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Fitting height of a finite group with a Frobenius group of automorphisms

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Abstract

Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that F acts without non-trivial fixed points (that is, such that $C_G(F) = 1$). It is proved that the Fitting height of G is equal to the Fitting height of the fixed-point subgroup $C_G(H)$ and the Fitting series of $C_G(H)$ coincides with the intersections of $C_G(H)$ with the Fitting series of G. As a corollary, it is also proved that for any set of primes π the π-length of G is equal to the π-length of $C_G(H)$.

Keywords: Frobenius group; automorphism; Fitting height; soluble; fixed-point-free

Introduction

Suppose that a finite group G admits a Frobenius group of automorphisms FH with complement H and kernel F such that the kernel acts fixed-point-freely: $C_G(F) = 1$. Note that the orders of G and FH are not assumed to be coprime. The condition $C_G(F) = 1$ alone implies strong restrictions on the structure of G. Since F is nilpotent being a Frobenius kernel, G is soluble by a theorem of Belyaev and Hartley [2, Theorem 0.11] based on the classification of finite simple groups. The Fitting height of G is bounded in terms of the number of primes whose product gives $|F|$ by a special case of Dade's theorem [4]. In the case of coprime orders of G and F, this result is in the earlier paper of Thompson [17]. The bounds in the latter paper were improved by various authors, including the linear bounds by Kurzweil [11] and Turull [19].

The 'additional' action of the Frobenius complement H suggests another approach to the study of G. By Clifford's theorem, all FH-invariant elementary abelian sections of G are free $\mathbb{F}_pH$-modules (for various p); see Lemma 1.3. Therefore it is natural to expect that many properties or parameters of G should be close to the corresponding properties or parameters of $C_G(H)$, possibly also depending on H. Several results of this nature were obtained recently [7, 8, 9, 10, 12, 13, 15, 16]; the properties and parameters in question being the order, the rank, the Fitting height, the nilpotency class, and the exponent.

The main result of the present paper is that the Fitting height of G is equal to the Fitting height of $C_G(H)$, and the Fitting series of $C_G(H)$ coincides with the intersections of $C_G(H)$ with the Fitting series of G (Theorem 2.1). Earlier this was proved by the author [9] under the additional condition $(|G|, |H|) = 1$. (We reproduce for the benefit of the reader some parts of the paper [9] that are used in the new proof of the present definitive result.)

As a corollary, it is also proved that for any set of primes π the π-length of G is equal to the π-length of $C_G(H)$, and the upper π-series of $C_G(H)$ coincides with the intersections of $C_G(H)$ with the upper π-series of G (Corollary 4.1).

It is worth mentioning that some parameters of G may be different from those of $C_G(H)$, like the nilpotency class or the exponent [1]; furthermore, the derived length of G may not be bounded in terms of $|FH|$ even if $C_G(H)$ is abelian [10]. There remain several open questions on how the structure of G depends on $C_G(H)$ and H, but at least they are now largely reduced to the case of nilpotent groups G by the main result of the present paper.

1 Fixed points and free modules

Here we discuss the question of covering the fixed points of a group of automorphisms in an invariant quotient by the fixed points in the group. Let $A \leqslant \operatorname{Aut} G$ for a finite group G and let N be a normal A-invariant subgroup of G. It is well-known that if $(|A|, |N|) = 1$, then $C_{G/N}(A) = C_G(A)N/N$.

But if we do not assume that $(|A|, |N|) = 1$, then the equality $C_{G/N}(A) = C_G(A)N/N$ may no longer be true. Recall that in this paper the order of $FH \leqslant \mathrm{Aut}\, G$ is not assumed to be coprime to $|G|$. Nevertheless we shall see that the fixed points in invariant sections will be covered by the fixed points in the group.

First we mention the following well-known result, which is a consequence of the properties of Carter subgroups (see, for example, [10, Lemma 2.2]).

Lemma 1.1. *Let G be a finite group admitting a nilpotent group of automorphisms F such that $C_G(F) = 1$. If N is a normal F-invariant subgroup of G, then $C_{G/N}(F) = 1$.*

We shall also need another consequence of the properties of Carter subgroups, which was noted in [10, Lemma 2.6].

Lemma 1.2. *Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. Then for each prime p dividing $|G|$ there is a unique FH-invariant Sylow p-subgroup of G.*

We are going to show that, furthermore, when G is a finite group admitting a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$, the fixed points of H in any FH-invariant section of G are also covered by the fixed points of H in G. This follows from the fact that every such section is a free $\mathbb{F}_pH$-module for a suitable prime p. (Earlier free $\mathbb{F}_pA$-modules were also used for proving, in certain special cases where both G and A are p-groups, that fixed points of $A \leqslant \mathrm{Aut}\, G$ in G cover fixed points of A in A-invariant sections in the papers by Thompson [18], Evens [5], and the author [6].)

