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L-PACKETS AND FORMAL DEGREES FOR $\mathrm{SL}_2(K)$ WITH K A LOCAL FUNCTION FIELD OF CHARACTERISTIC 2

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Abstract. Let $\mathcal{G} = \mathrm{SL}_2(K)$ with K a local function field of characteristic 2. We review Artin-Schreier theory for the field K, and show that this leads to a parametrization of L-packets in the smooth dual of $\mathcal{G}$. We relate this to a recent geometric conjecture. The L-packets in the principal series are parametrized by quadratic extensions, and the supercuspidal L-packets by biquadratic extensions. We compute the formal degrees of the elements in the supercuspidal packets.

Contents

1. Introduction

In this article we consider a local function field K of characteristic 2, namely $K = \mathbb{F}_q((\varpi))$, the field of Laurent series with coefficients in $\mathbb{F}_q$, with $q = 2^f$. This example is particularly interesting because there are countably many quadratic extensions of $\mathbb{F}_q((\varpi))$.

We consider $\mathcal{G} = \mathrm{SL}_2(K)$. Drawing on the accounts in [5, 16, 17], we review Artin-Schreier theory, adapted to the local function field $\mathbb{F}_q((\varpi))$. This leads to a parametrization of L-packets in the smooth dual of $\mathcal{G}$. In this article, we reserve the term L-packets for the ones which are not singletons.

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The L-packets in the principal series are parametrized by quadratic extensions, and the supercuspidal L-packets by biquadratic extensions. There are countably many supercuspidal packets.

By *canonical formal degree* we shall mean formal degree with respect to the Euler-Poicaré measure on $\mathcal{G}$, as in [12]. We compute the canonical formal degrees of the elements in the supercuspidal packets, relying on the Formal Degree component of the local Langlands correspondence, see [12, §6]. The canonical formal degrees are all dyadic rationals, in fact they are integer powers of 2. They depend on the residue degree f, and on the breaks in the lower ramification filtration of the Galois group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

The commutative triangle in Theorem 4.3, and the bijective maps 5.2, 5.3, 5.4 in §5, amount to a proof, for $\mathcal{G}$, of the *tempered* version of the geometric conjecture in [1].

Thanks to Anne-Marie Aubert for her careful reading of the file, and for sending us some valuable comments. The first author would like to thank Chandan Dalawat for a valuable exchange of emails and for the reference [5].

2. Artin-Schreier theory

Let K be a local field with positive characteristic p. The cyclic extensions of K whose degree n is coprime with p are described by Kummer theory. It is well known that any cyclic extension L/K of degree n, $(n,p) = 1$, is generated by a root α of an irreducible polynomial $x^n - a \in K[x]$. If $\alpha \in K^s$ is a root of $x^n - a$ then $K(\alpha)/K$ is a cyclic extension of degree n and is called a Kummer extension of K.

Artin-Schreier theory aims to describe cyclic extensions of degree equal to or divisible by $ch(K) = p$. It is therefore an analogue of Kummer theory, where the role of the polynomial $x^n - a$ is played by $x^n - x - a$. Essentially, every cyclic extension of K with degree $p = ch(K)$ is generated by a root α of $x^p - x - a \in K[x]$.

We fix an algebraic closure $\overline{K}$ of K and a separable closure K^s of K in $\overline{K}$. Let $\wp$ denote the Artin-Schreier endomorphism of the additive group K^s [9]:

$$\wp : K^s \to K^s, \quad x \mapsto x^p - x.$$

Given $a \in K$ denote by $K(\wp^{-1}(a))$ the extension $K(\alpha)$, where $\wp(\alpha) = a$ and $\alpha \in K^s$. We have the following characterization of finite cyclic Artin-Schreier extensions of degree p:

Theorem 2.1. (*i*) *Given $a \in K$, either $\wp(x) - a \in K[x]$ has one root in K in which case it has all the p roots are in K, or is irreducible.*

(*ii*) *If $\wp(x) - a \in K[x]$ is irreducible then $K(\wp^{-1}(a))/K$ is a cyclic extension of degree p, with $\wp^{-1}(a) \subset K^s$.*

(*iii*) *If L/K be a finite cyclic extension of degree p, then $L = K(\wp^{-1}(a))$, for some $a \in K$.*

(See [16, p.34] for more details.)

We fix now some notation. K is a local field with characteristic $p > 1$ with finite residue field k. The field of constants $k = \mathbb{F}_q$ is a finite extension of $\mathbb{F}_p$, with degree $[k : \mathbb{F}_p] = f$ and $q = p^f$.

Let $\mathfrak{o}$ be the ring of integers in K and denote by $\mathfrak{p} \subset \mathfrak{o}$ the (unique) maximal ideal of $\mathfrak{o}$. This ideal is principal and any generator of $\mathfrak{p}$ is called a uniformizer. A choice of uniformizer $\varpi \in \mathfrak{o}$ determines isomorphisms $K \cong \mathbb{F}_q((\varpi))$, $\mathfrak{o} \cong \mathbb{F}_q[[\varpi]]$ and $\mathfrak{p} = \varpi\mathfrak{o} \cong \varpi\mathbb{F}_q[[\varpi]]$.

A normalized valuation on K will be denoted by ν, so that $\nu(\varpi) = 1$ and $\nu(K) = \mathbb{Z}$. The group of units is denoted by $\mathfrak{o}^\times$.

2.1. **The Artin-Schreier symbol.** Let L/K be a finite Galois extension. Let $N_{L/K}$ be the norm map and denote $G^{ab}_{L/K} = Gal(L/K)^{ab}$ the abelianization of $Gal(L/K)$. The reciprocity map is a group isomorphism

$$K^\times/N_{L/K}L^\times \xrightarrow{\simeq} G^{ab}_{L/K}. \tag{2.1}$$

The Artin symbol is obtained by composing the reciprocity map with the canonical morphism $K^\times \to K^\times/N_{L/K}L^\times$

$$b \in K^\times \mapsto (b, L/K) \in G^{ab}_{L/K}. \tag{2.2}$$

From the Artin symbol we obtain a pairing

$$K \times K^\times \longrightarrow \mathbb{Z}/p\mathbb{Z}, (a, b) \mapsto (b, L/K)(\alpha) - \alpha, \tag{2.3}$$

where $\wp(\alpha) = a$, $\alpha \in K^s$ and $L = K(\alpha)$.

Definition 2.2. *Given $a \in K$ and $b \in K^\times$, the Artin-Schreier symbol is defined to be*

$$[a, b) = (b, L/K)(\alpha) - \alpha.$$

We summarize some important properties of the Artin-Schreier symbol.

