Hedging Strategies : Complete and Incomplete Systems of Markets

Papayiannis, Andreas

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HEDGING STRATEGIES: COMPLETE AND INCOMPLETE SYSTEMS OF MARKETS

A dissertation submitted to the University of Manchester for the degree of Master of Science in the Faculty of Humanities

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ANDREAS PAPAYIANNIS

MANCHESTER BUSINESS SCHOOL

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Abstract

We are motivated by the latest statistical facts that weather directly affects about 20% of the U.S. economy and, as a result energy companies experience enormous potential losses due to weather that is colder or warmer than expected for a certain period of a year. Incompleteness and illiquidity of markets renders hedging the exposure using energy as the underlying asset impossible. We attempt to price and hedge a written European call option with an asset that is highly correlated with the underlying asset; still, a significant amount of the total risk cannot be diversified. Yet, our analysis begins by considering hedging in a complete markets system that can be utilised as a theoretical point of reference, relative to which we can assess incompleteness. The Black-Scholes Model is introduced and the Monte Carlo approach is used to investigate the effects of three hedging strategies adopted; Delta hedging, Static hedging and a Stop-Loss strategy. Next, an incomplete system of markets is assumed and the Minimal Variance approach is demonstrated. This approach results in a non-linear PDE for the option price. We use the actuarial standard deviation principle to modify the PDE to account for the unhedgeable risk. Based on the derived PDE, two additional hedging schemes are examined: the Delta hedging and the Stop-loss hedging. We set up a risk-free bond to keep track of any money injected or removed from the portfolios and provide comparisons between the hedging schemes, based on the Profit/Loss distributions and their main statistical features, obtained at expiry.

Declaration

No portion of the work referred to in this dissertation has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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1 Introduction

1.1 Overview

Hedging can be considered as one of the most important investment strategies nowadays. These strategies aim to minimise the exposure to an unwanted business or investment risk, while allowing the business to be able to gain profits from any investment activities. In particular, a hedge is a position that is taken in one market to eliminate or even cancel out the risk associated to some opposite position taken in another market. If an investor decides to hedge a current position, he only protects himself from the effects of a negative event whilst the hedging strategy cannot stop the negative event from occurring; if the event does occur and the investor has hedged his position correctly, the impact is reduced [29].

Hedging techniques can be separated into four main categories [31].

- Direct Hedge: hedging an asset, such as a stock, with an asset that, if not the same, has similar price movements and trades in a similar manner.
- Dynamic Hedge: hedging a contingent claim, usually an option or a future, on an underlying asset by maintaining an offsetting position on the asset and changing its amount, according to certain conditions, as time progresses.
- Static Hedge: it is constructed at the beginning of the life of the claim in such a way that no further readjustment has to be made in the portfolio until expiry.
- Cross Hedge: hedging a contingent claim, in an underlying asset that cannot be traded on the exchange, by another asset that can be traded and it is highly correlated with the original one.

Hedging strategies usually involve the use of financial derivatives, which are securities whose values depend on the values of other underlying securities. The two most common derivative markets are the futures market and the options market. Portfolio managers, corporations and individual investors use financial derivatives to construct trading strategies where a loss in one investment is offset by a gain in a derivative and vice versa.

We denote a perfect hedge to be a hedge that completely eliminates the risk. Nevertheless, perfect hedges are very difficult to be constructed, thus very rare. Therefore, financial analysts are confined to finding ways of constructing hedging strategies so that they perform as close to perfect as possible [29].

Like any other wealth-building techniques, hedging has both advantages and disadvantages. It is important to note that a hedging technique can lead to either an increase or a decrease in a company's profits relative to the position it would hold with no hedging [29]. On the one hand, successful hedging protects the trader against price changes, inflation, currency exchange rate changes, but, on the other hand, every hedging strategy involves a cost and, as a result, an investor has to consider whether the benefits received from the hedge justify the expense [15].

1.2 The Problem

A complete market is one in which every agent is able to exchange every 'good'[1], either directly or indirectly, with every other agent [14].

However, in the real world, markets are usually not complete. Despite this, we begin by considering the complete market case because, as Flood said, "it can serve as a theoretical point of reference, relative to which incompleteness can be assessed" [14].

We will consider as an example a financial institution that has written a European call option, to buy one unit of stock, to a client in over-the-counter markets and is facing the problem of managing the risk exposure [29]. Throughout this project we assume discrete time and discrete time models. We introduce the Black-Scholes-Merton PDE and we price the option using the Black-Scholes model. We set up a riskless bond that will track down any money injected or taken out from the portfolio. We solve the problem by introducing the delta (Δ) and setting up the homonymous hedging strategy and then extend our analysis to other hedging techniques that corporation managers use, such as on Static hedging and on Stop-Loss strategy.

An incomplete market is one in which some payoffs cannot be replicated by trading in the marketed securities [46].

[1] We define 'good' to include the date and the environment in which a commodity is consumed, so that economists are able to consider consumption, production and investment choices in a multi-period world [14].