The presence of free $\mathbb{F}_pH$-modules is derived by Clifford's theorem. The following lemma must be known in folklore; similar lemmas are contained in [8, 10], but we present the proof for completeness. For a group A acting by linear transformations on a vector space V we use the right operator notation va for the image of $v \in V$ under $a \in A$. We also use the centralizer notation for the fixed-point subspace $C_V(A) = \{v \in V \mid va = v \text{ for all } a \in A\}$, and for the kernel $C_A(V) = \{a \in A \mid va = v \text{ for all } v \in V\}$. Recall that for a group A and a field k, a free kA-module of rank n is a direct sum of n copies of the group algebra kA each of which is regarded as a vector space over k of dimension $|A|$ with a basis $\{v_g \mid g \in A\}$ labelled by elements of A on which A acts in a regular permutation representation: $v_g h = v_{gh}$.

Lemma 1.3. *Suppose that V is a vector space over any field k admitting a finite Frobenius group of linear transformations FH with kernel F and complement H such that $C_V(F) = 0$. Then V is a free kH-module.*

Proof. First we perform reduction to the case of $|F|$ being coprime to the characteristic p of the field k. (No such reduction is needed in the case of characteristic zero.) Recall that F is nilpotent, and let $F = F_p \times F_{p'}$, where F_p is the Sylow p-subgroup of F. Since $C_V(F_{p'})$ is F_p-invariant, we must actually have $C_V(F_{p'}) = 0$. Otherwise the p-group F_p would have nontrivial fixed points on $C_V(F_{p'})$, and clearly $C_{C_V(F_{p'})}(F_p) = C_V(F)$. Thus, the hypothesis of the lemma also holds for V and the Frobenius group $F_{p'}H$, so we can assume that F is a p'-group.

We extend the ground field k to its algebraic closure $\overline{k}$, so that $\overline{V} = V \otimes_k \overline{k}$ is regarded as a $\overline{k}FH$-module. Note that we still have $C_{\overline{V}}(F) = 0$. It suffices to prove that $\overline{V}$ is a free $\overline{k}H$-module. Indeed, by the Deuring–Noether theorem [3, Theorem 29.7] two representations over a smaller field are equivalent if they are equivalent over a larger field. Being a free $\overline{k}H$-module, or a free kH-module, means having a basis, of the underlying vector space over the corresponding field, elements of which are permuted by H so that all orbits are regular. In such a basis H is represented by the corresponding permutational matrices, which are defined over k.

Consider an unrefinable series of $\overline{k}FH$-submodules

$$\overline{V} = U_1 > U_2 > \cdots > U_l > U_{l+1} = 0, \tag{1}$$

in which each factor U_i/U_{i+1} is an irreducible $\overline{k}FH$-module. Since an extension of a free $\overline{k}H$-module by a free $\overline{k}H$-module is again a free $\overline{k}H$-module, it is sufficient to prove that each factor U of (1) is a free $\overline{k}H$-module. Since $C_{\overline{V}}(F) = 0$ and the action of F is coprime (semisimple), we also have $C_U(F) = 0$.

By Clifford's theorem, U is the direct sum

$$U = W_1 \oplus \cdots \oplus W_t$$

of the Wedderburn components W_i with respect to F, which are homogeneous $\overline{k}F$-modules transitively permuted by H. None of the W_i is a trivial $\overline{k}F$-module, since $C_U(F) = 0$. Let H_1 be the stabilizer of W_1 in H in its action on the set $\{W_1, \dots, W_t\}$. If $H_1 = 1$, then the H-orbit of any $\overline{k}$-basis of W_1 gives a basis of U that is the union of several regular H-orbits, so then

U is a free $\overline{k}H$-module. But if $H_1 \neq 1$, then H_1 centralizes the centre $Z(F/C_F(W_1))$, which is represented on W_1 by scalar linear transformations. This centre is nontrivial, because $F/C_F(W_1)$ is a nontrivial nilpotent group, as W_1 is a nontrivial $\overline{k}F$-module. Because the action of H_1 on F is coprime, we then obtain nontrivial fixed points of H_1 on F, which is impossible in the Frobenius group FH. □

We now discuss the connections between free modules and fixed points.

Fixed points in free modules have a very clear description as the 'diagonal' elements. Indeed, let M be a free kA-module for a group A and any field k. There is a basis of M as a vector space over k consisting of regular orbits under the action of A. Choose one element in each of these regular A-orbits and let them span a subspace M_1. Then $M = \bigoplus_{a\in A} M_1 a$, where the direct summands $M_a = M_1 a$ are regularly permuted by A, so that $M_a b = M_{ab}$ for all $a, b \in A$. Obviously, the fixed-point subspace is $C_M(A) = \{\sum_{a\in A} ma \mid m \in M_1\}$. In particular,

$$\dim_k M = |A| \dim_k C_M(A). \tag{2}$$

Less obvious is the behaviour of fixed points 'above' free modules.

