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Standard tableaux and Klyachko's Theorem on Lie representations

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Abstract

We show that for all but two partitions λ of $n > 6$ there exists a standard tableau of shape λ with major index coprime to n . In conjunction with a deep result of Kraśkiewicz and Weyman this provides a new purely combinatorial proof of Klyachko's famous theorem on Lie representations of the general linear group.

Key words: standard tableau, free Lie algebra, Lie representation
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1 Introduction

Let λ be a partition of n ($\lambda \vdash n$). A tableau of shape λ is a numbering of the Young diagram of λ with the numbers from $\{1, \dots, k\}$, where $k \leq n$, such that the entries weakly increase along each row and strictly increase down each column. We say that a tableau is standard if each number in $\{1, \dots, n\}$ occurs exactly once. An entry i in a standard tableau is called a descent if $i + 1$ occurs in any row below i . For a standard tableau T we denote the set of all descents in T by $D(T)$ and define the sum of all descents to be the major index of T , $\text{maj}(T)$. For example, the tableau

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline 6 & & \\ \hline \end{array}$$

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of shape $\lambda = (3, 2, 1)$ has descent set $D(T) = \{2, 5\}$ and major index $\text{maj}(T) = 7$ (see [6, Section 8.3] for unexplained terminology and further examples).

In this paper we prove the following theorem.

Theorem 1 *Let $n \geq 3$ and let $\lambda \vdash n$. There exists a standard tableau of shape λ with major index coprime to n if and only if $\lambda \neq (1^n), (n), (2^2)$ or (2^3) .*

The significance of this theorem lies in the fact that when combined with a deep result of Kraśkiewicz and Weyman, it implies a celebrated theorem of Klyachko on Lie representations of the general linear group. Let V be a finite dimensional vector space over a field of characteristic zero and let $T = T(V)$ denote the tensor algebra of V . Thus $T = \bigoplus_{n \geq 0} T_n$ where $T_n = V^{\otimes n}$. It is known that each tensor power T_n is a semisimple module for the general linear group $GL(V)$, with the isomorphism types of the irreducible submodules of T_n parameterised by the partitions of n with at most $\dim(V)$ parts. Consider T as a Lie algebra via the multiplication $[x, y] = x \otimes y - y \otimes x$. Then, by a theorem of Witt, the Lie subalgebra $L = L(V)$ generated by V in T is the free Lie algebra on V . Moreover, $L = \bigoplus_{n \geq 1} L_n$, where each Lie power $L_n = T_n \cap L$ is a $GL(V)$ -submodule of T_n . Hence, the isomorphism types of the irreducible $GL(V)$ -submodules of L_n form a subset of those occurring in T_n . Klyachko's Theorem tells us that almost all of the irreducible modules occurring in T_n also appear in the n th Lie representation, L_n .

Theorem 2 (Klyachko [3]) *Let $n \geq 3$ and let $\lambda \vdash n$. There exists an irreducible $GL(V)$ -submodule of L_n with isomorphism type corresponding to λ if and only if λ has no more than $\dim(V)$ parts and $\lambda \neq (1^n), (n), (2^2)$ or (2^3) .*

Since publication in 1974, Klyachko's Theorem has attracted the attention of a number of authors. Recently new proofs have been given by Schocker [7] and Kovács and Stöhr [4]. While Klyachko's original proof was based on character theory, the arguments in [4] and [7] are mainly combinatorial.

In 1987, Kraśkiewicz and Weyman gave a beautiful combinatorial interpretation of the multiplicities of the irreducible $GL(V)$ -modules occurring in L_n .

Theorem 3 (Kraśkiewicz–Weyman [5]) *Let $a, n \in \mathbb{N}$ be fixed coprime numbers and let λ be a partition of n with at most $\dim(V)$ parts. The irreducible $GL(V)$ -module corresponding to λ occurs in L_n with multiplicity equal to the number of standard tableaux of shape λ with major index congruent to a modulo n .*

New proofs of this result have been given by Garsia [1] and Jöllenbeck and Schocker [2] (for another account of Garsia's proof see [6, Theorems 8.8 and 8.9]). It has been a longstanding challenge to deduce Klyachko's Theorem directly from Theorem 3. As Schocker [7] remarks "it seems to be rather difficult

to give a combinatorial proof ... by some analysis of the tableaux numbers ... and the Kraśkiewicz-Weyman Theorem only". Exactly that, however, is achieved in the present paper, as our Theorem 1 together with Theorem 3 clearly imply Theorem 2.

