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Rank and order of a finite group admitting a Frobenius group of automorphisms

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Abstract

Suppose that a finite group G admits a Frobenius group of automorphisms FH of coprime order with kernel F and complement H. In the case where G is a finite p-group such that $G = [G, F]$ it is proved that the order of G is bounded above in terms of the order of H and the order of the fixed-point subgroup $C_G(H)$ of the complement, and the rank of G is bounded above in terms of $|H|$ and the rank of $C_G(H)$. Earlier such results were known under the stronger assumption that the kernel F acts on G fixed-point-freely. As a corollary, in the case where G is an arbitrary finite group with a Frobenius group of automorphisms FH of coprime order with kernel F and complement H, estimates are obtained of the form $|G| \leqslant |C_G(F)| \cdot f(|H|, |C_G(H)|)$ for the order, and $\mathbf{r}(G) \leqslant \mathbf{r}(C_G(F)) + g(|H|, \mathbf{r}(C_G(H)))$ for the rank, where f and g are some functions of two variables.

to Victor Danilovich Mazurov on his 70th birthday

1 Introduction

Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H. Mazurov's problem 17.72 in Kourovka Notebook [1] generated several recent papers, which considered the case where the kernel F acts fixed-point-freely, $C_G(F) = 1$. The purpose of these results [2, 3, 4, 5, 6, 7, 8, 9, 10] is bounding the order, the rank, the Fitting height, the nilpotency class, and the exponent of the group G in terms of the corresponding properties and parameters of $C_G(H)$ and $|H|$.

The purpose of this note is to show that as far as the order and rank of G are concerned, similar results hold without this strong condition $C_G(F) = 1$, at least when the orders of G and FH are coprime. The proofs are essentially reduced to studying Sylow p-subgroups of G, for various primes p. Moreover, in the case where G is a p-group satisfying $G = [G, F]$ the results are most strong and depend only on $C_G(H)$ and H. Therefore it is convenient to state a separate theorem for p-groups.

Theorem 1. *Suppose that a finite p-group P admits a Frobenius group of automorphisms FH with kernel F and complement H such that the orders of G and FH are coprime*: $(|G|, |FH|) = 1$. *If $P = [P, F]$, then*

(a) *the nilpotency class of P is at most $2\log_p |C_P(H)|$*;
(b) *the order of P is bounded above in terms of the orders of $C_P(H)$ and H*;
(c) *the rank of P is bounded above in terms of $|H|$ and the rank of $C_P(H)$.*

In the present paper the *rank* of a finite group is the minimum number r such that every subgroup can be generated by r elements.

As a corollary we obtain a result on rank and order of an arbitrary finite group with a Frobenius group of automorphisms of coprime order. Let $\mathbf{r}(K)$ denote the rank of a finite group K.

Theorem 2. *Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that the orders of G and FH are coprime: $(|G|, |FH|) = 1$. Then*

(a) $|G| \leqslant |C_G(F)| \cdot f(|H|, |C_G(H)|)$ *for some function f of two variables*;
(b) $\mathbf{r}(G) \leqslant \mathbf{r}(C_G(F)) + g(|H|, \mathbf{r}(C_G(H)))$ *for some function g of two variables.*

Under the hypothses of Theorem 2 weaker bounds for the order and rank in terms of some functions depending on $|C_G(F)|$, $|H|$, $|C_G(H)|$) and $\mathbf{r}(C_G(F))$, $|H|$, $\mathbf{r}(C_G(H))$), respectively, could be obtained just by considering the action of FH on Thompson's critical subgroups in FH-invariant Sylow p-subgroups for various p. The new information is that the dependence on $C_G(F)$ is now in the form of a separate factor $|C_G(F)|$ or summand $\mathbf{r}(C_G(F))$.

All the functions mentioned in the theorems can be easily given explicit upper estimates.

The induced group of automorphisms of an invariant section is often denoted by the same letter. We use the abbreviation, say, "(m, n)-bounded" for "bounded above in terms of m, n only", that is, bounded above by a function depending only on m and n.

2 Preliminaries

Recall that if a group A is acting by automorphisms on a finite group G of coprime order, $(|A|, |G|) = 1$, then the fixed points of the induced action of A on the quotient G/N by an A-invariant normal subgroup are covered by fixed points of A in G:

$$C_{G/N}(A) = C_G(A)N/N.$$

In particular, $[[G, A], A] = [G, A]$. For every prime p, the group G has an A-invariant Sylow p-subgroup. We shall use these well-known properties of coprime action without special references.

