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Sheaves as essentially algebraic objects

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Abstract

We develop the notion of essentially algebraic theories from [1]. We associate with each Grothendieck site a corresponding essentially algebraic theory whose models are the sheaves on that site. This is used to classify locally finitely presented toposes, and to show that the category of modules over a ring object in a locally finitely presented topos is also locally finitely presentable.

1 Introduction

Our motivation comes from a paper by Prest and Ralph, [8], in which it was speculated that for a ring in a locally finitely presented topos, the category of modules is locally finitely presented, and it was shown that this is the case when the topos is a category of sheaves on a space. More generally, we can ask what it means for a sheaf to be finitely presented. Given a finite type Grothendieck topology on a small category, it is known that the finitely presented sheaves are precisely the sheafifications of finitely presented presheaves [10]. The notion of finite presentation can be made very concrete for presheaves, since we can of course represent a presheaf category by a multisorted equational theory. In this paper, we extend this approach to cover sheaves, so that given a Grothendieck site $(\mathcal{C}, J)$, we can write down an essentially algebraic theory (in the sense of [1]) whose models correspond to the sheaves on the site. This allows us to write presentations for sheaves in the same way as we would for presheaves, and the notion of finite presentation we get is the natural one. Furthermore, we can characterize locally finitely generated, locally finitely presented and locally coherent toposes using these ideas. These characterizations are known, but we believe this method of proving them is new. This leads to an immediate generalization of Prest and Ralph's result, since we just add (finitary) axioms to the finitary theory of the locally finitely presented topos. In the last section of the paper, we consider when the category of modules is locally coherent, rather than just locally finitely presented; this is a much rarer condition.

We will assume the reader is familiar with the general theory of Grothendieck toposes; an introduction to this topic is given in [6, III], and we will borrow notation from that book throughout this paper.

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2 Locally presentable categories

In this section we summarise facts about locally presentable categories from [1]. These definitions can be found for abelian categories in Appendix E of [7].

Let λ be a regular cardinal. A partially ordered set $(I, \leq)$ is said to be λ-*directed* if every subset of I of size less than λ has a least upper bound. Let $\mathcal{C}$ be a category. A λ-*directed system* in $\mathcal{C}$ is a functor $D : \mathcal{D} \to \mathcal{C}$ where $\mathcal{D}$ is a λ-directed partially ordered set, considered as a category. A λ-*directed colimit* in $\mathcal{C}$ is a colimit over a directed diagram. We denote the colimit over a diagram D with the notation $\varinjlim \mathcal{D}$. $\aleph_0$-directed colimits are called direct limits in many areas of mathematics. An object C in $\mathcal{C}$ is said to be λ-*presented* if the functor

$$\mathrm{Hom}(C, -) : \mathcal{C} \to \mathbf{Sets}$$

commutes with λ-directed colimits. Equivalently, given a λ-directed system

$$\{ D_i \xrightarrow{d_{ij}} D_j \mid i \leq j \in (I, \leq)\}$$

where $(I, \leq)$ is some λ-directed poset, and given a colimit cocone

$$\{ D_i \xrightarrow{d_i} L \}$$

we have that any map $f : C \to L$ factors through the cocone, that is, $f = d_i f'$ for some $i \in I$ and some $f' : C \to D_i$, and this factorization is essentially unique, in the sense that if $g : C \to D_j$ is some other map with $d_j g = f$, then for some $k \geq i, j$, we have $d_{ik} f' = d_{jk} g$. We say an object is *finitely presented* if it is $\aleph_0$-presented.

A set of objects $\mathcal{G}$ is said to *generate* $\mathcal{C}$ if for any pair of arrows $f \neq g : A \to B$ in $\mathcal{C}$, there is some map $x : G \to A$ with $G \in \mathcal{G}$ and $fx \neq gx$. Equivalently, if $\mathcal{C}$ has coproducts, for every object C there is an epimorphism

$$e : \coprod_i G_i \to C$$

where the objects G_i are all in $\mathcal{G}$.

An epimorphism $e : A \to B$ is said to be *strong* if given any commutative square

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ E & \xrightarrow[m]{} & C \end{array}$$

such that m is a monomorphism, there is a map $d : B \to E$ such that $md = g$ and $f = de$. In categories with pushouts, this is equivalent to stating that e is *extremal*, that is, it does not factor through any proper subobject of C.

If $\mathcal{C}$ has coproducts, a generating set $\mathcal{G}$ is said to *strongly generate* $\mathcal{C}$ if for every object C there is a strong epimorphism $e : \coprod_i G_i \to C$ as above. This is equivalent to the condition that whenever $s : S \to A$ is a proper monomorphism in $\mathcal{C}$, there is a map $x : G \to A$ with $G \in \mathcal{G}$, not admitting a factorization through s.

The category $\mathcal{C}$ is *locally finitely presented* if it is cocomplete and has a strong generating set of finitely presented objects.

Analogous to the above, we say an object C in $\mathcal{C}$ is λ-*generated* if the representable functor $\mathrm{Hom}(C, -)$ commutes with λ-directed colimits of diagrams where all the maps d_{ij} are monics. Such a diagram is called a λ-*directed union.* The category $\mathcal{C}$ is called *locally* λ-*generated* if it is cocomplete, co-wellpowered and has a strong generating set of λ-generated objects (note that one can prove locally λ-presentable categories are co-wellpowered using the definition above (eg [1, 1.58, 2.49]); it is an open question whether this condition is necessary in the definition of a locally λ-generated category). We say an object is *finitely generated* if it is $\aleph_0$-generated.

It follows immediately from the above definitions that the categories of λ-presented and λ-generated objects are closed under colimits of diagrams of size less than λ; in particular, the categories of finitely presented and generated objects are closed under finite colimits [1, 1.16].

Finally, an object C in $\mathcal{C}$ is *coherent* if it is finitely generated, and for any pullback diagram of the form

$$\begin{array}{ccc} B \times_C B' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array}$$

in which B and B' are finitely generated, we have that $B \times_C B'$ is also finitely generated. A category is *locally coherent* if it is cocomplete and has a strong generating set of coherent objects.

3 Essentially algebraic theories

Locally presentable categories can be characterized as categories of models of essentially algebraic theories. This gives a nice way of thinking about the objects in these categories. In this section, we will introduce essentially algebraic theories, and give an explicit description of an essentially algebraic theory associated with any given locally presentable category.

Recall the following from [1, 3.34].

Definition 1. *1. An essentially algebraic theory is given by a quadruple*

$$\Gamma = (\Sigma, E, \Sigma_t, \mathit{Def}\,)$$

Here Σ is a many sorted signature of algebras, over some set of sorts S.

Σ_t is a subset of Σ, denoting the set of function symbols we intend to view as total. We write Σ_p for the set $\Sigma - \Sigma_t$; these function symbols are to be interpreted as being partial. As usual, we view constant symbols as function symbols defined over the empty set of sorts.

The set E consists of equations over Σ - that is, pairs of terms in variables x_i, where each x_i has a sort $s_i \in S$. When considering the pair (t_1, t_2) as an equation, we will write it $t_1 = t_2$.

Finally Def is a function assigning to each partial function symbol

$$\sigma : \prod_{i \in I} s_i \to s$$

a collection of Σ_t-equations in variables $x_i \in s_i$, $(i \in I)$. These equations are taken to define the domain of definition for σ.

2. *We say that Γ is λ-ary for a regular cardinal λ if each function symbol in Σ takes fewer than λ arguments, and each Def (σ) contains fewer than λ equations.*

3. *By a model of Γ, we mean a partial Σ-algebra A such that A satisfies all equations of E, the total functions are everywhere defined, and a partial function $\sigma \in \Sigma_p$ is defined for a tuple $\mathbf{a}$ in the domain of σ if and only if the tuple $\mathbf{a}$ satisfies all the equations in Def (σ).*

4. *A morphism of Γ-models is a morphism of the underlying partial algebraic structures. We denote by $\mathbf{Mod}\Gamma$ the category of models of Γ.*

It is well-known [1, 3.36] that the categories of models of λ-ary essentially algebraic theories are precisely the locally λ-presentable categories. We will present a proof of this fact which we hope will make clear a sense in which an essentially algebraic object can be seen as being generated by a collection of its elements, in a similar manner to the way an ordinary algebraic object is.

Now assume we are given an essentially algebraic theory $\Gamma = (\Sigma, E, \Sigma_t, \mathrm{Def}\ )$ and a collection of variables $x_1 \in S_1$, …, $x_n \in S_n$, we construct terms in Γ over these variables as follows:

1. each variable x_i is a term of sort S_i.

2. given a total operation $f : S_1 \times \ldots \times S_n \to S$, and terms $t_1 \in S_1$, …, $t_n \in S_n$, $f(t_1, \ldots, t_n)$ is a term of sort S.