The rest of the paper is devoted to the proof of Theorem 1. It is easy to see that there are no standard tableaux of shape (1^n) , (n) , (2^2) or (2^3) with major index coprime to n . For example, the (only) two standard tableaux of shape (2^2) ,

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

have major indices 2 and 4, respectively. For all other partitions λ of $n \geq 3$ we show that there exists a standard tableau of shape λ with major index coprime to n . In Section 2 we consider partitions λ of n into two parts, Section 3 deals with rectangular partitions with more than two parts and in Section 4 we consider non-rectangular partitions of n into more than two parts.

2 Partitions with two parts

Let $n \geq 3$. For each two part partition $\lambda = (n - s, s)$ of n , where $1 \leq s \leq \lfloor n/2 \rfloor$ and $\lambda \neq (2^2)$, we define a standard tableau $T(\lambda)$ with major index coprime to n .

For n odd, $n = 2m + 1$ and $1 \leq s \leq m$, we define

$$T(n - s, s) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & \cdots & s & \cdots & m & m + s + 1 & \cdots & 2m + 1 \\ \hline m + 1 & \cdots & m + s & & & & & \\ \hline \end{array}$$

Then $\text{maj}(T(n - s, s)) = m$ which is coprime to n .

Now let n be even, $n = 2m$ and $1 \leq s \leq m$. For $s = 1$ we define

$$T(n - 1, 1) = \begin{array}{|c|c|c|c|} \hline 1 & 3 & \cdots & 2m \\ \hline 2 & & & \\ \hline \end{array}$$

and for $1 < s < m - 1$ we define $T(n - s, s)$ to be the standard tableau given below.

| | | | | | | | | | | |
|---|-----|-----|-------|-----|-----|-----|-----|-------|-----|----|
| 1 | 2 | ... | s | ... | m-1 | m+1 | m+2 | m+s+2 | ... | 2m |
| m | m+3 | ... | m+s+1 | | | | | | | |

When $s = m - 1$ we put

$$T(m+1, m-1) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & \cdots & m-1 & m+1 & m+2 \\ \hline m & m+3 & \cdots & 2m & & \\ \hline \end{array}$$

and, finally, for $s = m$, where $m > 2$, we set

$$T(m^2) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & \cdots & m-1 & m+2 \\ \hline m & m+1 & m+3 & \cdots & 2m-1 & 2m \\ \hline \end{array}$$

Hence, we have

$$\text{maj}(T(n-s, s)) = \begin{cases} 1 & \text{if } s = 1 \\ 2m+1 & \text{if } 1 < s \leq m \end{cases}$$

which is coprime to n for all values of s .

3 Rectangular Partitions with more than two parts

We consider rectangular partitions $\lambda = (m^k) \vdash n = mk$, where $k > 2$, $m > 1$ and $\lambda \neq (2^3)$. First suppose that $m = 2$ and let

$$T(2^k; j) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \vdots & \vdots \\ \hline 2j-1 & 2j \\ \hline 2j+1 & 2j+3 \\ \hline 2j+2 & 2j+5 \\ \hline 2j+4 & 2j+6 \\ \hline 2j+7 & 2j+8 \\ \hline \vdots & \vdots \\ \hline 2k-1 & 2k \\ \hline \end{array}$$

where $1 \leq j \leq k-3$ and $k > 3$. Then $T(2^k; j)$ is a standard tableau with descent set given by

$$\begin{aligned} \{2, \dots, 2j, 2j+1, 2j+3, 2j+5, 2j+6, \dots, 2k-2\} & \text{ if } 1 \leq j < k-3 \\ \{2, \dots, 2k-6, 2k-5, 2k-3, 2k-1\} & \text{ if } j = k-3 \end{aligned}$$

and it is easy to see that

$$\begin{aligned} \text{maj}(T(2^k; j)) &= 2j + 3 + 2 \sum_{i=1}^{k-1} i \\ &= 2j + 3 + k(k-1) \end{aligned}$$

for all $1 \leq j \leq k-3$.