Proposition 2.3. *The Artin-Schreier symbol is a bilinear map satisfying the following properties:*

(*i*) $[a_1 + a_2, b) = [a_1, b) + [a_2, b)$*;*
(*ii*) $[a, b_1b_2) = [a, b_1) + [a, b_2)$*;*
(*iii*) $[a, b) = 0, \forall a \in K \Leftrightarrow b \in N_{L/K}L^\times, L = K(\alpha)$ *and* $\wp(\alpha) = a$*;*
(*iv*) $[a, b) = 0, \forall b \in K^\times \Leftrightarrow a \in \wp(K)$*.*

(See [9, p.341])

2.2. **The groups $K/\wp(K)$ and $K^\times/K^{\times p}$.** In this section we recall some properties of the groups $K/\wp(K)$ and $K^\times/K^{\times p}$ and use them to redefine the pairing (2.3).

Consider the additive group K. The index of $\wp(K)$ in K is infinite [6, p.146]. Hence, $K/\wp(K)$ is infinite.

Proposition 2.4. *$K/\wp(K)$ is a discrete abelian torsion group.*

Proof. The ring of integers decomposes as a (direct) sum

$$\mathfrak{o} = \mathbb{F}_q + \mathfrak{p}$$

and we have

$$\wp(\mathfrak{o}) = \wp(\mathbb{F}_q) + \wp(\mathfrak{p}).$$

The restriction $\wp : \mathfrak{p} \to \mathfrak{p}$ is an isomorphism, see [5, Lemma 8]. Hence,

$$\wp(\mathfrak{o}) = \wp(\mathbb{F}_q) + \mathfrak{p}$$

and $\mathfrak{p} \subset \wp(K)$. It follows that $\wp(K)$ is an open subgroup of K and $K/\wp(K)$ is discrete. Since $\wp(K)$ is annihilated by p, $K/\wp(K)$ is a torsion group. $\square$

Now we concentrate on the multiplicative group $K^\times$. The subgroup $K^{\times p}$ is not open in $K^\times$ and the index $[K^\times : K^{\times p}]$ is infinite [6, Lemma p.115]. Hence, $K^\times/K^{\times p}$ is infinite. The next result gives a characterization of the topological group $K^\times/K^{\times p}$.

Proposition 2.5. *$K^\times/K^{\times p}$ is a profinite abelian p-torsion group.*

Proof. There is a canonical isomorphism $K^\times \cong \mathbb{Z} \times \mathfrak{o}^\times$. By [8, p.25], the group of units $\mathfrak{o}^\times$ is a direct product of countable many copies of the ring of p-adic integers

$$\mathfrak{o}^\times \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times ... = \prod_{\mathbb{N}} \mathbb{Z}_p.$$

Give $\mathbb{Z}$ the discrete topology and $\mathbb{Z}_p$ the p-adic topology. Then, for the product topology, $K^\times = \mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}_p$ is a topological group, locally compact, Hausdorff and totally disconnected.

$K^{\times p}$ decomposes as a product of countable many components

$$K^{\times p} \cong p\mathbb{Z} \times p\mathbb{Z}_p \times p\mathbb{Z}_p \times ... = p\mathbb{Z} \times \prod_{\mathbb{N}} p\mathbb{Z}_p.$$

Denote by $y = \prod_n y_n$ and element of $\prod_{\mathbb{N}} \mathbb{Z}_p$, where $y_n = \sum_{i=0}^{\infty} a_{i,n} p^i \in \mathbb{Z}_p$, for every n.

The map

$$\varphi : \mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}_p \to \mathbb{Z}/p\mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z}, (x, y) \mapsto (x(mod p), \prod_n pr_0(y_n))$$

where $pr_0(y_n) = a_{0,n}$ is the projection, is clearly a group homomorphism.

Now, $\mathbb{Z}/p\mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z} = \prod_{n=0}^{\infty} \mathbb{Z}/p\mathbb{Z}$ is a topological group for the product topology, where each component $\mathbb{Z}/p\mathbb{Z}$ has the discrete topology. Moreover, it is compact by Tyconoff Theorem, Hausdorff and totally disconnected [2, TGI.84, Prop. 10]. Therefore, $\prod_{n=0}^{\infty} \mathbb{Z}/p\mathbb{Z}$ is a profinite group.

Since

$$ker\varphi = p\mathbb{Z} \times \prod_{\mathbb{N}} p\mathbb{Z}_p,$$

it follows that there is an isomorphism of topological groups

$$K^\times/K^{\times p} \cong \prod_{\mathbb{N}} p\mathbb{Z}_p,$$

where $K^\times/K^{\times p}$ is given the quotient topology. Therefore, $K^\times/K^{\times p}$ is profinite. □

From Propositions 6.1 and 2.5, $K/\wp(K)$ is a discrete abelian group and $K/K^{\times p}$ is an abelian profinite group, both annihilated by $p = ch(K)$. Therefore, Pontryagin duality coincides with $Hom(-, \mathbb{Z}/p\mathbb{Z})$ on both of these groups, see [17]. See also [13] for more details on Pontryagin duality. The pairing (2.3) restricts to a pairing

$$[.,.) : K/\wp(K) \times K^\times/K^{\times p} \to \mathbb{Z}/p\mathbb{Z}. \tag{2.1}$$

which we refer from now on to the **Artin-Schreier pairing**. It follows from (*iii*) and (*iv*) of Proposition 2.3, the pairing is nondegenerate (see also [17, Proposition 3.1]). The next result shows that the pairing is perfect.

Proposition 2.6. *The Artin-Schreier symbol induces isomorphisms of topological groups*

$$K^\times/K^{\times p} \xrightarrow{\simeq} Hom(K/\wp(K), \mathbb{Z}/p\mathbb{Z}), bK^{\times p} \mapsto (a + \wp(K) \mapsto [a, b))$$

and

$$K/\wp(K) \xrightarrow{\simeq} Hom(K^\times/K^{\times p}, \mathbb{Z}/p\mathbb{Z}), a + \wp(K) \mapsto (bK^{\times p} \mapsto [a, b))$$

Proof. The result follows by taking $n = 1$ in Proposition 5.1 of [17], and from the fact that Pontryagin duality for the groups $K/\wp(K)$ and $K^\times/K^{\times p}$ coincide with $Hom(-, \mathbb{Z}/p\mathbb{Z})$ duality. Hence, there is an isomorphism of topological groups between each such group and its bidual. □

Let B be a subgroup of the additive group of K with finite index such that $\wp(K) \subseteq B \subseteq K$. The composite of two finite abelian Galois extensions of exponent p is again a finite abelian Galois extension of exponent p. Therefore, the composite

$$K_B = K(\wp^{-1}(B)) = \prod_{a \in B} K(\wp^{-1}(a))$$

is a finite abelian Galois extension of exponent p. On the other hand, if L/K is a finite abelian Galois extension of exponent p, then $L = K_B$ for some subgroup $\wp(K) \subseteq B \subseteq K$ with finite index.