Lemma 1.4. *Let A be a group of automorphisms of a finite group G, and M an A-invariant elementary abelian normal p-subgroup of G. If M, regarded as an $\mathbb{F}_pA$-module, is a free $\mathbb{F}_pA$-module, then $C_{G/M}(A) = C_G(A)M/M$.*

Proof. This fact could be derived from the triviality of the first cohomology group $H^1(A, M) = 0$. Indeed, the condition $H^1(A, M) = 0$ is equivalent to the fact that all complements of M in the natural semidirect product MA are conjugate (see, for example, [14, 11.4.7]). If $cM \in C_{G/M}(A)$, then A^c is another complement of M in MA, and if $A^c = A^{am} = A^m$ for some $a \in A$, $m \in M$, then $A^{cm^{-1}} = A$, whence $[A, cm^{-1}] \in A \cap \langle c\rangle M = 1$, so that $cm^{-1} \in C_G(A) \cap cM$.

The existence of a required fixed point can also be shown directly, without references to cohomology theory.[1] Let M have rank n as a free $\mathbb{F}_pA$-module. Then as an $\mathbb{F}_p$-space M has a basis $\{e_{i,g} \mid g \in A, \ \ i = 1, \dots, n\}$ freely permuted by A: $e_{i,g}h = e_{i,gh}$ for $g, h \in A$. Let $cM \in C_{G/M}(A)$. We need to

[1]The author is grateful to Gülin Ercan for pointing out an error in the induction step of the previous version of this calculation in [8].

find $x \in M$ such that $(cx)^h = c^h x^h = cx$ for all $h \in A$, that is, $[c,h]x^h = x$, which can be rewritten in the additive notation as

$$x - xh = [c,h] \quad \text{for all } h \in A. \tag{3}$$

Let $[c,a] = \sum_{i=1}^n \sum_{g\in A} \alpha_{i,g}(a)e_{i,g}$ for $a \in A$. From $c^{st} = (c^s)^t$ we obtain the formula

$$\alpha_{i,g}(st) = \alpha_{i,g}(t) + \alpha_{i,gt^{-1}}(s)$$

for any i, which also implies

$$\alpha_{i,g}(h^{-1}g) = \alpha_{i,g}(g) + \alpha_{i,1}(h^{-1}) \tag{4}$$

and, in particular,

$$0 = \alpha_{i,g}(g^{-1}g) = \alpha_{i,g}(g) + \alpha_{i,1}(g^{-1}),$$

whence,

$$\alpha_{i,1}(g^{-1}) = -\alpha_{i,g}(g). \tag{5}$$

We seek $x = \sum_{i=1}^n \sum_{g\in A} x_{i,g}e_{i,g}$ with unknown coefficients $x_{i,g}$. Equations (3) in coordinates give rise to the linear system of $n(|A|^2 - |A|)$ equations

$$x_{i,g} - x_{i,h} = \alpha_{i,g}(h^{-1}g); \qquad g,h \in A, \quad g \neq h, \quad i = 1,\dots,n. \tag{6}$$

We need to show that the system (6) is consistent. Since this system is the union of n disjoint subsystems in each of which the first index i of the unknowns $x_{i,g}$ is fixed, it is sufficient to show that each of these subsystems is consistent.

Thus, we now consider one of these subsystems of $|A|^2 - |A|$ equations

$$x_{i,g} - x_{i,h} = \alpha_{i,g}(h^{-1}g); \qquad g,h \in A, \quad g \neq h, \tag{7}$$

with i fixed. In order to show that it is consistent, we need to show that if $\beta(g,h)$ are any coefficients such that the corresponding linear combination of the rows of the coefficient matrix of (7) is trivial, then the right-hand sides of (7) also satisfy this dependence. In other words, if

$$\sum_h \beta(g,h) - \sum_h \beta(h,g) = 0 \qquad \text{for each } g \in A, \tag{8}$$

then $\sum_{g,h}\beta(g,h)\alpha_{i,g}(h^{-1}g)=0$. It is more convenient to rewrite (8) as

$$\sum_h \beta(g,h)=\sum_h \beta(h,g) \qquad \text{for each } g\in A. \tag{9}$$

Using formulae (4), (5), (9), we have

$$\begin{aligned}\sum_{g,h}\beta(g,h)\alpha_{i,g}(h^{-1}g) &= \sum_{g,h}\beta(g,h)(\alpha_{i,g}(g)+\alpha_{i,1}(h^{-1}))\\ &= \sum_{g,h}\beta(g,h)\alpha_{i,g}(g)+\sum_{g,h}\beta(g,h)\alpha_{i,1}(h^{-1})\\ &= \sum_{g}\alpha_{i,g}(g)\sum_h\beta(g,h)+\sum_h\alpha_{i,1}(h^{-1})\sum_g\beta(g,h)\\ &= \sum_{g}\alpha_{i,g}(g)\sum_h\beta(g,h)-\sum_h\alpha_{i,h}(h)\sum_g\beta(g,h)\\ &= \sum_{g}\alpha_{i,g}(g)\sum_h\beta(h,g)-\sum_h\alpha_{i,h}(h)\sum_g\beta(g,h)=0,\end{aligned}$$