For a group A acting by linear transformations on a vector space V we use the right operator notation va for the image of $v \in V$ under $a \in A$. We also use the centralizer notation for the fixed-point subspace $C_V(A) = \{v \in V \mid va = v \text{ for all } a \in A\}$ (just like for fixed-point subgroups). Recall that for a group A and a field k, a free kA-module of rank n is a direct sum of n copies of the group algebra kA each of which is regarded as a vector space over k of dimension $|A|$ with a basis $\{v_g \mid g \in A\}$ labelled by elements of A on which A acts in a regular permutation representation: $v_g h = v_{gh}$. Clearly, every free kA-module has nontrivial fixed-point for A given by the 'diagonal' elements — sums over A-orbits on these bases.

The following lemma is an easy consequence of Clifford's theorem.

Lemma 2.1 ([5, Lemma 2.5]). *If a Frobenius group FH with kernel F and complement H acts by linear transformations on a vector space V over a field k in such a way that $C_V(F)=0$, then V is a free kH-module.*

We immediately obtain the following.

Lemma 2.2. *If a Frobenius group FH with kernel F and complement H acts by automorphisms on a finite group G of coprime order in such a way that $[G,F]\neq 1$, then $C_G(H)\neq 1$.*

Proof. Choose an FH-invariant nontrivial Sylow p-subgroup P of G on which F acts nontrivially. Then F acts nontrivially also on the Frattini quotient $U=P/\Phi(P)$, which can be regarded as an $\mathbb{F}_pFH$-module. Then $V=[U,F]\neq 0$ is a free $\mathbb{F}_pFH$-module by Lemma 2.1. Hence there is a nontrivial fixed point of H in V, and therefore also in P. $\square$

A finite p-group P is said to be powerful if $[P,P]\leqslant P^p$ for $p\neq 2$, or $[P,P]\leqslant P^4$ for $p=2$. (Here, $A^n=\langle a^n\mid a\in A\rangle$.) Powerful p-groups is an indispensable tool in the study of ranks of finite p-groups.

Lemma 2.3 ([11]). (a) *If a powerful p-group P is generated by d elements, then the rank of P is at most d and P is a product of d cyclic subgroups.*

(b) *If P is a finite p-group of rank r, then P contains a characteristic powerful subgroup of index at most $p^{r(\log_2 r+2)}$.*

Lemma 2.4. *If a finite p-group P has rank r and exponent p^n, then $|P|\leqslant p^{nf(r)}$ for some r-bounded number $f(r)$.*

Proof. By Lemma 2.3(b) the group P can be assumed to be powerful; Lemma 2.3(a) completes the proof. $\square$

The following result was obtained by Kovács [12] for soluble groups on the basis of Hall–Higman type theorems and extended, with the use of the classification, to arbitrary finite groups by Longobardi and Maj [13] (with the bound $2d$) and Guralnik [14].

Lemma 2.5. *If d is the maximum of the ranks of the Sylow p-subgroups of a finite group (over all primes p), then the rank of this group is at most $d+1$.*

We also record the following well-known fact about nilpotent groups.

Lemma 2.6. *Let G be a nilpotent group of nilpotency class c.*

(a) *The order of G is bounded in terms of c and the order of $G/[G,G]$.*

(b) *The rank of G is bounded in terms of c and the rank of $G/[G,G]$.*

Proof. Let $\gamma_i=\gamma_i(G)$ denote terms of the lower central series of G. Both statements are consequences of the well-known fact that there is a homomorphism

$$\underbrace{\gamma_1/\gamma_2\otimes\cdots\otimes\gamma_1/\gamma_2}_{k}\ \rightarrow\ \gamma_k/\gamma_{k+1}$$

from the tensor power onto γ_k/γ_{k+1}. $\square$

3 Finite p-groups

Here we prove Theorem 1. Recall that P is a finite p-group admitting a Frobenius group FH of automorphisms of coprime order with kernel F and complement H such that $P = [P, F]$.

