The local Langlands correspondence for inner forms of SL_n

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2013

MIMS EPrint: **2013.43**

ISSN 1749-9097

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Warwick, May 2013

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Main example is $\mathbb{Q}_p$ in which each element looks like

$$x = \sum_{n \geq N} a_n p^n$$

The valuation is $\mathrm{val}_F(x) = N$ and the norm is $|x|_F = p^{-N}$.

From now on, F will be a local non-archimedean field.
In this talk: G will be $\mathrm{GL}_n(F)$, $\mathrm{SL}_n(F)$ or one of their inner forms $\mathrm{GL}_m(D)$, $\mathrm{SL}_m(D)$.
We will need the *Langlands dual group* $G^\vee$.
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This is a complex connected reductive group.
The dual of $\mathrm{GL}_n(F)$ is the complex reductive group $\mathrm{GL}_n(C)$.
The dual of $\mathrm{SL}_n(F)$ is the complex reductive group $\mathrm{PGL}_n(\mathbb{C})$.

We need the Weil group

$$\mathbf{W}_F = I_F \rtimes < \mathrm{Frob} >$$

Let $\overline{F}$ be a separable closure of F. The inertia group I_F fixes $F_\infty \subset \overline{F}$ and the Frobenius Frob generates the Galois group (a cyclic group) of each finite unramified extension $F_n \subset F_\infty$.

The Weil group $\mathbf{W}_F$ is a totally disconnected locally compact group. The inertia group I_F is a compact open subgroup, the quotient $\mathbf{W}_F/I_F = \mathbb{Z}$ in the discrete topology. The Weil group is a modified version of the absolute Galois group.

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but with a change of topology.
A Langlands parameter is a conjugacy class of morphisms

$$\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee$$

which are semisimple.

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$$\Phi(\mathrm{SL}_n(F)) := \mathrm{Hom}_{ss}(\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}), \mathrm{PGL}_n(\mathbb{C}))/\mathrm{PGL}_n(\mathbb{C})$$

Definition

A representation (π, V) on a complex vector space V is smooth if, for each $v \in V$, there exists an open subgroup of G which fixes v. Denote by $\mathbf{Irr}(G)$ *the set of equivalence classes of irreducible smooth representations of G.*

The local Langlands correspondence for $G = \mathrm{GL}_n(F)$:

$$\mathbf{Irr}(G) \simeq \Phi(G)$$

unique subject to several conditions
[Laumon-Rapoport-Stuhler; Harris-Taylor, Henniart].

In general the Langlands parameters do not suffice. It is necessary to refine the parameters, via irreducible representations of a certain finite group $\mathcal{S}$.

When the right choice is made for $\mathcal{S}$, it is possible to parametrize the union of the **Irr**-sets over the inner forms of $\mathrm{SL}_n(F)$.

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When the right choice is made for $\mathcal{S}$, it is possible to parametrize the union of the **Irr**-sets over the inner forms of $\mathrm{SL}_n(F)$.

We need to define *inner form* and the $\mathcal{S}$-group.

An associative algebra over a field is a division algebra if and only if it has a multiplicative identity element $1 \neq 0$ and every non-zero element a has a multiplicative inverse, i.e. an element x with $ax = xa = 1$.

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Let F be a local field, and let D be a division algebra with centre F, of dimension $\dim_F(D) = d^2$. The F-group $\mathrm{GL}_m(D)$ is an *inner form* of $\mathrm{GL}_{md}(F)$ and they all arise this way.

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The inner forms of $\mathrm{GL}_n(F)$ share the same dual group $\mathrm{GL}_n(\mathbb{C})$
Proofs of LLC for $\mathrm{GL}_m(D)$ are to be found in [ABPS1] and [HS]

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We write

$$\mathrm{GL}_m(D)^\sharp := \{g \in \mathrm{GL}_m(D) : Nrd(g) = 1\}$$

This is the derived group of $\mathrm{GL}_m(D)$. It is an inner form of $\mathrm{SL}_{md}(F)$ and every inner form of $\mathrm{SL}_n(F)$ arises in this way.
The inner forms of $\mathrm{SL}_n(F)$ share the same dual group $\mathrm{PGL}_n(\mathbb{C})$.

