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2012

MIMS EPrint: **2012.87**

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ISSN 1749-9097

# ON THE VOLUME OF TUBULAR NEIGHBOURHOODS OF REAL ALGEBRAIC VARIETIES

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ABSTRACT. The problem of determining the volume of a tubular neighbourhood has a long and rich history. Bounds on the volume of neighbourhoods of algebraic sets have turned out to play an important role in the probabilistic analysis of condition numbers in numerical analysis. We present a self-contained derivation of bounds on the probability that a random point, chosen uniformly from a ball, lies within a given distance of a real algebraic variety of any codimension. The bounds are given in terms of the degrees of the defining polynomials, and contain as special case an unpublished result by Oceanu.

## 1. INTRODUCTION

The purpose of these notes is to derive a bound on the volume of a tubular neighbourhood of a real algebraic variety in terms of the degrees of the defining polynomials. The problem is stated in probabilistic terms, namely, as the probability that a random point, uniformly distributed in a ball, falls within a certain neighbourhood of the variety.

**Theorem 1.1.** *Let  $V$  be the zero-set of multivariate polynomials  $f_1, \dots, f_s$  in  $\mathbb{R}^n$  of degree at most  $D$ . Let  $x$  be uniformly distributed in a ball  $B(p, \sigma)$  or radius  $\sigma$  around  $p \in \mathbb{R}^n$ . Then*

$$\mathbf{P}\{\text{dist}(x, V) \leq \varepsilon\} \leq 2 \left(\frac{2D\varepsilon}{\sigma}\right)^s \sum_{i=0}^m \binom{n}{s+i} \left(\frac{2D\varepsilon}{\sigma}\right)^i \left(1 + \frac{\varepsilon}{\sigma}\right)^{m-i}.$$

*If the polynomials  $f_1, \dots, f_s$  are homogeneous and  $p = 0$ , then*

$$\mathbf{P}\{\text{dist}(x, V) \leq \varepsilon\} \leq 2 \left(\frac{2D\varepsilon}{\sigma}\right)^s \sum_{i=0}^m \binom{n}{s+i} \left(\frac{2D\varepsilon}{\sigma}\right)^i.$$

The second of the stated equations is commonly attributed to A. Oceanu [8, Theorem 4.3], though a proof has not been published so far and does not seem available. From the proof of Theorem 1.1 we also get the following corollary, conjectured by J. Demmel [8, (4.15)].

**Corollary 1.1.** *Assume  $V$  is a complete intersection. Then for small enough  $\varepsilon$  we have*

$$\mathbf{P}\{\text{dist}(x, V) \leq \varepsilon\} = \text{vol}_{n-s}(V) \cdot \varepsilon^s \cdot \frac{2n\Gamma(n/2)}{\pi^{(n-s)/2}\Gamma(s/2)} + o(\varepsilon^s).$$

**1.1. History and applications.** In 1840, J. Steiner [23] showed that volume of an  $\varepsilon$ -neighbourhood of a convex body in  $\mathbb{R}^3$  could be written as a quadratic polynomial in  $\varepsilon$ . This result has become a staple of integral geometry and was the starting point of a myriad of generalisations in multiple directions. One such generalisation is a celebrated result by H. Weyl [25], who showed that for  $\varepsilon$  small enough, the volume of an  $\varepsilon$ -neighbourhood around a compact Riemannian submanifold of  $\mathbb{R}^n$  is given by a polynomial whose degree is the dimension of the manifold. Weyl's Tube Formula became an important ingredient in Allendoerfer and Weil's proof of the Gauss-Bonnet Theorem for hypersurfaces. For more on Weyl's tube formula and its ramification, see [11]. Tube formulas came into the radar of numerical analysis through the work of S. Smale [22], E. Kostlan [15], J. Renegar [19], and J. Demmel [8], among others, who were interested in the probabilistic analysis of condition numbers. It has been observed (see, e.g., [13, 7] and the references there) that the condition number of many numerical computation problems can be bounded by the inverse distance to a set of ill-posed inputs. In particular, if one can describe the set of ill-posed inputs as a subset of an algebraic variety, then a bound on the relative volume of its neighbourhood in terms of the degree of the variety directly translates into a result on the probability distribution of condition numbers. The results of Demmel [8] have been partially extended to the setting of *smoothed analysis* on the sphere in [4], by studying tubular neighbourhoods of hypersurfaces intersected with spherical caps. For a comprehensive survey of these ideas we refer to [2, 3].

**1.2. Main ideas.** Along the lines of previous work estimating the volume of tubes [4], the proof of Theorem 1.1 is based on three main ingredients: Weyl's tube formula, an integral-geometric kinematic formula, and Bézout-type bounds on the degree of Gauss maps. In what follows, let  $V$  be a complete intersection of dimension  $m = n - s$ . In a first step, based on Weyl's tube formula, a bound is derived in terms of *integrals of absolute curvature*:

$$\text{vol}_n T(V, \varepsilon) \leq \varepsilon^s \sum_{i=0}^m \frac{1}{s+i} |K_i|(V) \varepsilon^i.$$

The highest order term  $|K_m|(V)$  is intimately related to the *generalized Gauss map* of  $V$ , and can in fact be expressed in terms of the *degree* of this map. Using standard Bézout-type arguments it is possible to bound the degree of the Gauss map in terms of the degrees of the defining polynomials. The lower-order invariants  $|K_i|(V)$  can then be related to the highest order invariants  $|K_i|(V \cap L)$  of an intersection with a random linear subspace by means of *Crofton's Formula* from integral geometry:

$$|K_i|(V) \leq 2 \binom{n}{s+i} \int_{L \in \mathcal{E}_{s+i}^n} |K_i|(V \cap L) d\lambda_{s+i}^n.$$

where  $\mathcal{E}_{s+i}$  denotes the space of  $(s+i)$ -dimensional subspaces with suitable measure. Some care has to be taken when dealing with the uniform distribution on a sphere, and the fact that the algebraic variety in question may not be smooth or compact.

**1.3. Outline.** The paper is structured as follows. Section 2 gives a review of basic concepts of Riemannian geometry in Euclidean space, as well as Weyl's tube formula and results from integral geometry. Section 4 contains the proof of Theorem 1.1, using bounds on the integrals of absolute curvature in terms of degrees. The appendix is devoted to a complete proof of Weyl's tube formula in Euclidean space.

**1.4. Notation and Terminology.** We write  $B^n(p, \sigma)$  for the solid closed ball in  $\mathbb{R}^n$  with center  $p$  and radius  $\sigma > 0$ , and  $S^{n-1}(p, \sigma)$  for its boundary, and set  $S^{n-1} := S^{n-1}(0, 1)$  and  $B^n := B^n(0, 1)$ . We write  $\text{vol}_n M$  for the  $n$ -dimensional Lebesgue-measure of a measurable set  $M \subseteq \mathbb{R}^n$ , and often drop the subscript and simply write  $\text{vol } M$ . For an  $m$ -dimensional Riemannian manifold  $M$ , when we write  $\text{vol}_m M = \text{vol } M$  we mean  $\int_M \omega_M$ , with  $\omega_M$  the volume form associated to the Riemannian structure (see section 2.1.1). Whenever we say manifold, we mean smooth manifold.