Proof of Theorem 1(a). We begin with a bound for the nilpotency class of P. For brevity, let $\gamma_i = \gamma_i(P)$ denote terms of the lower central series of P. Let $|C_P(H)| = p^m$. By Lemma 2.2, whenever $[V, F] \neq 1$ for an FH-invariant section V, we must have $C_V(H) \neq 1$. Hence there are at most m factors of the lower central series of P where F acts nontrivially. Therefore, for some $i \leqslant 2m$ the group F acts trivially on the two consecutive factors γ_i/γ_{i+1} and $\gamma_{i+1}/\gamma_{i+2}$. We have

$$[F, \gamma_i, P] \leqslant [\gamma_{i+1}, P] = \gamma_{i+2}$$

and

$$[\gamma_i, P, F] = [\gamma_{i+1}, F] \leqslant \gamma_{i+2}.$$

Hence, by the Three Subgroup Lemma,

$$[[P, F], \gamma_i] = [P, \gamma_i] = \gamma_{i+1} \leqslant \gamma_{i+2}.$$

Consequently, $\gamma_{i+1} = 1$, since P is a nilpotent group. □

Proof of Theorem 1(b). Now that we have a bound for the nilpotency class of P, a bound for the order will follow by Lemma 2.6(a) if we obtain a bound for the order of P/γ_2.

Since the action is coprime, the abelian group P/γ_2 decomposes as

$$P/\gamma_2 = C_{P/\gamma_2}(F) \times \big[P/\gamma_2,\, F\big] = C_{P/\gamma_2}(F) \times P/\gamma_2,$$

whence $C_{P/\gamma_2}(F) = 1$. Therefore, $|P/\gamma_2| = |C_{P/\gamma_2}(H)|^{|H|}$ by [5, Theorem 2.7(a)] (this fact is an easy consequence of Lemma 2.1). Since the action is coprime, $|C_{P/\gamma_2}(H)| \leqslant |C_P(H)|$. Furthermore, the nilpotency class of P is at most $2\log_p |C_P(H)| \leqslant 2\log_2 |C_P(H)|$. Therefore the order $|P|$ is indeed bounded in terms of $|C_P(H)|$ and $|H|$ only. □

Proof of Theorem 1(c). We now obtain a bound for the rank of P. The crucial step is to show that P has a powerful p-subgroup of bounded rank and 'co-rank'. The construction of a powerful subgroup is similar to how it was done in [15] and [16]. But first we estimate the number of generators of P. Henceforth in this section we denote for brevity by $r = \mathbf{r}(C_P(H))$ the rank of $C_P(H)$.

Lemma 3.1. *The group P is generated by $r|H|$ elements.*

Proof. Consider the action of FH on the Frattini quotient $V = P/\Phi(P)$. As above in part (b), it is easy to see that $C_V(F) = 1$. Therefore, $|V| = |C_V(H)|^{|H|} \leqslant p^{r|H|}$ by [5, Theorem 2.7(a)] (or as a consequence of Lemma 2.1). □

Let M be some normal FH-invariant subgroup of P, which will be specified later.

Consider the quotient $\overline{P} = P/M^p$ (or P/M^4 if $p = 2$); let the bar denote the images.

Since $\overline{M} = M/M^p$ (or $\overline{M} = M/M^4$) has exponent p (or 4), the order of $C_{\overline{M}}(H)$ is at most p^f for some r-bounded number $f = f(r)$ by Lemma 2.4.

As usual, we denote terms of the upper central series by ζ_i, starting from the centre ζ_1.

Lemma 3.2. *We have $\overline{M} \leqslant \zeta_{2f+1}(\overline{P})$.*

Proof. Consider the following central series of $\overline{P}$:

$$M_1 = \overline{M} > M_2 > M_3 > \cdots > 1, \qquad \text{where } M_i = [...[\overline{M}, \underbrace{\overline{P}], \ldots, \overline{P}}_{i-1}].$$

All the M_i are normal FH-invariant subgroups of $\overline{P}$. Let $V_i = M_i/M_{i+1}$ and consider the action of FH on these sections.

Whenever $[V_i, F] \neq 1$ we have $C_{V_i}(H) \neq 1$ by Lemma 2.2. Since $|C_{\overline{M}}(H)| \leqslant p^f$, there can be at most f factors V_i with $[V_i, F] \neq 1$.