If k is odd then $k-1$ is even and, since $k > 3$, we have that $(k-1)/2$ is an integer in the range $1 < (k-1)/2 \leq k-3$. Hence

$$\begin{aligned} \text{maj}(T(2^k; \frac{k-1}{2})) &= k + 2 + k(k-1) \\ &\equiv k + 2 \pmod{n} \end{aligned}$$

since $n = 2k$ and $k-1$ is even. Thus, we have that $\text{maj}(T(2^k; (k-1)/2))$ is coprime to n .

Similarly, if k is even then, since $k > 3$, we have that $(k-2)/2$ is an integer in the range $1 \leq (k-2)/2 \leq k-3$. Hence

$$\begin{aligned} \text{maj}(T(2^k; \frac{k-2}{2})) &= k + 1 + k(k-1) \\ &= k^2 + 1 \\ &\equiv 1 \pmod{n} \end{aligned}$$

giving that $\text{maj}(T(2^k; (k-2)/2))$ is coprime to n . This proves Theorem 1 for all partitions $\lambda = (2^k)$, where $k > 3$.

Next suppose that $\lambda = (m^k)$ where $m, k > 2$ and let $T(m^k; i, s)$ denote the standard tableau given by

| | | | | | | |
|------------|------------|-----|--------|----------|-----|------------|
| 1 | ... | | | | | m |
| \vdots | | | | | | \vdots |
| $(i-1)m+1$ | ... | | | | | im |
| $im+1$ | $im+2$ | ... | $im+s$ | $im+s+2$ | ... | $(i+1)m+1$ |
| $im+s+1$ | $(i+1)m+2$ | ... | | | | $(i+2)m$ |
| \vdots | | | | | | \vdots |
| $(k-1)m+1$ | ... | | | | | km |

where $0 \leq i \leq k-2$, $1 \leq s \leq m-1$. Then $T(m^k; i, s)$ has descent set given by

$$\begin{aligned}
& \{s, m+1, 2m, \dots, (k-1)m\} && \text{if } i = 0 \\
& \{m, \dots, im, im+s, (i+1)m+1, (i+2)m, \dots, (k-1)m\} && \text{if } 0 < i < k-2 \\
& \{m, \dots, (k-2)m, (k-2)m+s, (k-1)m+1\} && \text{if } i = k-2
\end{aligned}$$

which gives

$$\begin{aligned}
\text{maj}(T(m^k; i, s)) &= m \sum_{j=1}^{k-1} j + im + s + 1 \\
&= \frac{mk(k-1)}{2} + im + s + 1
\end{aligned}$$

for all $0 \leq i \leq k-2$.

If n is odd, then k must be odd and it follows that

$$\text{maj}(T(m^k; i, s)) \equiv im + s + 1 \pmod{n}.$$

Hence, choosing $i = 0$, $s = 1$ gives $\text{maj}(T(m^k; 0, 1)) \equiv 2 \pmod{n}$, which is coprime to n .

If n is even, we consider two cases:

Case (i) If k is odd then $\text{maj}(T(m^k; i, s)) \equiv im + s + 1 \pmod{n}$ and, since n is even, we have that m must be even.

If $4|m$, choosing $i = (k-1)/2$ and $s = m/2$ gives

$$\begin{aligned}
\text{maj}(T(m^k; \frac{k-1}{2}, \frac{m}{2})) &\equiv \frac{k-1}{2}m + \frac{m}{2} + 1 \pmod{n} \\
&= \frac{mk}{2} + 1 \pmod{n} \\
&= \frac{n}{2} + 1 \pmod{n}
\end{aligned}$$

which is coprime to n .