Throughout this paper we denote by  $\mathcal{O}_{n-1} := 2\pi^{n/2}/\Gamma(\frac{n}{2})$  the  $(n-1)$ -dimensional volume of the unit sphere  $S^{n-1}$  in  $\mathbb{R}^n$ , and  $\omega_n := \mathcal{O}_{n-1}/n$  the  $n$ -dimensional volume of the solid unit ball in  $\mathbb{R}^n$ . The *flag coefficients* are defined as

$$(1) \quad \begin{bmatrix} n \\ k \end{bmatrix} := \binom{n}{k} \frac{\omega_n}{\omega_k \omega_{n-k}}$$

for  $n \geq 0$  and  $k \geq 0$ . They appear naturally in the study of invariant measures on Grassmannians [14].

## 2. PRELIMINARIES

We assume familiarity with the basic notions of Riemannian geometry, as described for example in [24, 9, 17]. The purpose of most of this section is to introduce notation and terminology.

**2.1. Riemannian manifolds in  $\mathbb{R}^n$ .** Given a Riemannian manifold  $M$  of dimension  $m$ , we denote by  $TM$  its tangent bundle, by  $C(M)$  the ring of smooth functions on  $M$ , and by  $\mathcal{X}(M)$  the  $C(M)$ -module of tangent vector fields on  $M$ . For  $p \in M$  we write  $T_p M$  for the tangent space at  $p$ . If  $v \in T_p \mathbb{R}^n$  and  $f \in C(\mathbb{R}^n)$ , then  $v(f)$  denotes the directional derivative of  $f$  in direction  $v$  at  $p$ .

In this article we are only concerned with submanifolds  $M$  of Euclidean space  $\mathbb{R}^n$ . As such, each  $T_p M$  can be identified with a subspace of  $T_p \mathbb{R}^n \cong \mathbb{R}^n$  in the obvious manner. Let  $NM := \{(p, v) \in T\mathbb{R}^n \mid p \in M, v \perp T_p M\}$  be the normal bundle to  $M$  in  $\mathbb{R}^n$  and denote by  $N_p M$  the fiber of  $NM$  over  $p \in M$ , i.e., the normal space to  $M$  at  $p$  in  $\mathbb{R}^n$ .

Let  $Y \in \mathcal{X}(\mathbb{R}^n)$  be a smooth vector field. For  $v \in T_p \mathbb{R}^n$  denote by  $\nabla_v Y := v(Y)$  the covariant derivative of  $Y$  along  $v$  at  $p$ . The covariant derivative satisfies  $v(\langle Y, Z \rangle) = \langle \nabla_v Y, Z \rangle + \langle Y, \nabla_v Z \rangle$ . In particular, for orthogonal fields  $Y$  and  $Z$  we have  $\langle \nabla_v Y, Z \rangle = -\langle Y, \nabla_v Z \rangle$ . For  $v \in N_p M$  and  $X, Y \in \mathcal{X}(M)$ , the second fundamental form  $S_v(X, Y)$  of  $X$  and  $Y$  along  $v$  is the symmetric, bilinear map  $T_p M \times T_p M \rightarrow \mathbb{R}$  defined by

$$S_v(X, Y) := \langle \nabla_{X(p)} Y, v \rangle,$$

where we assume the vector fields  $X, Y$  to be extended to a neighbourhood of  $M$  in  $\mathbb{R}^n$  for this definition to make sense. Given a normal vector field  $Z$  on  $M$  we have  $S_{Z(p)}(X, Y) = -\langle Y, \nabla_{X(p)} Z \rangle$  (since  $X, Y$  are orthogonal to  $Z$ ). Given an orthonormal frame field  $(E_1, \dots, E_m)$  on  $U \subset M$ , we will on occasion use the matrix  $S(Z)$  with entries in  $C(M)$  that represents this bilinear form with respect to that frame field. Its values at  $p \in U$  are given by the entries

$$(2) \quad S_{ij}(Z)(p) = S_{Z(p)}(E_i, E_j) = \langle \nabla_{E_i(p)} E_j, Z \rangle = -\langle E_j, \nabla_{E_i(p)} Z \rangle.$$

Note that we can also talk about  $S(v)$  for fixed  $v \in N_p M$ . Then we have  $S(v) = S(Z)(p)$  for any normal vector field such that  $Z(p) = v$ .

**2.1.1. A note on integration and orientation.** Given a Riemannian manifold  $M \subseteq \mathbb{R}^n$ , we denote by  $\omega_M$  the natural volume form on  $M$  associated to the Riemannian metric. Thus if  $U \subseteq M$  is an oriented coordinate neighbourhood and  $x^1, \dots, x^m: U \rightarrow \mathbb{R}^m$  are orthonormal coordinates (so that the tangent vectors  $\partial/\partial x^1, \dots, \partial/\partial x^m$  form a positively oriented, orthonormal basis at each  $p \in U$ ), then  $\omega_M = dx^1 \wedge \dots \wedge dx^m$  on  $U$ . All volume forms are densities (unsigned forms), though we will occasionally locally represent them as differential forms in an oriented coordinate neighbourhood  $U \subseteq M$  without always stating this explicitly. Given a differential map  $f$  from a manifold of the same dimension to  $M$ ,  $f^* \omega_M$  denotes the pull-back volume form.

**2.1.2. Curvature Invariants.** In this section we introduce the curvature invariants  $K_0(M), \dots, K_m(M)$  associated to a Riemannian manifold  $M$  in terms of the second fundamental form. These invariants are the key components in Weyl's formula (Section 3.1) for the volume of tubes around  $M$ . A variant of these invariants, the integrals of absolute curvatures  $|K_i|(M)$ , will feature prominently in much of this paper. For the case of spherical hypersurfaces these were introduced in [4].

Let  $M$  be an  $m$ -dimensional compact Riemannian manifold,  $U' \subseteq \mathbb{R}^n$  open and  $U = U' \cap M$ . Let  $(E_1, \dots, E_m)$  be an orthonormal frame field on  $U'$ , such that  $E_1(p), \dots, E_m(p)$  form an oriented basis of  $T_p M$  for all  $p \in U$ . For  $v \in N_p M$  let  $S(v)$  denote the  $m \times m$  matrix of the second fundamental form at  $p$  along  $v$  with respect to the frame field, as defined in (2).

For  $0 \leq i \leq m$  let  $\psi_i: N_p M \rightarrow \mathbb{R}$  be the homogeneous polynomial of degree  $i$  defined by

$$\det(\text{Id} - tS(v)) = \sum_{i=0}^m t^i \psi_i(v).$$

Note that the  $\psi_i(v)$  are, up to sign, the coefficients of the characteristic polynomial of  $S(v)$ . More precisely, we have

$$\psi_i(v) = (-1)^i \sigma_i(\kappa_1(v), \dots, \kappa_m(v)),$$

where the  $\kappa_i(v)$  are the eigenvalues of  $S(v)$ , i.e., the principal curvatures, and  $\sigma_i$  denotes the  $i$ -th elementary symmetric function. In particular,  $\psi_m(v) = (-1)^m \det S(v)$ . These quantities are, up to orientation, independent of the particular orthonormal frame used to define the matrix  $S(v)$ .

For  $p \in M$ , set  $S(N_p M) := \{v \in N_p M \mid \|v\| = 1\}$  and denote by  $S(NM)$  the corresponding normal sphere bundle. At a point  $p \in M$  define

$$I_i(p) = \int_{v \in S(N_p M)} \psi_i(v) \omega_{S(N_p M)}.$$

The  $I_i(p)$  are polynomial invariants of the second fundamental form in the sense of [12]. If  $v = \sum_{j=1}^s w^j E_{m+j}(p)$ ,  $s = n - m$ , then the  $I_i(p)$  are integrals over all  $u \in S^{s-1}$  of homogeneous polynomials of degree  $i$  in  $u^1, \dots, u^s$ . From this it follows that  $I_i(p) = 0$  for  $i$  odd.

