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Equivalent definitions:

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(c) $P = P_1 \rtimes \langle \varphi \rangle$, where $\varphi^p = 1$ and $x x^{\varphi} x^{\varphi^2} \cdots x^{\varphi^{p-1}} = 1$ for all $x \in P$ (splitting automorphism of P_1).

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(But there is no bound for the exponent of a p-group with a partition.)

Splitting automorphism approach

Splitting automorphism approach of condition (c) turned out to be most efficient. Recall:

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All groups with a splitting automorphism of order p form a variety of groups with operators defined by the laws $(*)$.

Analogues of theorems on group of exponent p

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As a corollary, a positive solution for the Hughes problem
was obtained for "almost all" finite p-groups.

Nilpotency class depending on automorphisms

EKh–Shumyatsky, 1995: if a finite group G of exponent p admits a soluble group of automorphisms A of coprime order such that the fixed-point subgroup $C_G(A)$ is soluble of derived length d, then G is nilpotent of $(p, d, |A|)$-bounded class.

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Theorem 1

Suppose that a finite p-group P with a partition admits a soluble group of automorphisms A of coprime order such that $C_P(A)$ has derived length d. Then any maximal subgroup of P containing $H_p(P)$ is nilpotent of $(p, d, |A|)$-bounded class.

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The bound for the nilpotency class of that maximal subgroup can be chosen the same as in EKh–Shumyatsky-95 for groups of exponent p.

Exponent

Theorem 2

If a finite p-group P with a partition admits a group of automorphisms A that acts faithfully on $P/H_p(P)$, then the exponent of P is bounded in terms of the exponent of $C_P(A)$.

Frobenius groups of automorphisms

Corollary

Suppose that a finite group G admits a Frobenius group of automorphisms FH with cyclic kernel $F = \langle \varphi \rangle$ of prime order p such that φ is a splitting automorphism, that is, $xx^{\varphi}x^{\varphi^2}\cdots x^{\varphi^{p-1}} = 1$ for all $x \in G$.

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(a) If $C_G(H)$ is soluble of derived length d, then G is nilpotent of (p, d)-bounded class.

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Plus p-bounded nilpotency class of $G_{p'}$ $\Rightarrow$ exponent of $G_{p'}$ is bounded in terms of p and exponent of $C_G(H)$.
So in (b) it remains to consider G_p. The result follows from Theorem 2 applied to $P = G_p\langle\varphi\rangle$ and $A = H$.

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Question remains open for the exponent, as well as for the derived length.

Proof of Theorem 1: elimination of automorphisms by nilpotency

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Theorem 1′

Suppose that a soluble group FA with normal Sylow p-subgroup $F = \langle\varphi\rangle$ of order p and Hall p'-subgroup A acts by automorphisms on a finite p-group G in such a manner that φ is a splitting automorphism, that is, $xx^{\varphi}x^{\varphi^2}\cdots x^{\varphi^{p-1}} = 1$ for all $x \in G$. If $C_G(A)$ is soluble of derived length d, then G is nilpotent of $(p, d, |A|)$-bounded class.

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Free *FA*-group

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Lemma

The subgroups C and S are invariant under any FA-endomorphism ϑ of X.

Trivialization of F

Since there is an FA-homomorphism $\xi : X \to G$ with $C, S \leqslant Ker\, \xi$, it is sufficient (and even necessary) to prove that

$[x_1, \dots, x_{c+1}] \in CS$, where c is the nilpotency class given by EKh-Shumyatsky theorem when $\varphi = 1$.

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Let $T = [X, F]F$ ("trivialization of F")

By EKh-Shumyatsky theorem, $[x_1, \dots, x_{c+1}] \in CST$,

that is, we need to eliminate T.

Higman's lemma

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One can show that we can furthermore assume that each c_i has the form

$$[[x_{i_1}^{a_*}, \ldots], [x_{i_2}^{a_*}, \ldots], \ldots, [x_{i_{c+1}}^{a_*}, \ldots]] \qquad (a_* \in A),$$

where $\{i_1, i_2, \ldots, i_{c+1}\} = \{1, 2, \ldots, c+1\}$ and there is at least one φ among "dots" in at least one of the subcommutators $[x_{i_k}^{a_*}, \ldots]$.