We are motivated by the latest statistical facts that weather directly affects about 20% of the U.S. economy and, as a result, many companies, such as energy companies, experience enormous potential losses due to weather that is colder or warmer than expected for a certain period of a year [8].

For years, companies have been using insurance to cover any catastrophic damages caused by unexpected weather conditions. However, insurance could not protect energy producers in the case of a reduced demand their company might had faced.

Fortunately, in 1996 the first weather derivatives were introduced in the over the counter markets. Weather derivatives either depend on

HDD: Heating degree days

or

CDD: Cooling degree days.

These two measures are calculated according to the average daily temperature in the following way;

$$HDD = \max(0, 65 - A)$$

$$CDD = \max(0, A - 65)$$

where A is the average of the highest and lowest temperature during a day, measured in degrees Fahrenheit [29].

The first derivatives were used when Aquila Energy created a dual-commodity hedge for Consolidated Edison Co [49]. The agreement provided the purchasing of energy from Aquila in August 1996 but the contract had to be drawn up in such a way that any unexpected weather changes during that month would be compensated. Weather derivatives contracts were soon traded with an $8-billion-a-year industry arising within a couple of years [7].

With these in mind, we will attempt to hedge a European call option on an underlying asset with another asset that is highly correlated with the first one. Nonetheless, due to imperfect correlation between the two assets, an unhedgeable residual risk arises, the basis risk [19].

Many researchers have attempted to extend the theory of complete markets to incomplete markets; among them is Xu who adopted a partial hedging technique that left him with some residual risk at expiration [50].

To price the option, we derive a non-linear PDE by using the modified standard deviation principle in infinitesimal time, according to Wang *et al* [19]. We will construct a best local hedge, one in which the residual risk is orthogonal to the risk which is hedged, by minimising the variance [19]. The actuarial standard deviation principle is then used to determine the price that takes into consideration the residual risk [19]. Based on the derived PDE, two additional hedging schemes are examined: Delta hedging and a Stop-Loss strategy.

Furthermore, we use C++ programming to code up the hedging strategies under consideration and, then, examine them by constructing the Profit/ Loss distributions that arise.

Lastly, having considered the case in both complete and incomplete markets we provide comparisons between the hedging techniques, based on their return distribution and their main statistical features: Value at risk, Conditional value at risk, Mean, Hedge Performance.

1.3 Outline

Chapter 2 is split in three main subchapters;

a. Fundamentals

   We provide the reader with some basic definitions that we will use and refer to in the entire project and explain some of the important statistical characteristics that our conclusion will rely on.

b. Hedging strategies in a complete market

   We introduce the Black-Scholes model and comment on some alternatives processes the asset price can follow. We introduce the Black-Scholes PDE and the option formulas that arise from it. We present the three hedging strategies to be studied and explicitly explain the theory they are based on.

c. Hedging strategies in an incomplete market

We briefly explain the notion incompleteness in markets and present some of the main causes of this phenomenon. Based on the risks that energy producers face, we recall alternative approaches that have been attempted by researchers in order to pricing and hedging such claims and comment on their performance and efficiency. We, then, introduce the hedging strategy used according to Wang et al [19] and derive the linear PDE along with the parameters involved.

Chapter 3 comprises the methodology, explaining all the methods and procedures followed and is divided into two subchapters: one on Complete and another on incomplete markets' systems. The parameters involved in both subchapters are listed in Table 3 while any additional ones, involved in the second subchapter, are listed in Table 4.

The methodology followed in Chapter 3 caused certain results to be deduced. The results yielded are depicted in a variety of forms, such as tables, diagrams and charts, throughout the fourth chapter, along with a commendation on them.

Finally, in Chapter 5, we conclude with an overall discussion and recommendations derived by the results obtained, as well as by presenting some future work and extensions that can be made.

2 Background theory

2.1 Fundamentals

2.1.1 Long and short positions

A long position is a position that involves the purchase of a security such as a stock, commodity, currency and derivative with the expectation that the asset will rise in value [33].

A short position is a position that involves the sale of a borrowed security such as a stock, commodity or currency with the expectation that the asset will fall in value [34]. In the context of derivatives such as options and futures, the short position is also known as the 'written' position.

2.1.2 European call option

A European call option is a financial contract between two parties that gives the holder the right to buy the underlying asset at a certain date at a predetermined price. In such a contract the date is known as the expiry date or maturity and the predetermined price as the strike or exercise price [29]. The holder of the option pays a cost of buying the call option, often called the call premium.