If $4 \nmid m$, choosing $i = (k-1)/2$ and $s = (m/2) + 1$ gives

$$\begin{aligned}
\text{maj}(T(m^k; \frac{k-1}{2}, \frac{m}{2} + 1)) &\equiv \frac{k-1}{2}m + \left(\frac{m}{2} + 1\right) + 1 \pmod{n} \\
&= \frac{mk}{2} + 2 \pmod{n} \\
&= \frac{n}{2} + 2 \pmod{n}
\end{aligned}$$

which is coprime to n .

Case (ii) If k is even then

$$\text{maj}(T(m^k; i, s)) \equiv \frac{mk}{2} + im + s + 1 \pmod{n}.$$

Hence choosing $i = (k/2) - 1$ and $s = m - 2$ gives

$$\begin{aligned}
\text{maj}(T(m^k; \frac{k}{2} - 1, m - 2)) &\equiv \frac{mk}{2} + \left(\frac{k}{2} - 1\right)m + (m - 2) + 1 \pmod{n} \\
&\equiv n - 1 \pmod{n}
\end{aligned}$$

which is coprime to n .

Hence, we have shown that every rectangular partition of n into k parts where $2 < k < n$ has a standard tableau with major index coprime to n , except for the rectangle $\lambda = (2^3)$.

4 Non-Rectangular Partitions with more than two parts

We begin with a few definitions and a technical lemma which are required for the final part of the proof. We say that a finite sequence of positive integers $\mu = (\mu_1, \dots, \mu_k)$ is a composition of n ($\mu \vDash n$) if $\mu_1 + \dots + \mu_k = n$. Moreover, we say that a tableau T of shape $\lambda \vdash n$ has weight μ if i occurs μ_i times in T . Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of n . We define the lower rim of λ to be the union of all boxes B in the Young diagram of λ such that there is no box directly below B . In other words, each box in the lower rim lies at the

end of a column. Hence, the number of boxes in the lower rim is equal to the number of columns in λ , which in turn is equal to λ_1 . Notice that if we remove a box from the end of a row in the lower rim of λ then the shape which remains is a Young diagram corresponding to a partition of $n - 1$. Hence we can successively remove up to λ_1 boxes from the lower rim of λ in such a way that the shape which remains is a Young diagram.

Lemma 4 *Let $n = mk + r$ where $m, k > 0$ and $0 \leq r < k$. Furthermore, let λ be a partition of n into k parts and $\mu = (\mu_1, \dots, \mu_k)$ be a composition of n with each $\mu_i \in \{m, m + 1\}$. Then there exists a tableau of shape λ with weight μ and first column with entries $1, \dots, k$.*

Proof. We prove the result by induction on k . When $k = 1$ we have that $n = m$ and $\lambda = \mu = (m)$. So the tableau consisting of a single row of 1's satisfies the conditions of the Lemma. For $k > 1$ there are two cases.

Case (i) $\mu_k = m$. Since λ is a partition of $n = mk + r$ into k parts, we have that $\lambda_1 \geq m$ and $\lambda_k \leq m$. In other words, there are at least m boxes in the lower rim of λ and at most m boxes in the k th row. Hence, we can remove m boxes from the lower rim of λ in such a way that all boxes are removed from the k th row and that the shape which remains is a Young diagram. Let $\tilde{\lambda}$ be the partition corresponding to this diagram. Then $\tilde{\lambda}$ is a partition of $\tilde{n} = \tilde{m}(k - 1) + \tilde{r}$ into $k - 1$ parts, where $\tilde{m} = m, \tilde{r} = r$ for $0 \leq r < k - 1$ and $\tilde{m} = m + 1, \tilde{r} = 0$ for $r = k - 1$.

Notice that if $r = k - 1$ we must have that $\mu_i = m + 1$ for all $i < k$, since $n = mk + k - 1 = (m + 1)(k - 1) + m$ and $\mu_k = m$. Applying induction to $\tilde{\lambda}$ and $\tilde{\mu} = (\mu_1, \dots, \mu_{k-1})$ gives a tableau \tilde{T} of shape $\tilde{\lambda}$ with weight $\tilde{\mu}$ and entries $1, \dots, k - 1$ in the first column. Finally, by returning the m boxes which we removed and entering a k into each of these we obtain the desired tableau.