as required. □

We are now ready to analyse the fixed points of the complement H of a Frobenius group of automorphisms FH. The following theorem was proved in [8] and [10], but we give a short proof here for completeness.

Theorem 1.5. *Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H. If N is an FH-invariant normal subgroup of G such that $C_N(F)=1$, then $C_{G/N}(H)=C_G(H)N/N$.*

Proof. Recall that N is soluble by [2, Theorem 0.11]. Consider an unrefinable FH-invariant normal series of G

$$G>N=N_1>N_2>\cdots>N_l>N_{l+1}=1 \tag{10}$$

connecting N with 1; its factors N_i/N_{i+1} are elementary abelian. Each factor of this series can be regarded as an $\mathbb{F}_pFH$-module for a suitable prime p, which is a free $\mathbb{F}_pH$-module by Lemma 1.3. We can now apply Lemma 1.4 in an easy induction on the length of the series (10) to find an element of $C_G(H)$ in any $gN\in C_{G/N}(H)$. For $l>1$ by induction there is $c_1N_l\in C_{G/N_l}(H)\cap gN/N_l$. By Lemma 1.4 applied with $M=N_l$ there is a required $c\in C_G(H)\cap c_1N_l\subseteq C_G(H)\cap gN$. □

2 Statement of the main result and initial reductions

Recall that the Fitting series starts with $F_0(G) = 1$ followed by the Fitting subgroup $F_1(G) = F(G)$, the largest normal nilpotent subgroup of a group G, and $F_{i+1}(G)$ is defined as the inverse image of $F(G/F_i(G))$. The smallest number l such that $F_l(G) = G$ is called the Fitting height of a soluble group G.

The following theorem was proved in [9] under the additional assumption $(|G|, |H|) = 1$. We are now able to prove the definitive result without any coprimeness assumptions on the orders of G and FH.

Theorem 2.1. *Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. Then*
(a) $F_i(C_G(H)) = F_i(G) \cap C_G(H)$ *for all* i;
(b) *the Fitting height of G is equal to the Fitting height of $C_G(H)$.*

Proof. Recall that the group G is soluble by [2, Theorem 0.11].

Since the image of $C_G(H)$ covers $C_{G/N}(H)$ for every FH-invariant normal subgroup N by Theorem 1.5, we have

$$F(C_G(H))N/N \leqslant F(C_{G/N}(H)). \tag{11}$$

It is also obvious that

$$F(C_G(H)) \cap N \leqslant F(C_N(H)). \tag{12}$$

These inclusions will be useful for showing that certain assumptions are inherited by subgroups or quotients of G. By Lemma 1.1 the hypothesis $C_G(F) = 1$ is also inherited by quotients by F-invariant subgroups and, obviously, by such subgroups.

It is sufficient to show that $F(C_G(H)) = F(G) \cap C_G(H)$. Then part (a) will follow in obvious fashion by induction, since $C_{G/F_i(G)}(H) = C_G(H)F_i(G)/F_i(G)$ for every i by Theorem 1.5. We shall see later that part (b) easily follows from part (a).

The inclusion $F(C_G(H)) \geqslant F(G) \cap C_G(H)$ is obvious. To prove the reverse inclusion we argue by contradiction: suppose that $F(C_G(H)) \not\leqslant F(G)$ and the semidirect product GFH with this condition has minimal possible order.

Since $F(G) = \bigcap_p O_{p',p}(G)$, there is a prime p such that $F(C_G(H)) \not\leqslant O_{p',p}(G)$. By (11) the image of $F(C_G(H))$ in $G/O_{p'}(G)$ is contained in $F(C_{G/O_{p'}(G)}(H))$. We also have $C_{G/O_{p'}(G)}(F) = 1$ by Lemma 1.1. By the minimality of our counterexample G, we must have $O_{p'}(G) = 1$, so that $F(G) = O_p(G)$.