The payoff from the call option is given by

$$C_T = (S_T - K)^+ = \begin{cases} S_T - K & if \;\; S_T \geq K \\ 0 & if \;\; S_T < K \end{cases} \tag{2.1.1}$$

where, S is the underlying asset price,

K is the strike price,

T is the expiry date

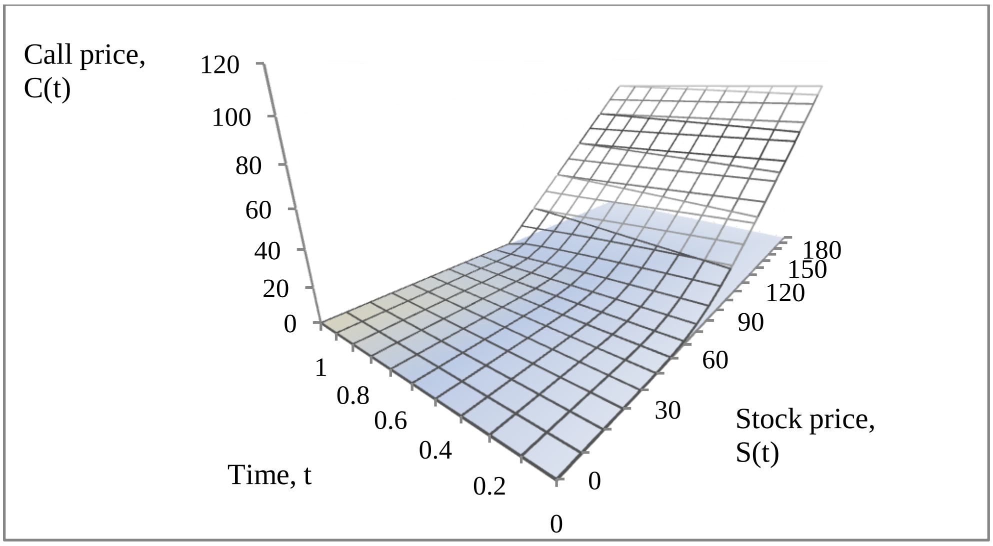

Figure 1: Call option price with T=1 and K=100

Asset prices greatly affect the value of the call option. As the asset price increases, the option becomes more valuable. A call option is always worth more than its expiry value because the holder can either maintain it until maturity or sell it for a price bigger than what he can receive if the option is exercised.

2.1.3 In-the-money, at-the-money, out-of-the money options

There are three possible states where an option can be at any time within its life: in-the-money, at-the-money and out-of-the-money.

An option is said to be in-the-money when the stock price is higher than the strike price. In this case the holder can exercise the option and realise their difference as a profit [29]. In-the-money call options can offer the holder unlimited potential gains, whereas the writer can face unlimited potential costs.

An option is said to be out-of-the-money when the stock price is below the strike price [29]. In this case the holder does not exercise the option and loses the entire call premium he had paid to buy the option at the first place. On the other hand, the writer of the option realises a profit equal to the call premium.

At-the-money options are options when the stock price is equal to the strike price [29].

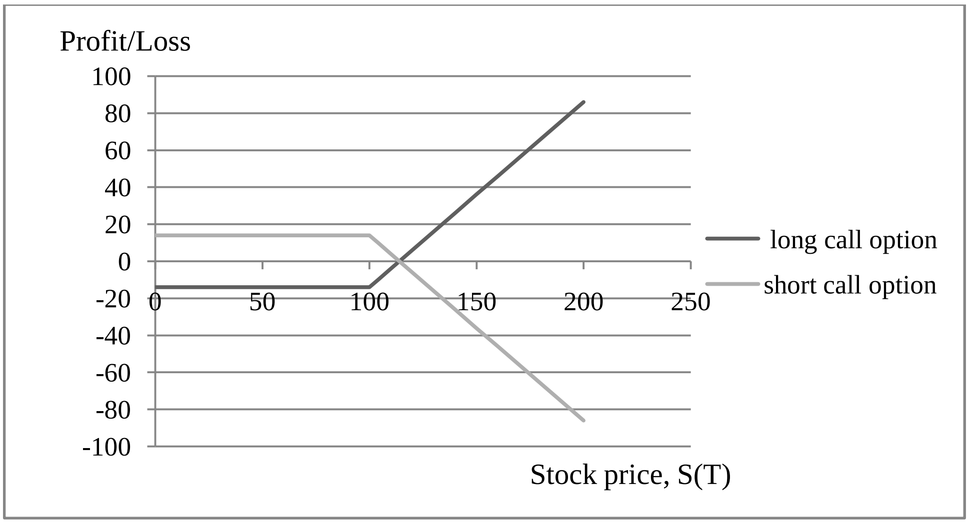

Figure 2: Profit/Loss function of the European call option with K=100

Figure 2 shows that the short party of the option is exposed to unlimited losses if the option closes in-the-money whereas realising a profit equal to the call premium if the option closes out-of-the money. Conversely, the option owner can realise unlimited profits if the option closes in-the-money and faces a loss equal to the call premium if it closes out-of-the-money.

2.1.4 Basis Risk

This is an unhedgeable risk, a market risk for which no suitable instruments are available [22]. Basis risk usually occurs when cross hedging techniques are in use. In particular, it is the risk that the change in the price of a hedge will not be the same as the change in the price of the asset it hedges [43].