3. if $\sigma : S_1 \times \ldots \times S_n \to S$ is a partial operation and $t_1 \in S_1$, …, $t_n \in S_n$ are terms such that the equations Def $(\sigma)(t_1, \ldots, t_n)$ hold in every model of (Σ, E), then $\sigma(t_1, \ldots, t_n)$ is a term of sort S.

Remark. The stipulation that the equations Def $(\sigma)(t_1, \ldots, t_n)$ hold in every model of (Σ, E) in step 3 above could be replaced, via a suitable Completeness Theorem, with the assertion that the axioms E admit a deduction of each of the equations in Def $(\sigma)(t_1, \ldots, t_n)$.

In a normal algebraic theory (that is, one where all the operations are assumed to be total), a presentation of an object is given by a collection of generators $\mathbf{x}$ and a collection of equations $\phi(\mathbf{x})$ which we assert is satisfied by $\mathbf{x}$. When we allow partial operations we add the complication that the equations which go together to form ϕ should include only terms over $\mathbf{x}$ which are well-defined. Consequently, we make the following definition: a *presentation* consists of a collection of variables $\mathbf{x} = \{x_i \in S_i\}_{i \in I}$ and a totally ordered collection of equations $R(\mathbf{x}) = \{\phi_i(\mathbf{x}) \mid i \in I\}$, where $(I, \leq)$ is some total order, and each ϕ_i is an equation in terms over $\mathbf{x}$. This has the restriction that if the partial operation σ is used to form a term in $\phi_i(\mathbf{x})$, then the arguments for σ satisfy the equations Def(σ) in every model of $E \cup \{\phi_j(\mathbf{x}) \mid j < i\}$.

We define terms over a presentation $\langle \mathbf{x} \mid R(\mathbf{x}) \rangle$ as follows:

1. each variable x_i is a term of sort S_i.

2. given a total operation $f : S_1 \times \ldots \times S_n \to S$, and terms $t_1 \in S_1, \ldots, t_n \in S_n$, $f(t_1, \ldots, t_n)$ is a term of sort S.

3. if $\sigma : S_1 \times \ldots \times S_n \to S$ is a partial operation and $t_1 \in S_1, \ldots, t_n \in S_n$ are terms such that the equations Def $(\sigma)(t_1, \ldots, t_n)$ are satisfied in every model of $(\Sigma, E \cup R(\mathbf{x}))$, then $\sigma(t_1, \ldots, t_n)$ is a term of sort S.

The collection of terms over a presentation is itself a model $\langle \mathbf{x} \mid R(\mathbf{x})\rangle$ of Γ, with the obvious operations. This is universal, in the sense that for any other Γ-model Y, and any tuple $\mathbf{y} \in Y$ satisfying $R(\mathbf{y})$, there is a unique map $\tilde{\mathbf{y}} : \langle \mathbf{x} \mid R(\mathbf{x})\rangle \to Y$ mapping the variables $\mathbf{x}$ to $\mathbf{y}$. The definition of $\tilde{\mathbf{y}}$ is given on the terms over the presentation by induction.

Furthermore, any Γ-model C admits such a presentation - we can take $\mathbf{x}$ to be all the elements of C, and $R(\mathbf{x})$ to be all the equations holding between them. We will of course usually be able to find smaller presentations than this.

Colimits can be determined in terms of these presentations. Let M and N be models of Γ, with presentations $\langle \mathbf{x}, R(\mathbf{x})\rangle$, $\langle \mathbf{y}, S(\mathbf{y})\rangle$. Then the coproduct $M \coprod N$ is the object with presentation $\langle (\mathbf{x}, \mathbf{y}), R(\mathbf{x}) \cup S(\mathbf{y})\rangle$. Given a Γ-model L and mappings $f : M \to L$, $g : N \to L$, we can find corresponding tuples $\mathbf{a}, \mathbf{b} \in L$, such that the equations $R(\mathbf{a})$ and $S(\mathbf{b})$ are satisfied. The tuple $(\mathbf{a}, \mathbf{b})$ in L corresponds to the coproduct factorization.

Similarly, given a parallel pair of morphisms $f, g : M \to N$, for every $m \in M$, the elements $f(m)$ and $g(m)$ can be expressed as terms $t^f_m(\mathbf{y})$, $t^g_m(\mathbf{y})$ over the generators $\mathbf{y}$ of N. A map $h : N \to L$ with $hf = hg$, corresponds to a tuple $\mathbf{z} \in L$, satisfying the equations $S(\mathbf{z})$, with the additional property that for each $m \in M$, $t^f_m(\mathbf{z}) = t^g_m(\mathbf{z})$ (this is the condition that $hf = hg$). In fact it suffices to require this just for the generators $\mathbf{x}$ of M. Thus the coequaliser of f and g admits the presentation $\langle \mathbf{y} \mid S(\mathbf{y}) \cup \{t^f_x(\mathbf{y}) = t^g_x(\mathbf{y})\}_{x \in \mathbf{x}}\rangle$.

In particular, given a Γ-model N and a presentation $\langle \mathbf{y} \mid S(\mathbf{y})\rangle$ of N, each equation τ in S is of the form $t^\tau_1(\mathbf{y}) = t^\tau_2(\mathbf{y})$. Furthermore t^τ_1 and t^τ_2 have the same sort, X_τ, say. Let N_1 be the free Γ-model on generators $\mathbf{x} = \{x_\tau \in X_\tau\}$, and N_2 the free model on generators $\mathbf{y}$. There is a pair of maps $f, g : N_1 \to N_2$ defined by $f : x_\tau \mapsto t^\tau_1(\mathbf{y})$, $g : x_\tau \mapsto t^\tau_2(\mathbf{y})$. Then N is the coequaliser of the maps f and g.

We summarize this information.

Lemma 2. *Every model M of an essentially algebraic theory Γ admits a presentation $\langle \mathbf{x} \mid R(\mathbf{x})\rangle$, and M can be expressed as the coequaliser of a diagram*

$$F \rightrightarrows G \longrightarrow M$$

where F and G are free models of Γ. The number of generators of G is bounded by the cardinality of $\mathbf{x}$, and the number of generators of F is bounded by the cardinality of $R(\mathbf{x})$.

For algebras over a signature Σ, the notions of finitely presented and finitely generated correspond to the usual notions we can define using generators and relations, this is proved in [1, 3.10]. We seek now to prove an analogous result for essentially algebraic objects. Let $\Gamma = (\Sigma, E, \Sigma_t, \mathrm{Def}\ )$ be an essentially algebraic theory. Let X be a model of Γ. We say that a tuple of elements $\mathbf{x} \in X$ *generates*

X if every element of X can be written as a term over the elements $\mathbf{x}$, using the term forming operations as described above.

Lemma 3. *Let $\Gamma = (\Sigma, E, \Sigma_t, Def\,)$ be a λ-ary essentially algebraic theory. Then the forgetful functor $U : \mathbf{Mod}(\Gamma) \to \mathbf{Mod}(\Sigma_t, E_t)$ preserves λ-directed colimits.*

Proof. It is sufficient to show that if $(D_i, \leq)$ is a λ-directed system of Γ-structures, then the colimit of the underlying (Σ_t, E_t)-structures is also a Γ-structure. To see this, let

$$D_i \xrightarrow{d_i} L$$

be the colimit cocone in the category of (Σ_t, E_t)-structures. Suppose we have some tuple $\mathbf{x}$ of elements of L that satisfy the equations Def (σ), for some partial operation σ. Then by the definition of a λ-directed colimit, there is some $i \in I$ and some $\mathbf{x}' \in D_i$ such that $D_i \models \text{Def }(\sigma)(\mathbf{x}')$ and $d_i(\mathbf{x}') = \mathbf{x}$. The well-defined term $d_i(\sigma(\mathbf{x}'))$ then gives us our definition for $\sigma(\mathbf{x})$. □

Lemma 4. *(cf. [1, 3.11], [10, 3.16(a)]) Let Γ be a λ-ary essentially algebraic theory. A model X is a λ-generated object in $\mathbf{Mod}(\Gamma)$ if and only if it has a generating set of size less than λ.*

Proof. Suppose X is λ-generated. For each set of elements $S \subseteq X$ with size less than λ, let $\bar{S}$ be the substructure of X generated by S (this is the set of all elements which can be written as terms over the elements of S). X can be written as the union of all the $\bar{S}$; therefore $X = \bar{S}$ for some S.