Since $C_{G/F(G)}(F) = 1$ by Lemma 1.1, by the minimality we have

$$F(C_{G/F(G)}(H)) \leqslant F_2(G)/F(G),$$

and therefore $F(C_G(H)) \leqslant F_2(G)$ by (11). Therefore, by the minimality, we have $G = F_2(G)$ in view of (12). Since $F(G) = O_p(G)$, the quotient $G/F(G) = F_2(G)/F(G)$ is a nilpotent p'-group. Therefore there is a prime $q \neq p$ dividing $|F(C_G(H))|$. Let $Q_0 = O_q(C_G(H))$. Since the quotient $G/O_p(G)$ is nilpotent, $C_G(H)/O_p(C_G(H))$ is also nilpotent. Therefore the Hall q'-subgroup of $C_G(H)$ centralizes Q_0. Consequently, $Q_0 \cap Z(C_G(H))) \neq 1$. We choose a nontrivial q-element $c \in Q_0 \cap Z(C_G(H)))$.

Consider $\langle c^{HF}\rangle = \langle c^F\rangle$, the minimal FH-invariant subgroup containing c. The subgroup $N = O_p(G)\langle c^F\rangle$ is also FH-invariant, and together with the (possibly not faithful) group of automorphisms FH is also a counterexample: $c \in F(C_N(H))$, since $c \in F(C_G(H))$, but $c \notin F(N) = O_p(G)$ (the latter equality holds because $F(G) = O_p(G)$ contains its centralizer). Therefore, by the minimality we have $G = O_p(G)\langle c^F\rangle$. Let Q be an FH-invariant Sylow q-subgroup of G, which exists by Lemma 1.2. Since all q-elements of G are contained in $O_p(G)Q$, we have $G = O_p(G)\langle c^F\rangle = O_p(G)Q$. It is easy to see that $C_G(H) = C_{O_p(G)}(H)C_Q(H)$; therefore $C_Q(H)$ is a Sylow q-subgroup of $C_G(H)$. Hence, $Q_0 = O_q(C_G(H)) \leqslant C_Q(H)$. In particular, $c \in Q$ and we must have the equality

$$Q = \langle c^F\rangle. \tag{13}$$

Since c acts faithfully on $O_p(G)/\Phi(O_p(G))$, in view of (11) and Lemma 1.1 by the minimality we must also have $\Phi(O_p(G)) = 1$, so that $V = O_p(G) = F(G)$ is an elementary abelian p-group, on which the quotient $G/V \cong Q$ acts faithfully. Since c acts non-trivially on the QFH-invariant subgroup $[V, Q]$, by similar minimality argument we must have $V = [V, Q]$, so that $C_V(Q) = 1$.

Note that c centralizes $C_V(H)$ but acts nontrivially on V. Our aim is a contradiction following from these assumptions.

We regard V as an $\mathbb{F}_pQFH$-module. At the same time we reserve the right to regard V as a normal subgroup of the semidirect product $VQFH$.

For example, we may use the commutator notation: the subgroup $[V, Q] = \langle [v, g] \mid v \in V,\ g \in Q \rangle$ coincides with the subspace spanned by $\{-v + vg \mid v \in V,\ g \in Q\}$. We also keep using the centralizer notation for fixed points, like $C_V(H) = \{v \in V \mid vh = v \text{ for all } h \in H\}$, and for kernels, like $C_Q(Y) = \{x \in Q \mid yx = y \text{ for all } y \in Y\}$ for a subset $Y \subseteq V$.

We now extend the ground field to a finite field k that is a splitting field for QFH and obtain a $kQFH$-module $\widetilde{V} = V \otimes_{\mathbb{F}_p} k$. Many of the above-mentioned properties of V are inherited by $\widetilde{V}$:

(V1) $\widetilde{V}$ is a faithful kQ-module;
(V2) $C_{\widetilde{V}}(Q) = 0$;
(V3) c acts trivially on $C_{\widetilde{V}}(H)$;
(V4) $C_{\widetilde{V}}(F) = 0$;

Our aim is to show that c centralizes $\widetilde{V}$, which will contradict (V1).

Consider an unrefinable series of $kQFH$-submodules

$$\widetilde{V} = V_1 > V_2 > \cdots > V_n > V_{n+1} = 0. \tag{14}$$

Let W be one of the factors of this series; it is a nontrivial irreducible $kQFH$-module. If c acts trivially on every such W, then c acts trivially on $\widetilde{V}$, as the order of c is coprime to the characteristic p of the field k — this contradicts (V1). Therefore in what follows we assume that c acts nontrivially on W.

The following properties hold for W:

(W1) c acts nontrivially on W;
(W2) $C_W(Q) = 0$;
(W3) c acts trivially on $C_W(H)$;
(W4) $C_W(F) = 0$;
(W5) W is a free kH-module.

Indeed, property (W1) has already been mentioned. Property (W2) follows from (V2) since the order of Q is coprime to the characteristic of the field k. Property (W3) follows from (V3) since $C_{\widetilde{V}}(H)$ covers $C_W(H)$. Property (W4) follows from (V4) by Lemma 1.1 as we can still regard $\widetilde{V}$ as a finite (additive) group on which F acts fixed-point-freely by property (V4). Property (W5) follows from (W4) by Lemma 1.3.