2.1.5 Value at risk (VaR) and Conditional Value at Risk (CVaR)

The Value at risk measures the potential loss in value of a portfolio over a defined period of time for a given confidence interval [29]. This consists of three key elements: the specified level of loss in value, the time horizon (N days) and the confidence level (X %). Thus, the Value at risk is the loss that can be made within an N-days time interval with the probability of only (100- X) % of being exceeded. Thus, if the 95% VaR of a portfolio is £50 million in one week's time it means that the portfolio has 95% chance of not losing more than £50 million in any given week (N=7, X=95).

Clearly, VaR focuses on the downside risk and it is frequently used by banks to measure their risk exposures and potential losses that might be facing due to adverse market movements over a certain period [47].

The Conditional value at risk, also known as the expected shortfall, is the expected loss during an N-day period conditional that an outcome in the (100-X) % left tail of the distribution in Figure 3 occurs [29]. It is another measure of risk that becomes more sensitive in the shape of the left tail of the profit/loss distribution. In other words, if things do turn out to be adverse for a company, the CVaR measures the expected magnitude of the loss that the company will face [51].

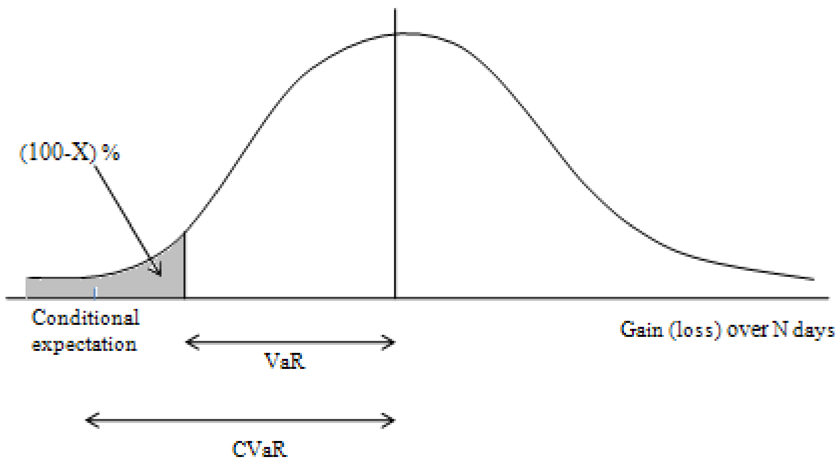

Figure 3: VaR and CVaR from the Profit/Loss distribution of the portfolio [29]

The X % VaR is simply the value at which X % of the portfolio values lie on the right of it and only (100-X) % lie on its left. The X% CVaR is the average of all the values that lie on the left of the X% VaR.

2.1.6 Hedge performance

We define the hedge performance to be the ratio of the standard deviation of the cost of writing and hedging the contingent claim to the theoretical price of the claim [29]. The hedge performance measure serves as an alternative to the relative error. A perfect hedge should have a hedge performance value theoretically equal to zero. We shall define a ‘decent’ hedge as one that should return values close to or zero.

2.2 Hedging strategies in a complete market

A financial market is said to be complete if every contingent claim is attainable. In other words, there is an equilibrium price for every asset in every possible state. As a result, traders can buy insurance contracts to protect and hedge themselves against any future time and state of the world [1].

In this section, we consider the problem of pricing and hedging a contingent claim in the case where the underlying asset can be traded. In particular, we adopt an option pricing method and investigate several hedging techniques that are used by traders and corporation managers and then we provide comparisons, based on their main statistical features, between the most common used ones.

For the next few sections we will consider as an example a financial institution that has written a European call option, to buy one unit of stock, to a client in over-the-counter markets and is facing the problem of managing the risk exposure [29]. We assume that there are no dividend payments on the stock. The parameters used are:

- stock price $S(0)=100$
- strike price $K = 100$
- risk free rate of interest $r = 0.05$
- expiry time $T = 1$ *year*
- the volatility $\sigma = 0.3$

One way of hedging its position is to buy the same option as it has sold, on the exchange [29]. Of course, an identical option might not be available on the exchange. Alternatively, the institution can use the underlying asset itself to maintain a position in such a way that it offsets any profits or losses incurred due to the short call option position.

In this chapter we examine the latter approach.

We assume that asset prices follow geometric Brownian motion, as indicated by the Black-Scholes model.

2.2.1 Alternatives to Black-Scholes model

However, there are several alternative processes that the asset price can follow other than the geometric Brownian motion.

For example, Richard Lu and Yi-Hwa Hsu used the Cox and Ross constant elasticity of variance model (CEV) to price options, which suggests that the stock price change dS has volatility $\sigma S^{\alpha}(t)$ rather than $\sigma S(t)$ [27]. The advantage of this model over the Black-Scholes one is that it can explain features such as the volatility smile.

Moreover, Kremer and Roendfeldt used Merton's jump-diffusion model to price warrants [39]. This model suggests that the stock returns are generated by two stochastic processes; a small continuous price change by a Wiener process and large infrequent price jumps produced by a Poisson process.