Given a parameter

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$$S_{\phi^\sharp} = \pi_0\, Z_{\mathrm{PGL}_n(\mathbb{C})}(\mathrm{im}\,\phi^\sharp)$$

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$$\mathcal{S}_{\phi^\sharp} = \pi_0 \, Z_{\mathrm{SL}_n(\mathbb{C})}(\mathrm{im}\, \phi^\sharp)$$

Short exact sequence

$$1 \to \mathcal{Z}_{\phi^\sharp} \to \mathcal{S}_{\phi^\sharp} \to S_{\phi^\sharp} \to 1$$

The irreducible reprsentations attached to *all the inner forms* of $\mathrm{SL}_n(F)$ will hopefully be parametrized by $\mathbf{Irr}(\mathcal{S}_{\phi^\sharp})$.

This kind of idea can be traced to Arthur, Vogan and Lusztig.

Define

$$\Phi^e(\mathrm{inn}\,\mathrm{SL}_n(F)) = \{(\phi^\sharp, \rho) : \phi^\sharp \in \Phi(\mathrm{SL}_n(F)), \rho \in \mathbf{Irr}(\mathcal{S}_{\phi^\sharp})\}$$

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Theorem (Hiraga-Saito; Aubert-Baum-Plymen-Solleveld)

Let F be a non-archimedean local field. There exists a bijection from

$$\Phi^e(\mathrm{inn}\ SL_n(F))$$

to

$$\{(G^\sharp, \pi) : G^\sharp \text{ standard inner form of } \mathrm{SL}_n(F), \pi \in \mathbf{Irr}(G^\sharp)\}$$

sending

$$(\phi^\sharp, \rho) \mapsto (G^\sharp_\rho, \pi(\phi^\sharp, \rho))$$

with several crucial properties.

Given $(\phi^\sharp, \rho)$. Given $\rho \in \mathbf{Irr}(\mathcal{S}_{\phi^\sharp})$ evaluate ρ on $\exp(2\pi i/n) \in \mathcal{Z}_{\phi^\sharp}$ to get a complex number $\exp(2\pi i/d)$. Let L/F be the unramified extension of degree d. Take the Frobenius automorphism $\sigma = \mathrm{Frob} \in \mathrm{Gal}(L/F)$ and construct the cyclic algebra

$$(L/F, \sigma, \varpi_F) := L[x]_\sigma/(x^d - \varpi_F).$$

in which $ux = x\sigma(u)$ for all $u \in L$, and $x^d = \varpi_F$. The cyclic algebra has dimension d^2 over F with basis

$$u_i x^i \qquad 1 \leq i, j \leq d$$

where $u_1, \ldots, u_d$ is an F-basis of L.

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Let

$$D = (L/F, \sigma, \varpi_F)$$

and

$$G_\rho^\sharp = \mathrm{GL}_m(D)^\sharp$$

with $n = md$.

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- If $\rho(\mathcal{Z}_{\phi^\sharp}) = 1$ then $d = 1, L = F, m = n, D = F, G^\sharp_\rho = \mathrm{SL}_n(F)$.
- If $\rho(\mathcal{Z}_{\phi^\sharp})$ is cyclic of order n then $d = n, m = 1, G^\sharp_\rho = \mathrm{SL}_1(D)$.

The central character of ρ determines the inner form $G^\sharp_\rho$ and will sometimes be denoted $\chi_{G^\sharp}$.

Now lift the parameter $\phi^\sharp$ from $\mathrm{PGL}_n(\mathbb{C})$ to $\mathrm{GL}_n(\mathbb{C})$:

$$\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$$

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Now ϕ is *relevant* for $\mathrm{GL}_m(D)$: the minimal Levi subgroup of $\mathrm{GL}_n(\mathbb{C})$ containing $\mathrm{im}\,\phi$ is of the form $\mathrm{GL}_{n_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{n_k}(\mathbb{C})$ with $d|n_j$. Recall that $n = md$.

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We obtain a singleton packet Π_ϕ for $\mathrm{GL}_m(D)$. Recall that the dual group of $\mathrm{GL}_m(D)$ is $\mathrm{GL}_n(\mathbb{C})$.

