Finite groups and Lie rings admitting a Frobenius group of automorphisms with fixed-point-free kernel

Khukhro, E. I.

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Evgeny Khukhro

March, 2012

Frobenius group

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Frobenius groups often occur in finite groups;
induce groups of automorphisms by conjugation

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thus, in what follows G is soluble;

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New approach:

proving that properties (or parameters) of G are close to the corresponding properties (parameters) of $C_G(H)$ (possibly also depending on H).

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Proofs of bounds for nilpotency class and exponent are by Lie ring methods, including similar results for Lie rings with such Frobenius groups of automorphisms.

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for nilpotent groups – bounds for nilpotency class...

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Proof is by Clifford's theorem.

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The idea is that something like this must also hold
in general, for any group G with $FH \leqslant \operatorname{Aut} G$ such that $C_G(F) = 1$.

Covering fixed points in quotient groups

Let $A \leqslant \mathrm{Aut}\, G$,

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(Useful because we do not assume $|G|$ to be coprime to $|FH|$.)

Lemma on fixed points “over a free module”

Let $A \leqslant \mathrm{Aut}\, G$, and let M be an A-invariant elementary abelian p-subgroup that is a free $\mathbb{F}_p A$-module.
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Proof: free module $\Rightarrow$ first cohomology group is trivial $H^1(A, M) = 0 \Leftrightarrow$ all complements of M in MA are conjugate.

If $cM \in C_{G/M}(A)$, then A^c is another complement of M in MA.
We have $A^c = A^{am} = A^m$ for $a \in A$, $m \in M$
$\Rightarrow A^{cm^{-1}} = A$
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(It is also possible to compute a fixed point directly, without using cohomology.)

Covering fixed points of the Frobenius complement

Theorem 1

Frobenius group $FH \leqslant \operatorname{Aut} G$ with kernel F and complement H.
If N is an FH-invariant normal subgroup such that $C_N(F) = 1$, then $C_{G/N}(H) = C_G(H)N/N$.

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Easy induction: unrefinable normal FH-invariant series

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Hypothesis is inherited by G/N_i by the Lemma on Carter subgroup.

Each factor N_i/N_{i+1} is a free $\mathbb{F}_p H$-module by Lemma on freedom.

Lemma on covering is applied to N_i/N_{i+1} and G/N_{i+1}.

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Proof is immediate from Theorem 1 on covering fixed points, Lemma on freedom, and description of fixed points for H in free H-modules.

Bounding the rank

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Fitting height (=nilpotent length)

Fitting series: $F_1(G) := F(G)$ is the Fitting subgroup, the largest normal nilpotent subgroup,

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Theorem 4

Frobenius group $FH \leqslant \mathrm{Aut}\, G$ with kernel F such that $C_G(F) = 1$. Then

(a) $F_i(C_G(H)) = F_i(G) \cap C_G(H)$ for all i;

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Clearly, (b) immediately follows from (a).
In part (a) the main case is proving that $F(C_G(H)) = F(G) \cap C_G(H)$.
(Then easy induction.)

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Questions: for other “radicals” $R(G)$ instead of $F(G)$?

Example of another "radical": π-length (π set of primes)

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Corollary

Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F)=1$. Then

(a) $O_\pi(C_G(H))=O_\pi(G)\cap C_G(H)$;

(b) π-length of G = π-length of $C_G(H)$;

(c) $O_{\pi_1,\pi_2,\dots,\pi_k}(C_G(H))=O_{\pi_1,\pi_2,\dots,\pi_k}(G)\cap C_G(H)$.

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Actually was proved earlier than Theorem 4.

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Theorem 4 largely reduces further study to the case of nilpotent groups.

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Without coprimeness condition: Dade's theorem, in particular, when A is nilpotent and $C_G(A) = 1$;

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In Theorem 4 the bound is best-possible, independent of $|H|$ and $|F|$, but of course under stronger hypotheses.

Bounding nilpotency class

Theorem 5 (EIKh–N. Yu. Makarenko–P. Shumyatsky)

Frobenius group $FH \leqslant \mathrm{Aut}\, G$ with cyclic kernel F such that $C_G(F) = 1$. If $C_G(H)$ is nilpotent of class c, then G is nilpotent of $(c, |H|)$-bounded class.

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Recall: G is soluble by Belyaev–Hartley+CFSG, and nilpotent by Theorem 4. Thus it is all about bounding the nilpotency class.

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Based on analogous theorem on Lie rings

Theorem 6 (EIKh–N. Yu. Makarenko–P. Shumyatsky)

Let L be a Lie ring satisfying certain conditions.
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Hence $C_{L(G)}(H)$ is also nilpotent of class c, and $C_{L(G)}(F) = 0$. By Theorem 6, $L(G)$ is nilpotent of $(c, |H|)$-bounded class, and therefore so is G.

Metacyclicity of FH is essential: simple Lie algebra

Example

The simple 3-dimensional Lie algebra L of characteristic $\neq 2$ with basis e_1, e_2, e_3 and structure constants $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2$

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$F = \{1, f_1, f_2, f_3\}$, where $f_i(e_i) = e_i$, $f_i(e_j) = -e_j$ for $i \neq j$, and $H = \langle h \rangle$ with $h(e_i) = e_{i+1 \pmod 3}$.