Therefore for some $k \leqslant 2f+1$ we must have both $[V_k, F] = 1$ and $[V_{k+1}, F] = 1$. In other words, we have

$$[[F, M_k], \overline{P}] \leqslant [M_{k+1}, \overline{P}] = M_{k+2}$$

and

$$[[M_k, \overline{P}], F] = [M_{k+1}, F] \leqslant M_{k+2}.$$

Hence, by the Three Subgroup Lemma,

$$[[\overline{P}, F], M_k] = [\overline{P}, M_k] = M_{k+1} \leqslant M_{k+2}.$$

Then $M_{k+1} = 1$, since $\overline{P}$ is nilpotent: $M_{k+1} \leqslant M_{k+2}$ implies $[M_{k+1}, \overline{P}] \leqslant [M_{k+2}, \overline{P}]$, that is, $M_{k+2} \leqslant M_{k+3}$, and so on, which becomes eventually the trivial subgroup due to the nilpotency of $\overline{P}$.

The equation $M_{k+1} = 1$ obtained above means precisely that $\overline{M} \leqslant \zeta_k(\overline{P}) \leqslant \zeta_{2f+1}(\overline{P})$. $\square$

We continue proving that P has $(r, |H|)$-bounded rank. We put $M = \gamma_{2f+1}$ (where, recall, $\gamma_{2f+1} = \gamma_{2f+1}(P)$). It is convenient to introduce the unified notation $p_* = p$ if $p \neq 2$, and $p_* = 4$ if $p = 2$.

Then by Lemma 3.2 we have $[\overline{M}, \overline{M}] \leqslant [\gamma_{2f+1}(\overline{P}), \zeta_{2f+1}(\overline{P})] = 1$, that is, $[M, M] \leqslant M^{p_*}$. This means precisely that $M = \gamma_{2f+1}(P)$ is a powerful p-subgroup of P.

The quotient $P/\gamma_{2f+1}^{p_*}$ is nilpotent of class $4f+1$, since $\gamma_{2f+1}/\gamma_{2f+1}^{p_*} \leqslant \zeta_{2f+1}(P/\gamma_{2f+1}^{p_*})$ by Lemma 3.2 and by the choice of M).

Since P is generated by $r|H|$ elements by Lemma 3.1 and $P/\gamma_{2f+1}^{p_*}$ is nilpotent of class $4f+1$, the rank of $P/\gamma_{2f+1}^{p_*}$ is $(|H|, r)$-bounded by Lemma 2.6(b).

In particular, the rank of $\gamma_{2f+1}/\gamma_{2f+1}^{p_*}$ is $(|H|, r)$-bounded. Since $\gamma_{2f+1}^{p_*} \leqslant \Phi(\gamma_{2f+1})$, we obtain that the number of generators of γ_{2f+1} is $(|H|, r)$-bounded. But in a powerful p-group the number of generators is equal to its rank (Lemma 2.3(a)), so that the rank of γ_{2f+1} is $(|H|, r)$-bounded.

Thus, both the rank of P/γ_{2f+1} and the rank of γ_{2f+1} are $(|H|, r)$-bounded, whence the rank of P is $(|H|, r)$-bounded, as required. $\square$

Remark. Both functions in parts (b) and (c) of Theorem 1 can of course be assumed to be non-decreasing in each of their arguments. Indeed, any function $f(x, y)$ of two positive integer variables can be replaced, for example, by the function

$$\bar{f}(x, y) = \sup_{\substack{u \leqslant x, \\ v \leqslant y}} f(u, v),$$

which already satisfies the required property.

4 General case

Let G be a finite group admitting a Frobenius group of automorphisms FH of coprime order with kernel F and complement H. Here we prove Theorem 2 on the order and rank of G.

For each prime p, let S_p be an FH-invariant Sylow p-subgroup of G (one for each p). We have $S_p = C_{S_p}(F)[S_p, F]$.

Proof of Theorem $2(a)$. By Theorem 1(b) we have $|[S_p, F]| \leqslant f_1(|H|, |C_{[S_p,F]}(H)|)$ for some function f_1 that is non-decreasing in each argument. Hence,

$$|S_p| \leqslant |C_{S_p}(F)| \cdot f_1(|H|, |C_{[S_p,F]}(H)|).$$

Note also that $S_p = C_{S_p}(F)$ if $[S_p, F] = 1$. Since $|G| = \prod_p |S_p|$ and $|C_G(F)| = \prod_p |C_{S_p}(F)|$, we obtain

$$|G| \leqslant \prod_p |C_{S_p}(F)| \cdot \prod_{[S_p,F]\neq 1} f_1(|H|, |C_{[S_p,F]}(H)|) = |C_G(F)| \cdot \prod_{[S_p,F]\neq 1} f_1(|H|, |C_{[S_p,F]}(H)|).$$

