Numerical Solution of the Equations Governing 2D Eulerian Gas-Solid Two-Phase Flow

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Abstract

This paper extends the 1D model discussed in Hudson & Harris [8] to 2D. A high resolution scheme is used to discretise two formulations of the 2D model and a variety of test cases are investigated to determine how the model behaves and the accuracy of the scheme.

1 Introduction

We adapt the 1D version of Eulerian gas-solid two-phase flow, see Hudson & Harris [8], to 2D using a 2D version of the high resolution scheme. In 2D, the equations are: the mass equations,

$$(\epsilon_k \rho_k)_t + \nabla \cdot (\epsilon_k \rho_k \mathbf{U}_k), = 0 \tag{1.1a}$$

momentum equations,

$$(\epsilon_k \rho_k \mathbf{U}_k)_t + \nabla \cdot (\epsilon_k \rho_k \mathbf{U}_k \otimes \mathbf{U}_k) + \omega_{1,3} \nabla p_g + \omega_{2,4} \nabla p_s = \epsilon_k \rho_k \mathbf{g} \mp \beta(\mathbf{U}_g - \mathbf{U}_s), \tag{1.1b}$$

and the fluctuation energy equation,

$$(\epsilon_s \rho_s T_s)_t + \nabla \cdot (\epsilon_s \rho_s T_s \mathbf{U}_s) = -\frac{2}{3} \left(p_s \nabla \cdot \mathbf{U}_s + 3\beta T_s \right). \tag{1.1c}$$

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The values of ω_k are chosen such that $\omega_1 + \omega_3$ and $\omega_2 + \omega_4$ are equal to either 0 or 1. As with the 1D version, the gas phase is assumed to be isentropic with pressure,

$$p_g(\rho_g) = C\rho_g^\gamma$$

and the solids phase has the simple pressure term

$$p_s(\epsilon_s, T_s) = \epsilon_s \rho_s T_s.$$

The interphase drag force is,

$$\beta = \frac{3}{4d_s} C_D \epsilon_g \epsilon_s \rho_g |\mathbf{U}_g - \mathbf{U}_s| \tag{1.2}$$

where C_D is a constant and

$$\epsilon_g + \epsilon_s = 1.$$

We include a gravity vector term $\mathbf{g} = (g_x, g_y)$. To close the models, we assume that the density of the solid is constant, ie $\rho_s = C$.

There are numerous choices of the values ω_k in the momentum equations (1.1b), but we are primarily interested in two standard models, which are well documented in the literature:

1. Model A, which is based on the work of Jackson [9] and modernised by Ding & Gidaspow [3], Boemer *et al.* [1] and Bouillard *et al.* [2]. This model can be obtained by setting,

   $$\omega_1 = \epsilon_g, \quad \omega_2 = 0, \quad \omega_3 = \epsilon_s \quad \text{and} \quad \omega_4 = 1.$$

   Unfortunately, Model A is conditionally hyperbolic and thus, may only be used in certain regions for the area of interest in this paper.

2. Model B, which is based on the work of Rudinger & Chang [15] and modernised by Bouillard *et al.* [2], Boemer *et al.* [1] and Gidaspow [4]. This model can be obtained by setting,

   $$\omega_1 = 1, \quad \omega_2 = 0, \quad \omega_3 = 0 \quad \text{and} \quad \omega_4 = 1.$$

   Unlike Model A, Model B is unconditionally hyperbolic, but the model is apparently less physical than Model A.

2 Formulations

To aid the numerical discretisation, we re-write the model (1.1) in system form as

$$\mathbf{w}_t + \mathbf{F}(\mathbf{w})_x + \mathbf{G}(\mathbf{w})_y = \mathbf{f} + \mathbf{g} + \mathbf{S}, \tag{2.1}$$

where $\mathbf{w}$ contains the conserved variables, $\mathbf{F}$ and $\mathbf{G}$ are the flux-functions, $\mathbf{f}$ and $\mathbf{g}$ contain the source terms with first order derivatives in the x and y direction respectively and $\mathbf{S}$ contains all the other source terms. We consider two formulations of the model.

2.1 Formulation S

Here, the model is written in system form with the gas pressure term rewritten as

$$\omega_{1,3}(p_g)_x = (p_g)_x - (1-\omega_{1,3})(p_g)_x,$$

thus obtaining

$$\mathbf{w} = \begin{bmatrix} \epsilon_g\rho_g \\ \epsilon_g\rho_g u_g \\ \epsilon_g\rho_g v_g \\ \epsilon_s \\ \epsilon_s u_s \\ \epsilon_s v_s \\ \epsilon_s T_s \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \epsilon_g\rho_g u_g \\ \epsilon_g\rho_g u_g^2 + p_g \\ \epsilon_g\rho_g u_g v_g \\ \epsilon_s u_s \\ \epsilon_s u_s^2 + \epsilon_s T_s \\ \epsilon_s u_s v_s \\ \epsilon_s T_s u_s \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \epsilon_g\rho_g v_g \\ \epsilon_g\rho_g u_g v_g \\ \epsilon_g\rho_g v_g^2 + p_g \\ \epsilon_s v_s \\ \epsilon_s v_s u_s \\ \epsilon_s v_s^2 + \epsilon_s T_s \\ \epsilon_s v_s T_s \end{bmatrix},$$

$$\mathbf{f} = \begin{bmatrix} 0 \\ (1-\omega_1)(p_g)_x - \omega_2(p_s)_x \\ 0 \\ 0 \\ -\omega_3\rho_s^{-1}(p_g)_x + (1-\omega_4)\rho_s^{-1}(p_s)_x \\ 0 \\ -\frac{2}{3\rho_s}p_s(u_s)_x \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} 0 \\ 0 \\ (1-\omega_1)(p_g)_y - \omega_2(p_s)_y \\ 0 \\ 0 \\ -\omega_3\rho_s^{-1}(p_g)_y + (1-\omega_4)\rho_s^{-1}(p_s)_y \\ -\frac{2}{3\rho_s}p_s(v_s)_y \end{bmatrix}$$

and

$$\mathbf{S} = \begin{bmatrix} 0 \\ \epsilon_g\rho_g g_x - \beta(u_g - u_s) \\ \epsilon_g\rho_g g_y - \beta(v_g - v_s) \\ 0 \\ \epsilon_s g_x + \frac{\beta}{\rho_s}(u_g - u_s) \\ \epsilon_s g_y + \frac{\beta}{\rho_s}(v_g - v_s) \\ -\frac{2}{\rho_s}\beta T_s \end{bmatrix}.$$

The Jacobian matrices of this system are

$$\mathbf{J}_x = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \epsilon_g^{-1} c_g^2 - u_g^2 & 2u_g & 0 & \rho_g \epsilon_g^{-1} c_g^2 & 0 & 0 & 0 \\ -u_g v_g & v_g & u_g & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_s T_s c_0^2 - u_s^2 & 2u_s & 0 & d_0 \\ 0 & 0 & 0 & -u_s v_s & v_s & u_s & 0 \\ 0 & 0 & 0 & -u_s T_s & T_s & 0 & u_s \end{bmatrix}$$

and

$$\mathbf{J}_y = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -u_g v_g & v_g & u_g & 0 & 0 & 0 & 0 \\ \epsilon_g^{-1} c_g^2 - v_g^2 & 0 & 2v_g & \rho_g \epsilon_g^{-1} c_g^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -u_s v_s & v_s & u_s & 0 \\ 0 & 0 & 0 & \epsilon_s T_s c_0^2 - v_s^2 & 0 & 2v_s & d_0 \\ 0 & 0 & 0 & -v_s T_s & 0 & T_s & v_s \end{bmatrix},$$

whose eigenvalues are

$$\lambda_{1,2} = u_g \eta_x + v_g \eta_y \pm c_g \sqrt{\epsilon_g^{-1}}, \quad \lambda_3 = u_g \eta_x + v_g \eta_y,$$

$$\lambda_{4,5} = u_s \eta_x + v_s \eta_y \pm \sqrt{T_s} \quad \text{and} \quad \lambda_{6,7} = u_s \eta_x + v_s \eta_y.$$

Here, η_x and η_y denote the x and y direction. Thus, for $\mathbf{J}_x$, $\eta_x = 1$ and $\eta_y = 0$ and for $\mathbf{J}_y$, $\eta_x = 0$ and $\eta_y = 1$.

2.2 Formulation P

An alternative formulation is presented in order to try and minimise the significance of an inhomogeneous term, $-p_s \nabla \cdot \mathbf{U}_s$, present in the fluctuation energy equation (1.1c). This term can create difficulties in obtaining an accurate numerical approximation, see Hudson & Harris [8]. Thus, we rewrite the term using the product rule, e.g.

$$-p_s (u_s)_x = -(p_s u_s)_x + u_s (p_s)_x.$$

By using this approach, we obtain

$$
\mathbf{w}=\begin{bmatrix}\epsilon_g\rho_g\\ \epsilon_g\rho_g u_g\\ \epsilon_g\rho_g v_g\\ \epsilon_s\\ \epsilon_s u_s\\ \epsilon_s v_s\\ \epsilon_s T_s\end{bmatrix},\quad \mathbf{F}=\begin{bmatrix}\epsilon_g\rho_g u_g\\ \epsilon_g\rho_g u_g^2+p_g\\ \epsilon_g\rho_g u_g v_g\\ \epsilon_s u_s\\ \epsilon_s u_s^2+\epsilon_s T_s\\ \epsilon_s u_s v_s\\ \frac{5}{3}\epsilon_s T_s u_s\end{bmatrix},\quad \mathbf{G}=\begin{bmatrix}\epsilon_g\rho_g v_g\\ \epsilon_g\rho_g u_g v_g\\ \epsilon_g\rho_g v_g^2+p_g\\ \epsilon_s v_s\\ \epsilon_s v_s u_s\\ \epsilon_s v_s^2+\epsilon_s T_s\\ \frac{5}{3}\epsilon_s v_s T_s\end{bmatrix},
$$

$$
\mathbf{f}=\begin{bmatrix}0\\ (1-\omega_1)(p_g)_x-\omega_2(p_s)_x\\ 0\\ 0\\ -\omega_3\rho_s^{-1}(p_g)_x+(1-\omega_4)\rho_s^{-1}(p_s)_x\\ 0\\ \frac{2}{3}u_s(\epsilon_s T_s)_x\end{bmatrix},\quad \mathbf{g}=\begin{bmatrix}0\\ 0\\ (1-\omega_1)(p_g)_y-\omega_2(p_s)_y\\ 0\\ 0\\ -\omega_3\rho_s^{-1}(p_g)_y+(1-\omega_4)\rho_s^{-1}(p_s)_y\\ \frac{2}{3}v_s(\epsilon_s T_s)_y\end{bmatrix}
$$

and

$$
\mathbf{S}=\begin{bmatrix}0\\ \epsilon_g\rho_g g_x-\beta(u_g-u_s)\\ \epsilon_g\rho_g g_y-\beta(v_g-v_s)\\ 0\\ \epsilon_s g_x+\frac{\beta}{\rho_s}(u_g-u_s)\\ \epsilon_s g_y+\frac{\beta}{\rho_s}(v_g-v_s)\\ -\frac{2}{\rho_s}\beta T_s\end{bmatrix}.
$$

The Jacobian matrices of this system are

$$
\mathbf{J}_x=\begin{bmatrix}0 & 1 & 0 & 0 & 0 & 0 & 0\\ \epsilon_g^{-1}c_g^2-u_g^2 & 2u_g & 0 & \rho_g\epsilon_g^{-1}c_g^2 & 0 & 0 & 0\\ -u_g v_g & v_g & u_g & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & \epsilon_s T_s c_0^2-u_s^2 & 2u_s & 0 & d_0\\ 0 & 0 & 0 & -u_s v_s & v_s & u_s & 0\\ 0 & 0 & 0 & -\frac{5}{3}u_s T_s & \frac{5}{3}T_s & 0 & \frac{5}{3}u_s\end{bmatrix}
$$

and

$$
\mathbf{J}_y=\begin{bmatrix}0 & 0 & 1 & 0 & 0 & 0 & 0\\ -u_g v_g & v_g & u_g & 0 & 0 & 0 & 0\\ \epsilon_g^{-1}c_g^2-v_g^2 & 0 & 2v_g & \rho_g\epsilon_g^{-1}c_g^2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -u_s v_s & v_s & u_s & 0\\ 0 & 0 & 0 & \epsilon_s T_s c_0^2-v_s^2 & 0 & 2v_s & d_0\\ 0 & 0 & 0 & -\frac{5}{3}v_s T_s & 0 & \frac{5}{3}T_s & \frac{5}{3}v_s\end{bmatrix},
$$

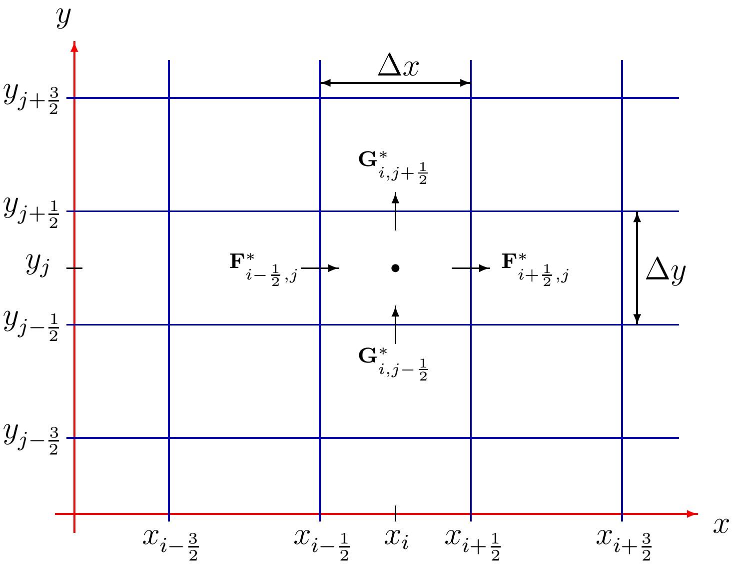

Figure 3.1: The two dimensional mesh.

whose eigenvalues are

$$\lambda_{1,2} = u_g\eta_x + v_g\eta_y \pm c_g\sqrt{\epsilon_g^{-1}}, \quad \lambda_3 = u_g\eta_x + v_g\eta_y,$$

$$\lambda_{4,5} = \frac{4}{3}(u_s\eta_x + v_s\eta_y) \pm \frac{1}{3}\sqrt{u_s^2\eta_x + v_s^2\eta_y + 15T_s} \quad \text{and} \quad \lambda_{6,7} = u_s\eta_x + v_s\eta_y.$$

Here, η_x and η_y denote the x and y direction. Thus, for $\mathbf{J}_x$, $\eta_x = 1$ and $\eta_y = 0$ and for $\mathbf{J}_y$, $\eta_x = 0$ and $\eta_y = 1$.

