5}{4}\Gamma_1^2 + 2\Gamma_1\Gamma_2\Big) \\ & + 2a^6\Big(\Gamma_1 + \frac{\Gamma_2}{4}\Big)\Big(\Gamma_1 - \Gamma_2\Big) + \frac{\Gamma_1 a^5}{4}(\Gamma_1 - 8\Gamma_2) \\ & + \frac{\Gamma_2 a^4}{16}(8\Gamma_2 + \Gamma_1) - \frac{\Gamma_1\Gamma_2}{32}\left(a^2 - \frac{1}{2}\right), \end{aligned}$$

and

$$A_2(\Gamma_1,\Gamma_2,a) = \Gamma_1 a^5 + \frac{1}{2}\Gamma_2 a^4 - \frac{5}{4}\Gamma_1 a^3 - \frac{11}{8}\Gamma_2 a^2 + \frac{1}{4}\Gamma_1 a - \frac{1}{8}\Gamma_2.$$

If $A(\Gamma_1,\Gamma_2,a) > 0$ *then* X_e *is* $SL(2,\mathbb{R})_\mu$*-stable. Conversely if* $A(\Gamma_1,\Gamma_2,a) < 0$ *then* X_e *is* $SL(2,\mathbb{R})$*-unstable.*



(a) (b)

Figure 7.2: $SL(2,\mathbb{R})$-stability for $\Gamma_1 = \Gamma_3 = 1$. The white region represents $SL(2,\mathbb{R})_\mu$-stable relative equilibria, and the red region represents $SL(2,\mathbb{R})$- unstable relative equilibria.

Proof. Consider the set of vortices (6.2) with $x_1 = x_3$, the determinant of the momentum value $\mu = J(X_1, X_2, X_3)$ is

$$\det \mu = 4\left(-\frac{\Gamma_2}{2} + \Gamma_1 a\right)^2, \tag{7.7}$$

therefore $\det \mu > 0$ for all $a \neq \frac{\Gamma_2}{2\Gamma_3}$. Thus every μ satisfying this condition is regular and elliptic with compact isotropy subgroup $SO(2,\mathbb{R})$. The stability results are due to corollaries 5.4.6 and 5.4.7, with (7.6) obtained by testing definiteness of H_ξ in the symplectic normal space N_1 to X_e given in Proposition 7.1.3. □

The complexity of (7.6) makes it hard to realise for which values of a, Γ_1 and Γ_2 we obtain $SL(2,\mathbb{R})_\mu$ and $SL(2,\mathbb{R})$-stability. In Figure 7.2 the $SL(2,\mathbb{R})_\mu$ and $SL(2,\mathbb{R})$-stability regions for $\Gamma_1 = \Gamma_3 = 1$ are plotted. Figure 7.2(b) shows in more detail the stability conditions when the vortices are close to each other, that is for small values of a. The dashed blue line represents $a = \frac{\Gamma_2}{2\Gamma_3}$, in which case $\mu = 0$ is not regular, so the stability results presented before can not be applied for that case. Despite this, we can conclude that X_e is $SL(2,\mathbb{R})$-stable as the angular velocity ξ is elliptic.

Lemma 7.2.3 shows that the momentum value μ is either elliptic or zero. By the analysis at the start of this section, $SL(2,\mathbb{R})$-stability is obtained for $\mu = 0$. For the elliptic case, having μ a compact isotropy subgroup, $SL(2,\mathbb{R})$- stability is obtained by definiteness of $\partial^2 \mathcal{H}_\xi|_{N_1}(X_e)$. However, for a geodesic configuration with three different lengths the computation of this Hessian is rather involved and further analysis is required to give a conclusion on the stability criteria of a relative equilibrium X_e.

Figure 7.3 shows a plot of the stability regions for the specific value of $\Gamma_2 = 1$. In that figure, the region under the surface represents the relative equilibrium points that are $SL(2,\mathbb{R})$-unstable. By a rescaling on time, any other relative equilibrium X_e of this type would have a similar stability conditions.

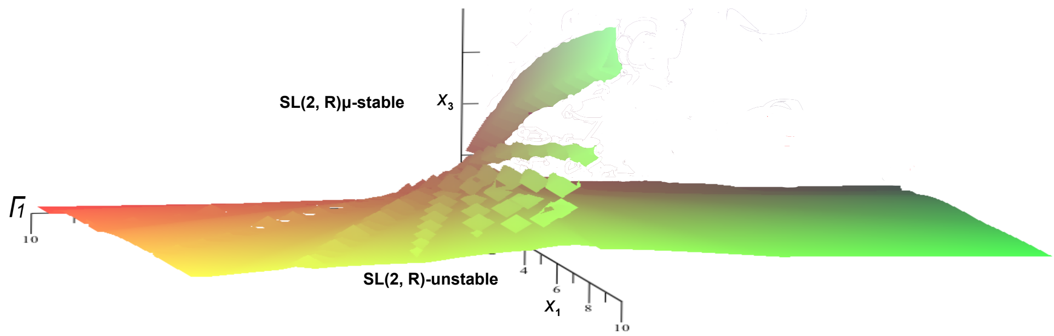


Figure 7.3: Stability graph for a geodesic configuration with three different lengths and $\Gamma_2 = 1$. The region under the surface is $SL(2,\mathbb{R})$-unstable.

Chapter 8

Conclusions

Given $X = (X_1, X_2, ..., X_N)$ a set of N point vortices on the hyperboloid with strength of vorticity $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_N)$ and momentum value $\mu = J(X) \neq 0$, we have identified trajectory of X based only on $\det \mu$. Here we provide a brief summary of our main results for $N = 2, 3$.

In Chapter 5 we have shown that every two point vortex configuration $X_e = (X_1, X_2)$ is a $SL(2, \mathbb{R})$-stable relative equilibrium. We classified the trajectory of X_e in terms of $\gamma = \frac{\Gamma_1}{\Gamma_2}$ and $c = d\left(\check{X}_1, \check{X}_2\right)$ in Section 5.2. In Corollary 5.4.9 we have additionally proved that if either $\gamma > -e^{-c}$ or $\gamma < -e^{c}$ then X_e is $SL(2, \mathbb{R})_\mu$- stable in $\mathcal{M}$. To our knowledge, none of these results have been derived before except for a discussion of the vortex dipole $\gamma = -1$ in [14].

Although the classification of relative equilibria of three point vortices given in Theorem 6.1.1 can be deduced from formulae (17) – (19) in [11], this classification has not been stated before. Using the geometric approach of symmetric Hamiltonian systems we have proved that a three point vortex relative equilibrium is either an equilateral or a geodesic configuration. Equally important, we have shown that every three-vortex equilateral configuration is a relative equilibrium. Furthermore, we have also given the explicit conditions of Γ_i for a geodesic configuration to be a relative equilibrium.

The stability criteria for three point vortices found in Chapter 7 has not been derived before. Remarkably, we have found that an equilateral three vortex configuration has the exact same stability conditions of those for the system on the plane and on the sphere. Finally we have proved that the momentum value of a geodesic relative equilibrium is either zero or elliptic, and provided some graphs showing the regions of stability.

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