But $C_{[S_p,F]}(H) \neq 1$ whenever $[S_p, F] \neq 1$ by Lemma 2.2. Hence in the product on the right-hand side the primes p divide $|C_G(H)|$. As a rough estimate, there are at most $\log_2 |C_G(H)|$ such primes. Therefore,

$$|G| \leqslant |C_G(F)| \cdot f_1(|H|, |C_G(H)|)^{\log_2 |C_G(H)|},$$

which is a required upper estimate for the order with the function

$$f(|H|, |C_G(H)|) = f_1(|H|, |C_G(H)|)^{\log_2 |C_G(H)|}.$$

□

Proof of Theorem $2(b)$. For each prime p, by Theorem 1(c) we have

$$\mathbf{r}([S_p, F]) \leqslant f_2(|H|, \mathbf{r}(C_{[S_p,F]}(H)))$$

for some function f_2 that is non-decreasing in each argument. Hence,

$$\mathbf{r}(S_p) \leqslant \mathbf{r}(C_{S_p}(F)) + f_2(|H|, \mathbf{r}(C_{[S_p,F]}(H))) \leqslant \mathbf{r}(C_G(F)) + f_2(|H|, \mathbf{r}(C_G(H))).$$

By Lemma 2.5, an upper estimate for the rank of G is obtained by adding 1 to the right-hand side. □

References

[1] *Unsolved Problems in Group Theory. The Kourovka Notebook*, no. 17, Institute of Mathematics, Novosibirsk, 2010.

[2] E. I. Khukhro, Graded Lie rings with many commuting components and an application to 2-Frobenius groups, *Bull. London Math. Soc.* **40**, (2008), 907–912.

[3] N. Y. Makarenko and P. Shumyatsky, Frobenius groups as groups of automorphisms, *Proc. Amer. Math. Soc.* **138** (2010), 3425–3436.

[4] E. I. Khukhro, Nilpotent length of a finite group admitting a Frobenius group of automorphisms with fixed-point-free kernel, *Algebra Logika* **49** (2010), 819–833; English transl., *Algebra Logic* **49** (2010), 551–560.

[5] E. I. Khukhro, N. Y. Makarenko, and P. Shumyatsky, Frobenius groups of automorphisms and their fixed points, *Forum Math.*, 2011; DOI: `10.1515/FORM.2011.152`; `arxiv.org/abs/1010.0343`.

[6] N. Yu. Makarenko, E. I. Khukhro, and P. Shumyatsky, Fixed points of Frobenius groups of automorphisms, *Dokl. Akad. Nauk* **437**, no. 1 (2011), 20–23; English transl., *Dokl. Math.* **83**, no. 2 (2011), 152–154.

[7] P. Shumyatsky, On the exponent of a finite group with an automorphism group of order twelve, *J. Algebra* **331** (2011), 482–489.

[8] P. Shumyatsky, Positive laws in fixed points of automorphisms of finite groups, *J. Pure Appl. Algebra* **215** (2011), 2550–2566.

[9] E. I. Khukhro, Fitting height of a finite group with a Frobenius group of automorphisms, *J. Algebra* **366** (2012), 1–11.

[10] E. I. Khukhro, Automorphisms of finite group admitting a partition, *Algebra Logika* **51**, no. 3 (2012), ??-??; English transl. in *Algebra Logic* **51**, no. 3 (2012), ??-??.

[11] A. Lubotzky and A. Mann, Powerful p-groups. I: Finite groups, *J. Algebra* **105** (1987), 484–505.

[12] L.G. Kovács, On finite soluble groups, *Math. Z.* **103** (1968), 37–39.

[13] P. Longobardi and M. Maj, On the number of generators of a finite group, *Arch. Math.* (*Basel*) **50** (1988), 110–112.

[14] R. Guralnick, On the number of generators of a finite group, *Arch. Math.* **53** (1989), 521–523.

[15] E. I. Khukhro and V. D. Mazurov, Finite groups with an automorphism of prime order whose centralizer has small rank, *J. Algebra* **301** (2006), 474–492.

[16] P. Shumyatsky, Involutory automorphisms of finite groups and their centralizers, *Arch. Math.* (*Basel*) **71** (1998), 425–432.