To prove the converse, suppose X has a generating set of size S less than λ, and let X be the λ-directed union of a collection of subobjects $X = \bigcup U_i$. Then each x_i can be written as a term over the elements of the U_i, and since the terms are λ-small, this term can only involve elements of fewer than λ of the objects U_i. Since there are fewer than λ elements x_i, the set of U_i's needed to write all the x_i as terms over the elements of the U_i's is also of cardinality less than λ. □

Lemma 5. *(cf [1, 3.12], [10, 3.16(b)]) The λ-presented objects in **Mod** (Γ) are precisely those which have a λ-small presentation.*

Proof. We show that a free model F of Γ on a single generator of sort X is λ-presented. This follows from the fact that λ-directed colimits in Mod (Γ) are calculated as in (Σ_t, E_t). Thus, given a directed colimit cocone

$$\{\, D_i \xrightarrow{d_i} L \mid i \in I\}$$

over a directed system $\{d_{ij} : D_i \to D_j \mid i \leq j \in (I, \leq)\}$ for some directed poset $(I, \leq)$, a map $F \to L$ corresponds to an element $x \in L$ of sort X. But since the directed colimit is the same as that for the underlying (Σ_t, E_t) structures, there is some $i \in I$ and some $x' \in D_i$ with $d_i(x') = x$. Thus the map $F \to D_i$ defined by x' is an appropriate factorization through the cocone.

Having established that a free object on a single generator is λ-presented, the result now follows from Lemma 2 and the fact that λ-presented objects are closed under λ-small colimits. □

Essentially algebraic theories characterise locally presentable categories; that is, a category $\mathcal{C}$ is locally presentable if and only if it is the category of models for an essentially algebraic theory. This is proved in [1, 3.36]. We will give a different proof of this, which will describe explicitly an essentially algebraic theory associated with a given locally presentable category.

To prove this result, we recall the following concept, from [1, 1.42]. For a small category $\mathcal{A}$ and a regular cardinal λ, denote by $\mathbf{Cont}_\lambda\mathcal{A}$ the category of all functors $\mathcal{A} \to \mathbf{Sets}$ preserving all λ-small limits in $\mathcal{A}$.

Theorem 6. *([1, 1.46]) If $\mathcal{C}$ is a locally λ-presentable category, and $\mathcal{A}$ is the subcategory of λ-presented objects in $\mathcal{C}$, then $\mathcal{C}$ is equivalent to $\mathbf{Cont}_\lambda\mathcal{A}$.*

Theorem 7. *A category $\mathcal{C}$ is locally λ-presentable if and only if it is equivalent to the category of models of a λ-ary essentially algebraic theory $(\Sigma, E, \Sigma_t, Def\,)$.*

Furthermore, this is a reflective subcategory of the category of models of the equational theory (Σ_t, E_t), where E_t is the subset of E containing those equations not using any of the function symbols from Σ_p and the inclusion functor preserves λ-directed colimits.

Proof. We have already proved the last part.

Suppose we are given a λ-ary essentially algebraic theory $\Gamma = (\Sigma, E, \Sigma_t, \mathrm{Def}\,)$. Let C be a model of (Σ_t, E_t). The reflection of C is just the Γ-structure given by the presentation $\langle \mathbf{c}, R(\mathbf{c})\rangle$, where the variables in $\mathbf{c}$ are the elements of C, and $R(\mathbf{c})$ is the set of in Σ_t holding for the elements of C (with an arbitrary ordering).

If $\mathcal{C}$ is a locally λ-presentable category and $\mathcal{A}$ is a reflective subcategory closed under λ-directed colimits, then $\mathcal{A}$ is also locally λ-presentable. The reflections of the λ-presentable objects in $\mathcal{C}$ are λ-presentable in $\mathcal{A}$, and form a strong generating set [1, 1.3].

To show that every locally λ-presentable category can be represented this way, let $\mathcal{C}$ be a locally λ-presented category, with $\mathcal{A}$ the category of λ-presented objects in $\mathcal{C}$.

Define a λ-ary essentially algebraic theory Γ as follows. The total part of Γ is just the category $\mathcal{A}^{\mathrm{op}}$, with equations those holding in $\mathcal{A}^{\mathrm{op}}$; that is, we consider the objects of $\mathcal{A}$ to be the sorts, with the function symbols $\mathcal{T}A \to A'$ between two sorts being the set of morphisms $A' \to A$ in $\mathcal{A}$ - this is described in more detail at the start of section 4.

For each λ-small diagram $\mathcal{D} = \{\, D_i \xrightarrow{f_{ij}^k} D_j \,\}$ in $\mathcal{A}$, with colimit cocone $\{\, D_i \xrightarrow{d_i} L \,\}$, the object L will be in $\mathcal{A}$, since λ-presentable objects are closed under λ-small colimits. Define partial operations $\sigma_{\mathcal{D}} : \prod_{i\in I} D_i \to L$, where Def (σ) is the set of equations $f_{ij}^k(x_j) = x_i$ for each arrow $f_{ij}^k : D_i \to D_j$ (note we can have $D_i = D_j$ for $i \neq j$). Add equations to our theory stating that if $\mathbf{x} = \{x_i \in D_i\}_{i\in I} \in \prod_{i\in I} D_i$ is a tuple, then $\mathcal{T}d_i\sigma_{\mathcal{D}}(\mathbf{x}) = \pi_i(x_i)$, for each $i \in I$, where $\mathcal{T}d_i$ is the function symbol corresponding to sd_i.

The category of models for Γ is the category of presheaves on $\mathcal{A}$ which preserve the λ-small limits existing in $\mathcal{A}^{\mathrm{op}}$; that is, the category $\mathbf{Cont}_\lambda\mathcal{A}^{\mathrm{op}}$ of λ-continuous set-valued functors on $\mathcal{A}^{\mathrm{op}}$. By Theorem 6, this is equivalent to $\mathcal{C}$. This proves that Γ is an essentially algebraic theory whose category of models is equivalent to $\mathcal{C}$. $\square$

Remark. The Yoneda embedding gives us a way to see the objects of $\mathcal{C}$ as models of the theory Γ in the obvious way: let C be an object of $\mathcal{C}$. Then for each object A in $\mathcal{A}$, the set of elements of C of that sort is the collection of maps $A \to C$; the functions in $\mathcal{A}$ act on this set by precomposition. For each diagram $\mathcal{D}$, the map $\sigma_{\mathcal{D}}$ sends a compatible cocone over the diagram with codomain C to the factorisation through the colimit.

It is clear from this that the free Γ-models in the given presentation are precisely the λ-presented objects in $\mathcal{C}$.

Lemma 8. *Let Γ be an essentially algebraic theory $(\Sigma, E, \Sigma_t, Def)$ such that every function symbol in Σ is finitary. Then the category of Γ-models is locally finitely generated.*

Proof. It suffices to show that for an essentially algebraic theory of the above form, if $(I, \leq)$ is a directed poset and $\{d_{ij} : D_i \to D_j \mid i \leq j \in I\}$ is a directed union of Γ-structures, then the colimit of the underlying (Σ_t, E_t)-structures is also a Γ-structure.

Write $\{d_i : D_i \to D \mid i \in I\}$ for the colimit cocone. Note that each d_i is also a monic map. In a locally presentable category, the monic maps are precisely the injective maps, so we can consider the D_i's to be essentially algebraic substructures of D.

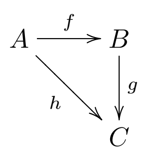
Now let σ be a partial operation, with domain of definition given by Def (σ). Since σ is a finitary operation, the set of equations Def (σ) uses only finitely many variables. Let $\mathbf{d}$ be a tuple in D such that $D \models$ Def $(\sigma)(\mathbf{d})$. Since the D_i's cover D, each element d_k from $\mathbf{d}$ occurs as an element of D_i for some i. Since the D_i's occur as a directed system, we can find some D_j containing the whole tuple $\mathbf{d}$. Then we define $\sigma(\mathbf{d})$ to be $\sigma^{D_j}(\mathbf{d})$. □

We conjecture that locally finitely generated categories are characterized by essentially algebraic theories of this form, but we do not have a proof of this.

4 Sheaves as essentially algebraic objects

Let $\mathcal{C}$ be a small category. The category $(\mathcal{C}, \mathbf{Sets})$ of set valued functors on $\mathcal{C}$ is described by a multi-sorted equational theory, which we will denote Γ_C. This theory is described as follows:

- For each object C of $\mathcal{C}$, we take a corresponding sort $\mathtt{C}$.
- For each morphism $f : C \to C'$ in $\mathcal{C}$, take a function symbol $\mathtt{f} : \mathtt{C} \to \mathtt{C}'$.
- For each commutative diagram in $\mathcal{C}$ of the form



add an equation to E in one variable $\mathtt{x}$ of sort $\mathtt{A}$ stating $\mathtt{gf}(\mathtt{x}) = \mathtt{h}(\mathtt{x})$.

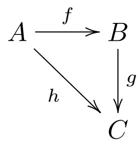
If $(\mathcal{C}, J)$ is a site, we can describe the presheaves on $\mathcal{C}$ as the set-valued functors on $\mathcal{C}^{\rm op}$ in the manner just described (so in this case, for each morphism $f : C \to C'$ in $\mathcal{C}$, we have a function symbol $\mathtt{f} : \mathtt{C}' \to \mathtt{C}$). Furthermore, we can extend the theory to an essentially algebraic theory $(\Sigma, E, \Sigma_t, \mathrm{Def}\ )$ whose models are the sheaves on the site.