All such extensions lie in the maximal abelian extension of exponent p, which we denote by $K_p = K(\wp^{-1}(K))$. The extension K_p/K is infinite and Galois. The corresponding Galois group $G_p = Gal(K_p/K)$ is an infinite profinite group and may be identified, under class field theory, with $K^\times/K^{\times p}$, see [17, Proposition 5.1]. The case $ch(K) = 2$ leads to $G_2 \cong K^\times/K^{\times 2}$ and will play a fundamental role in the sequel.

3. Quadratic characters

From now on we take K to be a local function field with $ch(K) = 2$. Therefore, K is of the form $\mathbb{F}_q((\varpi))$ with $q = 2^f$.

Recall that a character of $K^\times$ is a group homomorphism

$$\chi : K^\times \to \mathbb{T}$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle. Denote by $\widehat{K^\times}$ the group of characters of $K^\times$. There is a canonical isomorphism

$$\widehat{K^\times} \cong \widehat{\mathbb{Z} \times \mathfrak{o}^\times} \cong \mathbb{T} \times \widehat{\mathfrak{o}^\times}.$$

Therefore, given a character $\chi \in \widehat{K^\times}$, we may write $\chi = z^\nu \chi_0$, where $z \in \mathbb{T}$, ν is the valuation and $\chi_0 \in \widehat{\mathfrak{o}^\times}$. If $\chi_0 \equiv 1$ we say that χ is unramified. A character χ of $K^\times$ is called quadratic if $\chi^2 = 1$. Since the unique quadratic character of $\mathbb{Z}$ is $(n \mapsto (-1)^n)$, a nontrivial quadratic character has the form $\chi = (-1)^\nu \chi_0$, with $\chi_0^2 = 1$.

When $K = \mathbb{F}_q((\varpi))$, we have, according to [8, p.25],

$$\mathfrak{o}^\times \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times ... = \prod_{\mathbb{N}} \mathbb{Z}_2$$

with countably infinite many copies of $\mathbb{Z}_2$, the ring of 2-adic integers.

Artin-Schreier theory provides a way to parametrize all the quadratic extensions of $K = \mathbb{F}_q((\varpi))$. By Proposition 2.5, there is a bijection between the set of quadratic extensions of $\mathbb{F}_q((\varpi))$ and the group

$$\mathbb{F}_q((\varpi))^\times/\mathbb{F}_q((\varpi))^{\times 2} \cong \prod_{\mathbb{N}} 2\mathbb{Z}_2 = G_2$$

where G_2 is the Galois group of the maximal abelian extension of exponent 2. Since G_2 is an infinite profinite group, there are countably many quadratic extensions.

To each quadratic extension $K(\alpha)/K$, with $\alpha^2 - \alpha = a$, we associate the Artin-Schreier symbol

$$[a, .) : K^\times/K^{\times 2} \to \mathbb{Z}/2\mathbb{Z}.$$

Now, let φ denote the isomorphism $\mathbb{Z}/2\mathbb{Z} \cong \mu_2(\mathbb{C}) = \{\pm 1\}$ with the group of roots of unity. We obtain, by composing with the Artin-Schreier symbol, a unique multiplicative quadratic character $\chi_a = \varphi([a, .))$:

$$\chi_a : K^\times \to \mathbb{C}^\times. \tag{3.1}$$

Proposition 2.6 shows that every quadratic character of $\mathbb{F}_q((\varpi))^\times$ arises in this way.

Example 3.1. *The unramified quadratic extension of K is $K(\wp^{-1}(\mathfrak{o}))$, see* [5] *proposition* 12. *According to Dalawat, the group $K/\wp(K)$ may be regarded as an $\mathbb{F}_2$-space and the image of $\mathfrak{o}$ under the canonical surjection $K \to K/\wp(K)$ is an $\mathbb{F}_2$-line, i.e., isomorphic to $\mathbb{F}_2$. Since $\wp_{|\mathfrak{p}} : \mathfrak{p} \to \mathfrak{p}$ is an isomorphism, the image of $\mathfrak{p}$ in $K/\wp(K)$ is $\{0\}$, see lemma* 8 *in* [5]. *Now, choose any $a_0 \in \mathfrak{o}\backslash\mathfrak{p}$. The quadratic character $\chi_{a_0} = \varphi([a_0, .))$ associated with $K(\wp^{-1}(\mathfrak{o}))$ via class field theory is precisely the unramified character $(n \mapsto (-1)^n)$ from above. Note that any other choice $b_0 \in \mathfrak{o}\backslash\mathfrak{p}$, with $a_0 \neq b_0$, gives the same unique unramified character, since there is only one nontrivial coset $a_0 + \wp(K)$ for $a_0 \in \mathfrak{o}\backslash\mathfrak{p}$.*

Let $\mathcal{G}$ denote $\mathrm{SL}_2(K)$, let $\mathcal{B}$ be the standard Borel subgroup of $\mathcal{G}$, let $\mathcal{T}$ be the diagonal subgroup of $\mathcal{G}$. Let χ be a character of $\mathcal{T}$. Then, χ inflates to a character of $\mathcal{B}$. Denote by π_χ the (unitarily) induced representation $Ind_{\mathcal{B}}^{\mathcal{G}}(\chi)$. The representation space of V_χ of π_χ consists of locally constant complex valued functions $f : \mathcal{G} \to \mathbb{C}$ such that, for every $a \in K^\times$, $b \in K$ and $g \in \mathcal{G}$, we have

$$f\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = |a|\chi(a)f(g)$$

The action of $\mathcal{G}$ on V_χ is by right translation. The representations (π_χ, V_χ) are called (unitary) principal series of $\mathcal{G} = SL_2(K)$.

Let χ be a quadratic character of $K^\times$. The reducibility of the induced representation $\mathrm{Ind}_B^G(\chi)$ is well known in zero characteristic. W. Casselman proved that the same result holds in characteristic 2 and any other positive characteristic p.

Theorem 3.2. [3, 4] *The representation $\pi_\chi = \mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi)$ is reducible if, and only if, χ is either $|.|^{\pm}$ or a nontrivial quadratic character of $K^\times$.*

For a proof see [3, Theorems 1.7, 1.9] and [4, §9].

From now on, χ will be a quadratic character. It is a classical result that the unitary principal series for GL_2 are irreducible. For a representation of GL_2 parabolically induced by $1 \otimes \chi$, Clifford theory tells us that the dimension of the intertwining algebra of its restriction to SL_2 is 2. This is exactly the induced representation of SL_2 by χ:

$$\mathrm{Ind}_{\widetilde{B}}^{\mathrm{GL}_2(K)}(1 \otimes \chi)_{|\mathrm{SL}(2,K)} \xrightarrow{\simeq} \mathrm{Ind}_{B}^{\mathrm{SL}_2(K)}(\chi)$$

where $\widetilde{B}$ denotes the standard Borel subgroup of $\mathrm{GL}_2(K)$. This leads to reducibility of the induced representation $\mathrm{Ind}_B^G(\chi)$ into two inequivalent constituents. Thanks to M. Tadic for helpful comments at this point.