Our fundamental short exact sequence admits a different expression:

$$1 \to \mathcal{Z}_{\phi^\sharp} \to \mathcal{S}_{\phi^\sharp} \to X^G(\Pi_\phi(G)) \to 1$$

where

$$X^G(\Pi_\phi(G)) = \{\gamma \in \mathbf{Irr}(G/G^\sharp) : \Pi_\phi(G) \otimes \gamma \simeq \Pi_\phi(G)\}$$

The characters γ create intertwining operators I_γ. These intertwining operators span the commuting algebra $\mathrm{End}_{G^\sharp}(\Pi_\phi(G))$.
The map $\gamma \mapsto I_\gamma$ is a *projective representation* of $X^G(\Pi_\phi(G))$. The cocycle is trivial if and only if G is the split group $\mathrm{GL}_n(F)$.

Lemma (Hiraga–Saito, Lemma 12.5)

Denote the homomorphism $\mathcal{S}_{\phi^\sharp} \to X^G(\Pi_\phi(G)$ *by* α. *There exists a homomorphism*

$$\Lambda : \mathcal{S}_{\phi^\sharp} \to \mathrm{End}(\Pi_\phi)$$

such that

$$\Lambda(s) \in \mathbb{C}^\times \cdot I_{\alpha(s)}, \qquad s \in \mathcal{S}_{\phi^\sharp}$$

$$\Lambda(z) = \chi_{G^\sharp}(z) \cdot Id, \qquad z \in \mathcal{Z}_{\phi^\sharp}$$

This determines a representation of $\mathcal{S}_{\phi^\sharp} \times G^\sharp$ on V (the G-module on which $\Pi(\phi)$ acts). This, in turn, determines the decomposition

$$V \simeq \bigoplus \rho \otimes \pi(\phi^\sharp, \rho) \tag{1}$$

the sum taken over those $\rho \in \mathbf{Irr}(\mathcal{S}_{\phi^\sharp})$ with central character $\chi_{G^\sharp}$.
Eqn.(1) defines $\pi(\phi^\sharp, \rho)$. The multiplicity of $\pi(\phi^\sharp, \rho)$ in the module V is $\dim(\rho)$.
END OF PART 1

PART 2. Joint work with Sergio Mendes.

Let F be a local function field of characteristic 2, so that $F = \mathbb{F}_q((\varpi_F))$ with $q = 2^f$.

We have the following canonical homomorphism:

$$\mathbf{W}_F \to \mathbf{W}_F^{ab} \simeq F^\times \to F^\times/F^{\times 2}.$$

For these local function fields, we have, based on Artin-Schreier theory,

$$F^\times/F^{\times 2} \simeq \prod \mathbb{Z}/2\mathbb{Z}$$

the product over countably many copies of $\mathbb{Z}/2\mathbb{Z}$. Using the countable axiom of choice, we choose two copies of $\mathbb{Z}/2\mathbb{Z}$. This creates a homomorphism

$$\mathbf{W}_F \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

There are countably many such maps.

Following [Weil, *Exercises dyadiques*], denote by α, β, γ the images in $\mathrm{PSL}_2(\mathbb{C})$ of the elements

$$z_\alpha = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad z_\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z_\gamma = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

in $\mathrm{SL}_2(\mathbb{C})$. Denote by J the group generated by α, β, γ:

$$J := \{\epsilon, \alpha, \beta, \gamma\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

The group J is unique up to conjugacy in $\mathrm{PSL}_2(\mathbb{C})$.

The pre-image of J in $\mathrm{SL}_2(\mathbb{C})$ is the group $\{\pm 1, \pm z_\alpha, \pm z_\beta, \pm z_\gamma\}$ and is isomorphic to the group U_8 of unit quaternions $\{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$.
Define

$$\phi^\sharp : \mathbf{W}_F \to J \hookrightarrow \mathrm{PSL}_2(\mathbb{C})$$

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Note that

$$\mathcal{S}_{\phi^\sharp} := Z_{SL_2(\mathbb{C})}(\mathrm{im}\, \phi^\sharp) = U_8$$

as it contains the subgroup $\{I, -I\}$ of $\mathrm{SL}_2(\mathbb{C})$.
We have the short exact sequence

$$1 \to \mathcal{Z}_{\phi^\sharp} \to \mathcal{S}_{\phi^\sharp} \to S_{\phi^\sharp} \to 1$$

with $\mathcal{Z}_{\phi^\sharp} = \mathbb{Z}/2\mathbb{Z}$, $\mathcal{S}_{\phi^\sharp} \simeq U_8$, $S_{\phi^\sharp} \simeq J$.