- For every covering sieve J of an object C, we take a partial operation

$$\sigma_{\mathtt{J}} : \mathtt{C} \to \prod_{\mathtt{f} \in \mathtt{J}} \mathtt{dom(f)} \to \mathtt{C},$$

  in variables $\mathbf{x} = (\mathtt{x_f})_{\mathtt{f} \in \mathtt{J}}$ (each variable $(\mathtt{x_f})$ is of sort $\mathtt{dom(f)}$).

- The equations in Def $(\sigma_{\mathtt{J}})$ are those of the form $\mathtt{gf}(\mathtt{x_h}) = \mathtt{h}(\mathtt{x_h})$ whenever we have a commutative diagram



  such that g and h are in J.

- We add equations to E stating that $\mathtt{f}\sigma_{\mathtt{J}}(\mathbf{x}) = \mathtt{x_f}$ for every covering sieve J and every $f \in J$.

It is easily checked that the models of this essentially algebraic theory are just the sheaves for the topology. We write $\Gamma_{(\mathcal{C},J)}$ for this essentially algebraic theory.

In fact it suffices to take a basis for the Grothendieck topology in the above. That is, given a basis K for J, define an essentially algebraic theory by taking the partial operations $\sigma_{\{\mathtt{f_i}\}}$ defined for each covering family $\{f_i\}$ of morphisms in K as above. The models of this algebraic theory are again the sheaves for the topology. We denote this essentially algebraic theory by $\Gamma_{(\mathcal{C},K)}$.

Let J be a topology, such that there exists a regular cardinal λ and a basis K for the topology such that every covering family in K has less than λ elements. Then every function symbol in $\Gamma_{(\mathcal{C},K)}$ will take fewer than λ arguments, and by Lemma 8, the category $\mathbf{Sh}(\mathcal{C}, J)$ will be locally λ-generated.

5 Locally coherent and finitely presented toposes

By considering sheaves as essentially algebraic objects, we can understand the notions of finite presentability and coherence in a very concrete way. In this section, we will use this to characterize the different local generation conditions for toposes. The results here are mostly known, see e.g., [2, VI.2], but this approach gives us a different way of thinking about them.

To provide characterizations of toposes with these various local generation properties, we start by introducing the following form of the Comparison Lemma.

Lemma 9. *([6, p.589]) If $\mathcal{C}$ is a full subcategory of the Grothendieck topos, $\mathcal{E}$ whose objects form a generating set, and J is the topology on $\mathcal{C}$ in which the covering sieves on an object C are precisely those containing an epimorphic family of morphisms, then $\mathcal{E}$ is equivalent to* $\mathbf{Sh}(\mathcal{C}, J)$.

We use this result to find sites for a given topos. For instance, if we assume the topos $\mathcal{E}$ is locally finitely generated, we can take $\mathcal{C}$ to be the collection of finitely generated objects in $\mathcal{E}$. If J is a sieve on a finitely generated object in $\mathcal{E}$ which contains an epimorphic family of morphisms, than there is a finite family of morphisms in J which is also epimorphic. This observation leads to one half of the following result.

Proposition 10. *A topos $\mathcal{E}$ is locally finitely generated if and only if it is equivalent to $\mathbf{Sh}(\mathcal{C}, J)$, for some site $(\mathcal{C}, J)$, where every sieve in the topology J contains a dense finitely generated sieve.*

Proof. It remains to show that a topos of this form is locally finitely generated. But this follows immediately from Lemma 8, since for a site of this form, we can choose an essentially algebraic theory as in section 4 for which all the operation symbols are finitary. □

Definition 11. *If $(\mathcal{C}, J)$ is a site where every sieve in the topology J contains a dense finitely generated sieve, we say the topology J is of* finite type. *More generally, if every sieve in the topology J contains a dense λ-generated sieve, we say the topology J is of* λ-type.

A reflection functor $r : \mathcal{C} \to \mathcal{A}$ is said to be of finite type *(respectively, of* λ-type*) if the inclusion functor $i : \mathcal{A} \to \mathcal{C}$ preserves directed unions (respectively, λ-directed unions).*

There is some incosistency in the literature over whether, for a finite type localization, the inclusion functor $i : \mathcal{A} \to \mathcal{C}$ is required to preserve directed unions, or all directed colimits. This confusion is caused partly by the fact that if the small category $\mathcal{C}$ has pullbacks, the two definitions are equivalent. However, the definition of finite type topology given here is fairly universal, and in general, it is only equivalent to demanding that the inclusion functor preserve directed unions. We shall call a localization such that the inclusion functor preserves all directed colimits a *coherent type* localization.

It is shown in e.g., [10, 3.15] that for a Grothendieck topology J on a small category $\mathcal{C}$ with pullbacks, the topology J is of finite type if and only if the localization functor $a : \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{Sh}(\mathcal{C}, J)$ is.

The next result characterizes locally finitely presented toposes. To make the proof easier to follow, we make the following definition. Let $\mathcal{C}$ be a topos, and let $f : A \to C$, $g : B \to C$ be a pair of arrows in $\mathcal{C}$ with common codomain. By a *square in $\mathcal{C}$ over f and g*, we will mean a commutative diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{x_1} & B \\ {\scriptstyle x_2}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & C \end{array} .$$

We denote this square by (X, x_1, x_2). Given two squares over f and g, (X, x_1, x_2), (Y, y_1, y_2), we say a *factorisation* of X through Y is a map $x' : X \to Y$ with the property that $y_1 x' = x_1$, $y_2 x' = x_2$.

Theorem 12. *A topos $\mathcal{E}$ is locally finitely presented f to* $\mathbf{Sh}(\mathcal{C}, J)$*, for some site* $(\mathcal{C}, J)$*, where every sieve in the topology contains a dense sieve S with the property that (1) S is generated by a finite collection of arrows S'; and (2) every pair of arrows $f, g \in S'$ admits a finite collection of squares, $X_i = (X_i, x_1^i, x_2^i)$ with the property that every other square over f and g, $Y = (Y, y_1, y_2)$, factors through one of the X_i.*

Remark. A finite collection of the form given in condition (2) of this theorem is called a *weak multilimit* over the diagram

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & C \end{array}$$

Proof. Suppose the site $(\mathcal{C}, J)$ is of the form described. Take as a basis for the topology the finite collections of arrows generating each sieve S' as described in the statement of the theorem. Associate with this basis the essentially algebraic theory as described in section 4. For each sieve S', the equations stating that the squares X_i commute are sufficient to describe the essentially algebraic theory. This theory is finitary, and the category of its models is therefore locally finitely presented. This category is of course just $\mathbf{Sh}(\mathcal{C}, J)$.

It remains to show that if $\mathcal{E}$ is locally finitely presented, then it is equivalent to the category of sheaves on a site of this form. By the previous result, if $\mathcal{E}$ is locally finitely presented, it is equivalent to the category of sheaves on the site $(\mathcal{C}, J)$, where $\mathcal{C}$ is the category of finitely generated objects in $\mathcal{E}$, and J is the topology on $\mathcal{C}$ generated by the families of morphisms that are epimorphic families in $\mathcal{E}$. We claim that if J is not of the above form, then the objects in $\mathcal{C}$ are not finitely presented in $\mathcal{E}$.

Suppose C is in $\mathcal{C}$ and S is a finitely generated J-dense sieve on C, which is generated by a finite family of morphisms $\{s_i : S_i \to C\}_{1 \leq i \leq n}$. Suppose there is some pair s_i, s_j, which does not admit a finite family of squares with the property (2). Consider the collection of all squares (Y_k, y_1^k, y_2^k) over s_i, s_j.

Build up a directed system of functors as follows: each functor F is generated by a pair of elements $x_i \in S_i$, for each $1 \leq i \leq n$. The directed system consists of all functors with this generating set and containing finitely many of the relations satisfied by the generating set of elements s_i in S. Thus these functors form a directed system (the join of two functors F_1, F_2 in the system is the functor whose set of relations is just the union of those for F_1 and F_2). The colimit of this directed system is clearly S. However, there is no map from S to any of the functors F_i, since this would force F_i to satisfy extra relations. Since S is isomorphic to $\mathrm{Hom}(-, C)$ in the sheaf category, this contradicts the assertion that C is finitely presented in $\mathcal{E}$. □

The most obvious examples of sites which fulfil the condition given in Proposition 12 are those where the topology is trivial, i.e., for each object C in $\mathcal{C}$, $JC = \{\mathrm{Hom}(-, C)\}$ (this is the wholly obvious fact that presheaf categories are locally finitely presented) and those where the category $\mathcal{C}$ has pullbacks (this is the equally obvious fact that locally coherent toposes are locally finitely presented).