The two irreducible constituents

$$\pi_\chi = \mathrm{Ind}_B^G(\chi) = \pi_\chi^+ \oplus \pi_\chi^- \tag{3.2}$$

define an L-packet $\{\pi_\chi^+, \pi_\chi^-\}$ for SL_2.

4. A commutative triangle

In this section we confirm part of the geometric conjecture in [1] for $\mathrm{SL}_2(\mathbb{F}_q((\varpi)))$. We begin by recalling the underlying ideas of the conjecture.

Let $\mathcal{G}$ be the group of K-points of a connected reductive group over a nonarchimedean local field K. We have the *Bernstein decomposition*

$$\mathbf{Irr}(\mathcal{G}) = \bigsqcup \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$$

over all points $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$ the Bernstein spectrum of $\mathcal{G}$, see [14].

Let $\chi_{a_0} = \varphi([a_0, .))$ denote the unramified character of $K^\times$ associated with the unramified quadratic extension $K(\alpha_0) = K(\wp^{-1}(\mathfrak{o}))$ as in example 3.1. Fix a quadratic character $\chi_a = \varphi([a, .))$ associated via class field theory with the quadratic extension $K(\alpha)$ (in a fixed algebraic closure $\overline{K}$), where $\alpha^2 - \alpha = a$.

Proposition 4.1. *There is a unique quadratic extension $K(\beta)$ with associated character χ_{a_0+a}. Moreover, $\chi_{a_0+a} = \chi_{a_0}\chi_a$.*

Proof. The compositum $K(\alpha)K(\alpha_0)$ is Galoisian over K, with Galois group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Therefore, contains three subfields, which are quadratic extensions of K, namely $K(\alpha_0)$, $K(\alpha)$ and, say, $K(\beta)$. The extension $K(\beta)$ is such that $\beta^2 - \beta = a_0 + a$, and has an associated quadratic character given by χ_{a_0+a}. Hence

$$\chi_{a_0+a} = \varphi([a_0 + a, .)) = \varphi([a_0, .) + [a, .)) = \varphi([a_0, .))\varphi([a, .)) = \chi_{a_0}\chi_a.$$

□

By theorem 3.2, the induced representations

$$\pi_{a_0} = \mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi_{a_0})\ ,\ \pi_a = \mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi_a) \text{ and } \pi_{a_0+a} = \mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi_{a_0+a})$$

are reducible and split into a direct sum of two irreducible component.

Central to the geometric conjecture is the concept of extended quotient of the second kind, which we now define.

Let W be a finite group and let X be a complex affine algebraic variety. Suppose that W is acting on X as automorphisms of X. Define

$$\widetilde{X}_2 := \{(x, \tau) : \tau \in \mathbf{Irr}(W_x)\}.$$

Then W acts on $\widetilde{X}_2$:

$$\alpha(x, \tau) = (\alpha \cdot x, \alpha_* \tau).$$

Definition 4.2. *The extended quotient of the second kind is defined as*

$$(X/\!/W)_2 := \widetilde{X}_2/W.$$

Thus the extended quotient of the second kind is the ordinary quotient for the action of W on $\widetilde{X}_2$.

Theorem 4.3. *Let $\mathcal{G} = \mathrm{SL}_2(K)$ with $K = \mathbb{F}_q((\varpi)))$. Let $\mathfrak{s} = [\mathcal{T}, \chi]_G$ be a point in the Bernstein spectrum for the principal series of $\mathcal{G}$. Let $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ be the corresponding Bernstein component in $\mathbf{Irr}(\mathcal{G})$. Then the conjecture* [1] *is valid for $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ i.e. there is a commutative triangle of natural bijections*

$$\begin{tikzcd}
 & (T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2 \arrow[dl] \arrow[dr] & \\
\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \arrow[rr] & & \mathfrak{L}(G)^{\mathfrak{s}}
\end{tikzcd}$$

where $\mathfrak{L}(G)^{\mathfrak{s}}$ denotes the equivalence classes of enhanced parameters attached to $\mathfrak{s}$.

Proof. We recall that (G, T) are the complex dual groups of $(\mathcal{G}, \mathcal{T})$. Let $\mathbf{W}_K$ denote the Weil group of K. If φ is an L-parameter

$$\mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \to G$$

then $\mathcal{S}_\varphi$ is defined as follows:

$$\mathcal{S}_\varphi := \pi_0\, C_G(\mathrm{im}\, \varphi).$$

By an *enhanced* Langlands parameter, we shall mean a pair (φ, ρ) where φ is a parameter and $\rho \in \mathbf{Irr}(\mathcal{S}_\varphi)$. Following Reeder [12], we shall denote an enhanced Langlands parameter by $\varphi(\rho)$.

Case 1. Let χ be a quadratic character of $\mathcal{T}$: $\chi^2 = 1, \chi \neq 1$. Let L/K be the quadratic extension determined by χ. Now G contains a unique (up to conjugacy) subgroup $H \simeq \mathbb{Z}/2\mathbb{Z}$. Each quadratic extension L/K creates a parameter

$$\varphi_L : \mathbf{W}_K \to \mathrm{Gal}(L/K) \to G.$$

The map $\mathrm{Gal}(L/K) \to H$ factors through $K^\times/N_{L/K}L^\times$:

$$\varphi_L : \mathbf{W}_K \to \mathrm{Gal}(L/K) \simeq K^\times/N_{L/K}L^\times \to H \to G.$$

which shows that φ_L is the parameter attached to the packet π_χ.

To compute $\mathcal{S}_{\varphi_L}$, let $1, w$ be representatives of the Weyl group $W = W(G)$. Then we have

$$C_G(\operatorname{im}\varphi_L) = T \sqcup wT$$

So φ is a non-discrete parameter, and we have

$$\mathcal{S}_{\varphi_L} \simeq \mathbb{Z}/2\mathbb{Z}.$$

We have two enhanced Langlands parameters, namely $\varphi_L(\mathrm{triv})$ and $\varphi_L(\rho)$ where ρ is the nontrivial character of $\mathcal{S}_{\varphi_L}$.

Now define

$$\chi(\varpi) = \chi\left(\begin{array}{cc} \varpi & 0 \\ 0 & \varpi^{-1} \end{array}\right)$$

where ϖ is a uniformizer in K.