The *integrals of curvature* are defined as

$$(3) \quad K_i(M) := \int_M I_i(p) \omega_M = \int_{S(NM)} \psi_i(v) \omega_{S(NM)}.$$

It is easy to see that  $K_0(M) = \mathcal{O}_{s-1} \text{vol}_m M$ . Less trivial is the fact that  $K_m(M) = \mathcal{O}_{n-1} \chi(M)$ , where  $\chi(M)$  is the Euler characteristic of  $M$ . This is a consequence of the generalised Gauss-Bonnet Theorem (see [11] for a discussion of this result and its relation to Weyl's Tube Formula).

The *integrals of absolute curvature* are defined as

$$(4) \quad |K_i|(M) := \int_{S(NM)} |\psi_i(v)| \omega_{S(NM)}.$$

These are important for extending Weyl's tube formula to and inequality for the volume of  $\varepsilon$ -tubes for arbitrary  $\varepsilon$ .

**2.1.3. Manifolds with boundary.** The concepts introduced so far extend without difficulty to the setting of manifolds with boundary. In fact, we could easily extend the concepts to locally convex Whitney stratified spaces as in [1, Chapter 8], but we don't need this here. Instead of the tangent and normal space we have the more general notions of *tangent cone*  $T_p M$  and *normal cone*  $N_p M$  of  $M$  at a point  $p$ . The tangent cone is defined as

$$T_p M = \{\dot{c}(0) \mid c: (-1, 1) \rightarrow M, c(0) = p, \exists \delta > 0: c([0, \delta]) \subset M\}.$$

The tangent cone consists of the directions in which we can move from  $p$  while staying on  $M$ . Note that this is just the tangent space if  $p \in M$  is not on the boundary. If  $p \in \partial M$ , then in terms of local coordinates  $x: U \cap M \rightarrow \mathbb{R}_{x^m \geq 0}^m$  around  $p$  we have

$$T_p M = \left\{ \sum_{i=1}^m a_i \frac{\partial}{\partial x^i}(p) \mid a_m \geq 0 \right\}.$$

The normal cone is the dual cone of the tangent cone in  $\mathbb{R}^n$ :

$$(5) \quad N_p M := \{v \in T_p \mathbb{R}^n \mid \forall w \in T_p M: \langle v, w \rangle \leq 0\}.$$

Again, for  $p \in M \setminus \partial M$  this coincides with the normal space at  $p$ . The normal sphere bundle  $S(NM)$  and the integrals of curvature  $K_i(M)$  and  $|K_i|(M)$  are defined accordingly.

2.1.4. *The degree.* Let  $M, P$  be compact Riemannian manifolds and let  $f: M \rightarrow P$  be a smooth map. By Sard's Theorem [18, §2] almost all  $q \in P$  are regular values. For any such regular value  $q$ , the preimage  $f^{-1}(q)$  is either empty or a finite set with locally constant cardinality [18].

For measurable  $h: P \rightarrow \mathbb{R}$  we have

$$(6) \quad \int_{p \in M} h \circ f(p) f^* \omega_P = \int_{q \in P} h(q) \#f^{-1}(q) \omega_P,$$

where  $\#f^{-1}(q)$  denotes the cardinality of the preimage of  $q$ . Recall (Section 2.1.1) that we are dealing with unsigned forms, otherwise we would have to count the points in the fiber with signs, depending on whether  $f$  is orientation preserving or reversing at a point in the preimage.

We define the degree of  $f$  to be the maximum cardinality of the pre-image of a regular value under  $f$ :

$$\deg f := \max_{q \in \text{reg} P} \#f^{-1}(q).$$

With this definition we have

$$(7) \quad \int_M f^* \omega_P \leq \deg f \int_P \omega_P.$$

This notion of degree differs from the usual one from differential topology (see [18, §5]), which takes into account orientation.

These concepts generalise without difficulty to the setting of manifolds with boundary, provided  $f(\partial M) \subseteq \partial P$ . A regular value in this case is also required to be a regular value of  $f$  restricted to  $\partial M$ . If  $M$  is compact with compact boundary  $\partial M$ , then the preimage of a regular value is again finite, and constant on  $M \setminus \partial M$  if this set is connected. Since the boundary disappears when integrating  $m$ -forms, we can define the degree by restricting to  $M \setminus \partial M$ .

2.1.5. *Transversality.* The intersection of two manifolds  $M$  and  $P$  in  $\mathbb{R}^n$  of dimension  $m, \ell$  with  $m + \ell \geq n$  is called *transversal* at  $p \in M \cap P$ , if  $\dim T_p M \cap T_p P = m + \ell - n$ . The intersection is called transversal if it is transversal at every  $p \in M \cap P$ . In that case,  $M \cap P$  is a smooth  $(m + \ell - n)$ -dimensional manifold.

For the following lemma, recall that  $B^n(p, \sigma)$  denotes the closed ball of radius  $\sigma$  around  $p$  in  $\mathbb{R}^n$ , and  $S^{n-1}(p, \sigma) = \partial B^n(p, \sigma)$  is its boundary. By ‘‘almost all’’ we mean ‘‘up to a set of measure zero’’.

**Lemma 2.1.** *Let  $M$  be a Riemannian manifold. For almost all  $\sigma > 0$  the intersection  $B^n(p, \sigma) \cap M$  is a Riemannian manifold with boundary.*

PROOF. Use the fact that transversality is generic: for almost all  $\sigma > 0$  the intersection  $M \cap S^{n-1}(p, \sigma)$  is transversal, and therefore a smooth  $(n - 1)$ -dimensional manifold. To see this, consider the set

$$\Delta := \{(x, y, t) \in M \times S^{n-1}(p, 1) \times \mathbb{R} \mid x = p + ty\}$$

with the projection to the last component  $\pi_t: \Delta \rightarrow \mathbb{R}$ . Clearly,  $\pi_t^{-1}(\sigma)$  is diffeomorphic to  $M \cap S^{n-1}(p, \sigma)$ . Moreover,  $M \cap S^{n-1}(p, \sigma)$  is transverse if and only if  $\sigma$  is a regular value of  $\pi_t$ . The claim now follows from Sard's Lemma.  $\square$

## 3. GEOMETRY OF TUBES

**3.1. Weyl's tube formula.** References for the content of this section are [25, 11] and [1, Chapter 10]. Let  $M \subseteq \mathbb{R}^n$  be an oriented, compact Riemannian submanifold of dimension  $m < n$  and denote by  $s := n - m$  the codimension of  $M$  in  $\mathbb{R}^n$ . The (closed) *tube* of radius  $\varepsilon$  around  $M$  in  $\mathbb{R}^n$  is defined to be the set

$$(8) \quad T(M, \varepsilon) := \left\{ p \in \mathbb{R}^n \mid \begin{array}{l} \exists \text{ line of length } \leq \varepsilon \text{ from } p \\ \text{meeting } M \text{ orthogonally} \end{array} \right\}.$$

For compact manifolds this coincides with the  $\varepsilon$ -neighbourhood of  $M$  in  $\mathbb{R}^n$ , though in general this need not be the case.