If $(\mathcal{C}, J)$ is a site as described above, the finitely presented objects in $\mathbf{Sh}(\mathcal{C}, J)$ are described by the following result, which appears as [10, 3.16]. This can be seen as an immediate consequence of Lemma 5 applied to the description of sheaves as essentially algebraic objects given in Section 4.

Theorem 13. *Let J be a λ-type Grothendieck topology on a category $\mathcal{C}$.*

a If F is a λ-generated sheaf, there is a λ-generated presheaf P such that $F \cong aP$.

b If F is a λ-presented sheaf, there is a λ-presented presheaf P such that $F \cong aP$.

To look at coherent and locally coherent toposes, we will need the following result.

Lemma 14. *Let $\mathcal{C}$ be any category. Then the full subcategory of $\mathcal{C}$ consisting of coherent objects in $\mathcal{C}$ is closed under pullbacks.*

Proof. Suppose we are given a pullback diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{\pi_1} & B \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{f} & D \end{array}$$

in which the objects B, C and D are all assumed to be coherent. Since D is coherent and B and C are finitely generated, it follows that A is finitely generated as well.

Now suppose we have a further pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{p_1} & Y \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle h} \\ Z & \xrightarrow{f} & A \end{array}$$

in which Y and Z are finitely generated. Then the diagram

$$\begin{array}{ccc} X & \xrightarrow{p_1} & Y \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle g\pi_1 h} \\ Z & \xrightarrow{f\pi_2 k} & D \end{array}$$

is also a pullback diagram, and so X is finitely generated by coherence of D.

Thus A is coherent. □

The next result characterizes locally coherent toposes; it will require quite a bit of work to prove. This characterization was originally shown in [2, VI.2.1]. The proof we give here uses the idea of a presheaf as a model of an algebraic theory.

Recall from [6] the definition of the plus functor

$$(-)^+ : \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$$

and the associated sheaf functor

$$a = (-)^{+}+ : \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{Sh}(\mathcal{C}, J)$$

.

Proposition 15. *A topos $\mathcal{E}$ is locally coherent if and only if it is equivalent to $\mathbf{Sh}(\mathcal{C}, J)$, for some site $(\mathcal{C}, J)$, where $\mathcal{C}$ is closed under pullbacks and every sieve in the topology J is generated by a finite number of arrows.*

Such a topos is always cocomplete, so it suffices to prove that the functors $a\mathrm{Hom}(-, C)$ are coherent in such a topos. We will need to look at the notion of separated presheaves. A presheaf P on a site $(\mathcal{C}, J)$ is *separated* if for any object C in $\mathcal{C}$, and any cover S of C, if $x, y \in PC$ such that for all $f : D \to C$ in S, we have that if $Pf(x) = Pf(y)$, then $x = y$. That is, P is separated if elements of P agree on a cover only if they are the same. A presheaf P if separated if and only if P^{+} is a sheaf.

The separated presheaves form a reflective subcategory $\mathbf{Sep}(\mathcal{C})$ of $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$, and the associated sheaf functor factors through this reflection. On any presheaf P, we define, for each object C of $\mathcal{C}$ an equivalence relation R on PC given by

$$xRy \text{ if and only if } \exists S \in JC \text{ such that } \forall f \in S, Pf(x) = Pf(y).$$

Given a map $f : C \to C'$, the corresponding map $Pf : PC' \to PC$ respects this equivalence relation, so this defines a functor $(-)_{\mathrm{sep}} : \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{Sep}(\mathcal{C})$. We write P_{sep} for the image of P under this functor. This functor is a localization, so in particular the associated sheaf functor can be represented as $(-)_{\mathrm{sep}}$ followed by one application of the plus-functor. The details of this can be found in, for example, [10, p.32].

If F is a presheaf on a site $(\mathcal{C}, J)$, we say a subpresheaf $s : S \to F$ is *dense* if as is an isomorphism. The dense subpresheaves of a presheaf P are closed under intersections, so they form a directed system, denoted $\mathbf{D}(P)$. If F and G are presheaves then a map between $f : aF \to aG$ may be represented by a map $f' : F' \to G_{\mathrm{sep}}$ such that $af' = f$. Two maps $f : F' \to G_{\mathrm{sep}}$ and $g : F'' \to G_{\mathrm{sep}}$ represent the same map $aF \to aG$ if they agree on some dense subobject of $F' \cap F''$.

Lemma 16. *([10, 3.9]) For presheaves F and G on a site $(\mathcal{C}, J)$, there is a natural isomorphism*

$$Hom_{\mathbf{Sh}(\mathcal{C},J)}(aF, aG) \cong \varinjlim{}_{P' \in \mathbf{D}(P)} Hom_{\mathbf{Sets}^{\mathcal{C}^{op}}}(P', Q_{sep}).$$

Lemma 17. *Let $(\mathcal{C}, J)$ be a site where $\mathcal{C}$ is closed under pullbacks and every sieve in the topology J is generated by a finite number of arrows. Let $a : \mathbf{Sets}^{\mathcal{C}^{op}} \to \mathbf{Sh}(\mathcal{C}, J)$ be the localization functor. Then the functors $aHom(-, C)$ are coherent objects in the sheaf category.*

Proof. The functor $a\mathrm{Hom}(-, C)$ is finitely generated in the sheaf category by Theorem 13. It remains to prove the pullback property. We will show this by looking at the separated presheaf $\mathrm{Hom}(-, C)_{\mathrm{sep}}$. For each object C' in $\mathcal{C}$, the elements of $\mathrm{Hom}(-, C)_{\mathrm{sep}}(C')$ are equivalence classes of maps $f : C' \to C$, where f is equivalent to f' if there is some cover $\{g_i : G_i \to C'\}_{i \in I}$ such that $fg_i = f'g_i$, for all i.

Suppose we are given maps $\alpha : A \to a\mathrm{Hom}(-, C)$, $\beta : B \to \mathrm{Hom}(-, C)$ in the sheaf category, with A and B finitely generated. Since A and B are finitely generated in the sheaf category, they are isomorphic to sheaves aA', aB' for some finitely generated presheaves A' and B', by Theorem 13. The maps α and β can be represented by maps $\tilde{\alpha} : A^* \to \mathrm{Hom}(-, C)_{\mathrm{sep}}$ and $\tilde{\beta} : B^* \to \mathrm{Hom}(-, C)_{\mathrm{sep}}$ in $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ with A^* and B^* finitely generated dense subobjects of A' and B' respectively (we may assume A^* and B^* are finitely generated because the localization is of finite type). The presheaves A^*_{sep} and B^*_{sep} are finitely generated objects in the category $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$, since they are quotients of the finitely generated objects A^* and B^* respectively.

Thus the map α and β are given by maps $\alpha^* : A^*_{\mathrm{sep}} \to \mathrm{Hom}(-, C)_{\mathrm{sep}}$, $\beta^* : B^*_{\mathrm{sep}} \to \mathrm{Hom}(-, C)_{\mathrm{sep}}$, where A^*_{sep} and B^*_{sep} are finitely generated objects in $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$, and $a(\alpha^*) = \alpha$, $a(\beta^*) = \beta$.

Since A^*_{sep} and B^*_{sep} are objects in the presheaf category, we can assume they are generated as models of the algebraic theory $\Gamma_{\mathcal{C}}$ by elements $a_i \in A^*_{\mathrm{sep}}C_i$ and $b_j \in B^*_{\mathrm{sep}}C_j$. The map $\alpha^* : A^*_{\mathrm{sep}} \to \mathrm{Hom}(-, C)_{\mathrm{sep}}$ identifies each of the generators a_i with an equivalence class of arrows $C_i \to C$, and we choose a representative $a_i^* : C_i \to C$ for each equivalence class. Similarly, we choose representatives b_j^* for the image of each of the generators b_j.

For each pair of generators a_i of A^*_{sep} and b_j of B^*_{sep}, take the pullback square

$$\begin{array}{ccc} C_i \times_C C_j & \xrightarrow{\pi^2_{i,j}} & C_j \\ \Big\downarrow{\scriptstyle \pi^1_{i,j}} & & \Big\downarrow{\scriptstyle b_j^*} \\ C_i & \xrightarrow{a_i^*} & C \end{array}$$

Now define P to be the presheaf defined by taking generators (p_i, p_j) of sort $C_i \times_C C_j$ for each pair of generators a_i, b_j. Every term $t(p_i, p_j)$ of sort C' corresponds to a map $t : C' \to C_i \times_C C_j$, which then corresponds to a pair of maps $t_i : C' \to C_i$, $t_j : C' \to C_j$. The relations on P are defined by taking $t(p_{i_1}, p_{j_1}) = t'(p_{i_2}, p_{j_2})$ when the corresponding terms in A^*_{sep} and B^*_{sep} are equal, i.e., $t_{i_1}(a_{i_1}) = t'_{i_2}(a_{i_2})$ and $t_{j_1}(b_{j_1}) = t'_{j_2}(b_{j_2})$.