Since $\chi^2 = 1$, there is a point of reducibility. We have, at the level of elements,

$$\begin{tikzcd} & \{(\chi(\varpi), \mathrm{triv}), (\chi(\varpi), \rho)\} \arrow[dl] \arrow[dr] & \\ \{\pi_\chi^+, \pi_\chi^-\} \arrow[rr] & & \{\varphi_\chi(\mathrm{triv}), \varphi_\chi(\rho)\} \end{tikzcd}$$

Case 2. Let $\chi = 1$. The *principal parameter* is the composite map

$$\varphi_0 : \mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}(2, \mathbb{C}).$$

defined by extending the *principal* homomorphism $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C})$ trivially on $\mathbf{W}_K$, is a canonical discrete parameter for which $\mathcal{S}_{\varphi_0} = 1$. In the local Langlands correspondence for $\mathcal{G}$, the enhanced parameter $\varphi_0(\mathrm{triv})$ corresponds to the Steinberg representation $\mathrm{St}_{\mathcal{G}}$, see [12, 6.1.8].

Let φ_1 be the unique parameter for which $\varphi_1(\mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C})) = 1$. We have

$$\operatorname{im}\varphi_1 = 1, \quad C_G(\operatorname{im}\varphi_1) = G, \quad \mathcal{S}_{\varphi_1} = 1.$$

There is a unique enhanced parameter, namely $\varphi_1(\mathrm{triv})$. We have, at the level of elements, the commutative triangle

$$\begin{tikzcd} & \{(1, \mathrm{triv}), (1, \rho)\} \arrow[dl] \arrow[dr] & \\ \{\mathrm{St}_{\mathcal{G}}, 1_{\mathcal{G}}\} \arrow[rr] & & \{\varphi_0(\mathrm{triv}), \varphi_1(\mathrm{triv})\} \end{tikzcd}$$

Case 3. $\chi^2 \neq 1$. There are no points of reducibility, and we have a commutative triangle of sets, each with one element:

$$\begin{tikzcd} & (\psi(\varpi), \mathrm{triv}) \arrow[dl] \arrow[dr] & \\ \pi_\chi \arrow[rr] & & \varphi_\chi \end{tikzcd}$$

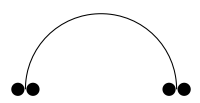
□

Corollary 4.4. *Let L/K be a quadratic extension of K. The L-parameters φ_L serve as parameters for the L-packets in the principal series of* $\mathrm{SL}_2(K)$.

It follows from §3 that there are countably many L-packets in the principal series of $\mathrm{SL}_2(K)$.

5. The tempered dual

The following picture



serves two purposes. First, it is an accurate portrayal of the extended quotient of the second kind

$$(\mathbb{T}//W)_2$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and the generator of $W = \mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{T}$ sending z to z^{-1}. Secondly, it is (conjecturally) an accurate portrayal of a connected component in the tempered dual of $\mathcal{G}$.

The topology on $(\mathbb{T}//W)_2$ comes about as follows. Let

$$\mathbf{Prim}(C(\mathbb{T}) \rtimes W)$$

denote the primitive ideal space of the noncommutative C^*-algebra $C(\mathbb{T}) \rtimes W$. By the classical Mackey theory for semidirect products, we have a canonical bijection

$$\mathbf{Prim}(C(\mathbb{T}) \rtimes W) \simeq (\mathbb{T}//W)_2. \tag{5.1}$$

The primitive ideal space on the left-hand side of (5.1) admits the Jacobson topology. So the right-hand side of (5.1) acquires, by transport of structure, a compact non-Hausdorff topology. The picture at the beginning of §6.1 is intended to portray this topology. We shall see that the Langlands parameters respect this topology. The double-points in the picture arise precisely when the corresponding induced representation has length 2.

The Plancherel Theorem of Harish-Chandra is valid for any local non-archimedean field, see Waldspurger [18]. This implies that, in the case at hand, the discrete series and the unitary principal series enter into the Plancherel formula. That is, the tempered dual of $\mathcal{G}$ comprises the discrete series and the irreducible constituents in the unitary principal series.

We now focus on the case of induced elements.

Suppose $\chi^2 \neq 1$, with $\mathfrak{s} = [\mathcal{T}, \chi]_{\mathcal{G}}$. Let ψ be an unramified unitary character of $\mathcal{T}$. Then we have a natural bijection

$$\mathbf{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}} \simeq \mathbb{T}, \qquad \mathrm{Ind}\, \pi_{\psi\chi} \mapsto \psi(\varpi). \tag{5.2}$$

Suppose $\chi^2 = 1, \chi \neq 1$, with $\mathfrak{s} = [T, \chi]_G$. Let $W = \mathbb{Z}/2\mathbb{Z}$. Then we have a bijective map

$$\mathbf{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}} \simeq (\mathbb{T}//W)_2. \tag{5.3}$$

This map is defined as follows. Let ρ denote the nontrivial character of W.

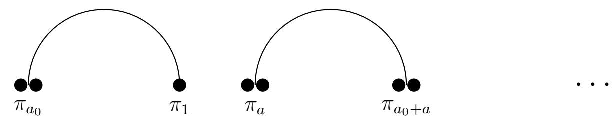
- If $\psi^2 \neq 1$, send $\mathrm{Ind}\, \pi_{\psi\chi}$ to $\psi(\varpi)$.
- If $\psi = 1$, send the pair of irreducible constituents π^+_χ, π^-_χ to the pair of points $(1, \mathrm{triv}),\ (1, \rho) \in (T//W)_2$.
- If $\psi = \epsilon$ the unique unramifed *quadratic* character of $\mathcal{T}$, send the pair of irreducible constituents $\pi^+_{\epsilon\chi}, \pi^-_{\epsilon\chi}$ to the pair of points $(-1, \mathrm{triv}),\ (-1, \rho) \in (T//W)_2$.

Suppose $\chi = 1$ and let $\mathfrak{s}_0 = [T, 1]_G$. Then we have a continuous bijection which is *not* a homeomorphism:

$$\mathbf{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}} \to (\mathbb{T}//W)_2. \tag{5.4}$$

- If $\psi^2 \neq 1$, send $\mathrm{Ind}\, \pi_{\psi\chi}$ to $\psi(\varpi)$.
- If $\psi = 1$, send the irreducible representations $\mathrm{triv}_{\mathcal{G}}, \mathrm{St}_{\mathcal{G}}$ to the pair of points $(1, \mathrm{triv}),\ (1, \rho) \in (T//W)_2$.
- If $\psi = \epsilon$ send the pair of irreducible constituents $\pi^+_\epsilon, \pi^-_\epsilon$ to the pair of points $(-1, \mathrm{triv}),\ (-1, \rho) \in (T//W)_2$.

By proposition 4.1 and the above argument, we may represent that part of the tempered dual $\mathbf{Irr}^{\mathrm{temp}}(\mathrm{SL}_2(\mathbb{F}_q((\varpi))))$ which corresponds to the unitary principal series in a diagram along the lines of [11, p.418].