We shall need the following simple remark.

Lemma 2.2. *Let FH be a Frobenius group with kernel F and complement H. In any action of FH with nontrivial action of F the complement H acts faithfully.*

Proof. Indeed, the action kernel K that does not contain F must intersect H trivially: otherwise $K\cap H$ acts trivially on $F/(K\cap F) \neq 1$ and therefore $K\cap H$ has nontrivial fixed points on F, since the action of K on F is coprime. □

3 Completion of the proof

The following lemma will be used repeatedly in the proof.

Lemma 3.1. *Suppose that $M = \bigoplus_{h\in H} M_h$ is a free kH-submodule of W, that is, the subspaces M_h form a regular H-orbit: $M_{h_1}h_2 = M_{h_1h_2}$ for $h_1, h_2 \in H$. If the element c leaves invariant each of the M_h, then c acts trivially on M.*

Proof. The fixed points of H in M are the diagonal elements $\sum_{h\in H} mh$ for any m in M_1, where $mh \in M_h$. Since c acts trivially on every such sum by property (W3) and leaves invariant every direct summand M_h, it must act trivially on each mh. Clearly, the elements mh run over all elements in all the summands $M_h = M_1h$, $h \in H$. □

By applying Clifford's theorem we consider the decomposition

$$W = W_1 \oplus \cdots \oplus W_t$$

of W into the direct sum of the Wedderburn components W_i with respect to Q. We consider the transitive action of FH on the set $\Omega = \{W_1, \dots, W_t\}$.

Lemma 3.2. *The element c acts trivially on the sum of components in any regular H-orbit in Ω.*

Proof. This follows from Lemma 3.1, because the sum of components in a regular H-orbit in Ω is obviously a free kH-submodule. □

Note that H transitively permutes the F-orbits in Ω. Let $\Omega_1 = W_1^F$ be one of these F-orbits and let H_1 be the stabilizer of Ω_1 in H in the action of H on F-orbits. If $H_1 = 1$, then all the H-orbits in Ω are regular, and then c acts trivially on W by Lemma 3.2. This contradicts our assumption (W1) that c acts nontrivially on W. Thus, we assume that $H_1 \neq 1$.

Lemma 3.3. *The subgroup H_1 has exactly one non-regular orbit in Ω_1 and this orbit is a fixed point.*

Proof. Let $\overline{F}$ be the image of F in its action on Ω_1. If $\overline{F} = 1$, then $\Omega_1 = \{W_1\}$ consists of a single Wedderburn component, and the lemma holds.

Thus, we can assume that $\overline{F} \neq 1$, and $\overline{F}H_1$ is a Frobenius group with complement H_1. By Lemma 2.2 the subgroup H_1 acts faithfully on Ω_1 and we use the same symbol for it in regard of its action on Ω_1.

Let S be the stabilizer of the point $W_1 \in \Omega_1$ in $\overline{F}H_1$. Since $|\Omega_1| = |\overline{F} : \overline{F} \cap S| = |\overline{F}H_1 : S|$ and the orders $|\overline{F}|$ and $|H_1|$ are coprime, S contains a conjugate of H_1; without loss of generality (changing W_1 and therefore S if necessary) we assume that $H_1 \leqslant S$. We already have a fixed point W_1 for H_1. It follows that H_1 acts on Ω_1 in the same way as H_1 acts by conjugation on the cosets of the stabilizer of W_1 in $\overline{F}$. But in a Frobenius group no non-trivial element of a complement can fix a non-trivial coset of a subgroup of the kernel. Otherwise there would exist such an element of prime order and, since this element is fixed-point-free on the kernel, its order would divide the order of that coset and therefore the order of the kernel, a contradiction. □

We now consider the H-orbits in Ω. Clearly, the H-orbits of elements of regular H_1-orbits in Ω_1 are regular H-orbits. Thus, by Lemma 3.3 there is exactly one non-regular H-orbit in Ω — the H-orbit of the fixed point W_1 of H_1 in Ω_1. Therefore by Lemma 3.2 we obtain the following.

Lemma 3.4. *The element c acts trivially on all the Wedderburn components W_i that are not contained in the H-orbit of W_1.*

Therefore, since c acts nontrivially on W, it must be nontrivial on the sum over the H-orbit of W_1. Moreover, since c commutes with H, the element c acts in the same way — and therefore nontrivially — on all the components in the H-orbit of W_1. In particular, c is nontrivial on W_1.

We employ induction on $|H|$ (which can be viewed for definiteness as 'enveloping' induction with respect to the secondary induction on the order of GFH). In the basis of this induction, $|H|$ is a prime, and either $H_1 = 1$, which gives a contradiction as described above, or $H_1 = H$, which is the case dealt with below.