In his influential paper [25], Weyl derived the expression

$$\text{vol } T(M, \varepsilon) = \mathcal{O}_{s-1} \varepsilon^s \sum_{\substack{i=0 \\ i \text{ even}}}^m \frac{(i-1)(i-3)\cdots 1}{(s+i)(s+i-2)\cdots s} \mu_i(M) \varepsilon^i$$

for the volume of a tube of radius  $\varepsilon$  around  $M$ , provided  $\varepsilon$  is small enough. The  $\mu_i(M)$  are the *curvature invariants* of  $M$ . The deeper part of Weyl's work consists of showing that these invariants are intrinsic, that is, they only depend on the curvature tensor of  $M$  and not on the particular embedding of  $M$  in  $\mathbb{R}^n$ . We will not need this feature here, however, and will be happy with expressing these invariants in terms of the second fundamental form.

The  $\mu_i(M)$  are just a different normalisation of the invariants  $K_i(M)$  introduced in Section 2.1.2, namely

$$(9) \quad K_i(M) = \mathcal{O}_{s-1} \frac{(i-1)(i-3)\cdots 1}{(s+i-2)(s+i-4)\cdots s} \mu_i(M).$$

for  $i$  even.

Note that the  $K_i(M)$  are no longer independent of the embedding, since the codimension enters the formula.

We will need a slight generalisation of Weyl's Tube Formula to manifolds with boundary that works for arbitrary  $\varepsilon$ . The definition of a tube (8) stays exactly the same.

**Theorem 3.1.** *Let  $M \subseteq \mathbb{R}^n$  be an oriented, compact,  $m$ -dimensional Riemannian manifold with boundary, and assume  $s := n - m > 0$ . Then for all  $\varepsilon > 0$  we have*

$$(10) \quad \text{vol } T(M, \varepsilon) \leq \varepsilon^s \sum_{i=0}^m \frac{1}{s+i} |K_i|(M) \varepsilon^i.$$

The proof is along the lines of [25, 11, 1]. For completeness we include it in the Appendix.

*Example 3.1.* Let  $M = S^m$  be the  $m$ -dimensional unit sphere in  $\mathbb{R}^n$ . Along the lines of the proof of the Tube Formula 3.1 we can derive

$$\text{vol } T(S^m, \varepsilon) = 2 \omega_{m+1} \varepsilon^s \sum_{i=0}^m \frac{\omega_{s+i}}{\omega_{i+1}} \binom{m+1}{i+1} \varepsilon^i.$$



for small  $\varepsilon$  (recall from Section 1.4 the definition of  $\omega_n$  and  $\mathcal{O}_n$ ). From this we get

$$K_i(S^m) = \frac{2\mathcal{O}_m\mathcal{O}_{s+i-1}}{\mathcal{O}_i} \binom{m}{i}$$

for  $i$  even. Note that  $K_0(S^m) = \mathcal{O}_m\mathcal{O}_{n-m-1}$  and that  $K_m(S^m) = 2\mathcal{O}_{n-1}$  for  $m$  even and  $K_m(S^m) = 0$  for  $m$  odd, in accordance with the Euler characteristic for spheres. Some special cases for the volume of tubes:

- (1) Setting  $m = n - 1$  we get  $\text{vol } T(S^{n-1}, \varepsilon) = \omega_n [(1 + \varepsilon)^n - (1 - \varepsilon)^n]$ , as was to be expected.
- (2) For  $m = 0$  we have  $\text{vol } T(S^0, \varepsilon) = 2 \varepsilon^n \omega_n$ .
- (3) For  $m = 1, n = 2$  we get the volume of the torus  $\text{vol } T(S^1, \varepsilon) = 2\pi^2\varepsilon^2$ .

**3.2. Integral Geometry.** In order to obtain upper bounds for the integrals of absolute curvature  $|K_i|(M)$ , we will first derive bounds for  $|K_m|(M)$  using the generalised Gauss map, and the case where  $0 \leq i < m$  is then handled by relating the  $i$ -th curvature invariants  $K_i(M)$  to the curvature invariants  $K_i(M \cap L)$  of the intersection of  $M$  with a random affine space of dimension  $s + i$ . Formulas relating invariant measures of a set to its intersection with random affine spaces are known by the name of *Crofton formulas* and play a central role in integral geometry. For an introduction to integral geometry and geometric probability we refer to [14, 21]. The version of Crofton's formula involving Weyl's curvature invariants is due to Chern [6] and Federer [10], see also [20, 15.95b] and [1, 13.1].

Let  $\mathcal{E}_k^n$  be the set of  $k$ -dimensional affine spaces in  $\mathbb{R}^n$  and  $\mathbb{G}(n, k)$  the Grassmannian of  $k$ -dimensional linear subspaces of  $\mathbb{R}^n$ . We can identify  $\mathcal{E}_k^n$  with the subset of those  $(V, p) \in \mathbb{G}(n, k) \times \mathbb{R}^n$  such that  $p \perp V$ , the one-to-one correspondence  $s$  being given by  $s(V, p) = p + V$  [14, Chapter 6].

Let  $\nu_k^n$  denote the  $O(n)$ -invariant measure on  $\mathbb{G}(n, k)$  induced by the identification  $\mathbb{G}(n, k) = O(n)/O(k) \times O(n - k)$ . It is well known that

$$\nu_k^n(\mathbb{G}(n, k)) = \frac{\mathcal{O}_{n-1} \cdots \mathcal{O}_{n-k}}{\mathcal{O}_{k-1} \cdots \mathcal{O}_0},$$

see also [5, 3.2] for a discussion. The product measure  $\nu_k^n \times \omega_{\mathbb{R}^n}$  gives rise to an invariant measure  $\bar{\lambda}_k^n$  on  $\mathcal{E}_k^n$ , defined by

$$\int_{V \in \mathbb{G}(n, k)} \left( \int_{p \in V^\perp} f \circ s(V, p) \omega_{V^\perp} \right) d\nu_k^n = \int_{L \in \mathcal{E}_k^n} f(L) d\bar{\lambda}_k^n.$$

for a measurable function  $f$  on  $\mathcal{E}_k^n$ . In particular, setting

$$f = \mathbf{1}_{B^n(p, \sigma)} = \begin{cases} 1 & L \cap B^n(p, \sigma) \neq \emptyset \\ 0 & \text{else} \end{cases}$$

we get

$$\begin{aligned} \bar{\lambda}_k^n(\{L \in \mathcal{E}_k^n \mid L \cap B^n(p, \sigma) \neq \emptyset\}) &= \int_{L \in \mathcal{E}_k^n} f(L) d\bar{\lambda}_k^n \\ &= \int_{V \in \mathbb{G}(n, k)} \left( \int_{p \in V^\perp} \mathbf{1}_{B^n(p, \sigma)} \omega_{V^\perp} \right) d\nu_k^n \\ &= \omega_{n-k} \sigma^{n-k} \nu_k^n(\mathbb{G}(n, k)). \end{aligned}$$

In the following we use the renormalised measure  $\lambda_k^n = \nu_k^n(\mathbb{G}(n, k))^{-1} \bar{\lambda}_k^n$ , so that

$$\lambda_k^n(\{L \in \mathcal{E}_k^n \mid L \cap B^n \neq \emptyset\}) = \omega_{n-k}.$$

The following theorem is merely a reformulation of [20, 15.95b] with a different normalisation of the measure, and after simplifying the constants, see also [1, Theorem 13.1.1]. Note also that with the parameters chosen here, it makes no difference whether we formulate this theorem with  $\mu_i(M)$  or with  $K_i(M)$ , since  $M \cap L$  has generically the same codimension  $s$  in  $L$  as  $M$  in  $\mathbb{R}^n$ . Recall the definition (1) of the flag coefficients in Section 1.4.