This presheaf P is clearly finitely generated, and we have a diagram

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & B^*_{\mathrm{sep}} \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \beta^*} \\ A^*_{\mathrm{sep}} & \xrightarrow{\alpha^*} & \mathrm{Hom}(-, C)_{\mathrm{sep}} \end{array}$$

This diagram is not in general a pullback diagram, but it suffices to show that it is mapped to one by the localization functor.

Let X be any functor, and suppose we have a commutative diagram

$$\begin{array}{ccc} aX & \xrightarrow{x_2} & B \\ \Big\downarrow{\scriptstyle x_1} & & \Big\downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\alpha} & a\mathrm{Hom}(-, C) \end{array}$$

The map $x_1 : aX \to A$ is given by a map $x'_1 : S \to A'_{\text{sep}}$, where S is a dense subobject of X. The sep-functor preserves monomorphisms, so A^*_{sep} is a subobject of A'_{sep}, and since both objects are mapped to aB by the associated sheaf functor, it is a dense subobject. The pullback of A^*_{sep} along x'_1 is also a dense subobject of X, so there is a map $x^*_1 : S \to A^*_{\text{sep}}$ with $a(x^*_1) = x_1$. Similarly, we may assume there is a map $x^*_2 : S \to B^*_{\text{sep}}$ with $a(x^*_2) = x_2$ (we can assume both maps have the same domain by taking the intersection of the two domains).

Suppose S is generated by elements s_k each of sort S_k. The maps x^*_1, x^*_2 send each of these generators s_k to $x^*_1(s_k) \in A^*_{\text{sep}}S_k$, $x^*_2(s_k) \in B^*_{\text{sep}}S_k$. This pair is represented by an element $(x^*_1(s_k), x^*_2(s_k)) \in PS_k$. Defining this on each generator gives us a transformation $\tilde{x} : S \to P$. We observe that this is indeed a transformation since any relations that are required to hold in P hold in each of its two components, by the assumption that x^*_1 and x^*_2 were transformations themselves.

We still need to show that the factorization is unique. Suppose there is another transformation $x' : S' \to P$, such that S' is dense in X, and $a(\pi_1 x') = x_1$ and $a(\pi_2 x') = x_2$. Then by Lemma 16, there is a subobject of T of S and S' on which $\pi_1 x'$ agrees with $\tilde{x}_1$, and $\pi_2 x'$ agrees with $\tilde{x}_2$; it follows by the definition of $\tilde{x}$ that $\tilde{x}_{|T} = x'_{|T}$. But if this is the case then x' and $\tilde{x}$ represent the same transformation $aX \to aP$. This concludes the proof. □

Proof of theorem 15: The topos of sheaves on a site $(\mathcal{C}, J)$ always has the representable functors $a\text{Hom}(-, C)$ as a generating set; we have just shown that these will be coherent. The converse follows immediately from the Comparison Lemma, 9 and Lemma 14.

6 Modules over a sheaf of rings

Let $\mathcal{E}$ be a topos of presheaves, i.e., $\mathcal{E} = \mathbf{Sets}^{\mathcal{C}^{\text{op}}}$ for some small category $\mathcal{C}$. A ring object in $\mathcal{E}$ is a *presheaf of rings* on $\mathcal{C}$ - this is a presheaf $R : \mathcal{C}^{op} \to \mathbf{Sets}$ such that RC has a ring structure for every object C in $\mathcal{C}$, and for each map $f : C \to C'$ in $\mathcal{C}$, the map $Rf : RC' \to RC$ is a morphism of rings. We write $\mathbf{Rings}(\mathcal{E})$ for the category of ring objects in $\mathcal{E}$.

If J is a topology on $\mathcal{C}$, then a ring object in $\mathbf{Sh}(\mathcal{C}, J)$ is a presheaf of rings such that the underlying presheaf of sets is a sheaf.

In particular, since the localization functor $a : \mathbf{Sets}^{\mathcal{C}^{\text{op}}} \to \mathbf{Sh}(\mathcal{C}, J)$ preserves finite products, we see that the localization of a presheaf of rings is also a sheaf of rings, and this is a reflection functor from $\mathbf{Rings}(\mathbf{Sets}^{\mathcal{C}^{\text{op}}}) \to \mathbf{Rings}(\mathbf{Sh}(\mathcal{C}, J))$ (and more generally, if $f : \mathcal{E} \to \mathcal{F}$ is a geometric morphism of toposes, this defines a 'geometric morphism' $\mathbf{Rings}(\mathcal{F}) \to \mathbf{Rings}(\mathcal{E})$, that is, an adjoint pair of functor between these two categories with the left adjoint preserving finite limits).

If $(R, 0_R, 1_R, -_R, +_R, \times_R)$ is a ring object in a topos, we define a *right R-module object* $(M, 0_M, -_M, +_M, \times_M)$ in $\mathcal{C}$ to be an abelian group object $(M, 0_M, -_M, +_M)$ together with a map $\times_M : M \times R \to M$ satisfying the commutativity conditions required by modules; for example, to show the multiplication is distributive over addition, we stipulate that the following diagram must commute:

$$
\begin{array}{ccc}
M \times R \times R & \xrightarrow{(\times)_M \times \mathrm{id}_R} & M \times R \\
{\scriptstyle \mathrm{id}_M \times (\times_R)} \downarrow & & \downarrow {\scriptstyle \times_M} \\
M \times R & \xrightarrow{\times_M} & M
\end{array}
$$

Morphisms of R-module objects are defined similarly.

If R is a presheaf of rings over some small category $\mathcal{C}$, then an R-module object M in $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ is a 'presheaf of R-modules' - for each object C in $\mathcal{C}$, MC will be an RC-module, and for a map $f : C \to C'$, the map $Mf : MC' \to MC$ will be an RC'-linear map, where MC is considered with the action of RC' on it defined by the map $Rf : RC' \to RC$. We denote the category of presheaves of R-modules over a presheaf of rings by **PreMod**-R. If R is a sheaf of rings we denote the category of sheaves of R-modules by Mod-R.

In particular, we see that since the localization functor $a : \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{Sh}(\mathcal{C}, J)$ preserves finite limits, if M is a presheaf of R-modules for some presheaf of rings R on a small category $\mathcal{C}$, then aM will be a sheaf of aR-modules.

Now let N be a sheaf of aR-modules. The presheaf of rings R has an action on N, given by

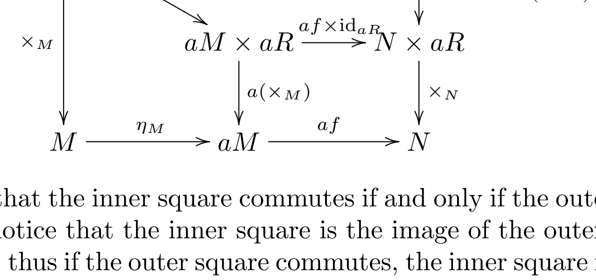
$$N \times R \xrightarrow{\eta_R \times \mathrm{id}_N} N \times aR \xrightarrow{\times_N} N$$

and one can easily show that N is a presheaf of R-modules with this action.

Theorem 18. *Let $(\mathcal{C}, J)$ be a site, with associated sheaf functor $a : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sets}^{\mathcal{C}^{op}}$, and inclusion functor $i : \mathbf{Sets}^{\mathcal{C}^{op}} \to \mathbf{Sh}(\mathcal{C}, J)$. Suppose R is a presheaf of rings on $\mathcal{C}$. Then a and i induce functors $a' :$ **PreMod**-$R \to$ **Mod**-aR, $i' :$ **Mod**-$aR \to$ **PreMod**-R, and this expresses* **Mod**-aR *as a localization of the category* **PreMod**-R.

Proof. We have already described the functors a' and i', and a' preserves finite limits since it commutes with the forgetful functor. It remains to show that a' is left adjoint to i', or equivalently, that given a presheaf of R-modules M and a sheaf of aR modules N, then a map $f : M \to N$ in $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ is a morphism in **PreMod**-R if and only if the corresponding map $af : aM \to N$ in $\mathbf{Sh}(\mathcal{C}, J)$ is a morphism in **Mod**-aR. We show that f commutes with the multiplication by R if and only if af commutes with the multiplication by aR; the proof that f is an abelian group map if and only if af is is similar.