3 High Resolution Scheme

We extend the numerical scheme used in Hudson & Harris [8] for 1D problems to 2D. The high resolution scheme is based on Roe's scheme [14] and was adapted by Hubbard & Garcia-Navarro [6]:

$$\mathbf{w}^{n+1}_{i,j} = \mathbf{w}^n_{i,j} - s_x\left(\mathbf{F}^*_{i+\frac{1}{2},j} - \mathbf{F}^*_{i-\frac{1}{2},j}\right) - s_y\left(\mathbf{G}^*_{i,j+\frac{1}{2}} - \mathbf{G}^*_{i,j-\frac{1}{2}}\right) + \Delta t(\mathbf{R}^*_{i,j} + \mathbf{S}^n_{i,j}), \qquad (3.1)$$

where the numerical flux-functions are

$$\mathbf{F}^*_{i+\frac{1}{2},j} = \frac{1}{2}(\mathbf{F}^n_{i+1,j} + \mathbf{F}^n_{i,j}) - \frac{1}{2}\sum_{k=1}^{5}\left[\tilde{\alpha}^x_k|\tilde{\lambda}^x_k|(1-\Phi(\tilde{\theta}^x_k)(1-|\tilde{\nu}^x_k|))\tilde{\mathbf{e}}^x_k\right]_{i+\frac{1}{2},j}$$

and

$$\mathbf{G}^*_{i,j+\frac{1}{2}} = \frac{1}{2}(\mathbf{G}^n_{i,j+1} + \mathbf{G}^n_{i,j}) - \frac{1}{2}\sum_{k=1}^{5}\left[\tilde{\alpha}^y_k|\tilde{\lambda}^y_k|(1-\Phi(\tilde{\theta}^y_k)(1-|\tilde{\nu}^y_k|))\tilde{\mathbf{e}}^y_k\right]_{i,j+\frac{1}{2}}.$$

The source terms not containing first order derivatives, $\mathbf{S}$, are approximated using a pointwise approach,

$$\mathbf{S}^n_i = \begin{bmatrix} 0 \\ \epsilon_g\rho_g g_x - \beta(u_g - u_s) \\ \epsilon_g\rho_g g_y - \beta(v_g - v_s) \\ 0 \\ \epsilon_s g_x + \frac{\beta}{\rho_s}(u_g - u_s) \\ \epsilon_s g_y + \frac{\beta}{\rho_s}(v_g - v_s) \\ -\frac{2}{\rho_s}\beta T_s \end{bmatrix}^n_i$$

and the source terms with first order derivatives are calculated using,

$$\Delta t\mathbf{R}^*_i = s_x(\mathbf{f}^-_{i+\frac{1}{2},j} + \mathbf{f}^+_{i-\frac{1}{2},j}) + s_y(\mathbf{g}^-_{i,j+\frac{1}{2}} + \mathbf{g}^+_{i,j-\frac{1}{2}}),$$

where

$$\mathbf{f}^\pm_{i+\frac{1}{2},j} = \frac{1}{2}\sum_{k=1}^{5}\left[\tilde{\beta}^x_k\tilde{\mathbf{e}}^x_k(1 \pm \operatorname{sgn}(\tilde{\lambda}^x_k)(1-\Phi(\tilde{\theta}^x_k)(1-|\tilde{\nu}^x_k|)))\right]_{i+\frac{1}{2},j}$$

and

$$\mathbf{g}^\pm_{i,j+\frac{1}{2}} = \frac{1}{2}\sum_{k=1}^{5}\left[\tilde{\beta}^y_k\tilde{\mathbf{e}}^y_k(1 \pm \operatorname{sgn}(\tilde{\lambda}^y_k)(1-\Phi(\tilde{\theta}^y_k)(1-|\tilde{\nu}^y_k|)))\right]_{i,j+\frac{1}{2}}.$$

The step sizes in space and time are Δx, Δy and Δt with i, j and n denoting the spatial and time grid number, respectively. The upstream and downstream boundaries are at (x_0, y_0) and (x_I, y_J) (I and J are the total number of spatial grid points in x and y), t_N is the final time,

$$s_x = \frac{\Delta t}{\Delta x}, \quad s_y = \frac{\Delta t}{\Delta y}, \quad \tilde{\nu}^{x,y}_k = s\tilde{\lambda}^{x,y}_k, \quad \tilde{\theta}^x_k = \frac{(\tilde{\alpha}^x_k)_{L+\frac{1}{2},j}}{(\tilde{\alpha}^x_k)_{L+\frac{1}{2},j}}, \quad L = i - \operatorname{sgn}(\tilde{\nu}^x_k)_{i+\frac{1}{2},j},$$

$$\tilde{\theta}^y_k = \frac{(\tilde{\alpha}^y_k)_{i,M+\frac{1}{2}}}{(\tilde{\alpha}^x_k)_{i,M+\frac{1}{2}}}, \quad M = j - \operatorname{sgn}(\tilde{\nu}^y_k)_{i,j+\frac{1}{2}},$$

and either the minmod flux-limiter [16],

$$\Phi(\theta) = \max(0, \min(1, \theta)), \tag{3.2}$$

or the van Leer [17] flux-limiter,

$$\Phi(\theta) = \frac{|\theta| + \theta}{1 + |\theta|}, \tag{3.3}$$

is used. To ensure the scheme remains stable, the time step is calculated using

$$\Delta t = \frac{\nu \min(\Delta x, \Delta y)}{2\max_{i,j}(|\lambda^x|, |\lambda^y|)}$$

where $\max(|\lambda|)$ is the maximum wave speed and $\nu \leq 1$ is the required Courant number.

The scheme is an adapted form of Roe's scheme [14] and requires Roe average values (denoted by $\tilde{}$) of the Jacobian matrices, eigenvalues ($\tilde{\lambda}$), eigenvectors ($\tilde{\mathbf{e}}$), wave strengths ($\tilde{\alpha}$) and inhomogeneous values ($\tilde{\beta}$). A full derivation of the Roe averages for the general system is presented in Appendix A and a summary is given in Tables 3.1 & 3.2, where the superscripts denote the corresponding system. The values of η_x and η_y are used to write the Roe averages in a compact form. Here, the $\tilde{a}^x$ values are obtained by setting $\eta_x = 1$ and $\eta_y = 0$ (in this case $\Delta w = w_{R,j} - w_{L,j}$) and the $\tilde{a}^y$ values are obtained by setting $\eta_x = 0$ and $\eta_y = 1$ (in this case $\Delta w = w_{i,R} - w_{i,L}$). The high resolution scheme can now be used to approximate the general system of equations for various multipliers ω_k.

4 Numerical Results

We are interested in investigating glass beads being transported by air. Thus, we take a solids density of $\rho_s = 2660$ kg/m^3 and particle diameter $d_s = 0.005$ m. For the gas phase, data corresponding to air at room temperature (20°C) with atmospheric pressure (100.0437 kPa), density $\rho_g = 1.2885$ kg/m^3 and viscosity $\mu_g = 1.58 \times 10^{-7}$ Pa.s are taken. Also $\gamma_g = 1.4$ and $C_p = 75916.16$ m$^{3.2}$/(kg$^{0.4}$s^2).

In 1D, Model A is conditionally hyperbolic (see [8]) thus, we expect the 2D version of Model A to also be conditionally hyperbolic. However, illustrating this is considerably more complicated in 2D. Thus, we only solve the 2D model in areas for which we anticipate the equations to be hyperbolic. Moreover, we only investigate the numerical solution of Model AP due to the robustness of the formulation in 1D.

We now investigate a variety of test cases using the above data. We only solve the general system when it is hyperbolic and require appropriate initial and boundary conditions

Roe Averages		
$\mathrm{roev}(a,b) = \frac{\sqrt{a_L}b_L + \sqrt{a_R}b_R}{\sqrt{a_L}+\sqrt{a_R}}$	$\tilde{\epsilon}_g = \frac{1}{2}((\epsilon_g)_R + (\epsilon_g)_L)$	$\tilde{c}_g = \begin{cases} \sqrt{\frac{\Delta p_g}{\Delta \rho_g}} & \text{if } \Delta\rho_g \neq 0 \\ c_g(\rho_g) & \text{otherwise} \end{cases}$
$\tilde{\rho}_g = \frac{1}{2}((\rho_g)_R + (\rho_g)_L)$	$\tilde{\epsilon}_s = \frac{1}{2}((\epsilon_s)_R + (\epsilon_s)_L)$	$\tilde{\mathbf{U}}_g = \mathrm{roev}(\epsilon_g\rho_g, \mathbf{U}_g)$
$\tilde{\mathbf{U}}_s = \mathrm{roev}(\epsilon_s, \mathbf{U}_s)$	$\tilde{T}_s = \mathrm{roev}(\epsilon_s, T_s)$	$\tilde{d}_k = \frac{(\tilde{\lambda}_k - \tilde{u}_g\eta_x - \tilde{v}_g\eta_y)^2\tilde{\epsilon}_g - \tilde{c}_g^2}{\tilde{\rho}_g\tilde{c}_g^2}$

Eigenvalues
$\tilde{\lambda}_{1,2} = \tilde{u}_g\eta_x + \tilde{v}_g\eta_y \mp \tilde{c}_g\sqrt{\tilde{\epsilon}_g^{-1}}, \quad \tilde{\lambda}_3 = \tilde{u}_g\eta_x + \tilde{v}_g\eta_y$
$\tilde{\lambda}^S_{4,5} = \tilde{u}_s\eta_x + \tilde{v}_s\eta_y \pm \sqrt{\tilde{T}_s}, \quad \tilde{\lambda}_{6,7} = \tilde{u}_s\eta_x + \tilde{v}_s\eta_y,$
$\tilde{\lambda}^P_{4,5} = \frac{4}{3}(\tilde{u}_s\eta_x + \tilde{v}_s\eta_y) \pm \frac{1}{3}\sqrt{\tilde{u}_s^2\eta_x + \tilde{v}_s\eta_y + 15\tilde{T}_s}$

Eigenvectors
$\tilde{\mathbf{e}}_{1,2} = \begin{bmatrix} 1 \\ \tilde{\lambda}_{1,2}\eta_x + \tilde{u}_g\eta_y \\ \tilde{v}_g\eta_x + \tilde{\lambda}_{1,2}\eta_y \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{\mathbf{e}}_3 = \begin{bmatrix} 0 \\ \eta_y \\ \eta_x \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{\mathbf{e}}_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \eta_y \\ \eta_x \\ 0 \end{bmatrix}$
$\tilde{\mathbf{e}}_{4,5,7} = \begin{bmatrix} 1 \\ \tilde{\lambda}_{4,5,7}\eta_x + \tilde{u}_g\eta_y \\ \tilde{v}_g\eta_x + \tilde{\lambda}_{4,5,7}\eta_y \\ \tilde{d}_{4,5,7} \\ (\tilde{\lambda}_{4,5,7}\eta_x + \tilde{u}_s\eta_y)\tilde{d}_{4,5,7} \\ (\tilde{v}_s\eta_x + \tilde{\lambda}_{4,5,7}\eta_y)\tilde{d}_{4,5,7} \\ \tilde{d}_{4,5,7}(\tilde{\lambda}_{4,5,7} - \tilde{u}_s\eta_x - \tilde{v}_s\eta_y)^2 \end{bmatrix}$

Table 3.1: Roe Average Values (superscripts denote formulation)

Wavestrengths $(k \neq a \neq b)$
$\tilde{\alpha}_k^{4,5,7} = \frac{(\tilde{\lambda}_a\tilde{\lambda}_b - \tilde{u}_s^2\eta_x - \tilde{v}_s^2\eta_y)\Delta\epsilon_s - (\tilde{\lambda}_a + \tilde{\lambda}_b - 2(\tilde{u}_s\eta_x + \tilde{v}_s\eta_y))\Delta(\epsilon_s(\eta_x u_s + \eta_y v_s)) + \Delta(\epsilon_s T_s)}{\tilde{d}_k(\tilde{\lambda}_k - \tilde{\lambda}_a)(\tilde{\lambda}_k - \tilde{\lambda}_b)}$
$\tilde{\alpha}_{1,2} = \mp\frac{\tilde{\lambda}_4\tilde{\alpha}_4 + \tilde{\lambda}_5\tilde{\alpha}_5 + \tilde{\lambda}_7\tilde{\alpha}_7 - (\tilde{\alpha}_4 + \tilde{\alpha}_5 + \tilde{\alpha}_7 - \Delta(\epsilon_g\rho_g))\tilde{\lambda}_{2,1} - \Delta(\epsilon_g\rho_g(\eta_x u_g + \eta_y v_g))}{\tilde{\lambda}_1 - \tilde{\lambda}_2}$
$\tilde{\alpha}_3 = \Delta(\rho_g\epsilon_g(\eta_x v_g + \eta_y u_g)) - (\eta_x\tilde{v}_g + \eta_y\tilde{u}_g)\Delta(\epsilon_g\rho_g)$
$\tilde{\alpha}_6 = \Delta(\epsilon_s(\eta_x v_s + \eta_y u_s)) - (\eta_x\tilde{v}_s + \eta_y\tilde{u}_s)\Delta\epsilon_s$

Inhomogeneous Terms
$\tilde{\beta}_k^{4,5,7} = \frac{(2(\eta_x\tilde{u}_s + \eta_y\tilde{v}_s) - \tilde{\lambda}_a - \tilde{\lambda}_b)\tilde{r}_4 + \tilde{r}_5}{\tilde{d}_k(\tilde{\lambda}_k - \tilde{\lambda}_a)(\tilde{\lambda}_k - \tilde{\lambda}_b)}, \quad \tilde{\beta}_{1,2} = \mp\frac{\tilde{\lambda}_4\tilde{\beta}_4 + \tilde{\lambda}_5\tilde{\beta}_5 + \tilde{\lambda}_7\tilde{\beta}_7 - (\tilde{\beta}_4 + \tilde{\beta}_5 + \tilde{\beta}_7)\tilde{\lambda}_{2,1} - \tilde{r}_2}{\tilde{\lambda}_1 - \tilde{\lambda}_2}$
and $\tilde{\beta}_{3,6} = 0$ where $\tilde{r}_4 = -\frac{\omega_3}{\rho_s}\Delta p_g + (1 - \omega_4)\Delta(\epsilon_s T_s)$
$\tilde{r}_2 = (1 - \omega_1)\Delta p_g - \rho_s\omega_2\Delta(\epsilon_s T_s) \quad \tilde{r}_5^S = -\frac{2}{3}\tilde{\epsilon}_s\tilde{T}_s\Delta(\eta_x u_s + \eta_y v_s).$
$\tilde{r}_5^P = \frac{2}{3}(\eta_x u_s + \eta_y v_s)\Delta(\epsilon_s T_s).$

Table 3.2: Roe Average Values (superscripts denote formulation)

for each test case. Unless stated otherwise, the high resolution scheme is used with $\Delta x = \Delta y = 1$, a Courant number $\nu = 0.8$ and free flow boundary conditions,

$$\mathbf{w}_{-i,j}^{n+1} = \mathbf{w}_{0,j}^{n}, \quad \mathbf{w}_{I+i,j}^{n+1} = \mathbf{w}_{I,j}^{n}, \quad \mathbf{w}_{i,-j}^{n+1} = \mathbf{w}_{j,0}^{n} \quad \text{and} \quad \mathbf{w}_{i,j+J}^{n+1} = \mathbf{w}_{i,J}^{n}.$$

All of the following test cases are for $\mathbf{g} = 0$.