**Theorem 3.2.** (*Crofton's Theorem*)

$$(11) \quad K_i(M) = \left[ \begin{array}{c} n \\ s+i \end{array} \right] \int_{L \in \mathcal{E}_{s+i}^n} K_i(M \cap L) d\lambda_{s+i}^n.$$

The standard way of proving this is to first show that Theorem 3.2 holds for some constant in front of the integral on the right-hand side and then to determine the constant by plugging in spheres for  $M$ , for which the  $K_i(M)$  can be determined by direct calculation from the definition.

Crofton's Theorem can be extended to a bound for integrals of absolute curvature.

**Theorem 3.3.**

$$(12) \quad |K_i|(M) \leq 2 \left[ \begin{array}{c} n \\ s+i \end{array} \right] \int_{L \in \mathcal{E}_{s+i}^n} |K_i|(M \cap L) d\lambda_{s+i}^n.$$

PROOF. Let  $M_+$  and  $M_-$  denote the parts of  $M$  on which  $I_i(p)$  is positive and negative, respectively. Then  $|K_i|(M) = |K_i(M_+)| + |K_i(M_-)|$  and consequently

$$\begin{aligned} |K_i|(M) &= \left[ \begin{array}{c} n \\ s+i \end{array} \right] \left( \left| \int_{L \in \mathcal{E}_{s+i}^n} K_i(M_+ \cap L) d\lambda_{s+i}^n \right| + \left| \int_{L \in \mathcal{E}_{s+i}^n} K_i(M_- \cap L) d\lambda_{s+i}^n \right| \right) \\ &\leq 2 \left[ \begin{array}{c} n \\ s+i \end{array} \right] \int_{L \in \mathcal{E}_{s+i}^n} |K_i|(M \cap L) d\lambda_{s+i}^n. \end{aligned}$$

which completes the proof.  $\square$

## 4. DEGREE BOUNDS

**4.1. Degree of the Gauss map.** In this section we interpret the expected value of the highest curvature invariant as the degree of a generalised Gauss map. Let  $S(NM)$  denote the normal sphere bundle over  $M$ . Note that  $S(NM)$  has codimension one in  $\mathbb{R}^n$ .

**Definition 4.1.** Let  $M \subseteq \mathbb{R}^n$  be a compact  $m$ -dimensional Riemannian manifold with boundary. The *generalized Gauss map* of  $M$  is defined as

$$\gamma: S(NM) \rightarrow S^{n-1}, \quad (p, v) \mapsto v.$$

The generalised Gauss map can be used to characterise the highest order curvature invariant. The following should be well known, but due to lack of a reference we sketch a proof.

**Lemma 4.1.** *The generalized Gauss map is surjective.*

PROOF. Let  $w \in S^{n-1}$ . For small enough  $\varepsilon$ , the map

$$\begin{aligned} h: S(NM) &\rightarrow \mathbb{R}, \\ (p, v) &\mapsto \langle v, w \rangle \end{aligned}$$

is differentiable everywhere. Since  $S(NM)$  is compact, this map attains a maximum at some  $(p, v)$ , and the gradient at this point is zero. A calculation (writing out the gradient) shows that the gradient is zero if and only if  $w = v$ .  $\square$

Recall not the definition (7) of the degree of a map.

**Lemma 4.2.** *Let  $M \subseteq \mathbb{R}^n$  be a compact Riemannian manifold of even dimension  $m$ . Then*

$$(13) \quad |K_m|(M) = \int_{v \in S^{n-1}} \#\gamma^{-1}(v) \omega_{S^{n-1}} \leq \mathcal{O}_{n-1} \deg \gamma.$$

PROOF. We need to show that

$$(14) \quad \gamma^* \omega_{S^{n-1}} = |\det S(v)| \omega_{S(NM)}$$

on  $U$ . Once this is shown, the claim of the lemma follows from the definition of the  $K_i(M)$  (3), namely,

$$|K_m|(M) = \int_{(p,u) \in S(NM)} |\det(S(u))| \omega_{S(NM)}.$$

Let  $U \subseteq M$  and let  $x^1, \dots, x^m: U \rightarrow \mathbb{R}^m$  be orthonormal coordinates on  $U$ . Let  $(E_1, \dots, E_n)$  be an orthonormal frame field defined in a neighbourhood of  $U$  in  $\mathbb{R}^n$ , such that on  $U$  we have  $E_i = \partial/\partial x^i$  for  $1 \leq i \leq m$ . The frame field  $E_1, \dots, E_n$  gives a local trivialisation of the sphere bundle

$$\begin{aligned} U \times S^{s-1} &\rightarrow S(NM) \\ (p, u) &\mapsto \left( p, \sum_{i=1}^s u^i E_{m+i} \right). \end{aligned}$$

An orthonormal coordinate system  $y^1, \dots, y^{s-1}$  for  $S^{s-1}$  thus gives rise to orthonormal coordinates  $x^1, \dots, x^m, y^1, \dots, y^{s-1}$  on  $S(NM)$ . With  $\omega_M = dx^1 \wedge \dots \wedge dx^m$  and  $dy = dy^1 \wedge \dots \wedge dy^{s-1}$  we have

$$(15) \quad \omega_{S(NM)} = \omega_M \wedge dy.$$

Similarly we have  $\omega_{S^{n-1}} = E_1^* \wedge \dots \wedge E_m^* \wedge dy^1 \wedge \dots \wedge dy^{s-1}$ . Let  $\phi$  be such that  $\gamma^* \omega_{S^{n-1}} = \phi(p, v) \omega_{S(NM)}$  as differential form. Then

$$\begin{aligned} \phi(p, v) &= \gamma^* \omega_{S^{n-1}} \left( \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^{s-1}} \right) \\ &= \omega_{S^{n-1}} \left( \gamma_* \frac{\partial}{\partial x^1}, \dots, \gamma_* \frac{\partial}{\partial x^m}, \gamma_* \frac{\partial}{\partial y^1}, \dots, \gamma_* \frac{\partial}{\partial y^{s-1}} \right). \end{aligned}$$

Note that

$$\begin{aligned}\gamma_* \frac{\partial}{\partial x^i} &= \sum_{\ell=1}^s u^\ell \frac{\partial}{\partial x^i} E_{m+\ell}(p), \\ \gamma_* \frac{\partial}{\partial y^j} &= \sum_{\ell=1}^s \frac{\partial}{\partial y^j} u^\ell E_{m+\ell}(p),\end{aligned}$$

from which we obtain

$$\phi(p, v) = \omega_M \left( \frac{\partial}{\partial x^1} \gamma, \dots, \frac{\partial}{\partial x^m} \gamma \right) \cdot dy \left( \frac{\partial}{\partial y^1} \gamma, \dots, \frac{\partial}{\partial y^{s-1}} \gamma \right).$$

A direct calculation shows that

$$\left\langle \frac{\partial}{\partial x^i} \gamma, E_j \right\rangle = -S_{ij}(v).$$

From this it follows that

$$\omega_M \left( \frac{\partial}{\partial x^1} \gamma, \dots, \frac{\partial}{\partial x^m} \gamma \right) = (-1)^m \det S(v).$$

Clearly

$$dy \left( \frac{\partial}{\partial y^1} \gamma, \dots, \frac{\partial}{\partial y^{s-1}} \gamma \right) = 1$$

from which the claim follows.  $\square$

For an affine subspace  $L \in \mathcal{E}_{s+i}^n$  in general position, the intersection  $M \cap L$  is either empty or an  $i$ -dimensional submanifold of  $L \cong \mathbb{R}^{s+i}$ . In the latter case we can define the degree of  $M$  with respect to  $L$  as the degree of the Gauss map of  $M \cap L$  in  $L$ , that is,

$$\deg(M; L) := \deg \gamma|_{M \cap L} \leq \max_{v \in S^{s+i-1}} \#\gamma|_{M \cap L}^{-1}(v).$$

Define the  $i$ -th degree  $\deg_i(M)$  of  $M$  to be the maximum of  $\deg(M; L)$  over all  $L \in \mathcal{E}_{s+i}^n$  that intersect  $M$ :

$$\deg_i(M) := \sup_{L \in \mathcal{E}_{s+i}^n} \deg(M; L).$$

We can now bound the volume of an  $\varepsilon$ -tube around  $M$  by a function of the  $i$ -th degrees of  $M$  and of  $\varepsilon$ . In what follows we use the notation

$$\mu_{p,\sigma}(A) := \frac{\text{vol}(A \cap B^n(p, \sigma))}{\text{vol} B^n(p, \sigma)}$$

for the relative volume of a measurable set  $A \subseteq \mathbb{R}^n$  with respect to the ball  $B^n(p, \sigma)$ .