Carr *et al* used the variance gamma model that has two additional parameters, drift of the Brownian motion and the volatility of the time change and the show that these parameters control the skewness and the kurtosis of the return distribution [9].

Nevertheless, the advantages that these processes have over the Black-Scholes model are irrelevant[2] to our present study, which is why we will be considering the Black-Scholes model.

2.2.2 Asset prices in the Black-Scholes model

In the Black-Scholes model the risk-free bond value is given by

$$db(t) = rb(t)dt \tag{2.2.1}$$

for all $t \in [0,T]$

$$b(t) = b(0)e^{rt} \tag{2.2.2}$$

with r be the risk free rate of interest [44].

The share price, when assuming a real-world drift, is described by the stochastic differential equation:

$$dS(t) = \mu S(t)dt + \sigma S(t)dZ(t) \tag{2.2.3}$$

for all $t \in [0,T]$

where $\mu \in \Re$ is the mean rate of return, $\sigma \in \Re_+$ is the volatility and $dZ(t)$ is the increment of a Wiener process.

[2] This project aims to compare the different hedging strategies and not to examine which stock price model represents the real price process more accurately.

The above stochastic differential equation can be solved exactly to give:

$$S(t) = S(0)\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + \sigma dZ(t)\right] \tag{2.2.4}$$

for all $t \in [0,T]$

where $S(0)$ is the current share price at $t = 0$.

This indicates that the share price $S(t)$ at time t follows a log-normal distribution, implying that the log- share price changes (returns) obey

$$\log\left(\frac{S(t)}{S(0)}\right) \underline{\underline{low}}\ N\left(\left(\mu - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right) \tag{2.2.5}$$

In order to obtain the expressions that refer to the risk-neutral world we define

$$W(t) = Z(t) + \frac{\mu - r}{\sigma}t \tag{2.2.6}$$

where $W(t)$ is another Wiener process according to Girsanov's theorem [45].

Substituting (2.2.6) to (2.2.3) we get:

$$dS(t) = rS(t)dt + \sigma S(t)dW(t) \tag{2.2.7}$$

This stochastic differential equation can be solved exactly to give:

$$S(t) = S(0)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma dW(t)\right] \tag{2.2.8}$$

or

$$S(t) = S(0)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\Phi\sqrt{T-t}\right] \tag{2.2.9}$$

where Φ is a variable drawn at random from a Normal distribution, N(0,1) [35].

2.2.3 Black-Scholes-Merton differential equation

The model was first introduced in 1973 by Fisher Black, Robert Merton and Myron Scholes and it is widely used in finance to determine fair prices of European options.

The differential equation, as indicated by Hull' [29], suggests that

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0 \qquad (2.2.10)$$

where f is the price of a call option or other derivative contingent on the stock price S,

r is the risk-free interest rate and

σ is the volatility of the stock.

2.2.3.1 Assumptions

- The stock price is described as the solution of the stochastic differential equation $dS = \mu S dS + \sigma S dW$ where μ is called the mean rate of return, σ the volatility and W is a Brownian motion.
- Short selling of securities with full use of proceed is permitted.
- There are no transaction costs or taxes. All securities are perfectly divisible.
- There are no dividends during the life of the derivative.
- There are no riskless arbitrage opportunities.
- Security trading is continuous.
- The risk-free rate of interest, r, is constant and the same for all maturities.

2.2.3.2 Derivation of the Black-Scholes-Merton differential equation [29]

From Ito's lemma it follows that

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2} \right) dt + \frac{\partial f}{\partial S}\sigma S dW \qquad (2.2.11)$$

The discrete versions of the above equations are

$$\Delta S = \mu S \Delta S + \sigma S \Delta W \qquad (2.2.12)$$

and

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2} \right) \Delta t + \frac{\partial f}{\partial S}\sigma S \Delta W \qquad (2.2.13)$$

where Δf and ΔS are the changes in f and S in a small time interval Δt.

Since the stochastic (random) term in the two equations is the same, one can construct a portfolio involving both the derivative and the stock in such a way that the stochastic term is eliminated. The value of the portfolio is

$$\Pi = -f + \frac{df}{dS} S \tag{2.2.14}$$

meaning that the portfolio holder takes a short position in the derivative and a long position in df/dS amount of shares.