Consider the diagram below, where $f = af.\eta_M$:



Our claim is that the inner square commutes if and only if the outer square does. To see this, notice that the inner square is the image of the outer square under the functor a; thus if the outer square commutes, the inner square must commute also. Now suppose the inner square commutes. Then $\times_N.\eta_{(N\times R)}.(f\times \mathrm{id}_R) = \times_N.(af\times \mathrm{id}_{aR}).\eta_{(M\times R)} = af.a(\times_M).\eta_{M\times R} = af.\eta_M.\times_M$, and thus the outer square commutes, as required. □

In the paper of Prest and Ralph, [8], the following result was shown: let X be a topological space with a basis of compact open sets, and let R be a sheaf of rings on X (we refer to a topological space with a sheaf of rings as a *ringed space*). Then the category of R-modules is locally finitely presentable.

In that paper, it was asked under what conditions this result generalizes to an arbitrary Grothendieck topos. This question can be answered using the characterizations given up to this point.

Theorem 19. *If $\mathcal{E}$ is a locally finitely presentable topos (respectively locally finitely generated) and R is a ring object in $\mathcal{E}$, then Mod-R, the category of R-module objects in $\mathcal{E}$, is locally finitely presentable (respectively locally finitely generated).*

Proof. Let $\mathcal{E}$ be a locally finitely presentable topos. Then by Theorem 12, $\mathcal{E}$ is equivalent to $\mathbf{Sh}(\mathcal{C}, J)$, where J is a topology on a small category $\mathcal{C}$ with the property that every covering sieve on an object C in $\mathcal{C}$ contains a finitely presentable covering sieve.

We use the essentially algebraic theory describing the objects in $\mathbf{Sh}(\mathcal{C}, J)$ described in section 4. Sheaves of R-modules can be described by adding more functions and equations to this theory.

To define the category of R-module objects in $\mathbf{Sh}(\mathcal{C}, J)$, we add total operation symbols to the above signature. For a given sort $\mathtt{C}$ (that is, each object C in $\mathcal{C}$), we add a function symbol $\mathtt{r} : \mathtt{C} \to \mathtt{C}$, for each element $r \in RC$. We also add a constant symbol $\mathtt{0}_\mathtt{C}$ of sort $\mathtt{C}$, and function symbols $+_\mathtt{C} : \mathtt{C}\times\mathtt{C} \to \mathtt{C}$ and $-\mathtt{C} : \mathtt{C} \to \mathtt{C}$. We expand E to include equations stating that with the operations so defined, the collection of elements of the sort $\mathtt{C}$ is an RC-module for each object C, and each function symbol $\mathtt{f} : \mathtt{B} \to \mathtt{C}$ is an RB-linear map.

This gives us a description of Mod-R as a finitary essentially algebraic category; it is therefore locally finitely presentable.

To get the corresponding result for locally finitely generated toposes, we use a similar argument, but the sets Def (σ) are allowed to contain infinitely many equations. The rest of the argument is unchanged. □

It is well-known that for a topological space X, the category of sheaves on X is locally finitely presentable if and only if the space has a basis $\mathcal{B}$ of compact open sets, see e.g., [5, D3.3.14]. This can be seen as a consequence of [6, II.2.3], which states that a sheaf on the lattice $\mathrm{Op}(X)$ is equivalent to a sheaf on the sublattice consisting of the basis elements $\mathcal{B}$. If this basis consists of compact open sets, then the essentially algebraic theory described in section 3.1 will be finitary.

Remark. It has been pointed out to us that results in section D5 of [5] can be used to provide a straightforward proof that local λ-presentability of a topos $\mathcal{E}$ implies local λ-presentability of any category of modules in $\mathcal{E}$. Our proof has a much more model theoretic flavour, and we hope it will be more straightforward to those with experience in this field.

We now turn our attention to the question of when the category of modules on a ringed space is locally coherent. It should be noted at this point that the definition of coherence we are using is distinct from (and a lot stronger than) the definition of a coherent sheaf of modules used by algebraic geometers, in for example [4, II.5].

Let X be a topological space, and let R be a presheaf of rings on X. Denote by $\mathrm{Op}(X)$ the lattice of open sets of X. Denote by Γ_R the essentially algebraic theory of presheaves of modules over R.

For each open set $U \in \mathrm{Op}(X)$, define a presheaf $R_U : \mathrm{Op}(X)^{\mathrm{op}} \to \mathbf{Sets}$ by

$$R_U(V) = R(V) \text{ if } V \subseteq U; \; 0 \text{ if } V \not\subseteq U.$$

Each presheaf-of-modules R_U is the free model of the theory Γ_R generated by a single element of sort U. It follows that the collection $\{R_U \mid U \in \mathrm{Op}(X)\}$ is a generating set of objects for **PreMod**-R - if $\alpha \neq \beta : F \to G$ are distinct maps in **PreMod**-R, then there is some $U \in \mathrm{Op}(X)$ and some $x \in FU$ such that $\alpha_U(x) \neq \beta_U(x)$. The element x represents a map $\tilde{x} : R_U \to F$, and the inequality $\alpha\tilde{x} \neq \beta\tilde{x}$ holds.

Every presheaf R_U is finitely generated, and is a subobject of the presheaf R, considered as an object in **PreMod**-R.

Lemma 20. *Let R be a presheaf of rings on a space X. The category* **PreMod**-R *is locally coherent if and only if R, considered as a sheaf of modules, is coherent as an object in the category.*

Proof. If R is coherent as a module over itself, the objects R_U form a generating set of coherent objects. We claim that any finitely generated subobject of a coherent object is itself coherent. If C is coherent and $s : S \to C$ is a finitely generated subobject, and there is a pullback diagram

$$\begin{array}{ccc} B \times_S B' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ B & \longrightarrow & S \end{array}$$

then the diagram

$$\begin{array}{ccc} B \times_S B' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array}$$

obtained by composing with s is also a pullback. Thus if B and B' are finitely generated, $B \times_S B'$ must also be finitely generated, by coherence of C.

This shows that the objects R_U form a coherent generating set for the category **PreMod**-R.

Conversely, if the category **PreMod**-R is locally coherent; then every finitely presentable object is coherent [7, E.1.18], and since R is finitely presentable in **PreMod**-R, it must be coherent also. □

We look for conditions under which a presheaf of rings R is coherent.

Lemma 21. *If R is a coherent presheaf of rings on a space X, then for every open set U in X, RU is a coherent ring.*

Proof. Let R be a presheaf of rings on a space, and let U be an open set in X such that RU is not a coherent ring. Then there is some finitely generated ideal $\langle x_1, \ldots, x_n \rangle = I \subseteq RU$, such that I is not finitely presentable. But I defines a finitely generated subobject of R - just take the subobject of R generated by $x_1, \ldots, x_n$. This is not finitely presentable; if I had a presentation with only finitely many relations, then taking the relations which occur between terms of sort U would give a finite presentation of IU. □

A version of the next result appeared as Theorem 2.18 in [9]. In this paper, the result was proved for an arbitrary presheaf of rings R, as long as RU was coherent for each open set $U \in \mathrm{Op}(X)$. As it turns out, the argument in [9] contains a mistake: if F is a finitely presentable module and G a finitely generated submodule, then the sheaf-of-rings structure of R can force G to have infinitely many relations. The argument can be made to work if we insist that the presheaf-of-rings R be finitely presentable, however.

Theorem 22. *(cf [9, 2.18]) Let R be a presheaf of rings on the space X. If R is finitely presentable, and for each open set $U \in Op(X)$, RU is a coherent ring, then the presheaf of rings R is coherent as an object in* **PreMod**-R.

Proof. Let $\langle x_1, \ldots, x_n \mid r_1(\mathbf{x}), \ldots, r_m(\mathbf{x}) \rangle$ be a presentation of R in the language of presheaves of rings over $\mathcal{C}$. Each element x_i has sort U_i where U_i is some open set in $\mathrm{Op}(X)$. Suppose for some open set U_i, we have an open set $U \subseteq U_i$; then denote by x_i^U the restriction of the element x_i to U (i.e., the image of x_i under the restriction map $RU_i \to RU$). Each relation $r_j(\mathbf{x})$ is an equation between terms in the variables $\mathbf{x} = x_1, \ldots, x_n$, and each equation is between terms in some sort V_j, where $V_j \in \mathrm{Op}(X)$.

On any open set $U \in \mathrm{Op}(X)$, the ring RU is generated by the set of elements x_i^U, for those elements x_i where $U \subseteq U_i$. The relations that hold on these elements $\{x_i^U \mid U \subseteq U_i\}$ are precisely those induced by those relations $r_j(\mathbf{x})$ which have $U \subseteq U_j$. In particular, if U and U' are two open sets that are contained in precisely the same open sets U_i and V_j (that is, for every $i = 1, \ldots, n$, $U \subseteq U_i$ if and only if $U' \subseteq U_i$, and for every $j = 1, \ldots, m$, $U \subseteq U_j$ if and only if $U' \subseteq U_j$) then the rings RU and RV are isomorphic (since they have the same presentation).