4.1 Dome Advection

For the first test case, we consider a simple solids advection problem, which has an analytical solution. The analytical solution is derived by assuming that the gas density and phase velocities are constants,

$$\rho_g(x,y,t) = R, \quad u_g(x,y,t) = u_s(x,y,t) = U \quad \text{and} \quad v_g(x,y,t) = v_s(x,y,t) = V,$$

then the model simplifies to

$$(\epsilon_s)_t + U(\epsilon_s)_x + V(\epsilon_s)_y = 0, \quad p_s(x,y,t) = P \quad \text{and} \quad (T_s)_t + U(T_s)_x + V(T_s)_y = 0,$$

where P is a constant. Thus, we obtain the analytical solution

$$\epsilon_s(x,y,t) = \epsilon_s(x - Ut, y - Vt, 0) \quad \text{and} \quad T_s(x,y,t) = \frac{P}{\epsilon_s}.$$

To simulate a dome propagating at 45°, we use the initial conditions,

$$\epsilon_s(x,y,0) = \begin{cases} 0.1 + 0.1\sin^2(\frac{\pi}{20}(x-5))\sin^2(\frac{\pi}{20}(y-5)) & \text{if } 5 \leq (x,y) \leq 25 \\ 0.1 & \text{otherwise} \end{cases}$$

with

$$R = 1.2885, \quad U = V = 5 \quad \text{and} \quad P = 0.01.$$

Figure 4.1 illustrates the results at $t = 10$ s compared to the analytical solution. The scheme propagated the dome to the correct location and preserved the values of the constants. However, dissipation is present in the results of the volume fraction and granular temperature. The dissipation is considerably less than the first order version of the scheme and can be minimised by using a smaller mesh, but at the expense of longer computational run times.

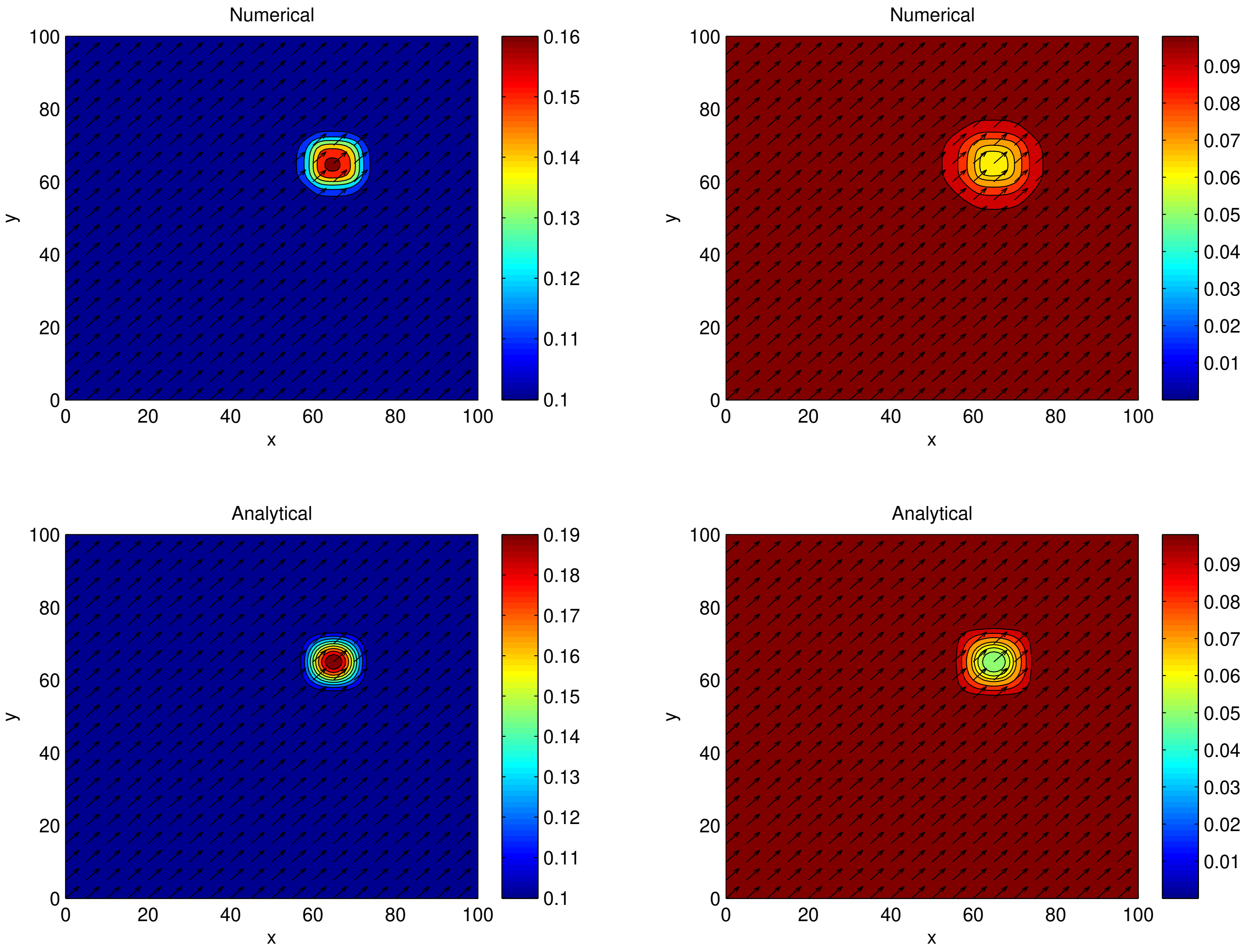

Figure 4.1: Dome Advection Test Case at $t = 10$ s.

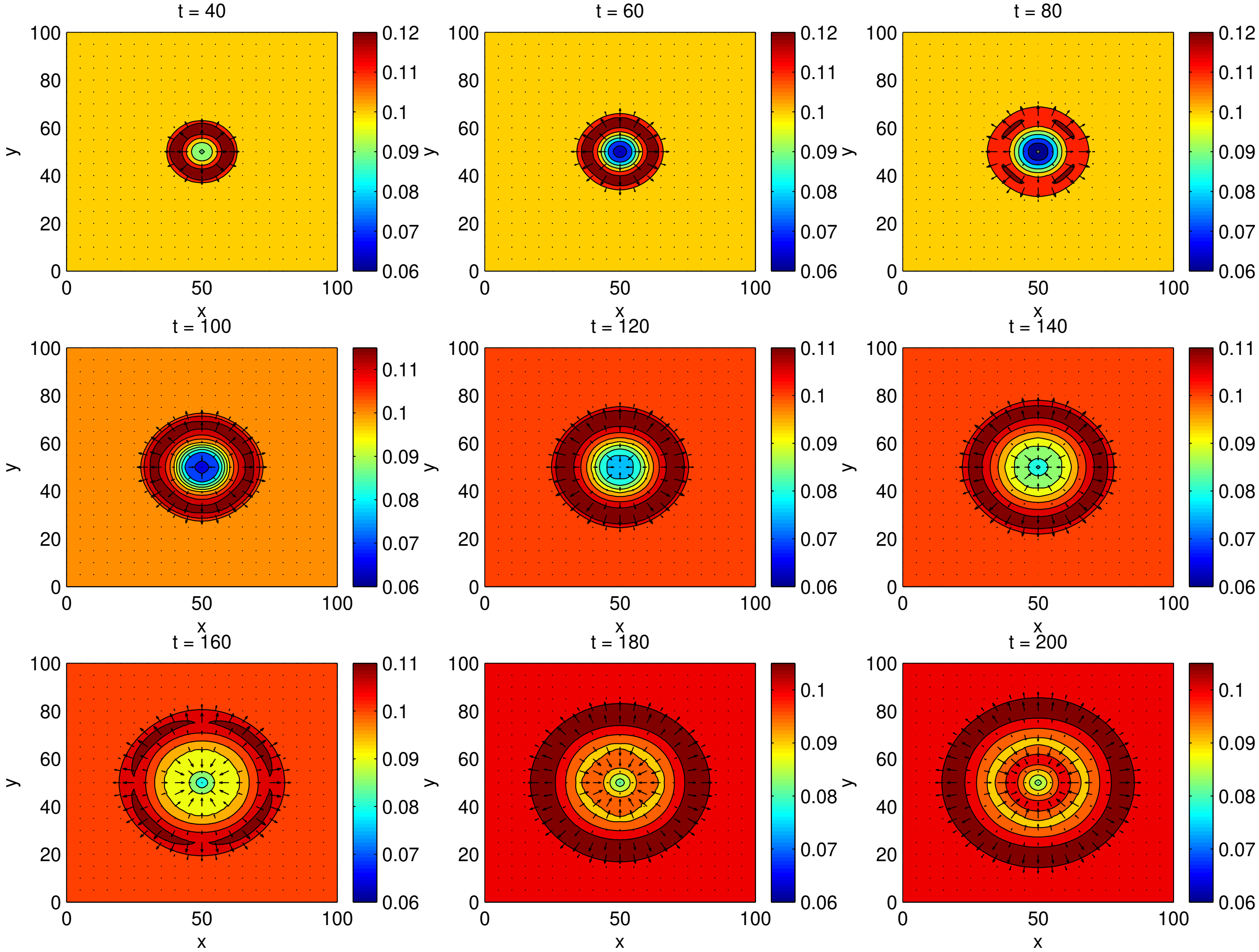

Figure 4.2: Dome Test Case (ϵ_s, u_s & v_s).

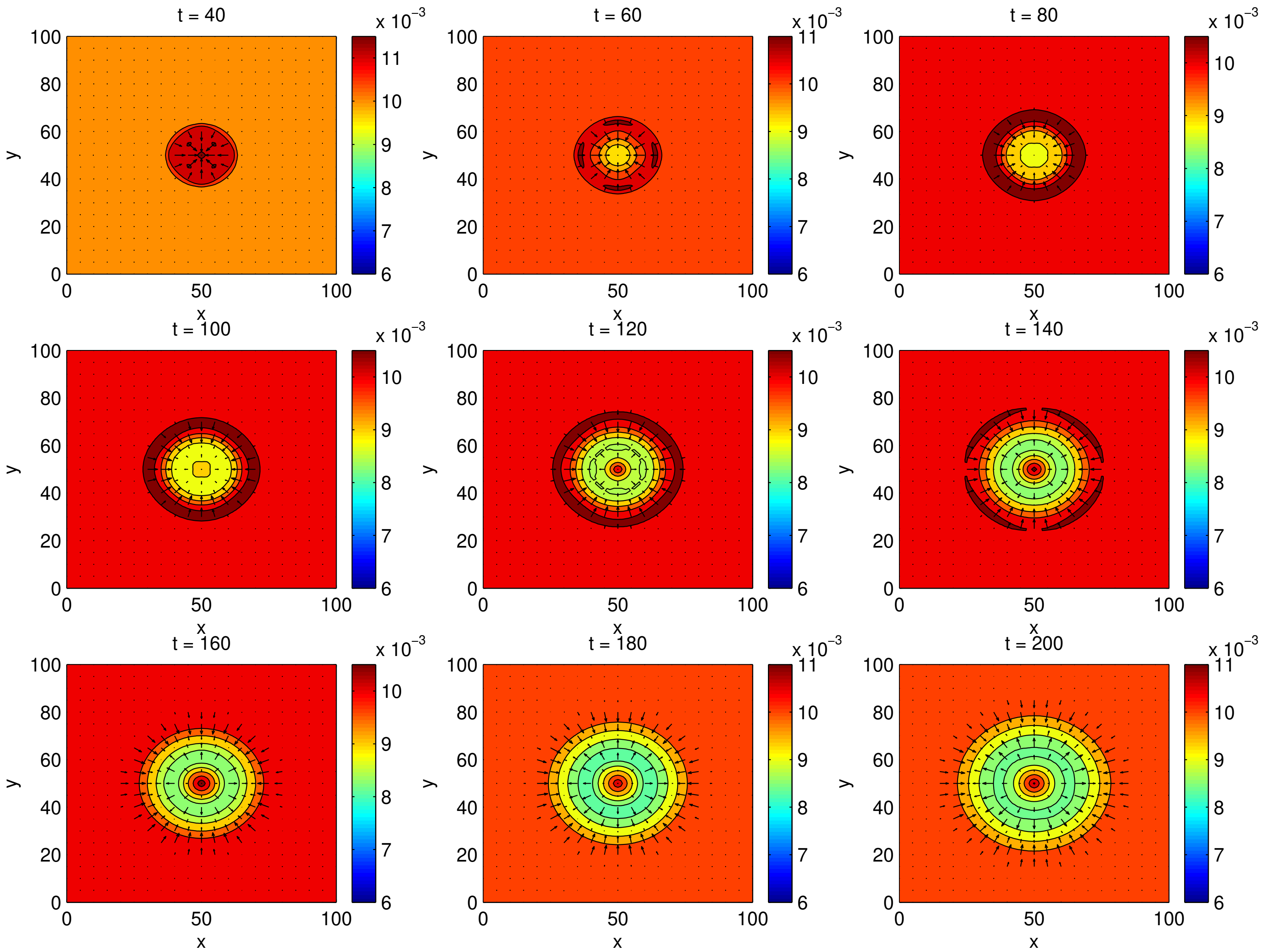

Figure 4.3: Dome Test Case (T_s, u_g & v_g).