By definition, the change in the portfolio in the time interval Δt is

$$\Delta\Pi = -\Delta f + \frac{df}{dS} \Delta S \tag{2.2.15}$$

Substituting equations (2.2.12) and (2.2.13) into (2.2.15) we get

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t \tag{2.2.16}$$

The resulting equation does not include any random terms, thus the portfolio will be riskless during the time interval Δt. In order for arbitrage opportunities to be eliminated we must have that

$$\Delta\Pi = r\Pi\Delta t \tag{2.2.17}$$

Substituting (2.2.14) and (2.2.16) into (2.2.17) we obtain

$$r\left(f - \frac{\partial f}{\partial S} S \right) \Delta t = \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t \tag{2.2.18}$$

and after rearranging we obtain

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0 \tag{2.2.19}$$

2.2.4 Black-Scholes pricing formulas

According to Hull' [29], the formula for the fair price of a European call option on a non-dividend paying stock is considered as a function of the asset price S and time t where all the other parameters are assumed constant. That is;

$$C(S,t) = S_t N(d_1) - Ke^{-r(T-t)} N(d_2) \tag{2.2.20}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{2.2.21}$$

and

$$d_2 = \frac{\ln\left(\frac{S_t}{K}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t} \tag{2.2.22}$$

and

$$N(x) = \frac{1}{2\pi} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy \tag{2.2.23}$$

is the cumulative probability distribution function for a standard normal distribution.

2.2.5 Risk-neutral valuation

In a risk-neutral world all individuals are indifferent to risk. In such a world, the expected return on all securities is the risk-free rate of interest. The Black-Scholes-Merton differential equation does not involve any variables that can change according to the risk preferences of the investors. All the variables that appear in the equation are independent of any risk preferences (the mean rate of return does not appear in the equation). This key property gives rise to the term risk-neutral valuation, meaning that we can assume that the world is risk-neutral when pricing options and consider the resulting prices to be correct when moving back to the real world. In a risk-neutral world, we obtain the current prices of any cash flow by just discounting its expected value at the risk-free rate, a procedure that simplifies the analysis of derivatives [29].

2.2.6 Naked and Covered positions

One strategy that can be followed by the financial institution is to do nothing. This strategy is called a naked position. This means that the institution realises a profit equal to the call premium when the option closes out-of-the-money and faces costs when it closes in-the-money [29].

Alternatively, the financial institution can take a covered position. This means that as soon as the option is sold, the institution takes a long position in the underlying asset. If the call option closes in-the-money the strategy works in favour of the institution; however if it closes out-of-the-money the institution loses with the long position [29].

2.2.7 Stop-Loss strategy

This hedging strategy suggests that a financial institution should buy a unit of stock as soon as the stock price rises above the strike price and should sell it as soon as the stock price drops below the strike price [29]. In this way, the institution succeeds in having a covered position when the stock price is bigger than the strike price and a naked position when it is less than the strike price. This technique ensures that the institution owns the stock when the option closes in-the-money and does not own the stock when the option closes out-of-the-money.

To illustrate this idea I have created an example, based on Hull' [29], where two possible asset price paths are being considered when the strike price is $K = 100$.

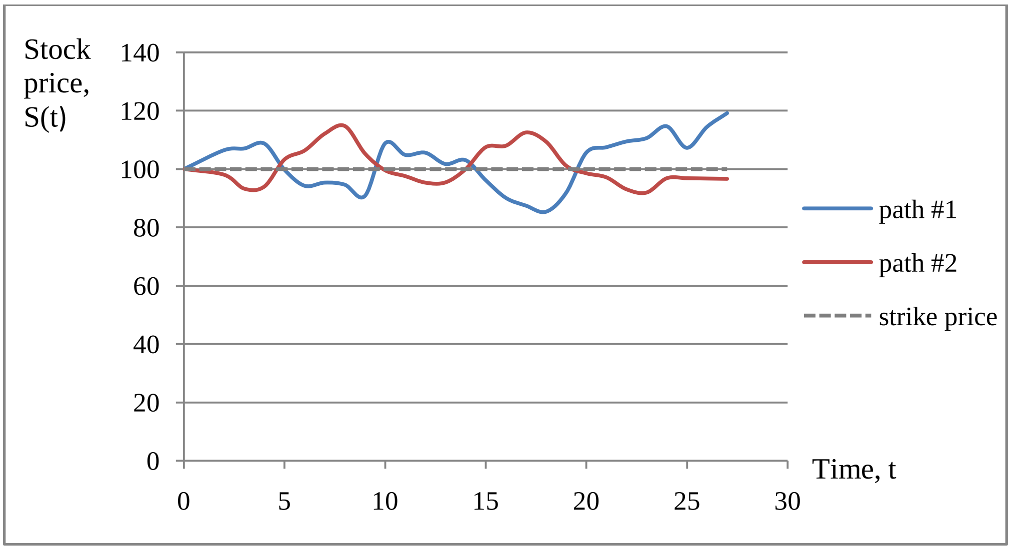

Figure 4: Stop-Loss strategy example

If path #1 is followed by the stock price, the stop-loss strategy involves buying the stock at $t = 0$, selling it at $t = 5$, buying it at $t = 10$, selling it at $t = 15$ and buying it at $t = 20$. The option closes in-the-money and the institution has a covered position.

Alternatively, if the stock price changes according to path #2, the stop-loss strategy involves selling the stock at $t = 0$, buying it at $t = 5$, selling it at $t = 10$, buying it at $t = 15$ and selling it at $t = 20$. The option closes out-of-the-money and the institution has a naked position.