The $\mathcal{S}$-group has order 8 and admits 4 characters $\rho_1, \ldots, \rho_4$ of order 1 and one irreducible representation ρ_5 of degree 2:

$$1^2 + 1^2 + 1^2 + 1^2 + 2^2 = 8$$

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$$1^2 + 1^2 + 1^2 + 1^2 + 2^2 = 8$$

- The central character of ρ_j when $1 \leq j \leq 4$ is trivial, and so in this case $G_\rho^\sharp = \mathrm{SL}_2(F)$.
- The central character of ρ_5 is of order 2 and we have $G_\rho^\sharp = \mathrm{SL}_1(D)$.

In the local Langlands correspondence for the inner forms of $\mathrm{SL}_2(F)$, the parameter

$$\phi^\sharp : \mathbf{W}_F \to \mathrm{PGL}_2(\mathbb{C})$$

creates an L-packet for $\mathrm{SL}_2(F)$ with 4 elements, and a singleton packet for the inner form $\mathrm{SL}_1(D)$:

$$\{\pi(\phi^\sharp, \rho_1), \pi(\phi^\sharp, \rho_2), \pi(\phi^\sharp, \rho_3), \pi(\phi^\sharp, \rho_4)\}$$

$$\{\pi(\phi^\sharp, \rho_5)\}$$

For the field $F = \mathbb{F}_q((\varpi_F))$ with $q = 2^f$, this phenomenon happens infinitely often.
Informally: the parameter $\phi^\sharp$ creates a "big packet" and the elements $\rho \in \mathbf{Irr}(\mathcal{S}_{\phi^\sharp})$ partition this big packet into two smaller packets, one for $\mathrm{SL}_2(F)$ and one for $\mathrm{SL}_1(D)$.

For $\mathrm{SL}_2(F)$ this is a *supercuspidal* L-packet.
Each parameter $\phi^\sharp : \mathbf{W}_F \to \mathrm{PGL}_2(\mathbb{C})$ lifts to a representation

$$\phi : \mathbf{W}_F \to \mathrm{GL}_2(\mathbb{C}).$$

This representation is *triply imprimitive*, as in [BH,p. 255].
Let $\mathfrak{T}(\phi)$ be the group of characters χ of $\mathbf{W}_F$ such that $\chi \otimes \phi \simeq \phi$. Then $\mathfrak{T}(\phi)$ is non-cyclic of order 4, as in [BH,p. 257].

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The characters in $\mathfrak{T}(\phi)$ can be distinguished, to some extent, see [BH, p.257].

Formal degrees via gamma ratios

Let (π, V) be a discrete series representation of G and let μ be a Haar measure on G. There exists a positive real constant $deg_\mu(\pi)$ such that for all $v \in V$

$$deg_\mu(\pi) \cdot \int_G |(v, \pi(g)v)|^2 d\mu(g) = (v, v)^2$$

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We will normalize μ so that the formal degree of the Steinberg is 1. This is the *Euler–Poincaré measure*.

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The formal degree, w.r.t. this measure, will be denoted $Deg(\pi)$.

Formula for the formal degree [Mark Reeder]: with $\varphi = \phi^\sharp$, we have

$$Deg(\pi(\varphi,\rho)) = \frac{\dim \rho}{|S_\varphi|} \cdot \left| \frac{\gamma(\varphi)}{\gamma(\varphi_0)} \right|$$

where

$$\gamma(\varphi) = \frac{L(\mathrm{Ad}\,\varphi, 1) \cdot \epsilon(\mathrm{Ad}\,\varphi)}{L(\mathrm{Ad}\,\varphi, 0)}.$$

The Artin-Deligne L-function $L(\mathrm{Ad}\,\varphi, s)$

$$L(\mathrm{Ad}\,\varphi, s) = \det(1 - q^{-s} F_q | \mathfrak{h}^u)^{-1}$$

where

$$F_q = \mathrm{Ad}\,\varphi \left(\mathrm{Frob} \times \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} \right)$$

and $\mathfrak{h}$ is the be the subspace of $\mathfrak{g}$ fixed by the inertia subgroup I_F under $\mathrm{Ad}\,\varphi$ and $\mathfrak{h}^u$ is the fixed set of $\mathrm{Ad}\,\varphi(u)$.