First suppose that $H_1 \neq H$. Then we consider $U = \bigoplus_{f\in F} W_1 f$, the sum of components in Ω_1, which is a $kQFH_1$-module.

Lemma 3.5. *The element c acts trivially on $C_U(H_1)$.*

Proof. Indeed, c acts trivially on the sum over any regular H_1-orbit, because such an H_1-orbit is a part of a regular H-orbit, on the sum over which c acts trivially by Lemma 3.2. By Lemma 3.3 it remains to show that c is trivial on $C_{W_1}(H_1)$.

For $x \in C_{W_1}(H_1)$ and some right transversal $\{t_i \mid 1 \leqslant i \leqslant |H : H_1|\}$ of H_1 in H we have $\sum_i xt_i \in C(H)$. Indeed, for any $h \in H$ we have $\sum_i xt_ih = \sum_i xh_{1i}t_{j(i)}$ for some $h_{1i} \in H_1$ and some permutation of the same transversal $\{t_{j(i)} \mid 1 \leqslant i \leqslant |H : H_1|\}$, and the latter sum is equal to $\sum_i xt_{j(i)} = \sum_i xt_i$ as $xh_{1i} = x$ for all i. Since c acts trivially on $\sum_i xt_i \in C(H)$ by property (W3), it must also act trivially on each summand, as they are in different c-invariant components; in particular, $xc = x$. □

In order to use induction on $|H|$, we consider the additive group U on which the group QFH_1 acts as a group of automorphisms. The action of Q and F may not be faithful, but the action of H_1 is faithful by Lemma 2.2, because FH_1 is a Frobenius group and the action of F is nontrivial since $C_U(F) = 0$ by property (W4). Switching to multiplicative notation also for the additive group of U, we now have the semidirect product $G_1 = UQ$ admitting the Frobenius group of automorphisms $(F/C_F(G_1))H_1$ such that $C_{G_1}(F/C_F(G_1)) = 1$. By Lemma 3.5 we obtain that $c \in F(C_{G_1}(H_1))$, because $C_{G_1}(H_1) = C_U(H_1)C_Q(H_1)$ is a $\{p, q\}$-group, in which $C_U(H_1)$ is a normal p-subgroup centralized by the q-element c. If $H_1 \neq H$, then by induction on $|H|$ we must have $c \in F(G_1)$, which implies that c acts trivially on U. This contradicts the fact that c acts nontrivially on $W_1 \subseteq U$.

Thus, it remains to consider the case where $H_1 = H$, that is, W_1 is H-invariant, which is assumed in what follows.

We introduce the notation $K_i = C_Q(W_i)$ for the kernels of the action of Q on the Wedderburn components W_i.

We now focus on the action on W_1. Let F_1 denote the stabilizer of W_1 in F, so that the stabilizer of W_1 in FH is equal to F_1H. In particular, the group F_1H acts by automorphisms on Q/K_1.

Then for any element $f \in F \setminus F_1$ the component W_1^f is outside the H-orbit of W_1, which is equal to $\{W_1\}$ in the case under consideration. By Lemma 3.4, the element c acts trivially on all the Wedderburn components outside the H-orbit of W_1. Thus, c acts trivially on W_1^f, which is equivalent to $c^{f^{-1}}$ acting trivially on W_1. In other words, $c^x \in K_1$ for any $x \in F \setminus F_1$. (Note that it does not matter that W_1 is not F-invariant: for any $g \in F$ the element c^g belongs to Q, which acts on W_1.) Furthermore, we also obviously have

$$c^{yx} \in K_1 \quad \text{for any } y \in F_1 \text{ and } x \in F \setminus F_1. \tag{15}$$

Since $Q = \langle c^F \rangle$ by (13), we obtain that $Q = \langle c^{F_1} \rangle K_1$.

The group FH (even F) acts transitively on the W_i and therefore on the set of the kernels K_i. Therefore, by Remak's theorem, the nilpotency class of $Q/\bigcap K_i$ is equal to the nilpotency class of Q/K_1. To lighten the notation we denote $K = \bigcap K_i$. Note also that $K = C_Q(W)$ is an FH-invariant normal subgroup of Q, the kernel of the action of Q on W, so that FH acts on Q/K.

We choose a simple commutator $z = [c^{y_1}, \dots, c^{y_n}]$ of maximal weight in the elements c^y, $y \in F_1$, that does not belong to K_1. Since $Q = \langle c^{F_1} \rangle K_1$, the weight of this commutator is equal to the nilpotency class of Q/K_1, which is equal to the nilpotency class of Q/K. Therefore the image of z in Q/K belongs to the centre of this FH-invariant section. (Note that we do not exclude the case of Q/K_1 being abelian; then we can simply take $z = c$.)