**Theorem 4.1.** *Let  $M$  be a compact Riemannian manifold of dimension  $m$  in  $\mathbb{R}^n$  and set  $s = n - m$ . Then for  $\varepsilon > 0$  we have*

$$\text{vol} T(M, \varepsilon) \leq 2\omega_n \varepsilon^s \sum_{i=0}^m \binom{n}{s+i} \deg_i(M) \varepsilon^i,$$

with equality for  $\varepsilon$  small enough. For the portion of  $T(M, \varepsilon)$  contained in a ball  $B^n(p, \sigma)$  we have, for almost all  $\sigma > 0$ ,

$$\mu_{p,\sigma}(T(M, \varepsilon)) \leq 2 \left(\frac{\varepsilon}{\sigma}\right)^s \sum_{i=0}^m \binom{n}{s+i} \deg_i(M) \left(\frac{\varepsilon}{\sigma}\right)^i \left(1 + \frac{\varepsilon}{\sigma}\right)^{m-i}.$$

PROOF. By the bound (12) we have

$$\begin{aligned}
\frac{1}{s+i}|K_i|(M) &= \frac{2}{s+i} \left[ \begin{matrix} n \\ s+i \end{matrix} \right] \int_{L \cap M \neq \emptyset} |K_i|(M \cap L) d\lambda_{s+i}^n \\
&\leq \frac{2\omega_n}{(s+i)\omega_{s+i}\omega_{m-i}} \left( \begin{matrix} n \\ s+i \end{matrix} \right) \int_{L \cap M \neq \emptyset} \mathcal{O}_{s+i-1} \deg_i(M; L) d\lambda_{s+i}^n \\
&= \frac{2\omega_n \omega_{m-i} \omega_{s+i}}{\omega_{s+i} \omega_{m-i}} \left( \begin{matrix} n \\ s+i \end{matrix} \right) \deg_i(M) \\
&= 2\omega_n \left( \begin{matrix} n \\ s+i \end{matrix} \right) \deg_i(M).
\end{aligned}$$

Plugging this into the Tube Formula (10) the first claim follows.

For the second claim, set  $M' := M \cap B(p, \sigma)$ . For almost all  $\sigma$ ,  $M'$  will be a smooth compact Riemannian manifold with smooth  $(m-1)$ -dimensional boundary  $\partial M$  (Lemma 2.1). Moreover, it is easy to see that  $T(M, \varepsilon) \cap B^n(p, \sigma) \subseteq T(M', \varepsilon + \sigma)$ . We can therefore bound  $\text{vol}(T(M, \varepsilon) \cap B^n(p, \sigma))$  by  $\text{vol} T(M', \varepsilon + \sigma)$  and thus, as we just showed, in terms of the  $\deg_i(M')$ .

It follows that

$$\begin{aligned}
\deg_i(M') &:= \frac{1}{\omega_{m-i}} \int_{L \in \mathcal{E}_{s+i}^n} \deg(M'; L) d\lambda_{s+i}^n \\
&\leq \frac{1}{\omega_{m-i}} \int_{L \cap B^n(p, \sigma + \varepsilon)} \deg(M'; L) d\lambda_{s+i}^n \\
&\leq \frac{1}{\omega_{m-i}} \lambda_{s+i}^n(\{L \in \mathcal{E}_{s+i}^n \mid L \cap B^n(p, \sigma + \varepsilon) \neq \emptyset\}) \cdot \deg_i(M').
\end{aligned}$$

By our normalisation,

$$\lambda_{s+i}^n(\{L \in \mathcal{E}_{s+i}^n \mid L \cap B^n(p, \sigma + \varepsilon) \neq \emptyset\}) = (\sigma + \varepsilon)^{m-i} \omega_{m-i}.$$

We conclude that

$$\text{vol} T(M', \varepsilon) = \omega_n \varepsilon^s \sum_{i=0}^m \left( \begin{matrix} n \\ s+i \end{matrix} \right) \deg_i(M') (\varepsilon + \sigma)^{m-i} \varepsilon^i.$$

The claim follows by dividing by  $\text{vol} B^n(p, \sigma) = \omega_n \sigma^n$  and noting that  $\deg_i(M') \leq \deg_i(M)$ .  $\square$

**4.2. Complete intersections.** Let  $f_1, \dots, f_s \in \mathbb{R}[X_1, \dots, X_n]$  be polynomials such that their common zero set  $M$  is a compact complete intersection, i.e., for every  $p \in M$  the gradients  $\nabla f_1, \dots, \nabla f_s$  are linearly independent. The gradients determine an orientation of  $M$ .

**Lemma 4.3.** *Let  $M$  be a compact complete intersection defined as the zero-set of polynomials  $f_1, \dots, f_s$ . Let  $\deg f_i \leq D$  for  $1 \leq i \leq s$ . Then the degree of the generalized Gauss map  $\varphi: S(NM) \rightarrow S^{n-1}$  is bounded by*

$$\deg \varphi \leq (2D)^n.$$

PROOF. Let  $f = \sum_{i=1}^s f_i^2$ , so that in particular,  $Z(f) = Z(f_1, \dots, f_s)$ . Let  $\varepsilon$  be such that  $\varepsilon^2$  is a regular value of  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  and set  $f_\varepsilon = f - \varepsilon^2$ . Then  $V_\varepsilon = Z(f_\varepsilon)$  is a compact hypersurface with associated Gauss map  $\gamma_\varepsilon(x) = \nabla f_\varepsilon(x) / \|\nabla f_\varepsilon(x)\|$ . By a standard argument using Bézout's

Theorem (c.f., [16]), the degree of  $\gamma_\varepsilon$  is bounded by  $(2D)^n$ . Now consider the set

$$V = \{((p, v), t) \in S(N\mathbb{R}^n) \times \mathbb{R} \mid f(p) = t \text{ and } v \in N_p f^{-1}(p)\},$$

with projections  $\pi_1((p, v), t) = v$  and  $\pi_2((p, v), t) = t$ . We will show:

- (1)  $V$  is a smooth  $n$ -dimensional variety.
- (2) For every  $w \in S^{n-1}$ ,  $\pi_1^{-1}(w)$  is a smooth curve.
- (3) 0 is a regular value of  $\pi_2: \pi_2^{-1}(w)$ .