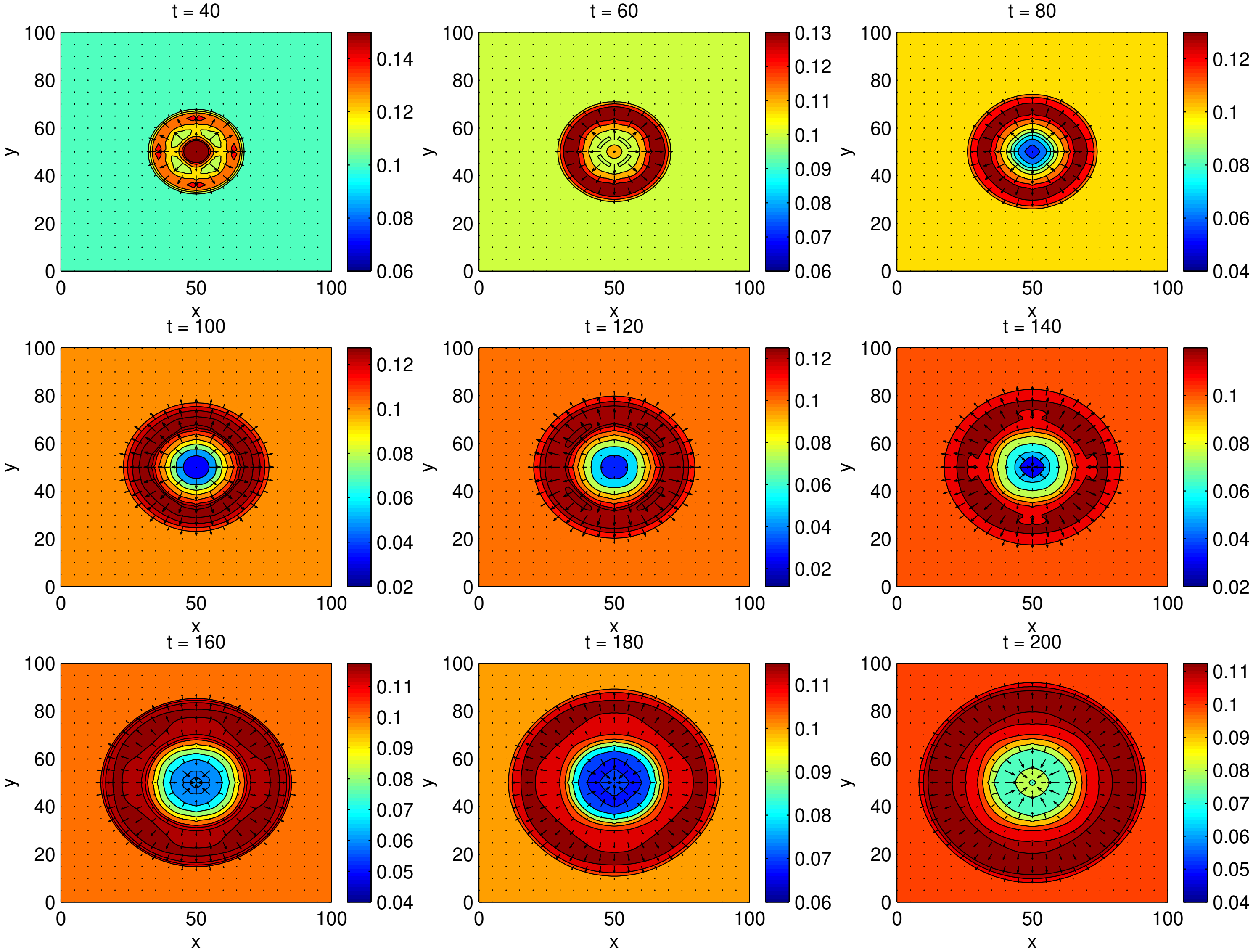

Figure 4.4: Cylinder Test Case (ϵ_s, u_s & v_s).

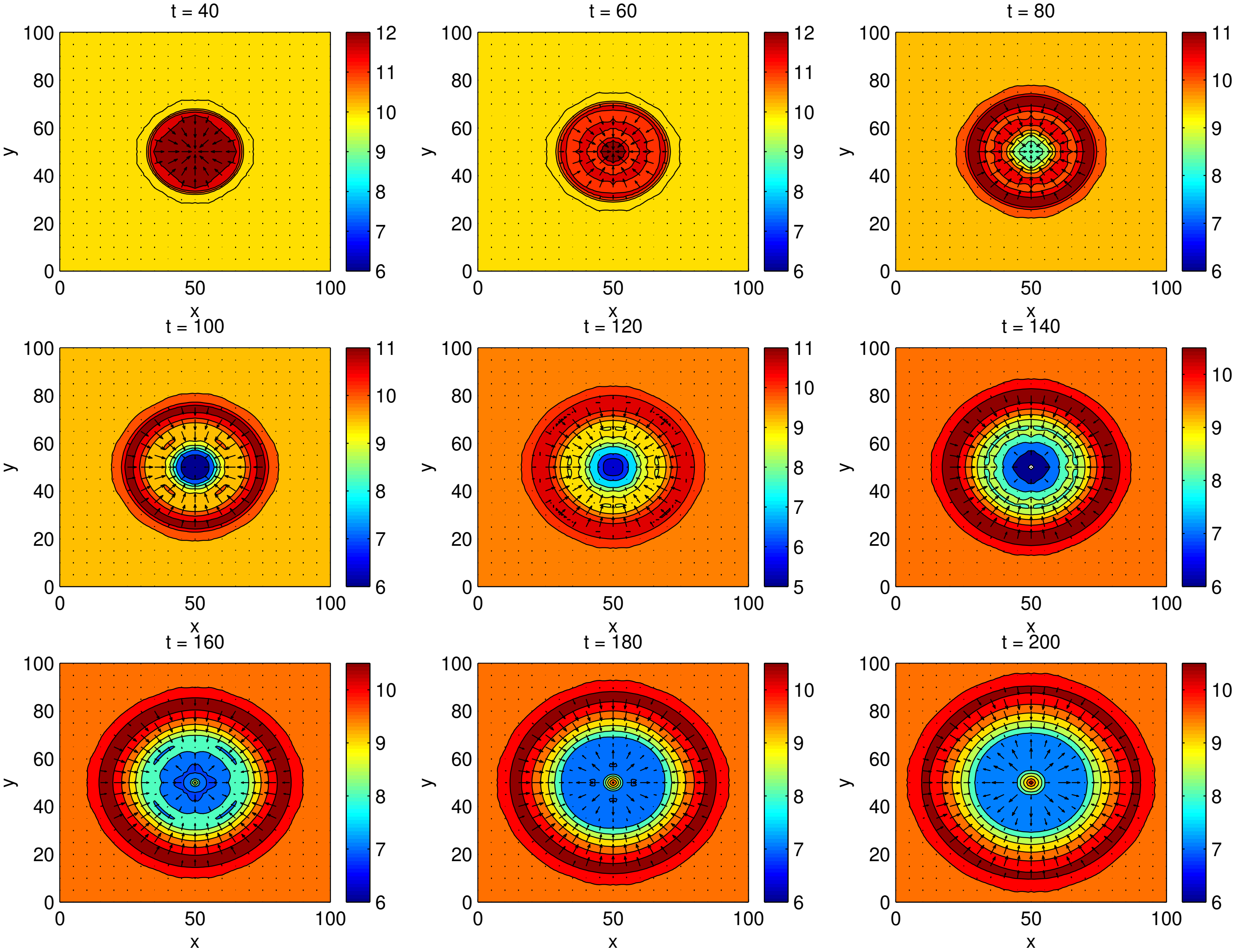

Figure 4.5: Cylinder Test Case (T_s^*, u_g & v_g). Here, $T_s = T_s^* \times 10^{-3}$.

4.2 Dome & Cylinder

For the second and third test cases, we consider a dome or cylinder of solids in the centre of the domain. The initial conditions for this test case are,

$$\rho_g(x,y,0) = 1.2885, \quad u_k(x,y,0) = v_k(x,y,0) = 0, \quad T_s(x,y,0) = 0.1\epsilon_s$$

and either

$$\epsilon_s(x,y,0) = \begin{cases} 0.1 + 0.1\sin^2(\frac{\pi}{20}(x-40))\sin^2(\frac{\pi}{20}(y-40)) & \text{if } 40 \leq (x,y) \leq 60 \\ 0.1 & \text{otherwise} \end{cases}$$

for the dome test case or

$$\epsilon_s(x,y,0) = \begin{cases} 0.2 & \text{if } (x-50)^2 + (y-50)^2 \leq 100 \\ 0.1 & \text{otherwise} \end{cases}$$

for the cylinder test case.

Figures 4.2 & 4.3 and Figures 4.4 & 4.5 illustrate the results until $t = 200$ s for the dome and cylinder test cases, respectively. As expected, the solids volume fraction spreads out in a circular pattern and produces a *pebble effect*, which the scheme accurately captures. Moreover, the results are smooth and free from any numerical problems (e.g. spurious oscillations or kinks) and the scheme accurately captures the shocks. The numerical problem associated with stagnation points (as discussed in [8]) is also not present in the results here due to the problematic term being rewritten and discretised in a different manner.

4.3 Cube

For the forth test case, we consider a cube of solids in the centre of the domain. The initial conditions for this test case are,

$$\epsilon_s(x,y,0) = \begin{cases} 0.2 & \text{if } 40 \leq (x,y) \leq 60 \\ 0.1 & \text{otherwise} \end{cases}$$

with

$$\rho_g(x,y,0) = 1.2885, \quad u_k(x,y,0) = v_k(x,y,0) = 0 \quad \text{and} \quad T_s(x,y,0) = 0.1\epsilon_s.$$

Figures 4.6 & 4.7 illustrate the results until $t = 200$ s. Again, the results are smooth and free from any numerical problems (e.g. spurious oscillations or kinks). Notice that initially, four shock waves propagate away from the cube in different directions, but they eventually merge together into a circular pattern.

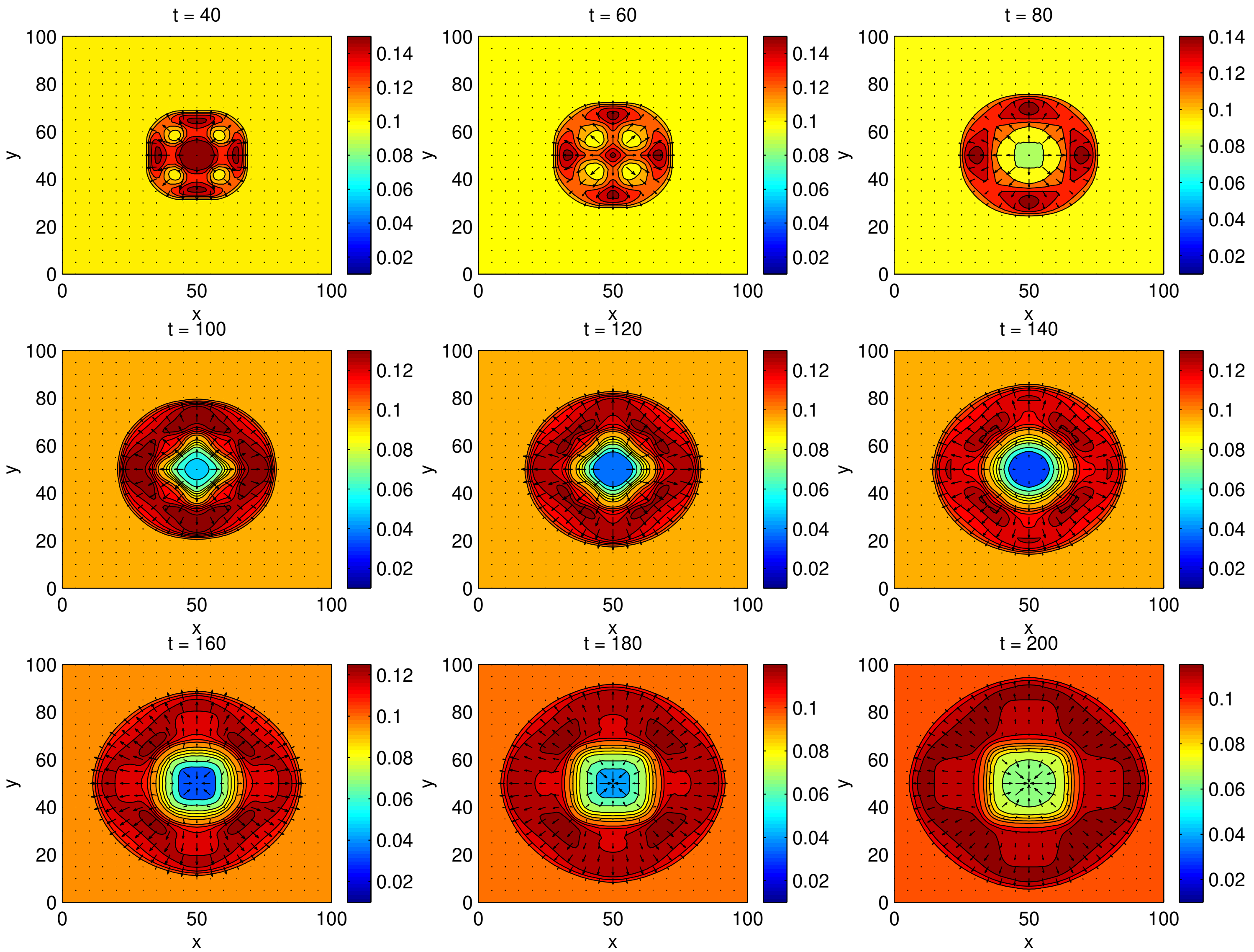

Figure 4.6: Cube Test Case (ϵ_s, u_s & v_s).