A little complication is to due to the possibility that the order F may not be coprime to q. This is why we switch to the Hall q'-subgroup $F_{q'}$ of F. We must have

$$C_Q(F_{q'}) = 1; \tag{16}$$

otherwise the Sylow q-subgroup F_q of $F = F_{q'} \times F_q$ would have non-trivial fixed points on $C_Q(F_{q'})$, and these fixed points would be fixed points for F, contrary to the hypothesis. To lighten the notation, we set $X = F_{q'}$.

Since the image $\bar{z}$ of z in Q/K is central, the product $\prod_{x\in X} \bar{z}^x$ is obviously a fixed point of X on Q/K, and therefore must be trivial by (16) and Lemma 1.1. Thus, we have

$$1 \equiv \prod_{x\in X} z^x \equiv \prod_{y\in X\cap F_1} z^y \cdot \prod_{x\in X\setminus F_1} z^x \pmod{K}. \tag{17}$$

In the second product on the right, each element $z^x = [c^{y_1 x}, \dots, c^{y_n x}]$ belongs to K_1, since $c^{y_i x} \in K_1$ for all i by (15). For each element in the first product on the right of (17) we have $z^y \equiv z \pmod{K_1}$ for any $y \in X \cap F_1$. Indeed, the image of z in Q/K_1 is also central and therefore is represented by a scalar linear transformation in the action on the homogeneous Wedderburn component W_1 for Q; also note that W_1 and therefore K_1 are F_1-invariant, so that the action of $y \in X \cap F_1$ on the image of z in Q/K_1 is well-defined.

As a result, by passing to congruences modulo K_1 in (17) we obtain

$$1 \equiv \prod_{y\in X\cap F_1} z^y \equiv z^{|X\cap F_1|} \pmod{K_1}.$$

Hence, $z \in K_1$, since $|X \cap F_1|$ is coprime to q. This is a contradiction with the choice of z, which completes the proof of part (a).

We now derive part (b) of Theorem 2.1. We already know that $F_i(C_G(H)) = F_i(G) \cap C_G(H)$ for all i. In order to obtain that the Fitting height of $C_G(H)$ is equal to the Fitting height of G, it remains to show that if $C_G(H) \leqslant F_n(G)$, then $F_n(G) = G$. Suppose the opposite, $F_n(G) \neq G$. By Lemmas 1.1 and 1.3 any elementary abelian FH-invariant section S of $G/F_n(G)$ is a free $\mathbb{F}_pH$-module (for some prime p), on which H has nontrivial 'diagonal' fixed points by (2). Since $C_G(H)$ covers the fixed points of H in S by Theorem 1.5, this contradicts the inclusion $C_G(H) \leqslant F_n(G)$. □

4 Upper π-series

Recall that for a set of primes π the maximal normal π-subgroup of a finite group G is denoted by $O_\pi(G)$. For any sets of primes $\pi_1, \pi_2, \dots, \pi_k$, by induction, $O_{\pi_1,\pi_2,\dots,\pi_k}(G)$ is the full inverse image of $O_{\pi_k}\big(G/O_{\pi_1,\pi_2,\dots,\pi_{k-1}}(G)\big)$. The π-length of a π-soluble group G is the minimum number of symbols π in the equation $O_{\pi',\pi,\pi',\pi\dots,\pi,\pi'}(G) = G$, where π' stands for the complementary set of primes of π.

Corollary 4.1. *Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. Then*

(a) $O_\pi(C_G(H)) = O_\pi(G) \cap C_G(H)$ *for any set of primes* π;

(b) *the π-length of G is equal to the π-length of $C_G(H)$;*

(c) $O_{\pi_1,\pi_2,\dots,\pi_k}(C_G(H)) = O_{\pi_1,\pi_2,\dots,\pi_k}(G) \cap C_G(H)$ *for any sets of primes* $\pi_1, \pi_2, \dots, \pi_k$.

Proof. We only need to prove part (a); then both parts (b) and (c) follow by induction using part (a) with various sets π, because $C_G(H)$ covers the fixed points of H in any FH-invariant quotient by Theorem 1.5.

The inclusion $O_\pi(C_G(H)) \geqslant O_\pi(G) \cap C_G(H)$ is obvious. To prove the reverse inclusion we argue by contradiction. Suppose that $O_\pi(C_G(H)) \nleqslant O_\pi(G) \cap C_G(H)$ and consider $\bar G = G/O_\pi(G)$. Then there is a prime $q \in \pi$ such that $O_q(C_{\bar G}(H)) \neq 1$ (recall that G is soluble). We have $O_q(C_{\bar G}(H)) \leqslant F(C_{\bar G}(H)) = F(\bar G) \cap C_{\bar G}(H)$ by Theorem 2.1(a), so that $O_q(\bar G) \neq 1$, which is impossible, as $O_\pi(\bar G) = 1$. □

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