The first double point represent the L-packet $\{\pi^+_{a_0}, \pi^-_{a_0}\}$. The second and third double-points represent, respectively, the L-packets $\{\pi^+_a, \pi^-_a\}$ and $\{\pi^+_{a_0+a}, \pi^-_{a_0+a}\}$. The second half-circle is repeated countably many times, and is parametrized by L-parameters $\{\varphi_a\}_{a+\wp(K)}$, see theorem 4.4.

Topology on the tempered dual. Let $\mathcal{G} = \mathrm{SL}_2(\mathbb{F}_q(\varpi)))$. The tempered dual of $\mathcal{G}$ is the disjoint union $X = X_{\mathcal{G}}$ of the discrete series and the irreducible constituents in the principal series. We equip X with the following topology $\mathfrak{T}$: The topology $\mathfrak{T}$ must induce the standard topologies on each point, each copy of $\mathbb{T}$, and each copy (except one) of $(\mathbb{T}//W)_2$, *all of which (except one)* must become $\mathfrak{T}$-open sets. On the exceptional copy of $(\mathbb{T}//W)_2$ the Steinberg point $\mathrm{St}_{\mathcal{G}}$ must be $\mathfrak{T}$-isolated. Then $\mathfrak{T}$ is a locally compact topology on X. It is not Hausdorff.

In the space X, each L-packet in the unitary principal series will feature as a $\mathfrak{T}$-double-point.

There will be countably many double-points, one for each quadratic extension $K(\alpha)$; *cf.* the diagram in [11] for the tempered dual of $\mathrm{SL}_2(\mathbb{Q}_p)$ with $p > 2$. In that diagram, there are just three double-points. For $\mathrm{SL}_2(\mathbb{Q}_2)$ there would be seven double-points.

Each supercuspidal L-packet will feature as four $\mathfrak{T}$-isolated points in X.

We conjecture that $\mathfrak{T}$ coincides with the Jacobson topology on the primitive ideal space of the reduced C^*-algebra of $\mathcal{G}$.

6. Biquadratic extensions of $\mathbb{F}_q((\varpi))$

Quadratic extensions L/K are obtained by adjoining an $\mathbb{F}_2$-line $D \subset K/\wp(K)$. Therefore, $L = K(\wp^{-1}(D)) = K(\alpha)$ where $D = span\{a + \wp(K)\}$, with $\alpha^2 - \alpha = a$. In particular, if a_0 is integer, the $\mathbb{F}_2$-line $V_0 = span\{a_0 + \wp(K)\}$ contains all the cosets $a_i + \wp(K)$ where a_i is an integer and so $K(\wp^{-1}(\mathfrak{o})) = K(\wp^{-1}(V_0)) = K(\alpha_0)$ where $\alpha_0^2 - \alpha_0 = a_0$ gives the unramified quadratic extension.

Biquadratic extensions are computed the same way, by considering planes $W = span\{a + \wp(K), b + \wp(K)\} \subset K/\wp(K)$. Therefore, if $a+\wp(K)$ and $b+\wp(K)$ are $\mathbb{F}_2$-linearly independent then $K(\wp^{-1}(W)) := K(\alpha, \beta)$ is biquadratic, where $\alpha^2 - \alpha = a$ and $\beta^2 - \beta = b$, $\alpha, \beta \in K^s$. Therefore, $K(\alpha, \beta)/K$ is biquadratic if $b - a \notin \wp(K)$.

A biquadratic extension containing the line V_0 is of the form $K(\alpha_0, \beta)/K$. There are countably many quadratic extensions L_0/K containing the unramified quadratic extension. They have ramification index $e(L_0/K) = 2$. And there are countably many biquadratic extensions L/K which do not contain the unramified quadratic extension. They have ramification index $e(L/K) = 4$.

So, there is a plentiful supply of biquadratic extensions $K(\alpha, \beta)/K$.

6.1. **Ramification.** The space $K/\wp(K)$ comes with a filtration

$$0 \subset_1 V_0 \subset_f V_1 = V_2 \subset_f V_3 = V_4 \subset_f \ldots \subset K/\wp(K) \tag{6.1}$$

where V_0 is the image of $\mathfrak{o}_K$ and V_i $(i > 0)$ is the image of $\mathfrak{p}^{-i}$ under the canonical surjection $K \to K/\wp(K)$. For $K = \mathbb{F}_q((\varpi))$ and $i > 0$, each inclusion $V_{2i} \subset_f V_{2i+1}$ is a sub-$\mathbb{F}_2$-space of codimension f. The $\mathbb{F}_2$-dimension of V_n is

$$dim_{\mathbb{F}_2} V_n = 1 + \lceil n/2 \rceil, \tag{6.2}$$

where $\lceil x \rceil$ is the smallest integer not less than x.

Let L/K denote a Galois extension with Galois group G. For each $i \geq -1$ we define the i^{th}-ramification subgroup of G (in the lower numbering) to be:

$$G_i = \{\sigma \in G : \sigma(x) - x \in \mathfrak{p}_L^{i+1}, \forall x \in \mathfrak{o}_L\}.$$

An integer t is a *break* for the filtration $\{G_i\}_{i\geq -1}$ if $G_t \neq G_{t+1}$. The study of ramification groups $\{G_i\}_{i\geq -1}$ equivalent to the study of breaks of the filtration.

There is another decreasing filtration with upper numbering $\{G^i\}_{i\geq -1}$ and defined by the Hasse-Herbrand function $\psi = \psi_{L/K}$:

$$G^u = G_{\psi(u)}.$$

In particular, $G^{-1} = G_{-1} = G$ and $G^0 = G_0$, since $\psi(0) = 0$.

Let $G_2 = Gal(K_2/K)$ be the Galois group of the maximal abelian extension of exponent 2, $K_2 = K(\wp^{-1}(K))$. Since $G_2 \cong K^\times/K^{\times 2}$ (Proposition 2.5), the pairing $K^\times/K^{\times 2} \times K/\wp(K) \to \mathbb{Z}/2\mathbb{Z}$ from (2.1) coincides with the pairing $G_2 \times K/\wp(K) \to \mathbb{Z}/2\mathbb{Z}$.

The profinite group G_2 comes equipped with a ramification filtration $(G_2^u)_{u\geq -1}$ in the upper numbering, see [5, p.409]. For $u \geq 0$, we have an orthogonal relation [5, Proposition 17]

$$(G_2^u)^\perp = \overline{\mathfrak{p}^{-\lceil u\rceil+1}} = V_{\lceil u\rceil -1} \tag{6.3}$$

under the pairing $G_2 \times K/\wp(K) \to \mathbb{Z}/2\mathbb{Z}$.

Since the upper filtration is more suitable for quotients, we will first compute the upper breaks and then use the Hasse-Herbrand function to compute the lower breaks in order to obtain the lower ramification filtration.