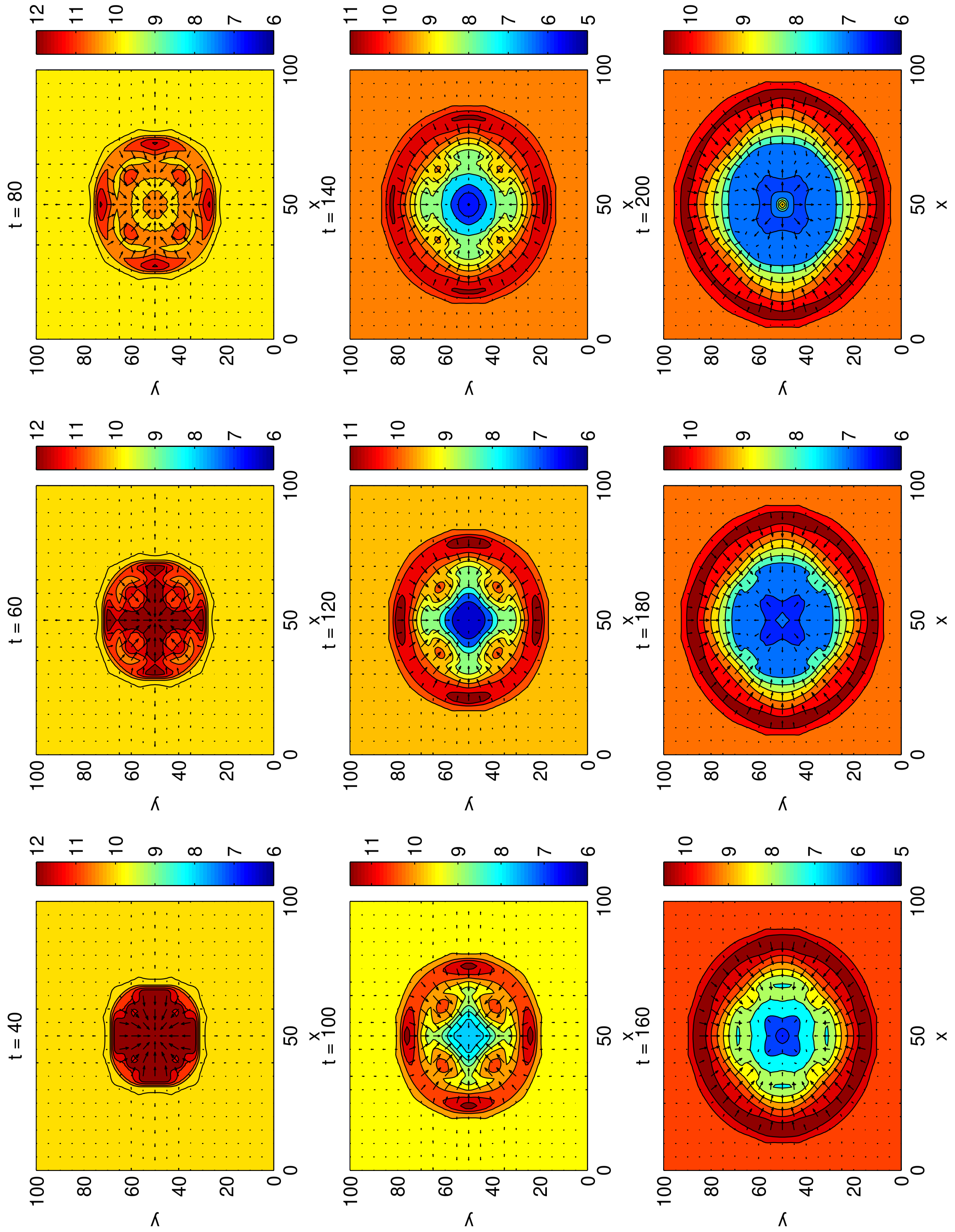

Figure 4.7: Cube Test Case (T_s^*, u_g & v_g). Here, $T_s = T_s^* \times 10^{-3}$.

5 Conclusion

We have extended the 1D isentropic Eulerian gas-solid two-phase model (as discussed by Hudson & Harris [8]) to 2D and presented two different formulations of the model. The high resolution scheme was successfully adapted to 2D and the results obtained for a variety of test cases were promising due to no spurious numerical oscillations occurring in the results. A variety of test cases were investigated to try and determine the behaviour of the model in 2D.

The high resolution scheme accurately captured the circular motion / spreading for the dome and cylinder test cases even though a cartesian mesh was used. However, for such test cases a more appropriate mesh would help improve the accuracy of the results. Thus, an extension of the scheme to a finite volume scheme for variable meshes is desirable.

We only considered the results of Formulation AP since Formulation AS produces a numerical error at stagnation points. Moreover, the results of Model B are practically identical to Model A. This is due to the gas density ρ_g remaining almost constant for the test cases under investigation, which results in the gradient of gas pressure being small. Since Models A and B only differ due to the gas pressure derivatives, there is little difference between them if the gradient of the gas pressure term is negligible.

Thus, we have demonstrated that the high resolution scheme can be used to obtain an accurate numerical solution of Models A & B in 2D.

Acknowledgements

The first author would like to thank the **E**ngineering and **P**hysical **S**cience **R**esearch **C**ouncil for their funding.

A Roe Averages

The numerical scheme discussed in Section 3 is based on Roe's scheme [14] and thus, requires Roe averaged values. In this section, we briefly outline the derivation of the Roe averaged values for the general system.

We require Roe averaged Jacobian matrices (for both $\mathbf{F}$ and $\mathbf{G}$), $\tilde{\mathbf{J}}(\mathbf{w}_L, \mathbf{w}_R)$, which satisfies the following u-properties [14],

- $\tilde{\mathbf{J}}(\mathbf{w}_L, \mathbf{w}_R)$ must be diagonalisable with real eigenvalues (hyperbolicity);
- $\tilde{\mathbf{J}}(\mathbf{w}_L, \mathbf{w}_R) \to \tilde{\mathbf{J}}(\mathbf{w})$ as $\mathbf{w}_L$,$\mathbf{w}_R \to \mathbf{w}$ (consistency);
- $\Delta\mathbf{F} = \tilde{\mathbf{J}}(\mathbf{w}_L, \mathbf{w}_R)\Delta\mathbf{w}$ (conservation).

We obtain such a Roe averaged Jacobian by using the conservation u-property to obtain Roe averaged values of the variables, which can then be used to obtain the eigenvalues and eigenvectors.

We now derive the Roe average values for each flux-function.

A.1 $\mathbf{F}(\mathbf{w})$

We first derive the Roe averages for the Jacobian matrix associated with $\mathbf{F}$, i.e.

1. Formulation S,

$$\tilde{\mathbf{J}}_x = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \tilde{\epsilon}_g^{-1}\tilde{c}_g^2 - \tilde{u}_g^2 & 2\tilde{u}_g & 0 & \tilde{\rho}_g\tilde{\epsilon}_g^{-1}\tilde{c}_g^2 & 0 & 0 & 0 \\ -\tilde{u}_g\tilde{v}_g & \tilde{v}_g & \tilde{u}_g & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \tilde{\epsilon}_s\tilde{T}_s - \tilde{u}_s^2 & 2\tilde{u}_s & 0 & 1 \\ 0 & 0 & 0 & -\tilde{u}_s\tilde{v}_s & \tilde{v}_s & \tilde{u}_s & 0 \\ 0 & 0 & 0 & -\tilde{u}_s\tilde{T}_s & \tilde{T}_s & 0 & \tilde{u}_s \end{bmatrix} \quad \text{and} \quad \mathbf{F} = \begin{bmatrix} \epsilon_g\rho_g u_g \\ \epsilon_g\rho_g u_g^2 + p_g \\ \epsilon_g\rho_g u_g v_g \\ \epsilon_s u_s \\ \epsilon_s u_s^2 + \epsilon_s T_s \\ \epsilon_s u_s v_s \\ \epsilon_s T_s u_s \end{bmatrix}.$$

2. Formulation P,

$$\tilde{\mathbf{J}}_x = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \tilde{\epsilon}_g^{-1}\tilde{c}_g^2 - \tilde{u}_g^2 & 2\tilde{u}_g & 0 & \tilde{\rho}_g\tilde{\epsilon}_g^{-1}\tilde{c}_g^2 & 0 & 0 & 0 \\ -\tilde{u}_g\tilde{v}_g & \tilde{v}_g & \tilde{u}_g & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \tilde{\epsilon}_s\tilde{T}_s - \tilde{u}_s^2 & 2\tilde{u}_s & 0 & 1 \\ 0 & 0 & 0 & -\tilde{u}_s\tilde{v}_s & \tilde{v}_s & \tilde{u}_s & 0 \\ 0 & 0 & 0 & -\frac{5}{3}\tilde{u}_s\tilde{T}_s & \frac{5}{3}\tilde{T}_s & 0 & \frac{5}{3}\tilde{u}_s \end{bmatrix} \quad \text{and} \quad \mathbf{F} = \begin{bmatrix} \epsilon_g\rho_g u_g \\ \epsilon_g\rho_g u_g^2 + p_g \\ \epsilon_g\rho_g u_g v_g \\ \epsilon_s u_s \\ \epsilon_s u_s^2 + \epsilon_s T_s \\ \epsilon_s u_s v_s \\ \frac{5}{3}\epsilon_s T_s u_s \end{bmatrix}.$$

A.1.1 Values

We require Roe average values, which can be obtained by satisfying

$$\Delta\mathbf{F} = \tilde{\mathbf{J}}_x \Delta\mathbf{w}.$$

Both formulations result in ensuring that the following equations are satisfied:

$$\Delta(\epsilon_g\rho_g u_g) = \Delta(\epsilon_g\rho_g u_g), \tag{A.1a}$$

$$\Delta(\epsilon_g\rho_g u_g^2) + \Delta p_g = (\tilde{c}_g^2\tilde{\epsilon}_g^{-1} - \tilde{u}_g^2)\Delta(\epsilon_g\rho_g) + 2\tilde{u}_g\Delta(\epsilon_g\rho_g u_g) + \tilde{\rho}_g\tilde{\epsilon}_g^{-1}\tilde{c}_g^2\Delta\epsilon_s, \tag{A.1b}$$

$$\Delta(\epsilon_g\rho_g u_g v_g) = -\tilde{u}_g\tilde{v}_g\Delta(\epsilon_g\rho_g) + \tilde{v}_g\Delta(\epsilon_g\rho_g u_g) + \tilde{u}_g\Delta(\epsilon_g\rho_g v_g), \tag{A.1c}$$

$$\Delta(\epsilon_s u_s) = \Delta(\epsilon_s u_s), \tag{A.1d}$$

$$\Delta(\epsilon_s u_s^2) + \Delta(\epsilon_s T_s) = -\tilde{u}_s^2\Delta\epsilon_s + 2\tilde{u}_s\Delta(\epsilon_s u_s) + \Delta(\epsilon_s T_s), \tag{A.1e}$$

$$\Delta(\epsilon_s u_s v_s) = -\tilde{u}_s\tilde{v}_s\Delta\epsilon_s + \tilde{v}_s\Delta(\epsilon_s u_s) + \tilde{u}_s\Delta(\epsilon_s v_s) \tag{A.1f}$$

and

$$\Delta(\epsilon_s u_s T_s) = -\tilde{u}_s\tilde{T}_s\Delta\epsilon_s + \tilde{T}_s\Delta(\epsilon_s u_s) + \tilde{u}_s\Delta(\epsilon_s T_s). \tag{A.1g}$$

Clearly, equations (A.1a) and (A.1d) are automatically satisfied and by letting

$$\tilde{u}_g^2\Delta(\epsilon_g\rho_g) - 2\tilde{u}_g\Delta(\epsilon_g\rho_g u_g) + \Delta(\epsilon_g\rho_g u_g^2) = 0$$

and

$$\tilde{u}_s^2\Delta\epsilon_s - 2\tilde{u}_s\Delta(\epsilon_s u_s) + \Delta(\epsilon_s u_s^2) = 0,$$

we obtain

$$\tilde{u}_g = \frac{\sqrt{(\epsilon_g\rho_g)_L}(u_g)_L + \sqrt{(\epsilon_g\rho_g)_R}(u_g)_R}{\sqrt{(\epsilon_g\rho_g)_L} + \sqrt{(\epsilon_g\rho_g)_R}} \quad \text{and} \quad \tilde{u}_s = \frac{\sqrt{(\epsilon_s)_L}(u_s)_L + \sqrt{(\epsilon_s)_R}(u_s)_R}{\sqrt{(\epsilon_s)_L} + \sqrt{(\epsilon_s)_R}}.$$

From (A.1c), (A.1f) and (A.1g) by substitution we can easily obtain

$$\tilde{T}_s = \frac{\sqrt{(\epsilon_s)_L}(T_s)_L + \sqrt{(\epsilon_s)_R}(T_s)_R}{\sqrt{(\epsilon_s)_L} + \sqrt{(\epsilon_s)_R}}, \quad \tilde{v}_s = \frac{\sqrt{(\epsilon_s)_L}(v_s)_L + \sqrt{(\epsilon_s)_R}(v_s)_R}{\sqrt{(\epsilon_s)_L} + \sqrt{(\epsilon_s)_R}}$$

and

$$\tilde{v}_g = \frac{\sqrt{(\epsilon_g\rho_g)_L}(v_g)_L + \sqrt{(\epsilon_g\rho_g)_R}(v_g)_R}{\sqrt{(\epsilon_g\rho_g)_L} + \sqrt{(\epsilon_g\rho_g)_R}}.$$

Now, we are left with

$$\tilde{\epsilon}_g\Delta p_g = \tilde{c}_g^2(\Delta(\epsilon_g\rho_g) - \tilde{\rho}_g\Delta\epsilon_g)$$

By substituting $\tilde{c}_g^2 = \frac{\Delta p_g}{\Delta \rho_g}$, we obtain

$$\Delta(\epsilon_g \rho_g) = \tilde{\epsilon}_g \Delta \rho_g + \tilde{\rho}_g \Delta \epsilon_g.$$

Here, we have two options:

$$\tilde{\epsilon}_g = \frac{1}{2}((\epsilon_g)_L + (\epsilon_g)_R) \quad \text{and} \quad \tilde{\rho}_g = \frac{1}{2}((\rho_g)_L + (\rho_g)_R) \tag{A.2}$$

or

$$\tilde{\epsilon}_g = \sqrt{(\epsilon_g)_L(\epsilon_g)_R} \quad \text{and} \quad \tilde{\rho}_g = \frac{\sqrt{(\epsilon_g)_L}(\rho_g)_L + \sqrt{(\epsilon_g)_R}(\rho_g)_R}{\sqrt{(\epsilon_g)_L} + \sqrt{(\epsilon_g)_R}}. \tag{A.3}$$

Thus, all the Roe averages have been obtained.