2.2.8 Delta hedging

We introduce the ‘Greeks’, which are variables mostly denoted by Greek letters and they are of great importance in risk management. Each variable represents the sensitivity of the price of a derivative, a call option in our case, to a small change in the value of one of the following parameters: underlying asset, time, volatility, interest rate [48]. Consequently, since the portfolio faces various component risks, these are treated in isolation so that by rebalancing the portfolio accordingly, the desired exposure can be achieved [48].

In this section we introduce the Delta (Δ) of the option. It is the rate of change of the option price with respect to the price of the underlying asset. Particularly, this is the slope of the curve that indicates the relationship between the option and the corresponding stock price [29].

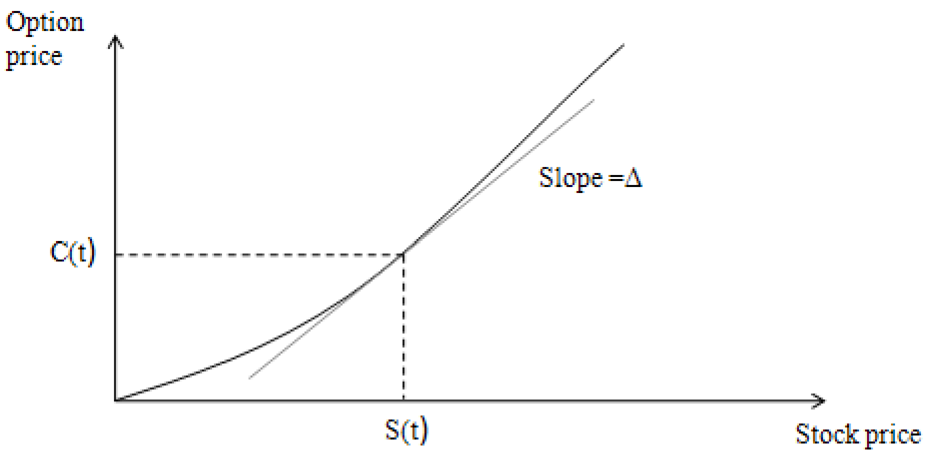

Figure 5: Calculation of Delta [29]

Therefore,

$$\Delta = \frac{\partial C}{\partial S} \tag{2.2.24}$$

The Delta (Δ) always takes values within the range [0, 1] for a European Call option.

The above relation means that if the stock price changes by a small amount then the option price will change by about Δ times that amount. For instance, let Δ=0.4 then if the stock price goes up by 2 pounds then the option price will go up by approximately 0.4 * 2 = 0.8 pounds, and if the stock price drops by 2 pounds then the option price will also drop by about 0.8 pounds.

The delta (Δ) of the European call option on a stock that pays no dividends can be proved to be

$$\Delta = N(d_1) \qquad (2.2.25)$$

where d_1 and $N(x)$ are defined in (2.2.21) and (2.2.23) respectively.

The above formula calculates the delta of a long position in one European call option. Similarly, the delta of a short position in one European call option is $-N(d_1)$.

Returning to our example, the financial institution has written a European call option on one share. Suppose that at the time of the agreement, the delta of the short call position is $-N(d_1)$. In order for the institution to hedge its short position it has to take a long position in $N(d_1)$ shares. The long position in the shares will then tend to offset any gains or losses realised due to the written call position. Thus, the portfolio becomes

$$\Pi = -C + N(d_1)S \qquad (2.2.26)$$

and the delta of the overall position is kept at zero. An investor's position with delta zero is called 'delta neutral'.

The delta of the option changes in a continuous basis because d_1 in the formula of delta depends on many parameters that change every day. As a result, a portfolio stays delta neutral for only a small period of time and thus if the investor wants to continue keeping the portfolio delta hedged he has to adjust his position in shares very frequently [29]. This procedure is referred to as hedge rebalancing.