According to [5, Proposition 17], the positive breaks in the filtration $(G^v)_v$ occur precisely at integers prime to p. So, for $ch(K) = 2$, the positive breaks will occur at odd integers. The lower numbering breaks are also integers. If G is cyclic of prime order, then there is a unique break for any decreasing filtration $(G^v)_v$ (see [5], Proposition 14). In general, the number of breaks depends on the possible filtration of the Galois group.

Given a plane $W \subset K/\wp(K)$, the filtration (6.1) $(V_i)_i$ on $K/\wp(K)$ induces a filtration $(W_i)_i$ on W, where $W_i = W \cap V_i$. There are three possibilities for the filtration breaks on a plane and we will consider each case individually.

Case 1 : W contains the line V_0, i.e. $L_0 = K(\wp^{-1}(W))$ contains the unramified quadratic extension $K(\wp^{-1}(V_0)) = K(\alpha_0)$ of K. The extension has residue degree $f(L_0/K) = 2$ and ramification index

$e(L_0/K) = 2$. In this case, there is an integer $t > 0$, necessarily odd, such that the filtration $(W_i)_i$ looks like

$$0 \subset_1 W_0 = W_{t-1} \subset_1 W_t = W.$$

By the orthogonality relation (6.3), the upper ramification filtration on $G = Gal(L_0/K)$ looks like

$$\{1\} = ... = G^{t+1} \subset_1 G^t = ... = G^0 \subset_1 G^{-1} = G$$

Therefore, the upper ramification breaks occur at -1 and t. The lower ramification breaks can be computed using the Hasse-Herbrand function. The table for the index of G^u in G^0 is as follows:

$u \in$	$[0,t]$	$]t,+\infty[$
$G^u =$	G^0	$\{1\}$
$(G^0 : G^u) =$	1	2

We have, $\psi(t) = \int_0^t (G^0 : G^u)du = t$, and the lower ramifications breaks occur at -1 and t. It follows that the **lower filtration** is

$$(6.4)\quad G_{-1} = G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\ ;\ G_0 = ... = G_t = \mathbb{Z}/2\mathbb{Z}\ ;\ G_{t+1} = \{1\}$$

The number of such W is equal to the number of planes in V_t containing the line V_0 but but not contained in the subspace V_{t-1}. Note that this number can be computed and equals the number of biquadratic extensions of K containing the unramified quadratic extensions and with a pair of upper ramification breaks $(-1, t)$, $t > 0$ and odd.

Example 6.1. *The number of biquadratic extensions containing the unramified quadratic extension and with a pair of upper ramification breaks* $(-1, 1)$ *is equal to the number of planes in an* $1 + f$*-dimensional* $\mathbb{F}_2$*-space, containing the line* V_0*. There are precisely*

$$1 + 2 + 2^2 + ... + 2^{f-1} = \frac{1 - 2^f}{1 - 2} = q - 1$$

of such biquadratic extensions.

Case 2.1 : W does not contains the line V_0 and the induced filtration on the plane W looks like

$$0 = W_{t-1} \subset_2 W_t = W$$

for some integer t, necessarily odd.

The number of such W is equal to the number of planes in V_t whose intersection with V_{t-1} is $\{0\}$. Note that, there are no such planes when $f = 1$. So, for $K = \mathbb{F}_2((\varpi))$, **case 2.1** does not occur.

Suppose $f > 1$. By the orthogonality relation, the upper ramification ramification filtration on $G = Gal(L/K)$ looks like

$$\{1\} = ... = G^{t+1} \subset_2 G^t = ... = G^{-1} = G$$

Therefore, there is a single upper ramification break occur at $t > 0$ and necessarily odd. The lower ramification breaks occurs at the same t, since we have:

$u \in$	$[0, t]$	$]t, +\infty[$
$G^u =$	G^0	$\{1\}$
$(G^0 : G^u) =$	1	2^2

and so, $\psi(t) = \int_0^t (G^0 : G^u)du = t$, and the lower ramifications breaks occur at -1 and t. It follows that the **lower filtration** is

$$G_{-1} = G = ... = G_t = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \ ; \ G_{t+1} = \{1\} \tag{6.5}$$

For $f = 1$ there is no such biquadratic extension. For $f > 1$, the number of these biquadratic extensions equals the number of planes W contained in an $\mathbb{F}_2$-space of dimension $1 + fi$, $t = 2i - 1$, which are transverse to a given codimension-f $\mathbb{F}_2$-space.

Case 2.2 : W does not contains the line V_0 and the induced filtration on the plane W looks like

$$0 = W_{t_1-1} \subset_1 W_{t_1} = W_{t_2-1} \subset_1 W_{t_2} = W$$

for some integers t_1 and t_2, necessarily odd, with $0 < t_1 < t_2$.

The orthogonality relation for this case implies that the upper ramification filtration on $G = Gal(L/K)$ looks like

$$\{1\} = ... = G^{t_2+1} \subset_1 G^{t_2} = ... = G^{t_1+1} \subset_1 G^{t_1} = ... = G$$

The upper ramification breaks occur at odd integers t_1 and t_2.

Now, index of G^u in G^0 is:

$u \in$	$[0, t_1]$	$]t_1, t_2]$	$]t_2, +\infty[$
$(G^0 : G^u) =$	1	2	2^2

The lower breaks occur at

$$\psi(t_1) = \int_0^{t_1} (G^0 : G^u)du = t_1.$$

and at

$$\psi(t_2) = \int_0^{t_2} (G^0 : G^u)du = \int_0^{t_1} (G^0 : G^u)du + \int_{t_1}^{t_2} (G^0 : G^u)du$$

$$= t_1 + 2(t_2 - t_1) = 2t_2 - t_1.$$

In this case, the lower breaks occur at t_1 and $2t_2 - t_1$, with $0 < t_1 < t_2$ the odd integers where the upper ramification breaks occur.

We conclude that the **lower filtration** is given by

$$G = G_0 = ... = G_{t_1} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \tag{6.6}$$

$$G_{t_1+1} = ... = G_{2t_2-t_1} = \mathbb{Z}/2\mathbb{Z}\ ;\ G_{2t_2-t_1+1} = \{1\} \tag{6.7}$$

There is only a finite number of such biquadratic extensions, for a given pair of upper breaks (or lower breaks) (t_1, t_2).

7. Formal degrees

In this section, we are influenced by the lecture notes of Reeder [12], and the preceding three talks in Washington, DC. For $\mathcal{G} = \mathrm{SL}_2(K)$, the dual group $G = \mathrm{SO}_3(\mathbb{C})$ contains a unique (up to conjugacy) subgroup $J \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$, whose nontrivial elements are 180-degree rotations about three orthogonal axes. One can check that the centralizer and normalizer of J are given by

$$C_G(J) = J, \quad N_G(J) = O$$

where $O \simeq S_4$ is the rotatation group of the octahedron whose vertices are the unit vectors on the given orthogonal axes. The quotient $O/J \simeq \mathrm{GL}_2(\mathbb{Z}/2)$ is the full automorphism group of J.