Case (ii) $\mu_k = m + 1$. Since μ is a composition of $n = mk + r$ into k parts with each $\mu_i \in \{m, m + 1\}$, we have that $r \geq 1$. Moreover, since λ is a partition of $n = mk + r$ into k parts, we have that $\lambda_1 \geq m + 1$ and $\lambda_k \leq m$. In other words, there are at least $m + 1$ boxes in the lower rim of λ and at most m boxes in the k th row. Hence, we can remove $m + 1$ boxes from the lower rim of λ in such a way that all boxes are removed from the k th row and that the shape which remains is a Young diagram. Let $\tilde{\lambda}$ be the partition corresponding to this diagram. Then $\tilde{\lambda}$ is a partition of $\tilde{n} = \tilde{m}(k - 1) + \tilde{r}$ into $k - 1$ parts, where $\tilde{m} = m, \tilde{r} = r - 1$ for $0 < r < k$.

Applying induction to $\tilde{\lambda}$ and $\tilde{\mu} = (\mu_1, \dots, \mu_{k-1})$ gives a tableau \tilde{T} of shape $\tilde{\lambda}$ with weight $\tilde{\mu}$ and entries $1, \dots, k - 1$ in the first column. Again, by returning the $m + 1$ boxes which we removed and entering a k into each of these we obtain the desired tableau. \square

We are now in a position to complete the proof of Theorem 1. Consider the non-rectangular partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ of $n = mk + r$ into k parts where $m > 0$, $k > 2$ and $0 \leq r < k < n$. We claim that there exists a standard tableau T of shape λ , with major index coprime to n and descent set of the form

$$D(T) = \{\mu_1, \mu_1 + \mu_2, \dots, \mu_1 + \mu_2 + \dots + \mu_{k-1}\}$$

where (i) $\mu_i \in \{m, m + 1\}$ if $k \nmid n$ and (ii) $\mu_i \in \{m - 1, m\}$ if $k|n$.

(i) If $k \nmid n$ then from the Lemma it is clear that for all such $\lambda \vdash n$ and for all $\mu = (\mu_1, \dots, \mu_k) \vDash n$ where $\mu_i \in \{m, m + 1\}$ there exists a standard tableau T of shape λ with descent set as above. We simply replace the μ_i occurrences of i in the tableau of the Lemma by the entries

$$\sum_{j=1}^{i-1} \mu_j + 1, \sum_{j=1}^{i-1} \mu_j + 2, \dots, \sum_{j=1}^i \mu_j$$

from left to right to obtain a standard tableau T of shape λ . Let \mathfrak{T} denote the set of all standard tableaux obtained in this way (from the Lemma we see that there is at least one for every choice of μ) and let $T \in \mathfrak{T}$ be one such standard tableau corresponding to the choice $\mu = (\mu_1, \dots, \mu_k)$. Since $\mu \vDash n$ we have that exactly r of the μ_i 's are equal to $m + 1$; $\mu_{i_1}, \dots, \mu_{i_r}$ say. Hence, the major index of T is given by

$$\begin{aligned} \text{maj}(T) &= (k-1)\mu_1 + (k-2)\mu_2 + \dots + \mu_{k-1} = \sum_{i=1}^{k-1} (k-i)\mu_i \\ &= \sum_{i=1}^{k-1} (k-i)m + \sum_{j=1}^r (k-i_j) \\ &= \frac{mk(k-1)}{2} + rk - \sum_{j=1}^r i_j \end{aligned}$$

The sum of the i_j ranges from $1 + 2 + \dots + (r-1) + r = r(r+1)/2$ at the smallest to $(k-r+1) + (k-r+2) + \dots + (k-1) + k = rk - r(r-1)/2$ at the greatest, giving $r(k-r) + 1$ consecutive values of $\text{maj}(T)$ for $T \in \mathfrak{T}$. Hence, for each integer z such that

$$\frac{n(k-1)}{2} - \frac{r(k-r)}{2} \leq z \leq \frac{n(k-1)}{2} + \frac{r(k-r)}{2}$$

there exists a tableau $T \in \mathfrak{T}$ with $\text{maj}(T) = z$.