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Then $C_L(F) = 0$, while $C_L(H)$ is one-dimensional (hence abelian).

Metacyclicity of FH is essential: nilpotent Lie rings L of unbounded derived length.

Example

Let the additive group of L be the direct sum of three copies of $\mathbb{Z}/p^m\mathbb{Z}$ for a prime $p \neq 2$ with generators e_1, e_2, e_3; let the structure constants be $[e_1, e_2] = pe_3, \ [e_2, e_3] = pe_1, \ [e_3, e_1] = pe_2$.

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It is easy to see that L is nilpotent of class m, and its derived length is $\approx \log m$.

Metacyclicity of FH is essential: nilpotent groups

Example

That nilpotent Lie ring can be turned into a nilpotent group:

If in the preceding example $p > m$, then the Lazard correspondence can be applied based on the "truncated" Baker–Campbell–Hausdorff formula. Then L becomes a finite p-group P of the same derived length admitting the same group of automorphisms FH with $C_P(F) = 1$ and with cyclic $C_P(H)$.

About the proof of Theorem 6 on Lie rings

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Let $|F| = n$. Extend the ground ring by a primitive nth root of unity ω.

Define the eigenspaces for $F = \langle \varphi \rangle$:

$L_i = \{x \in L \mid x^{\varphi} = \omega^i x\}$. Roughly speaking,

$$L = L_1 \oplus \cdots \oplus L_{n-1} \quad \text{and} \quad [L_i, L_j] \subseteq L_{i+j\,(\mathrm{mod}\,n)},$$

which is a $(\mathbb{Z}/n\mathbb{Z})$-grading. Plus we have the condition $L_0 = C_L(F) = 0$.

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Kreknin's theorem: then L is soluble of n-bounded derived length. But we need nilpotency, and of class bounded in terms of $C_L(H)$ and $|H|$.

The simplest case of abelian $C_L(H)$

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The sum over an H-orbit belongs to $C_L(H)$, which is abelian. Therefore

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Expand brackets. If $[x_k, x_l] \neq 0$, there must be other terms in the same component L_{k+l} for cancellation to happen.

Therefore, $k + l = kr^i + lr^j$, so that $l = -\dfrac{r^i - 1}{r^j - 1}k$.

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Hence, for a given k there are at most $|H|^2$ values of l such that $[L_k, L_l] \neq 0$.

“Selective nilpotency” conditions

Theorem 7

Let $L = \bigoplus_{i=0}^{n-1} L_i$ be a $(\mathbb{Z}/n\mathbb{Z})$-graded Lie ring such that $L_0 = 0$ and for some m every grading component L_k may not commute with at most m components:

$$|\{i \mid [L_k, L_i] \neq 0\}| \leqslant m.$$

(a) Then L is soluble of m-bounded derived length.

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This works for the case of abelian $C_L(H)$ in Theorem 6.

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The proof uses versions of Kreknin’s theorem (due to Shalev and EIKh), when there are only few non-zero grading components.

In the general case, when $C_L(H)$ is non-abelian but nilpotent of class c, a more complicated technical "selective nilpotency" condition arises, from which the required result is derived by rather difficult arguments.

Some open questions

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The same type of questions can be asked for other properties and parameters of a finite group G with a Frobenius group of automorphisms FH such that $C_G(F) = 1$: if $C_G(H)$ is supersoluble? satisfies other laws, like Engel?

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Proof is easily reduced to the case where G is a finite p-group.

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Subalgebra $L_p(G) = \langle D_1/D_2 \rangle$ generated by D_1/D_2.

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Therefore G can be assumed to be powerful, which are easy to handle: if a powerful p-group is generated by elements of given order e, then the group is of exponent e. Use again $G = \langle C_G(H)^F \rangle$.

Some further results.
First step for exponent with non-metacyclic FH:

P. Shumyatsky

Frobenius group $FH \leqslant \mathrm{Aut}\, G$ of order $|FH| = 12$ with kernel F such that $C_G(F) = 1$. Then the exponent of G is bounded in terms of the exponent of $C_G(H)$ (and "12").

Combination of exponent and nilpotency:

P. Shumyatsky

Frobenius group $FH \leqslant \mathrm{Aut}\, G$ with cyclic kernel F such that $C_G(F) = 1$. If $C_G(H)$ satisfies a positive law of degree k, then G satisfies a positive law of degree bounded in terms of k and $|FH|$.

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Positive law: $v = w$, where group words v, w involve only positive powers of variables.

A positive law of degree k for a finite group implies that it is an extension of a nilpotent group of k-bounded class by a group of k-bounded exponent. Conversely, every such an extension satisfies a positive law of bounded degree.

Splitting automorphism instead of fixed-point-free:

Theorem 9

Frobenius group $FH \leqslant \mathrm{Aut}\, G$ with kernel $F = \langle \varphi \rangle$ of prime order p such that φ is a splitting automorphism, that is, $xx^{\varphi}x^{\varphi^2} \cdots x^{\varphi^{p-1}} = 1$ for all $x \in G$.

(a) If $C_G(H)$ is soluble of derived length d, then G is nilpotent of (p, d)-bounded class.

(b) The exponent of G is bounded in terms of p and the exponent of $C_G(H)$.