Once this is shown, it follows that for  $\delta > 0$  small enough, the Gauss map on  $V_\delta$  and on  $V$  have the same degree.  $\square$

**Proposition 4.1.** *If the polynomials  $f_1, \dots, f_s$  are homogeneous and  $p = 0$ , then*

$$\mu_{p,\sigma}(T(M, \varepsilon)) \leq 2 \left(\frac{\varepsilon D}{\sigma}\right)^s \sum_{i=0}^m \binom{n}{s+i} \left(\frac{\varepsilon D}{\sigma}\right)^i.$$

In general we have

$$\mu_{p,\sigma}(T(M, \varepsilon)) \leq 2 \left(\frac{\varepsilon D}{\sigma}\right)^s \sum_{i=0}^m \binom{n}{s+i} \left(\frac{\varepsilon D}{\sigma}\right)^i \left(1 + \frac{\varepsilon}{\sigma}\right)^{m-i}.$$

PROOF. As in the proof of Theorem 4.1, let  $M' = M \cap L$ . Because of the homogeneity of  $M$ , we have  $T(M, \varepsilon) \cap B^n(p, \sigma) \subseteq T(M', \varepsilon)$ . From the proof of Theorem 4.1 it follows that

$$\deg_i(M') \leq \sigma^{m-i} \cdot \deg_i(M),$$

and we have the Tube Formula

$$\mu_{p,\sigma}(T(M, \varepsilon)) \leq 2 \left(\frac{\varepsilon}{\sigma}\right)^s \sum_{i=0}^m \binom{n}{s+i} \left(\frac{\varepsilon}{\sigma}\right)^i \deg_i(M).$$

It remains to bound  $\deg_i(M)$ , for which we use Bézout's Theorem. We want to bound the degree of  $M \cap L$ , and after a change of coordinates we can assume that  $L$  is given by  $x_{s+i+1} = 0, \dots, x_n = 0$ . The  $f_i$  can therefore be seen as polynomials in  $s+i$  variables, denoted by  $\bar{x}$ . The claim now follows from Lemma 4.3.  $\square$

### 4.3. Nonsmooth varieties.

#### APPENDIX

In this appendix we give a proof of the Tube formula. For convenience we restate it here.

**Theorem 4.2.** *Let  $M \subseteq \mathbb{R}^n$  be a compact, oriented,  $m$ -dimensional Riemannian manifold with boundary, and assume  $s := n - m > 0$ . Then for all  $\varepsilon > 0$  we have*

$$(16) \quad \text{vol } T(M, \varepsilon) \leq \varepsilon^s \sum_{i=0}^m \frac{1}{s+i} |K_i|(M) \varepsilon^i,$$

where the  $|K_i|(M)$  are the integrals of absolute curvature. For small enough  $\varepsilon$  we have the equality

$$(17) \quad \text{vol } T(M, \varepsilon) = \varepsilon^s \sum_{\substack{i=0 \\ i \text{ even}}}^m \frac{1}{s+i} |K_i|(M) \varepsilon^i.$$

PROOF OF THEOREM 3.1. We prove the first inequality and point out on the way how the equality for small  $\varepsilon$  is obtained. We will first prove the theorem for compact manifolds without boundary, and at the end indicate the necessary modifications for dealing with boundaries.

Consider the surjective map

$$f: S(NM) \times [0, \varepsilon] \rightarrow T(M, \varepsilon) \subseteq \mathbb{R}^n \\ (p, v, t) \mapsto p + tv$$

of compact manifolds with boundary. For  $(p, v) \in S(NM)$  the critical radius is defined as

$$\rho_M(p, v) = \sup\{t \mid \text{dist}(p + tv, M) = t\},$$

and set  $\rho_M = \inf_{(p,v) \in S(NM)} \rho_M(p, v)$ . Then the map  $f$  is injective provided  $\varepsilon \leq \rho_M$ .

By Sard's Theorem the set of critical values of  $f$  has Lebesgue measure zero and the fibers of  $f$  at regular values are finite and locally constant [18, §1]. Given the natural volume form  $\omega_{\mathbb{R}^n}$  on  $\mathbb{R}^n$  we thus have, by (6),

$$(18) \quad \text{vol } T(M, \varepsilon) \leq \int_{p \in T(M, \varepsilon)} \#f^{-1}(p) \omega_{\mathbb{R}^n} = \int_{S(NM) \times (0, \varepsilon)} f^* \omega_{\mathbb{R}^n},$$

with equality if  $\varepsilon \leq \rho_M$ . Recall that we are dealing with unsigned (volume) forms.

The problem reduces to evaluating the right-hand side. We claim that

$$(19) \quad f^* \omega_{\mathbb{R}^n} = t^{s-1} |\det(\text{Id} - tS(v))| \omega_{S(NM)} \wedge dt.$$

Assuming this to hold for the moment, the claimed inequality for the volume of tubes follows by integrating

$$\begin{aligned} \int_{S(NM) \times (0, \varepsilon)} f^* \omega_{\mathbb{R}^n} &= \int_{S(NM)} \left( \int_0^\varepsilon t^{s-1} |\det(\text{Id} - tS(v))| dt \right) \omega_{S(NM)} \\ &\leq \int_{S(NM)} \left( \int_0^\varepsilon t^{s-1} \sum_{i=0}^m t^i |\psi_i(v)| dt \right) \omega_{S(NM)} \\ &= \sum_{i=0}^m \left( \int_0^\varepsilon t^{s-1+i} dt \right) \left( \int_{S(NM)} |\psi_i(v)| \omega_{S(NM)} \right) \\ &= \sum_{i=0}^m \frac{1}{s+i} \varepsilon^{s+i} |K_i|(M). \end{aligned}$$

It therefore remains to prove (19). Note that is  $\varepsilon < \rho_M$ , then the map  $f$  is injective and, with the right choice of orientation, the determinant  $\det(\text{Id} - tS(v))$  is always positive. We can therefore omit the absolute value and obtain an equality with the integrals of curvature.

Let  $(x^1, \dots, x^m): U \rightarrow \mathbb{R}^m$  be orthonormal coordinates on  $U = U' \cap M$ . Let  $(E_1, \dots, E_n)$  be an orthonormal frame field on  $U'$  such that  $E_i := \frac{\partial}{\partial x^i}$  on  $U \subseteq M$  for  $1 \leq i \leq m$ . Set  $\omega_M := E_1^* \wedge \dots \wedge E_m^*$  and  $\omega_N := E_{m+1}^* \wedge \dots \wedge E_n^*$  ( $E_i^*$  denoting the dual of  $E_i$ ). We then have  $\omega_{\mathbb{R}^n} = \omega_M \wedge \omega_N$ , and for the restriction to  $M$ ,  $\omega_M|_{TM} = dx^1 \wedge \dots \wedge dx^m$ .