If k is odd then for all $T \in \mathfrak{T}$ we have $\text{maj}(T) \equiv a \pmod n$ where

$$-\frac{r(k-r)}{2} \leq a \leq \frac{r(k-r)}{2}$$

and since $0 < r < k$ and $k > 2$ we must have that $\text{maj}(T) \equiv 1$ for some standard tableau $T \in \mathfrak{T}$.

If k is even then for all $T \in \mathfrak{T}$ we have $\text{maj}(T) \equiv a \pmod n$ where

$$\frac{n}{2} - \frac{r(k-r)}{2} \leq a \leq \frac{n}{2} + \frac{r(k-r)}{2}.$$

One of these values is coprime to n . Indeed, since $r > 0$ we have that $r(k-r) \geq k-1$ giving that $(r(k-r))/2 \geq 2$ for $k > 4$. For $k = 4$ it is easy to check that $((mk+r) + r(k-r))/2 = [n/2] + 2$. Hence, for $r > 0$, k even and greater than 2 we have that there exist standard tableaux in \mathfrak{T} with major index congruent to $[n/2] + 1$ and $[n/2] + 2$ modulo n , one of which is coprime to n .

(ii) If $k|n$ then we have that $r = 0$ and $n = mk$. Since λ is non-rectangular, we must have that $\lambda_1 \geq m+1$ and $\lambda_k \leq m-1$. Recall that λ_1 is equal to the number of boxes in the lower rim of λ and λ_k is the number of boxes in the k th row of λ . Hence we can remove $m+1$ boxes from the lower rim of λ , including all λ_k boxes from the k th row, in such a way that the shape that remains is a Young diagram. Let $\tilde{\lambda}$ denote the partition corresponding to this Young diagram. Then $\tilde{\lambda}$ is a partition of $\tilde{n} = mk - (m+1) = (m-1)(k-1) + (k-2)$ into $k-1$ parts.

Since $k > 2$, we have that $(k-1) \nmid \tilde{n}$ and, applying the reasoning of Case (i), we see that for each $\mu = (\mu_1, \dots, \mu_{k-1}) \vDash \tilde{n}$ there exist standard tableaux of shape $\tilde{\lambda}$ with descent set given by

$$D(T) = \{\mu_1, \mu_1 + \mu_2, \dots, \mu_1 + \mu_2 + \dots + \mu_{k-2}\}$$

where $\mu_i \in \{m-1, m\}$. Let $\tilde{\mathfrak{T}}$ denote the set of all such standard tableaux. Recall that there are $k-1$ consecutive values for the major indices of the tableaux in $\tilde{\mathfrak{T}}$.

For each $\tilde{T} \in \tilde{\mathfrak{T}}$, replacing the $m+1$ boxes which we removed earlier and entering the numbers $mk-m, \dots, mk$ from left to right gives rise to a standard tableau T of shape λ with

$$\begin{aligned} D(T) &= D(\tilde{T}) \cup \{mk-m-1\} \\ &= \{\mu_1, \mu_1 + \mu_2, \dots, \mu_1 + \mu_2 + \dots + \mu_{k-2} + \mu_{k-1}\} \end{aligned}$$

where $\mu_i \in \{m-1, m\}$. Hence

$$\text{maj}(T) = \text{maj}(\tilde{T}) + mk - m - 1.$$

We show that one of these $k-1$ values is coprime to n . Indeed, for all T obtained in this way we have

$$\frac{m(k-1)(k-2)}{2} + (m-1)(k-1) \leq \text{maj}(T) \leq \frac{m(k-1)(k-2)}{2} + mk - m - 1.$$

Moreover, if k is odd then $\text{maj}(T) \equiv a \pmod{n}$ where

$$n - k + 1 \leq a \leq n - 1$$

giving $n-1$ coprime to n , for example. Finally, if k is even then $\text{maj}(T) \equiv a \pmod{n}$ where

$$\frac{n}{2} - k + 1 \leq a \leq \frac{n}{2} - 1$$

with one of $(n/2) - 1$ and $(n/2) - 2$ coprime to n . This completes the proof of Theorem 1.

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