A.1.2 Eigenvalues & Eigenvectors

The Roe averaged eigenvalues are the same as the original values,

1. Formulation S,

$$\tilde{\lambda}_{1,2} = \tilde{u}_g \pm \tilde{c}_g \sqrt{\tilde{\epsilon}_g^{-1}}, \quad \tilde{\lambda}_3 = \tilde{u}_g, \quad \tilde{\lambda}_{4,5} = \tilde{u}_s \pm \sqrt{\tilde{T}_s} \quad \text{and} \quad \tilde{\lambda}_{6,7} = \tilde{u}_s.$$

2. Formulation P,

$$\tilde{\lambda}_{1,2} = \tilde{u}_g \pm \tilde{c}_g \sqrt{\tilde{\epsilon}_g^{-1}}, \quad \tilde{\lambda}_3 = \tilde{u}_g, \quad \tilde{\lambda}_{4,5} = \frac{4}{3}\tilde{u}_s \pm \frac{1}{3}\sqrt{\tilde{u}_s^2 + 15\tilde{T}_s} \quad \text{and} \quad \tilde{\lambda}_{6,7} = \tilde{u}_s.$$

For simplicity (and computational efficiency), the eigenvectors are determined in terms of the eigenvalues:

1. Formulation S,

$$\begin{bmatrix} -\tilde{\lambda} & 1 & 0 & 0 & 0 & 0 & 0 \\ \tilde{\epsilon}_g^{-1}\tilde{c}_g^2 - \tilde{u}_g^2 & 2\tilde{u}_g - \tilde{\lambda} & 0 & \tilde{\rho}_g \tilde{\epsilon}_g^{-1}\tilde{c}_g^2 & 0 & 0 & 0 \\ -\tilde{u}_g \tilde{v}_g & \tilde{v}_g & \tilde{u}_g - \tilde{\lambda} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tilde{\lambda} & 1 & 0 & 0 \\ 0 & 0 & 0 & \tilde{\epsilon}_s \tilde{T}_s - \tilde{u}_s^2 & 2\tilde{u}_s - \tilde{\lambda} & 0 & 1 \\ 0 & 0 & 0 & -\tilde{u}_s \tilde{v}_s & \tilde{v}_s & \tilde{u}_s - \tilde{\lambda} & 0 \\ 0 & 0 & 0 & -\tilde{u}_s \tilde{T}_s & \tilde{T}_s & 0 & \tilde{u}_s - \tilde{\lambda} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{bmatrix} = 0.$$

2. Formulation P,

$$\begin{bmatrix} -\tilde{\lambda} & 1 & 0 & 0 & 0 & 0 & 0 \\ \tilde{\epsilon}_g^{-1}\tilde{c}_g^2 - \tilde{u}_g^2 & 2\tilde{u}_g - \tilde{\lambda} & 0 & \tilde{\rho}_g\tilde{\epsilon}_g^{-1}\tilde{c}_g^2 & 0 & 0 & 0 \\ -\tilde{u}_g\tilde{v}_g & \tilde{v}_g & \tilde{u}_g - \tilde{\lambda} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tilde{\lambda} & 1 & 0 & 0 \\ 0 & 0 & 0 & \tilde{\epsilon}_s\tilde{T}_s - \tilde{u}_s^2 & 2\tilde{u}_s - \tilde{\lambda} & 0 & 1 \\ 0 & 0 & 0 & -\tilde{u}_s\tilde{v}_s & \tilde{v}_s & \tilde{u}_s - \tilde{\lambda} & 0 \\ 0 & 0 & 0 & -\frac{5}{3}\tilde{u}_s\tilde{T}_s & \frac{5}{3}\tilde{T}_s & 0 & \frac{5}{3}\tilde{u}_s - \tilde{\lambda} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{bmatrix} = 0.$$

Thus,

$$e_2 = \tilde{\lambda} e_1, \tag{A.4a}$$

$$(\tilde{\epsilon}_g^{-1}\tilde{c}_g^2 - \tilde{u}_g^2)e_1 + (2\tilde{u}_g - \tilde{\lambda})e_2 + \tilde{\rho}_g\tilde{\epsilon}_g^{-1}\tilde{c}_g^2 e_4 = 0, \tag{A.4b}$$

$$-\tilde{u}_g\tilde{v}_g e_1 + \tilde{v}_g e_2 + (\tilde{u}_g - \tilde{\lambda})e_3 = 0, \tag{A.4c}$$

$$e_5 = \tilde{\lambda} e_4, \tag{A.4d}$$

$$-\tilde{u}_s^2 e_4 + (2\tilde{u}_s - \tilde{\lambda})e_5 + e_7 = 0, \tag{A.4e}$$

$$-\tilde{u}_s\tilde{v}_s e_4 + \tilde{v}_s e_5 + (\tilde{u}_s - \tilde{\lambda})e_6 = 0 \tag{A.4f}$$

and either

$$-\tilde{u}_s\tilde{T}_s e_4 + \tilde{T}_s e_5 + (\tilde{u}_s - \tilde{\lambda})e_7 = 0 \tag{A.4g}$$

for Formulation S or

$$-\frac{5}{3}\tilde{u}_s\tilde{T}_s e_4 + \frac{5}{3}\tilde{T}_s e_5 + (\frac{5}{3}\tilde{u}_s - \tilde{\lambda})e_7 = 0 \tag{A.4h}$$

for Formulation P. Now, by substituting (A.4a) into (A.4b), we obtain

$$e_4 = \left(\frac{\tilde{u}_g^2\tilde{\epsilon}_g - \tilde{c}_g^2 - (2\tilde{u}_g - \tilde{\lambda})\tilde{\epsilon}_g\tilde{\lambda}}{\tilde{\rho}_g\tilde{c}_g^2} \right) e_1 = \left(\frac{(\tilde{\lambda} - \tilde{u}_g)^2\tilde{\epsilon}_g - \tilde{c}_g^2}{\tilde{\rho}_g\tilde{c}_g^2} \right) e_1$$

and similarly for (A.4c),

$$(\tilde{u}_g - \tilde{\lambda})(e_3 - \tilde{v}_g e_1) = 0.$$

Also, by substituting (A.4d) into (A.4e) and (A.4f), we obtain

$$(\tilde{u}_s - \tilde{\lambda})(e_6 - \tilde{v}_s e_4) = 0 \quad \text{and} \quad e_7 = e_4(\tilde{\lambda} - \tilde{u}_s)^2,$$

respectively.

Thus, we obtain

$$\tilde{\mathbf{e}}_{1,2} = \begin{bmatrix} 1 \\ \tilde{\lambda}_{1,2} \\ \tilde{v}_g \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{\mathbf{e}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{\mathbf{e}}_{4,5} = \begin{bmatrix} 1 \\ \tilde{\lambda}_{4,5} \\ \tilde{v}_g \\ \tilde{d}_{4,5} \\ \tilde{\lambda}_{4,5}\tilde{d}_{4,5} \\ \tilde{v}_s\tilde{d}_{4,5} \\ \tilde{d}_{4,5}(\tilde{\lambda}_{4,5} - \tilde{u}_s)^2 \end{bmatrix},$$

$$\tilde{\mathbf{e}}_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{e}}_7 = \begin{bmatrix} 1 \\ \tilde{u}_s \\ \tilde{v}_g \\ \tilde{d}_7 \\ \tilde{u}_s\tilde{d}_7 \\ \tilde{v}_s\tilde{d}_7 \\ 0 \end{bmatrix},$$

where

$$\tilde{d}_k = \frac{(\tilde{\lambda}_k - \tilde{u}_g)^2\tilde{\epsilon}_g - \tilde{c}_g^2}{\tilde{\rho}_g\tilde{c}_g^2}.$$

A.1.3 Wave Strengths

We obtain wave strengths are determined through the decomposition,

$$\Delta\mathbf{w} = \sum_{k=1}^{7} \tilde{\alpha}_k\tilde{\mathbf{e}}_k,$$

which for both formulations results in solving

$$\Delta(\epsilon_g\rho_g) = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_4 + \tilde{\alpha}_5 + \tilde{\alpha}_7, \tag{A.5a}$$

$$\Delta(\epsilon_g\rho_g u_g) = \tilde{\lambda}_1\tilde{\alpha}_1 + \tilde{\lambda}_2\tilde{\alpha}_2 + \tilde{\lambda}_4\tilde{\alpha}_4 + \tilde{\lambda}_5\tilde{\alpha}_5 + \tilde{\lambda}_7\tilde{\alpha}_7, \tag{A.5b}$$

$$\Delta(\epsilon_g\rho_g v_g) = \tilde{v}_g\tilde{\alpha}_1 + \tilde{v}_g\tilde{\alpha}_2 + \tilde{\alpha}_3 + \tilde{v}_g\tilde{\alpha}_4 + \tilde{v}_g\tilde{\alpha}_5 + \tilde{v}_g\tilde{\alpha}_7, \tag{A.5c}$$

$$\Delta\epsilon_s = \tilde{d}_4\tilde{\alpha}_4 + \tilde{d}_5\tilde{\alpha}_5 + \tilde{d}_7\tilde{\alpha}_7 \tag{A.5d}$$

$$\Delta(\epsilon_s u_s) = \tilde{\lambda}_4\tilde{d}_4\tilde{\alpha}_4 + \tilde{\lambda}_5\tilde{d}_5\tilde{\alpha}_5 + \tilde{\lambda}_7\tilde{d}_7\tilde{\alpha}_7, \tag{A.5e}$$

$$\Delta(\epsilon_s v_s) = \tilde{v}_s\tilde{d}_4\tilde{\alpha}_4 + \tilde{v}_s\tilde{d}_5\tilde{\alpha}_5 + \tilde{\alpha}_6 + \tilde{v}_s\tilde{d}_7\tilde{\alpha}_7 \tag{A.5f}$$

and

$$\Delta(\epsilon_s T_s) = \tilde{d}_4(\tilde{\lambda}_4 - \tilde{u}_s)^2\tilde{\alpha}_4 + \tilde{d}_5(\tilde{\lambda}_5 - \tilde{u}_s)^2\tilde{\alpha}_5 \tag{A.5g}$$

for $\tilde{\alpha}_k$. Unfortunately, by solving these set of simultaneous equations, algebraic expressions of $\tilde{\alpha}_{1,2}$ are complicated. To keep the algebraic expressions in a compact form, we solve (A.5d), (A.5e) and (A.5g) to obtain $\tilde{\alpha}_{4,5,7}$,

$$\tilde{\alpha}_k = \frac{(\tilde{\lambda}_a\tilde{\lambda}_b - \tilde{u}_s^2)\Delta\epsilon_s - (\tilde{\lambda}_a + \tilde{\lambda}_b - 2\tilde{u}_s)\Delta(\epsilon_s u_s) + \Delta(\epsilon_s T_s)}{\tilde{d}_k(\tilde{\lambda}_k - \tilde{\lambda}_a)(\tilde{\lambda}_k - \tilde{\lambda}_b)}$$

where $k \neq a \neq b$, and then solve $\tilde{\alpha}_{1,2}$ in terms of $\tilde{\alpha}_{4,5,7}$,

$$\tilde{\alpha}_{1,2} = \mp\frac{\tilde{\lambda}_4\tilde{\alpha}_4 + \tilde{\lambda}_5\tilde{\alpha}_5 + \tilde{\lambda}_7\tilde{\alpha}_7 - (\tilde{\alpha}_4 + \tilde{\alpha}_5 + \tilde{\alpha}_7 - \Delta(\epsilon_g\rho_g))\tilde{\lambda}_{2,1} - \Delta(\epsilon_g\rho_g u_g)}{\tilde{\lambda}_1 - \tilde{\lambda}_2}.$$

We can then easily obtain the last two wavestrengths by solving,

$$\tilde{\alpha}_3 = \Delta(\epsilon_g\rho_g v_g) - \tilde{v}_g\Delta(\epsilon_g\rho_g) = \sqrt{(\epsilon_g\rho_g)_R(\epsilon_g\rho_g)_L}\Delta v_g$$

and

$$\tilde{\alpha}_6 = \Delta(\epsilon_s v_s) - \tilde{v}_s\Delta\epsilon_s = \sqrt{(\epsilon_s)_R(\epsilon_s)_L}\Delta v_s.$$