Each bi-quadratic extension L/K gives a surjective homomorphism

$$\varphi_L : \mathbf{W}_F \to J$$

which is a discrete parameter with $S_{\varphi_L} = J$, since $C_G(J) = J$, and whose conjugacy class depends only on L, since $O/J = \mathrm{Aut}(J)$.

Since

$$|S_{\varphi_L}| = 4$$

the L-packet Π_{φ_L} has 4 constituents. There are countably many bi-quadratic extensions, therefore there are countably many L-packets with 4 constituents.

None of these packets contains the Steinberg representation $\mathrm{St}_{\mathcal{G}}$ and so, according to Conjecture 6.1.4 in [12], these are all supercuspidal L-packets, each with 4 elements.

Consider the principal parameter:

$$\mathrm{Ad}\,\varphi_0 : \mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{C}))$$

The *adjoint gamma value* is given by

$$\gamma(\varphi_0) = \frac{q}{1 + q^{-1}}$$

where $q = 2^f$.

Concerning the adjoint gamma value $\gamma(\varphi)$ we have

$$\mathrm{Ad}\,\varphi : \mathbf{W}_K \to J \to \mathrm{SO}_3(\mathbb{C}) \to^{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{so}_3(\mathbb{C}))$$

The adjoint representation of $\mathrm{SO}_3(\mathbb{C})$ is equivalent to the standard representation of $\mathrm{SO}_3(\mathbb{C})$ on $\mathbb{C}^3$ and so we replace the above sequence of maps by

$$\mathrm{Ad}\,\varphi : \mathbf{W}_K \to J \to \mathrm{SO}_3(\mathbb{C}).$$

For the L-function, we have

$$L(\mathrm{Ad}\,\varphi, s) = \frac{1}{1+q^{-s}}$$

and so we have

$$\gamma(\varphi) = \frac{2}{1+q^{-1}} \cdot \varepsilon(\varphi)$$

where

$$\varepsilon(\varphi) = \pm q^{\alpha(\varphi)/2}.$$

Note that we have

$$\left|\frac{\gamma(\varphi)}{\gamma(\varphi_0)}\right| = \frac{2}{q} \cdot \varepsilon(\varphi). \tag{7.1}$$

Now $\alpha(\varphi)$ is the Weil-Deligne version of the Artin conductor which is give here by

$$\alpha(\varphi) = \sum_{i \geq 0} \frac{\dim(\mathfrak{g}/\mathfrak{g}^{D_i})}{[D_0 : D_i]}$$

see [12], Reeder's notation.

We have to take the cases separately, beginning with (6.4).

Case 1: We have

$$G_{-1} = G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\ ;\ G_0 = ... = G_t = \mathbb{Z}/2\mathbb{Z}\ ;\ G_{t+1} = \{1\}$$

We have

$$\alpha(\varphi) = (1+t)2$$

According to Conjecture 6.1(1) in [14], we have

$$\mathrm{Deg}(\pi_{\varphi_L}(\rho)) = \frac{1}{4} \cdot \left|\frac{\gamma(\varphi)}{\gamma(\varphi_0)}\right| = \frac{1}{4} \cdot \frac{2}{q} \cdot |\varepsilon(\varphi)| = 2^{t-f}$$

the *canonical formal degree* of each supercuspidal constituent in the packet Π_{φ_L}, i.e. the formal degree w.r.t. the Euler-Poincaré measure on $\mathcal{G}$. If we fix the field K, then the formal degree tends to ∞ as the break number t tends to ∞.

The least allowed value of t is $t = 1$. When $t = f = 1$, the canonical formal degree of each element in the packet Π_{φ_L} is equal to 1. The lower ramification filtration is

$$G_{-1} = G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\ ;\ G_0 = G_1 = \mathbb{Z}/2\mathbb{Z}\ ;\ G_2 = \{1\}$$

and so, according to 6.1(5) in [12], the elements in this packet are not of depth zero.

Case 2.1: The lower ramification filtration is

$$G_{-1} = G = \ldots = G_t = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\ ;\ G_{t+1} = \{1\}$$

We have

$$\alpha(\varphi) = \sum_{i \geq 0} \frac{\dim(\mathfrak{g}/\mathfrak{g}^{D_i})}{[D_0 : D_i]} = (t+1)3$$

According to 6.1(1) in [12], we have

$$\begin{aligned}
\mathrm{Deg}(\pi_{\varphi_L}(\rho)) &= \frac{1}{4} \cdot \left| \frac{\gamma(\varphi)}{\gamma(\varphi_0)} \right| \\
&= \frac{1}{4} \cdot \frac{2}{q} \cdot |\varepsilon(\varphi)| \\
&= \frac{1}{2q} \cdot 2^{\alpha(\varphi)/2} \\
&= \frac{1}{2q} \cdot 2^{3(1+t)/2} \\
&= 2^{3(1+t)/2-f-1}
\end{aligned}$$

Note that t is odd, therefore the formal degree is a *rational* number.

Case 2.2: This case admits the following lower ramification filtration:

$$G = G_0 = ... = G_{t_1} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$
$$G_{t_1+1} = ... = G_{2t_2-t_1} = \mathbb{Z}/2\mathbb{Z} \; ; \; G_{2t_2-t_1+1} = \{1\}$$

We have

$$\alpha(\varphi) = \sum_{i \geq 0} \frac{\dim(\mathfrak{g}/\mathfrak{g}^{D_i})}{[D_0 : D_i]} = (t_1+1)3 + \frac{(2t_2)2}{2} = 3 + 3t_1 + 2t_2$$

and, according to 6.1(1) in [12], we have

$$\begin{aligned}
\mathrm{Deg}(\pi_{\varphi_L}(\rho)) &= \frac{1}{4} \cdot \left| \frac{\gamma(\varphi)}{\gamma(\varphi_0)} \right| \\
&= \frac{1}{4} \cdot \frac{2}{q} \cdot |\varepsilon(\varphi)| \\
&= \frac{1}{2q} \cdot 2^{\alpha(\varphi)/2} \\
&= \frac{1}{2q} \cdot 2^{3(1+t_1)/2+t_2} \\
&= 2^{3(1+t_1)/2+t_2-f-1}
\end{aligned}$$

the canonical formal degree of each supercuspidal in the packet Π_{φ_L}. If we fix f, then the formal degree tends to ∞ as the break numbers tend to ∞.

Note that t_1 is odd, therefore all the formal degrees are *rational* numbers, in conformity with the rationality of the gamma ratio [7, Prop. 4.1].

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