$\mathrm{Ad}\,\varphi_0$ is given by

$$\mathrm{Ad}\,\varphi_0 : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C}) \to Aut(\mathfrak{s}l_2(\mathbb{C}))$$

and

$$\gamma(\varphi_0) = \frac{q}{1+q^{-1}}$$

where $q = 2^f$. We have

$$L(\mathrm{Ad}\,\varphi, s) = \frac{1}{1+q^{-s}}$$

and so we have

$$\gamma(\varphi) = \frac{2}{1+q^{-1}} \cdot \varepsilon(\varphi)$$

where

$$\varepsilon(\varphi) = \pm q^{\alpha(\varphi)/2}.$$

We have

$$\left|\frac{\gamma(\varphi)}{\gamma(\varphi_0)}\right| = \frac{2}{q}\cdot\varepsilon(\varphi). \tag{2}$$

To compute the epsilon number $\epsilon(\mathrm{Ad}\,\varphi)$.

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Recall the data:

$$\mathrm{Ad} \circ \varphi : \mathrm{Gal}(L/F) \simeq J \hookrightarrow \mathrm{PSL}_2(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{g})$$

where L/F is a biquadratic extension.

We have

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Now $\alpha(\varphi)$ is the Weil-Deligne version of the Artin conductor

$$\alpha(\varphi) = \sum_{i\geq 0}\frac{\dim(\mathfrak{g}/\mathfrak{g}^{J_i})}{[J_0 : J_i]}$$

The formal degrees of the elements in the supercuspidal packets for $\mathrm{SL}_2(F)$ depend on the breaks in the lower ramification filtration of the Galois group J attached to the biquadratic extensions of F. Here are the answers:

Case 1: We have

$$J = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \ ; \ J_0 = ... = J_t = \mathbb{Z}/2\mathbb{Z} \ ; \ J_{t+1} = \{1\}$$

We have

$$\alpha(\varphi) = (1+t)2$$

We have

$$\mathrm{Deg}(\pi(\varphi, \rho)) = \frac{1}{4} \cdot \left| \frac{\gamma(\varphi)}{\gamma(\varphi_0)} \right| = \frac{1}{4} \cdot \frac{2}{q} \cdot |\varepsilon(\varphi)| = 2^{t-f}$$

Case 2.1: The lower ramification filtration is (with t an odd number)

$$J = \ldots = J_t = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\ ;\ J_{t+1} = \{1\}$$

We have

$$\alpha(\varphi) = \sum_{i \geq 0} \frac{\dim(\mathfrak{g}/\mathfrak{g}^{J_i})}{[J_0 : J_i]} = (t+1)3$$

We have

$$\mathrm{Deg}(\pi(\varphi, \rho)) = \frac{1}{4} \cdot \left| \frac{\gamma(\varphi)}{\gamma(\varphi_0)} \right| = 2^{3(1+t)/2 - f - 1}$$

The formal degree is a *rational* number.

Case 2.2: This case admits the following lower ramification filtration (with t_1 odd):

$$J = J_0 = \dots = J_{t_1} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$
$$J_{t_1+1} = \dots = J_{2t_2-t_1} = \mathbb{Z}/2\mathbb{Z} \ ; \ J_{2t_2-t_1+1} = \{1\}$$

We have

$$\alpha(\varphi) = \sum_{i \geq 0} \frac{\dim(\mathfrak{g}/\mathfrak{g}^{J_i})}{[J_0 : J_i]} = (t_1+1)3 + \frac{(2t_2)2}{2} = 3 + 3t_1 + 2t_2$$

and we have

$$\mathrm{Deg}(\pi(\varphi,\rho)) = \frac{1}{4} \cdot \left| \frac{\gamma(\varphi)}{\gamma(\varphi_0)} \right| = 2^{3(1+t_1)/2+t_2-f-1}$$

the formal degree of each supercuspidal in the packet Π_ϕ.

The formal degrees depend on the residue degree f and the breaks in the lower ramification filtration of the Galois group $\mathrm{Gal}(L/F)$ of a biquadratic extension L/F. In fact, the set of formal degrees is

$$\{2^n : n \in \mathbb{Z}\}$$

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Reference: Mendes-Plymen, arXiv:1302.6038[math.RT]