A.1.4 Inhomogeneous Terms

The values of $\tilde{\beta}_k$ are also determined from the decomposition,

$$\Delta x\tilde{\mathbf{f}} = \sum_{k=1}^{7}\tilde{\beta}_k\tilde{\mathbf{e}}_k,$$

which for both formulations results in solving

$$0 = \tilde{\beta}_1 + \tilde{\beta}_2 + \tilde{\beta}_4 + \tilde{\beta}_5 + \tilde{\beta}_7, \tag{A.6a}$$

$$\tilde{r}_2 = \tilde{\lambda}_1\tilde{\beta}_1 + \tilde{\lambda}_2\tilde{\beta}_2 + \tilde{\lambda}_4\tilde{\beta}_4 + \tilde{\lambda}_5\tilde{\beta}_5 + \tilde{\lambda}_7\tilde{\beta}_7, \tag{A.6b}$$

$$0 = \tilde{v}_g\tilde{\beta}_1 + \tilde{v}_g\tilde{\beta}_2 + \tilde{\beta}_3 + \tilde{v}_g\tilde{\beta}_4 + \tilde{v}_g\tilde{\beta}_5 + \tilde{v}_g\tilde{\beta}_7, \tag{A.6c}$$

$$0 = \tilde{d}_4\tilde{\beta}_4 + \tilde{d}_5\tilde{\beta}_5 + \tilde{d}_7\tilde{\beta}_7, \tag{A.6d}$$

$$\tilde{r}_5 = \tilde{\lambda}_4\tilde{d}_4\tilde{\beta}_4 + \tilde{\lambda}_5\tilde{d}_5\tilde{\beta}_5 + \tilde{\lambda}_7\tilde{d}_7\tilde{\beta}_7, \tag{A.6e}$$

$$0 = \tilde{v}_s\tilde{d}_4\tilde{\beta}_4 + \tilde{v}_s\tilde{d}_5\tilde{\beta}_5 + \tilde{\beta}_6 + \tilde{v}_s\tilde{d}_7\tilde{\beta}_7 \tag{A.6f}$$

and

$$\tilde{r}_7 = \tilde{d}_4(\tilde{\lambda}_4 - \tilde{u}_s)^2\tilde{\beta}_4 + \tilde{d}_5(\tilde{\lambda}_5 - \tilde{u}_s)^2\tilde{\beta}_5 \tag{A.6g}$$

for $\tilde{\beta}_k$. Here,

$$\tilde{r}_2 = (1-\omega_1)\Delta p_g - \rho_s\omega_2\Delta(\epsilon_s T_s),$$
$$\tilde{r}_4 = -\frac{\omega_3}{\rho_s}\Delta p_g + (1-\omega_4)\Delta(\epsilon_s T_s),$$
$$\tilde{r}_7^S = -\frac{2}{3}\tilde{\epsilon}_s\tilde{T}_s\Delta u_s \quad \text{and} \quad \tilde{r}_7^P = \frac{2}{3}\tilde{u}_s\Delta(\epsilon_s T_s).$$

As with the wave strengths, algebraic expressions of $\tilde{\beta}_{1,2}$ are complicated. Thus, we solve (A.6d), (A.6e) and (A.6g) to obtain $\tilde{\beta}_{4,5,7}$,

$$\tilde{\beta}_k = \frac{(2\tilde{u}_s - \tilde{\lambda}_a - \tilde{\lambda}_b)\tilde{r}_5 + \tilde{r}_7}{\tilde{d}_k(\tilde{\lambda}_k - \tilde{\lambda}_a)(\tilde{\lambda}_k - \tilde{\lambda}_b)}$$

where $k \neq a \neq b$, and then solve $\tilde{\beta}_{1,2}$ in terms of $\tilde{\beta}_{4,5,7}$,

$$\tilde{\beta}_{1,2} = \mp\frac{\tilde{\lambda}_4\tilde{\beta}_4 + \tilde{\lambda}_5\tilde{\beta}_5 + \tilde{\lambda}_7\tilde{\beta}_7 - (\tilde{\beta}_4 + \tilde{\beta}_5 + \tilde{\beta}_7)\tilde{\lambda}_{2,1} - \tilde{r}_2}{\tilde{\lambda}_1 - \tilde{\lambda}_2}.$$

The last two values of $\tilde{\beta}$ can now be solved,

$$\tilde{\beta}_3 = 0 \quad \text{and} \quad \tilde{\beta}_6 = 0.$$

A.2 G(w)

We first derive the Roe averages for the Jacobian matrix associated with **G**, i.e.

1. Formulation S,

$$\tilde{\mathbf{J}}_y = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\tilde{u}_g\tilde{v}_g & \tilde{v}_g & \tilde{u}_g & 0 & 0 & 0 & 0 \\ \tilde{\epsilon}_g^{-1}\tilde{c}_g^2 - \tilde{v}_g^2 & 0 & 2\tilde{v}_g & \tilde{\rho}_g\tilde{\epsilon}_g^{-1}\tilde{c}_g^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\tilde{u}_s\tilde{v}_s & \tilde{v}_s & \tilde{u}_s & 0 \\ 0 & 0 & 0 & \tilde{\epsilon}_s\tilde{T}_s - \tilde{v}_s^2 & 0 & 2\tilde{v}_s & 1 \\ 0 & 0 & 0 & -\tilde{v}_s\tilde{T}_s & 0 & \tilde{T}_s & \tilde{v}_s \end{bmatrix} \quad \text{and} \quad \mathbf{G} = \begin{bmatrix} \epsilon_g\rho_g v_g \\ \epsilon_g\rho_g u_g v_g \\ \epsilon_g\rho_g v_g^2 + p_g \\ \epsilon_s v_s \\ \epsilon_s v_s u_s \\ \epsilon_s v_s^2 + \epsilon_s T_s \\ \epsilon_s v_s T_s \end{bmatrix}.$$

2. Formulation P,

$$\tilde{\mathbf{J}}_y = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\tilde{u}_g\tilde{v}_g & \tilde{v}_g & \tilde{u}_g & 0 & 0 & 0 & 0 \\ \tilde{\epsilon}_g^{-1}\tilde{c}_g^2 - \tilde{v}_g^2 & 0 & 2\tilde{v}_g & \tilde{\rho}_g\tilde{\epsilon}_g^{-1}\tilde{c}_g^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\tilde{u}_s\tilde{v}_s & \tilde{v}_s & \tilde{u}_s & 0 \\ 0 & 0 & 0 & \tilde{\epsilon}_s\tilde{T}_s - \tilde{v}_s^2 & 0 & 2\tilde{v}_s & 1 \\ 0 & 0 & 0 & -\frac{5}{3}\tilde{v}_s\tilde{T}_s & 0 & \frac{5}{3}\tilde{T}_s & \frac{5}{3}\tilde{v}_s \end{bmatrix} \quad \text{and} \quad \mathbf{G} = \begin{bmatrix} \epsilon_g\rho_g v_g \\ \epsilon_g\rho_g u_g v_g \\ \epsilon_g\rho_g v_g^2 + p_g \\ \epsilon_s v_s \\ \epsilon_s v_s u_s \\ \epsilon_s v_s^2 + \epsilon_s T_s \\ \frac{5}{3}\epsilon_s v_s T_s \end{bmatrix}.$$

A.2.1 Values

We require Roe average values, which can be obtained by satisfying

$$\Delta\mathbf{G} = \tilde{\mathbf{J}}_y\Delta\mathbf{w}.$$

Thus, for both formulations we must ensure that

$$\Delta(\epsilon_g\rho_g v_g) = \Delta(\epsilon_g\rho_g v_g), \tag{A.7a}$$

$$\Delta(\epsilon_g\rho_g u_g v_g) = -\tilde{u}_g\tilde{v}_g\Delta(\epsilon_g\rho_g) + \tilde{v}_g\Delta(\epsilon_g\rho_g u_g) + \tilde{u}_g\Delta(\epsilon_g\rho_g v_g), \tag{A.7b}$$

$$\Delta(\epsilon_g\rho_g v_g^2) + \Delta p_g = (\tilde{c}_g^2\tilde{\epsilon}_g^{-1} - \tilde{v}_g^2)\Delta(\epsilon_g\rho_g) + 2\tilde{v}_g\Delta(\epsilon_g\rho_g v_g) + \tilde{\rho}_g\tilde{\epsilon}_g^{-1}\tilde{c}_g^2\Delta\epsilon_s, \tag{A.7c}$$

$$\Delta(\epsilon_s v_s) = \Delta(\epsilon_s v_s), \tag{A.7d}$$

$$\Delta(\epsilon_s u_s v_s) = -\tilde{u}_s\tilde{v}_s\Delta\epsilon_s + \tilde{v}_s\Delta(\epsilon_s u_s) + \tilde{u}_s\Delta(\epsilon_s v_s) \tag{A.7e}$$

$$\Delta(\epsilon_s v_s^2) + \Delta(\epsilon_s T_s) = -\tilde{v}_s^2\Delta\epsilon_s + 2\tilde{v}_s\Delta(\epsilon_s v_s) + \Delta(\epsilon_s T_s), \tag{A.7f}$$

and

$$\Delta(\epsilon_s v_s T_s) = -\tilde{v}_s\tilde{T}_s\Delta\epsilon_s + \tilde{T}_s\Delta(\epsilon_s v_s) + \tilde{v}_s\Delta(\epsilon_s T_s). \tag{A.7g}$$

Clearly, equations (A.7a) and (A.7d) are automatically satisfied and by letting

$$\tilde{v}_g^2\Delta(\epsilon_g\rho_g) - 2\tilde{v}_g\Delta(\epsilon_g\rho_g v_g) + \Delta(\epsilon_g\rho_g v_g^2) = 0$$

and

$$\tilde{v}_s^2\Delta\epsilon_s - 2\tilde{v}_s\Delta(\epsilon_s v_s) + \Delta(\epsilon_s v_s^2) = 0,$$

we obtain

$$\tilde{v}_g = \frac{\sqrt{(\epsilon_g\rho_g)_L}(v_g)_L + \sqrt{(\epsilon_g\rho_g)_R}(v_g)_R}{\sqrt{(\epsilon_g\rho_g)_L} + \sqrt{(\epsilon_g\rho_g)_R}} \quad \text{and} \quad \tilde{v}_s = \frac{\sqrt{(\epsilon_s)_L}(v_s)_L + \sqrt{(\epsilon_s)_R}(v_s)_R}{\sqrt{(\epsilon_s)_L} + \sqrt{(\epsilon_s)_R}}.$$

From (A.7b), (A.7e) and (A.7g) by substitution we can easily obtain

$$\tilde{T}_s = \frac{\sqrt{(\epsilon_s)_L}(T_s)_L + \sqrt{(\epsilon_s)_R}(T_s)_R}{\sqrt{(\epsilon_s)_L} + \sqrt{(\epsilon_s)_R}}, \quad \tilde{u}_s = \frac{\sqrt{(\epsilon_s)_L}(u_s)_L + \sqrt{(\epsilon_s)_R}(u_s)_R}{\sqrt{(\epsilon_s)_L} + \sqrt{(\epsilon_s)_R}}$$

and

$$\tilde{u}_g = \frac{\sqrt{(\epsilon_g\rho_g)_L}(u_g)_L + \sqrt{(\epsilon_g\rho_g)_R}(u_g)_R}{\sqrt{(\epsilon_g\rho_g)_L} + \sqrt{(\epsilon_g\rho_g)_R}}.$$

Now, we are left with

$$\tilde{\epsilon}_g \Delta p_g = \tilde{c}_g^2(\Delta(\epsilon_g\rho_g) - \tilde{\rho}_g\Delta\epsilon_g)$$

By substituting $\tilde{c}_g^2 = \frac{\Delta p_g}{\Delta \rho_g}$ into (**??**), we obtain

$$\Delta(\epsilon_g\rho_g) = \tilde{\epsilon}_g\Delta\rho_g + \tilde{\rho}_g\Delta\epsilon_g.$$

Here, we have two options:

$$\tilde{\epsilon}_g = \frac{1}{2}((\epsilon_g)_L + (\epsilon_g)_R) \quad \text{and} \quad \tilde{\rho}_g = \frac{1}{2}((\rho_g)_L + (\rho_g)_R) \tag{A.8}$$

or

$$\tilde{\epsilon}_g = \sqrt{(\epsilon_g)_L(\epsilon_g)_R} \quad \text{and} \quad \tilde{\rho}_g = \frac{\sqrt{(\epsilon_g)_L}(\rho_g)_L + \sqrt{(\epsilon_g)_R}(\rho_g)_R}{\sqrt{(\epsilon_g)_L} + \sqrt{(\epsilon_g)_R}}. \tag{A.9}$$

Thus, all the Roe averages have been obtained.

A.2.2 Eigenvalues & Eigenvectors

The Roe averaged eigenvalues are the same as the original values,

1. Formulation S,

$$\tilde{\lambda}_{1,2} = \tilde{v}_g \pm \tilde{c}_g\sqrt{\tilde{\epsilon}_g^{-1}}, \quad \tilde{\lambda}_3 = \tilde{v}_g, \quad \tilde{\lambda}_{4,5} = \tilde{v}_s \pm \sqrt{\tilde{T}_s} \quad \text{and} \quad \tilde{\lambda}_{6,7} = \tilde{v}_s.$$

2. Formulation P,

$$\tilde{\lambda}_{1,2} = \tilde{v}_g \pm \tilde{c}_g\sqrt{\tilde{\epsilon}_g^{-1}}, \quad \tilde{\lambda}_3 = \tilde{v}_g, \quad \tilde{\lambda}_{4,5} = \frac{4}{3}\tilde{v}_s \pm \frac{1}{3}\sqrt{\tilde{v}_s^2 + 15\tilde{T}_s} \quad \text{and} \quad \tilde{\lambda}_{6,7} = \tilde{v}_s.$$

For simplicity (and computational efficiency), the eigenvectors are determined in terms of the eigenvalues, which are obtained by solving:

1. Formulation S,

$$\tilde{\mathbf{J}}_y = \begin{bmatrix} -\tilde{\lambda} & 0 & 1 & 0 & 0 & 0 & 0 \\ -\tilde{u}_g\tilde{v}_g & \tilde{v}_g - \tilde{\lambda} & \tilde{u}_g & 0 & 0 & 0 & 0 \\ \tilde{\epsilon}_g^{-1}\tilde{c}_g^2 - \tilde{v}_g^2 & 0 & 2\tilde{v}_g - \tilde{\lambda} & \tilde{\rho}_g\tilde{\epsilon}_g^{-1}\tilde{c}_g^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tilde{\lambda} & 0 & 1 & 0 \\ 0 & 0 & 0 & -\tilde{u}_s\tilde{v}_s & \tilde{v}_s - \tilde{\lambda} & \tilde{u}_s & 0 \\ 0 & 0 & 0 & \tilde{\epsilon}_s\tilde{T}_s - \tilde{v}_s^2 & 0 & 2\tilde{v}_s - \tilde{\lambda} & 1 \\ 0 & 0 & 0 & -\tilde{v}_s\tilde{T}_s & 0 & \tilde{T}_s & \tilde{v}_s - \tilde{\lambda} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{bmatrix} = 0.$$