The frame field also gives a local trivialisation of the sphere bundle

$$U \times S^{s-1} \rightarrow S(NM)$$

$$(p, u) \mapsto \left( p, \sum_{i=1}^s u^i E_{m+i}(p) \right).$$

An orthonormal coordinate system  $y^1, \dots, y^{s-1}$  for  $S^{s-1}$  then gives rise to orthonormal coordinates  $(x^1, \dots, x^m, y^1, \dots, y^{s-1}, t)$  on  $S(NM) \times (0, \varepsilon)$ . Setting  $dx = dx^1 \wedge \dots \wedge dx^m$  and  $dy = dy^1 \wedge \dots \wedge dy^{s-1}$  we have

$$(20) \quad \omega_{S(NM)} \wedge dt = dx \wedge dy \wedge dt.$$

Let  $\phi(p, v, t)$  be such that  $f^* \omega_{\mathbb{R}^n} = \phi(p, v, t) \omega_{S(NM)} \wedge dt$  as differential form. By Equation (20) we obtain

$$\begin{aligned} \phi(p, v, t) &= f^* \omega_{\mathbb{R}^n} \left( \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^{s-1}}, \frac{\partial}{\partial t} \right) \\ &= \omega_{\mathbb{R}^n} \left( f_* \frac{\partial}{\partial x^1}, \dots, f_* \frac{\partial}{\partial x^m}, f_* \frac{\partial}{\partial y^1}, \dots, f_* \frac{\partial}{\partial y^{s-1}}, f_* \frac{\partial}{\partial t} \right). \end{aligned}$$

We next observe that, using the definition of  $f$ ,

$$\begin{aligned} f_* \frac{\partial}{\partial x^i} &= \frac{\partial}{\partial x^i} p + t \sum_{\ell=1}^s u^\ell \frac{\partial}{\partial x^i} E_{m+\ell}(p), \\ f_* \frac{\partial}{\partial y^j} &= t \sum_{\ell=1}^s \frac{\partial}{\partial y^j} u^\ell E_{m+\ell}(p), \\ f_* \frac{\partial}{\partial t} &= v. \end{aligned}$$

In particular,  $f_*(T_v S^{s-1} \times T_t \mathbb{R}) \subseteq N_p M = (T_p M)^\perp$ , so that

$$\phi(p, v, t) = \omega_M \left( \frac{\partial}{\partial x^1} f, \dots, \frac{\partial}{\partial x^m} f \right) \cdot \omega_N \left( \frac{\partial}{\partial y^1} f, \dots, \frac{\partial}{\partial y^{s-1}} f, \frac{\partial}{\partial t} f \right).$$

A straight-forward calculation (using the definitions from Section (2.1)) shows that

$$\left\langle \frac{\partial}{\partial x^i} f, E_j \right\rangle = \left\langle E_i + t \frac{\partial}{\partial x^i} Z, E_j \right\rangle = \delta_{ij} - t S_{ij}(v),$$

where  $Z$  is a normal vector field with  $Z(p) = v$ . From this it follows that

$$\omega_M \left( \frac{\partial}{\partial x^1} f, \dots, \frac{\partial}{\partial x^m} f \right) = \det(\text{Id} - tS(v)).$$

Similarly one obtains

$$\omega_N \left( \frac{\partial}{\partial y^1} f, \dots, \frac{\partial}{\partial y^{s-1}} f, \frac{\partial}{\partial t} f \right) = t^{s-1}.$$



This completes the proof of the claimed inequality in the case of manifolds without boundary.

Now let  $M$  be a manifold with boundary. We can write the tube  $T(M, \varepsilon)$  as a union

$$(21) \quad T(M, \varepsilon) = T_M(\partial M, \varepsilon) \cup T_M(M \setminus \partial M, \varepsilon),$$

where for a subset  $P \subseteq M$  we define

$$T_M(P, \varepsilon) = \bigcup_{p \in P} \{p + tv \mid 0 \leq t \leq \varepsilon, v \in S(N_p M)\}$$

and  $S(N_p M)$  denotes the set of vectors in the normal cone of unit length. Note that  $T_M(\partial M, \varepsilon)$  is *not* the  $\varepsilon$ -tube around  $\partial M$  in  $\mathbb{R}^n$ , since we only look at points that are in the direction of the normal cone. Moreover, for  $\varepsilon$  smaller than the critical radius, the union is disjoint. We therefore have

$$\text{vol } T(M, \varepsilon) \leq \text{vol } T_M(\partial M, \varepsilon) + \text{vol } T_M(M \setminus \partial M, \varepsilon).$$

The volume of  $T_M$  is computed just as in the case of manifolds without boundary, with the difference that in the definition of the invariants  $K_i(M)$  we only integrate over normal vectors in the normal cone to  $p$ .  $\square$

#### REFERENCES

- [1] Robert J. Adler and Jonathan E. Taylor. *Random fields and geometry*. Springer Monographs in Mathematics. Springer, New York, 2007.
- [2] Peter Bürgisser. Smoothed Analysis of Condition Numbers. In *Foundations of Computational Mathematics 2008*. Cambridge University Press, 2008.
- [3] Peter Bürgisser. Smoothed analysis of condition numbers. In *Proceedings of the International Congress of Mathematicians. Volume IV*, pages 2609–2633, New Delhi, 2010. Hindustan Book Agency.
- [4] Peter Bürgisser, Felipe Cucker, and Martin Lotz. The probability that a slightly perturbed numerical analysis problem is difficult. *Math. Comp.*, 77(263):1559–1583, 2008.
- [5] Peter Bürgisser, Felipe Cucker, and Martin Lotz. Coverage processes on spheres and condition numbers for linear programming. 2009.
- [6] Shiing-shen Chern. On the kinematic formula in integral geometry. *J. Math. Mech.*, 16:101–118, 1966.
- [7] James Demmel. On condition numbers and the distance to the nearest ill-posed problem. *Numer. Math.*, 51(3):251–289, 1987.
- [8] James W. Demmel. The probability that a numerical analysis problem is difficult. *Math. Comp.*, 50(182):449–480, 1988.
- [9] Manfredo Perdigão do Carmo. *Riemannian geometry*. Mathematics: Theory & Applications.
- [10] Herbert Federer. Curvature measures. *Trans. Amer. Math. Soc.*, 93:418–491, 1959.
- [11] Alfred Gray. *Tubes*, volume 221 of *Progress in Mathematics*. Second edition.
- [12] Ralph Howard. The kinematic formula in Riemannian homogeneous spaces. *Mem. Amer. Math. Soc.*, 106(509):vi+69, 1993.
- [13] W. Kahan. *Conserving Confluence Curbs Ill-Condition*. Computer Science Dept., University of California, Berkeley, 1972. Technical Report 6.
- [14] Daniel A. Klain and Gian-Carlo Rota. *Introduction to geometric probability*. Lezioni Lincee. [Lincei Lectures]. Cambridge University Press, Cambridge, 1997.
- [15] Eric James Kostlan. *STATISTICAL COMPLEXITY OF NUMERICAL LINEAR ALGEBRA*. ProQuest LLC, Ann Arbor, MI, 1985. Thesis (Ph.D.)—University of California, Berkeley.
- [16] John Milnor. On the Betti numbers of real varieties. *Proc. Amer. Math. Soc.*, 15:275–280, 1964.

- [17] John W. Milnor. *Morse theory*. Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.
- [18] John W. Milnor. *Topology from the differentiable viewpoint*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Based on notes by David W. Weaver, Revised reprint of the 1965 original.
- [19] J. Renegar. On the efficiency of Newton's method in approximating all zeros of a system of complex polynomials. *Math. Oper. Res.*, 12(1):121–148, 1987.
- [20] Luis A. Santaló. *Integral geometry and geometric probability*.
- [21] Rolf Schneider and Wolfgang Weil. *Stochastic and integral geometry*. Probability and its Applications (New York). Springer-Verlag, Berlin, 2008.
- [22] Steve Smale. The fundamental theorem of algebra and complexity theory. *Bull. Amer. Math. Soc. (N.S.)*, 4(1):1–36, 1981.
- [23] Jacob Steiner. Ueber parallele Flächen. In Karl Weierstrass, editor, *Jacob Steiner's gesammelte Werke*, pages 173–176. 1881.
- [24] Shlomo Sternberg. *Lectures on differential geometry*. Second edition.
- [25] Hermann Weyl. On the Volume of Tubes. *Amer. J. Math.*, 61(2):461–472, 1939.

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