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then substitute the result into right-hand side of the original $(*)$.

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$$[x_1, \ldots, x_{c+1}] \equiv c_1^{k_1} \cdots c_m^{k_m} \pmod{CS} \qquad (*)$$

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And so on, at each step we double the number of occurrences of φ in the new c_i.

Since $X\langle\varphi\rangle$ is nilpotent (being a finite p-group!), in the end we get $\equiv 1$, as required.

Proof of exponent theorem.

By known results, proof of Theorem 2 reduces to the following result.

Theorem 2′

If a finite p-group G admits a Frobenius group of automorphisms FA with kernel $F = \langle\varphi\rangle$ of order p and complement A such that φ is a splitting automorphism, that is, $xx^{\varphi}x^{\varphi^2}\cdots x^{\varphi^{p-1}} = 1$ for all $x \in G$, then the exponent of P is bounded in terms of the exponent of $C_P(A)$.

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It remains to obtain a bound for the exponent of $V = G/[G, G]$.

Abelian case: eigenspaces.

Consider $V = G/[G, G]$ as a $\mathbb{Z}FA$-module, with additive notation. In particular, $v + v\varphi + v\varphi^2 + \cdots + v\varphi^{p-1} = 0$ for all $v \in V$ by hypothesis.

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Extend the ground ring by a primitive pth root of unity ω, forming $W = V \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$. Still have $w + w\varphi + w\varphi^2 + \cdots + w\varphi^{p-1} = 0$ for all $w \in W$.

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Define analogues of eigenspaces for the "linear transformation" φ:

$$W_i = \{w \in W \mid w\varphi = \omega^i w\}.$$

Then W is an "almost direct sum" of the W_i:

$$pW \subseteq W_0 + W_1 + \cdots + W_{p-1}$$

and

if $w_0 + w_1 + \cdots + w_{p-1} = 0$ for $w_i \in W_i$, then $pw_i = 0$ for all i.

A-orbits.

First: since $\varphi = 1$ on W_0, for $x \in W_0$ we have $px = x + x\varphi + \cdots + x\varphi^{p-1} = 0$, so that $pW_0 = 0$.

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$W_i\alpha \subseteq W_{ri}$ for all $i \in \mathbb{Z}/p\mathbb{Z}$. Indeed, if $x_i \in W_i$, then
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Given $u_k \in W_k$ for $k \neq 0$, to lighten the notation we denote $u_k\alpha^i$ by
u_{r^ik}; note that $u_{r^ik} \in W_{r^ik}$.

Let p^e be the exponent of $C_G(A)$. Claim: W_i are annihilated by p^{1+e}.

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In multiplicative notation, the exponent of $G/[G,G]$ divides p^{2+e}, so the exponent of G divides $p^{c(2+e)}$, where c is the nilpotency class of G, which is bounded in terms of p.

Remark

If, for some reason, it is known that the derived length s of the group G in Theorems 1 or 2, or in the Corollary, is relatively small, then EKh-81 can be used instead to give a possibly better estimate

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A smaller bound for the nilpotency class would also imply a smaller bound for the exponent.

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Similarly, the same arguments as above prove

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Suppose that a soluble group FA with normal Sylow p-subgroup F and Hall p'-subgroup A acts by automorphisms on a finite p-group G in such a manner that for some fixed $\varphi_1, \dots, \varphi_p \in F$ we have $x^{\varphi_1} x^{\varphi_2} \cdots x^{\varphi_p} = 1$ for all $x \in G$. If $C_G(A)$ is soluble of derived length d, then G is nilpotent of $(p, d, |A|)$-bounded class.

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Generalizations

There is also local nilpotency theorem in EKh-93, which may also have generalizations in the context of additional group of automorphisms...