To prove that the category **PreMod**-R is locally coherent, it suffices to prove that the presheaf-of-modules R is a coherent object of this category. Since **PreMod**-R is an abelian category, it suffices to show that any finitely generated subobject of R is finitely presentable. To see this, suppose that $I \subseteq R$ is a finitely generated subobject of R; let $y_1, \ldots, y_l$ be a generating set of elements, where each element y_k is of sort W_k. Suppose we have two open sets $U \subseteq U'$ that are contained in precisely the same open sets from the collection $U_1, \ldots, U_n, V_1, \ldots, V_m, W_1, \ldots, W_l$. Then RU and RU' are isomorphic, and this isomorphism restricts to an isomorphism $IU \cong IU'$, since IU will be the subobject of RU generated by the elements y_k^U for each k with $U \subseteq W_k$, and IU' will be the generated by the elements $y_k^{U'}$ for precisely the same values of k from $1, \ldots, l$.

Thus the presheaf-of-rings I is completely described by its presentation on the open sets which are intersections of subsets of the set

$$\{U_1, \ldots, U_n, V_1, \ldots, V_m, W_1, \ldots, W_l\}.$$

There are only finitely many such intersections, and since RU is coherent on every open set U, IU is finitely presentable on each open set U that is such an intersection. Combining the finite presentation of I on each intersection from this set, we can write down a finite presentation for the whole presheaf I. □

The next result gives us a condition for categories of sheaves of modules (as opposed to just presheaves) to be locally coherent.

Theorem 23. *Let $(\mathcal{C}, J)$ be a site such that the inclusion functor $i : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sets}^{\mathcal{C}^{op}}$ preserves directed colimits, and let R be a presheaf of modules on $\mathcal{C}$ such that* **PreMod**-*R is locally coherent. Then the category* **Mod**-*aR is locally coherent also.*

Proof. Suppose $(I, \leq)$ is some directed partial order, and we are given a directed diagram in **Mod**-aR,

$$\{ D_i \xrightarrow{d_i j} D_j \mid i \leq j \in (I, \leq)\}.$$

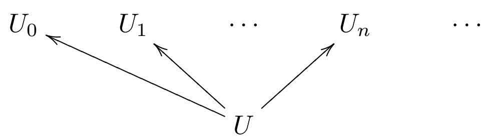
This is a directed diagram in **PreMod**-R also, and the directed colimit of this diagram exists in **PreMod**-R, and has as its underlying presheaf of sets the colimit of the underlying presheaves of sets (see [1, 3.4(4), 3.6(6)]). Let L be the directed colimit in **PreMod**-R, so we have an action of R on L denoted $\times_L : L \times R \to L$.

Since the inclusion functor $i : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sets}^{\mathcal{C}^{op}}$ preserves directed colimits, L is a sheaf, and the map $a(\times_L) : L \times aR \to L$ is an action of aR on L, with respect to which L is the colimit of the diagram in **Mod**-aR. □

It is nevertheless possible to find a ring object R in a locally coherent topos, such that the category of R-modules is not locally coherent, as demonstrated by the next result.

Theorem 24. *Let X be a topological space with infinitely many open sets. Then there is a sheaf of rings R on X, with the property that RU is a coherent ring for every open set U in X, but the category Mod-R is not locally coherent.*

Proof. If X has infinitely many open sets, then the distributive lattice $\mathrm{Op}(X)$ has infinitely many elements. Recall that such a lattice must contain an infinite chain. For suppose it does not. Then the lattice has finite height n. Since the lattice has infinitely many elements, there is a least height $i < n$ such that $\mathrm{Op}(X)$ has infinitely many elements of height i. Note that since the empty set is the unique element of height 0, $i > 0$. There are only finitely many elements of height $i-1$, so there must be one element, U say, with infinitely many elements of height i above U. Choose a countable collection of these, $\{U_i\}_{i\in\mathbb{N}_0}$.



For a given i, consider whether U_i is contained in the union of the other elements $\bigvee_{j\neq i} U_j$. Only finitely many of the U_i can have this property - if infinitely many did, than we could construct an infinite chain from taking their finite unions. So there exists at least one open set which does not - U_0, say. So we may assume $U_0 \subseteq \bigvee_{i \geq 1} U_i$. For each $i \geq 1$, $U_0 \cap U_i = U$. So we have the equalities

$$\bigvee_{i \geq 1} (U_0 \cap U_i) = U$$

and

$$U_0 \cap \bigvee_{i \geq 1} U_i = U_0.$$

Together, these equalities contradict the fact that $\mathrm{Op}(X)$ is a Heyting algebra. So $\mathrm{Op}(X)$ must have an infinite chain.

Now suppose the lattice $\mathrm{Op}(X)$ contains an infinite chain. Then in particular, it contains a chain isomorphic to ω, or a chain isomorphic to ω^{op}. We deal with these two cases separately.

First of all, we look at the case when $\mathrm{Op}(X)$ contains a sequence of open sets isomorphic to ω, say

$$U_0 \longrightarrow U_1 \longrightarrow \cdots \qquad U_n \longrightarrow \cdots \longrightarrow U_\infty$$

We may assume that U_0 is the empty set, that $U_\infty = \bigcup_{i=0}^{\infty} U_i$, and that every open set U_α is connected (if U_α is the smallest disconnected set, replace it with the connected component of U_α that contains $U_{\alpha-1}$).

In this case, for each U_i, we define $RU_i = \mathbb{Z}/2^i\mathbb{Z}$. We define RU_∞ to be $\mathbb{Z}_2$, the ring of 2-adic integers. For an arbitrary connected open set $V \subseteq X$, there is a smallest number $i_V \in \mathbb{N} \cup \{\infty\}$ such that $V \subseteq U_{i_V}$. Define $RV = RU_{i_V}$. For a disconnected subset V, define RV to be the product over the connected components.

The presheaf R so described is a sheaf. It suffices to check the sheaf condition on connected subsets V of X, since R sends disjoint unions to the appropriate product by definition.

Suppose V is a connected subset of X, and let $\{V_j\}_{j \in J}$ be a cover of V, and take a matching family $x_j \in RV_j$ of elements of R. Suppose $RV = RU_\alpha$ (that is, U_α is the smallest set in the chain such that $V \subseteq U_\alpha$). But then there is some point $p \in V$ with $p \notin U_i$ for any $i < \alpha$. But the point p is in V_j for some $j \in J$, and in particular $RV_j = RU_\alpha = RV$. Now the amalgamation for the matching family will be the element $x_j \in RV_j = RV$.

The sheaf R is finitely presentable as a module over itself. As before, we consider the subpresheaf M generated by $2 \in RX$. On each open set V in X, MV consists of those elements of RV which admit a division by two. The same argument as for R shows that this is a sheaf.

We need to show that M is not finitely presentable. The argument above shows that if V is covered by subsets V_j then $RV_j = RV$ for some j, and consequently $MV_j = MV$ also. Thus if the presentation includes a relation on some open subset V, this cannot be used to derive relations on subsets strictly containing V. Now suppose there is a finite presentation for M. There is some $i \in \mathbb{N}$ such that the finite presentation does not induce relations on MU_i. But MU_i is not free. So there cannot be any finite presentation for M.

Now we examine the case when $\mathrm{Op}(X)$ contains an infinite chain isomorphic to ω^{op}, say

$$0 = U_\infty \longrightarrow \ldots \longrightarrow U_n \longrightarrow \ldots \longrightarrow U_1 \longrightarrow U_0 = X\,.$$

Let k be a field. Define a sheaf of rings R on X as follows: let $RX = k[x_0]$. For an open set V that is not contained in any U_i for $i \in \mathbb{N}$, set $RV = \mathbf{0}$, the one element ring. For any other open set $V \subseteq U$, let n be the smallest natural number such that $V \subseteq U_n$ (if V is contained in every U_n, set $n = \infty$). Then define

$$RV = k[x_0, x_1, \ldots, x_n]/\langle x_0 x_i = x_0 \rangle_{1 \le i \le n}.$$

The restriction maps are the canonical inclusions of the polynomial rings. Every ring RV is coherent (since we take a quotient of a finite polynomial ring by a finitely generated ideal).

We can show that R is a sheaf in a similar way to the first case: if V is covered by a collection of open sets V_i, there is some V_i with $RV_i = RV$, and this is how we find an amalgamation whenever we have a matching family of elements of R for the cover.

Now let M be the subobject of R in Mod-R generated by the object x_0 of sort X. For each open set V of X with $RV = k[x_0, \ldots, x_n]/\langle x_0 x_i = x_0 \rangle_{1 \le i \le n}$, MV is given by the presentation

$$MV = \langle y | y x_i = y \rangle_{1 \le i \le n}.$$

The same argument as for R shows that M is a sheaf. The object M is finitely generated in Mod-R (by x_0), but a finite presentation for M would only mention finitely many of the variables x_i in M, so there would be some x_k that is not mentioned in the presentation. But then MU_k would not to satisfy $y x_k = y$. So M is not finitely presentable. $\square$

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