2. Formulation P,

$$\tilde{\mathbf{J}}_y = \begin{bmatrix} -\tilde{\lambda} & 0 & 1 & 0 & 0 & 0 & 0 \\ -\tilde{u}_g\tilde{v}_g & \tilde{v}_g - \tilde{\lambda} & \tilde{u}_g & 0 & 0 & 0 & 0 \\ \tilde{\epsilon}_g^{-1}\tilde{c}_g^2 - \tilde{v}_g^2 & 0 & 2\tilde{v}_g - \tilde{\lambda} & \tilde{\rho}_g\tilde{\epsilon}_g^{-1}\tilde{c}_g^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tilde{\lambda} & 0 & 1 & 0 \\ 0 & 0 & 0 & -\tilde{u}_s\tilde{v}_s & \tilde{v}_s - \tilde{\lambda} & \tilde{u}_s & 0 \\ 0 & 0 & 0 & \tilde{\epsilon}_s\tilde{T}_s - \tilde{v}_s^2 & 0 & 2\tilde{v}_s - \tilde{\lambda} & 1 \\ 0 & 0 & 0 & -\frac{5}{3}\tilde{v}_s\tilde{T}_s & 0 & \frac{5}{3}\tilde{T}_s & \frac{5}{3}\tilde{v}_s - \tilde{\lambda} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{bmatrix} = 0.$$

Thus,

$$e_3 = \tilde{\lambda}e_1, \tag{A.10a}$$

$$-\tilde{u}_g\tilde{v}_g e_1 + (\tilde{v}_g - \tilde{\lambda})e_2 + \tilde{u}_g e_3 = 0, \tag{A.10b}$$

$$(\tilde{\epsilon}_g^{-1}\tilde{c}_g^2 - \tilde{v}_g^2)e_1 + (2\tilde{v}_g - \tilde{\lambda})e_3 + \tilde{\rho}_g\tilde{\epsilon}_g^{-1}\tilde{c}_g^2 e_4 = 0, \tag{A.10c}$$

$$e_6 = \tilde{\lambda}e_4, \tag{A.10d}$$

$$-\tilde{u}_s\tilde{v}_s e_4 + (\tilde{v}_s - \tilde{\lambda})e_5 + \tilde{u}_s e_6 = 0 \tag{A.10e}$$

$$-\tilde{v}_s^2 e_4 + (2\tilde{v}_s - \tilde{\lambda})e_6 + e_7 = 0, \tag{A.10f}$$

and either

$$-\tilde{v}_s\tilde{T}_s e_4 + \tilde{T}_s e_6 + (\tilde{v}_s - \tilde{\lambda})e_7 = 0 \tag{A.10g}$$

for Formulation S or

$$-\frac{5}{3}\tilde{v}_s\tilde{T}_s e_4 + \frac{5}{3}\tilde{T}_s e_6 + (\frac{5}{3}\tilde{v}_s - \tilde{\lambda})e_7 = 0 \tag{A.10h}$$

for Formulation P. Now, by substituting (A.10a) into (A.10c), we obtain

$$e_4 = \left(\frac{\tilde{v}_g^2\tilde{\epsilon}_g - \tilde{c}_g^2 - (2\tilde{v}_g - \tilde{\lambda})\tilde{\epsilon}_g\tilde{\lambda}}{\tilde{\rho}_g\tilde{c}_g^2}\right)e_1 = \left(\frac{(\tilde{\lambda} - \tilde{v}_g)^2\tilde{\epsilon}_g - \tilde{c}_g^2}{\tilde{\rho}_g\tilde{c}_g^2}\right)e_1$$

and similarly for (A.10b),

$$(\tilde{v}_g - \tilde{\lambda})(e_2 - \tilde{u}_g e_1) = 0.$$

Also, by substituting (A.10d) into (A.10e) and (A.10f), we obtain

$$(\tilde{v}_s - \tilde{\lambda})(e_5 - \tilde{u}_s e_4) = 0 \quad \text{and} \quad e_7 = e_4(\tilde{\lambda} - \tilde{v}_s)^2,$$

respectively.

Thus, we obtain

$$\tilde{\mathbf{e}}_{1,2} = \begin{bmatrix} 1 \\ \tilde{u}_g \\ \tilde{\lambda}_{1,2} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{\mathbf{e}}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{\mathbf{e}}_{4,5} = \begin{bmatrix} 1 \\ \tilde{u}_g \\ \tilde{\lambda}_{4,5} \\ \tilde{d}_{4,5} \\ \tilde{u}_s \tilde{d}_{4,5} \\ \tilde{\lambda}_{4,5} \tilde{d}_{4,5} \\ \tilde{d}_{4,5}(\tilde{\lambda}_{4,5} - \tilde{v}_s)^2 \end{bmatrix},$$

$$\tilde{\mathbf{e}}_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{e}}_7 = \begin{bmatrix} 1 \\ \tilde{u}_g \\ \tilde{v}_s \\ \tilde{d}_7 \\ \tilde{u}_s \tilde{d}_7 \\ \tilde{v}_s \tilde{d}_7 \\ 0 \end{bmatrix},$$

where

$$\tilde{d}_k = \frac{(\tilde{\lambda}_k - \tilde{v}_g)^2 \tilde{\epsilon}_g - \tilde{c}_g^2}{\tilde{\rho}_g \tilde{c}_g^2}.$$

A.2.3 Wave Strengths

We obtain wave strengths are determined through the decomposition,

$$\Delta \mathbf{w} = \sum_{k=1}^{7} \tilde{\alpha}_k \tilde{\mathbf{e}}_k,$$

which for both formulations results in solving

$$\Delta(\epsilon_g \rho_g) = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_4 + \tilde{\alpha}_5 + \tilde{\alpha}_7, \tag{A.11a}$$

$$\Delta(\epsilon_g \rho_g u_g) = \tilde{u}_g \tilde{\alpha}_1 + \tilde{u}_g \tilde{\alpha}_2 + \tilde{\alpha}_3 + \tilde{u}_g \tilde{\alpha}_4 + \tilde{u}_g \tilde{\alpha}_5 + \tilde{u}_g \tilde{\alpha}_7, \tag{A.11b}$$

$$\Delta(\epsilon_g \rho_g v_g) = \tilde{\lambda}_1 \tilde{\alpha}_1 + \tilde{\lambda}_2 \tilde{\alpha}_2 + \tilde{\lambda}_4 \tilde{\alpha}_4 + \tilde{\lambda}_5 \tilde{\alpha}_5 + \tilde{\lambda}_7 \tilde{\alpha}_7, \tag{A.11c}$$

$$\Delta\epsilon_s = \tilde{d}_4 \tilde{\alpha}_4 + \tilde{d}_5 \tilde{\alpha}_5 + \tilde{d}_7 \tilde{\alpha}_7 \tag{A.11d}$$

$$\Delta(\epsilon_s u_s) = \tilde{u}_s \tilde{d}_4 \tilde{\alpha}_4 + \tilde{u}_s \tilde{d}_5 \tilde{\alpha}_5 + \tilde{\alpha}_6 + \tilde{u}_s \tilde{d}_7 \tilde{\alpha}_7 \tag{A.11e}$$

$$\Delta(\epsilon_s v_s) = \tilde{\lambda}_4 \tilde{d}_4 \tilde{\alpha}_4 + \tilde{\lambda}_5 \tilde{d}_5 \tilde{\alpha}_5 + \tilde{\lambda}_7 \tilde{d}_7 \tilde{\alpha}_7, \tag{A.11f}$$

and

$$\Delta(\epsilon_s T_s) = \tilde{d}_4 (\tilde{\lambda}_4 - \tilde{v}_s)^2 \tilde{\alpha}_4 + \tilde{d}_5 (\tilde{\lambda}_5 - \tilde{v}_s)^2 \tilde{\alpha}_5 \tag{A.11g}$$

for $\tilde{\alpha}_k$. Unfortunately, by solving these set of simultaneous equations, algebraic expressions of $\tilde{\alpha}_{1,2}$ are complicated. To keep the algebraic expressions in a compact form, we solve (A.5d), (A.5e) and (A.5g) to obtain $\tilde{\alpha}_{4,5,7}$,

$$\tilde{\alpha}_k = \frac{(\tilde{\lambda}_a \tilde{\lambda}_b - \tilde{v}_s^2)\Delta\epsilon_s - (\tilde{\lambda}_a + \tilde{\lambda}_b - 2\tilde{v}_s)\Delta(\epsilon_s v_s)\Delta(\epsilon_s T_s)}{\tilde{d}_k (\tilde{\lambda}_k - \tilde{\lambda}_a)(\tilde{\lambda}_k - \tilde{\lambda}_b)}$$

where $k \neq a \neq b$, and then solve $\tilde{\alpha}_{1,2}$ in terms of $\tilde{\alpha}_{4,5,7}$,

$$\tilde{\alpha}_{1,2} = \mp \frac{\tilde{\lambda}_4 \tilde{\alpha}_4 + \tilde{\lambda}_5 \tilde{\alpha}_5 + \tilde{\lambda}_7 \tilde{\alpha}_7 - (\tilde{\alpha}_4 + \tilde{\alpha}_5 + \tilde{\alpha}_7 - \Delta(\epsilon_g \rho_g))\tilde{\lambda}_{2,1} - \Delta(\epsilon_g \rho_g v_g)}{\tilde{\lambda}_1 - \tilde{\lambda}_2}.$$

We can then easily obtain the last two wavestrengths by solving,

$$\tilde{\alpha}_3 = \Delta(\epsilon_g \rho_g u_g) - \tilde{u}_g \Delta(\epsilon_g \rho_g)$$

and

$$\tilde{\alpha}_6 = \Delta(\epsilon_s u_s) - \tilde{u}_s \Delta\epsilon_s$$

A.2.4 Inhomogeneous Terms

The values of $\tilde{\beta}_k$ are also determined from the decomposition,

$$\Delta y \tilde{\mathbf{g}} = \sum_{k=1}^{7} \tilde{\beta}_k \tilde{\mathbf{e}}_k,$$

which for both formulations results in solving

$$0 = \tilde{\beta}_1 + \tilde{\beta}_2 + \tilde{\beta}_4 + \tilde{\beta}_5 + \tilde{\beta}_7, \tag{A.12a}$$

$$0 = \tilde{u}_g \tilde{\beta}_1 + \tilde{u}_g \tilde{\beta}_2 + \tilde{\beta}_3 + \tilde{u}_g \tilde{\beta}_4 + \tilde{u}_g \tilde{\beta}_5 + \tilde{u}_g \tilde{\beta}_7, \tag{A.12b}$$

$$\tilde{r}_3 = \tilde{\lambda}_1\tilde{\beta}_1 + \tilde{\lambda}_2\tilde{\beta}_2 + \tilde{\lambda}_4\tilde{\beta}_4 + \tilde{\lambda}_5\tilde{\beta}_5 + \tilde{\lambda}_7\tilde{\beta}_7, \tag{A.12c}$$

$$0 = \tilde{d}_4\tilde{\beta}_4 + \tilde{d}_5\tilde{\beta}_5 + \tilde{d}_7\tilde{\beta}_7, \tag{A.12d}$$

$$0 = \tilde{u}_s\tilde{d}_4\tilde{\beta}_4 + \tilde{u}_s\tilde{d}_5\tilde{\beta}_5 + \tilde{\beta}_6 + \tilde{u}_s\tilde{d}_7\tilde{\beta}_7 \tag{A.12e}$$

$$\tilde{r}_6 = \tilde{\lambda}_4\tilde{d}_4\tilde{\beta}_4 + \tilde{\lambda}_5\tilde{d}_5\tilde{\beta}_5 + \tilde{\lambda}_7\tilde{d}_7\tilde{\beta}_7, \tag{A.12f}$$

and

$$\tilde{r}_7 = \tilde{d}_4(\tilde{\lambda}_4 - \tilde{v}_s)^2\tilde{\beta}_4 + \tilde{d}_5(\tilde{\lambda}_5 - \tilde{v}_s)^2\tilde{\beta}_5 \tag{A.12g}$$

for $\tilde{\beta}_k$. Here,

$$\tilde{r}_3 = (1-\omega_1)\Delta p_g - \rho_s\omega_2\Delta(\epsilon_s T_s),$$

$$\tilde{r}_6 = -\frac{\omega_3}{\rho_s}\Delta p_g + (1-\omega_4)\Delta(\epsilon_s T_s),$$

$$\tilde{r}_7 = -\frac{2}{3}\tilde{\epsilon}_s\tilde{T}_s\Delta v_s \quad \text{and} \quad \tilde{r}_7 = \frac{2}{3}\tilde{v}_s\Delta(\epsilon_s T_s).$$

As with the wave strengths, algebraic expressions of $\tilde{\beta}_{1,2}$ are complicated. Thus, we solve (A.6d), (A.6f) and (A.6g) to obtain $\tilde{\beta}_{4,5,7}$,

$$\tilde{\beta}_k = \frac{(2\tilde{v}_s - \tilde{\lambda}_a - \tilde{\lambda}_b)\tilde{r}_6 + \tilde{r}_7}{\tilde{d}_k(\tilde{\lambda}_k - \tilde{\lambda}_a)(\tilde{\lambda}_k - \tilde{\lambda}_b)}$$

where $k \neq a \neq b$, and then solve $\tilde{\beta}_{1,2}$ in terms of $\tilde{\beta}_{4,5,7}$,

$$\tilde{\beta}_{1,2} = \mp\frac{\tilde{\lambda}_4\tilde{\beta}_4 + \tilde{\lambda}_5\tilde{\beta}_5 + \tilde{\lambda}_7\tilde{\beta}_7 - (\tilde{\beta}_4 + \tilde{\beta}_5 + \tilde{\beta}_7)\tilde{\lambda}_{2,1} - \tilde{r}_3}{\tilde{\lambda}_1 - \tilde{\lambda}_2},$$

where

$$\tilde{r}_3 = (1-\omega_1)\Delta p_g - \rho_s\omega_2\Delta(\epsilon_s T_s), \quad \tilde{r}_6 = -\frac{\omega_3}{\rho_s}\Delta p_g + (1-\omega_4)\Delta(\epsilon_s T_s)$$

and

$$\tilde{r}_7 = -\frac{2}{3}\tilde{\epsilon}_s\tilde{T}_s\Delta v_s.$$

The last two values of $\tilde{\beta}$ can now be solved,

$$\tilde{\beta}_3 = 0 \quad \text{and} \quad \tilde{\beta}_6 = 0.$$

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