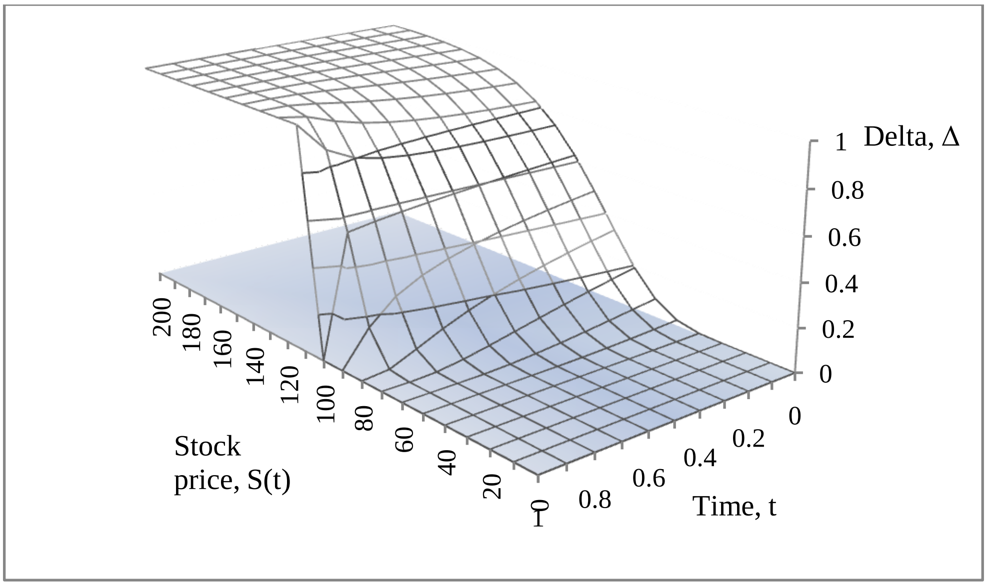

Figure 6: Delta of a call option with T=1 and K=100

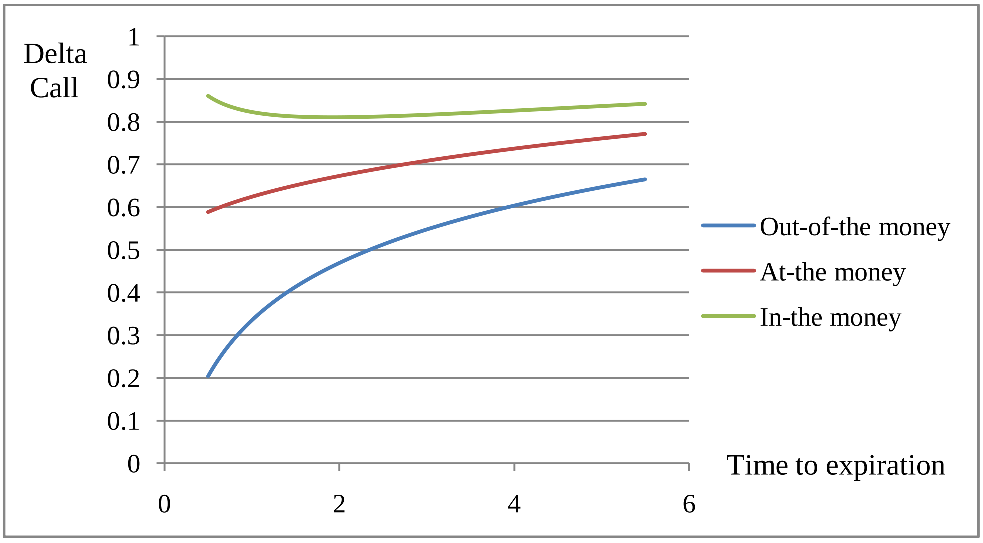

Figure 7: Patterns for variation of delta with time to maturity for a call option

Figure 6 shows that as the asset price increases, the delta approaches unity. As options expire, the delta is either zero or one, depending on whether the asset price closes above or below the strike price. Figure 7 shows the behaviour of the Delta of a call option in the three possible states as the time to expiration increases. As T increases, a significant convergence is observed; however, an out-of-the-money call option declines, as opposed to in-the-money and at-the-money ones.

There are several ways of constructing the delta hedging strategy. Fliess and Join utilise the existence of trends in financial time series in order to propose a model-free setting for delta hedging [13]. Nonetheless, we assume lognormality of prices, as proposed by Black and Scholes, and we shall adjust the long position in the shares according to the new value of delta at each time interval and use a bond to track the cost (in money) that arises for accomplishing this.

The following two tables illustrate the idea of how the delta-hedging procedure works throughout the life of the option, when the hedge is rebalanced every two weeks. Table 1 consists of a European call option that closes in-the-money and Table 2 consists of an identical option that closes out-of-the- money.

Table 1: Call option closes out-of-the-money

week	S(t)	Delta	C(t)	Total cost of purchasing the shares	Total value of the portfolio
0	100	0.62425	14.23135	-48.19382	0
2	93.61149	0.53431	10.21881	-39.86687	-0.06836
4	85.97174	0.41287	6.31482	-29.50342	-0.32310
6	88.05065	0.43913	6.93478	-31.87252	-0.14153
8	94.87611	0.54010	9.98245	-41.51357	-0.25328
10	91.44893	0.48128	7.92091	-36.21394	-0.12267
12	91.61463	0.47802	7.69399	-35.98553	0.11427
14	101.0477	0.62223	12.55882	-50.62674	-0.31056
16	97.04307	0.55695	9.84091	-44.38901	-0.18195
18	95.20838	0.52139	8.50088	-41.08908	0.05081
20	86.88466	0.36355	4.48374	-27.45463	-0.35106
22	84.60318	0.31092	3.43216	-23.05484	-0.18175
24	82.04486	0.25251	2.45487	-18.30630	-0.04430
26	90.10772	0.39469	4.75641	-31.15326	-0.34515
28	91.45834	0.41176	4.94739	-32.77682	-0.06301
30	90.51311	0.38025	4.20821	-29.98524	0.22384
32	83.91198	0.22772	1.88707	-17.24439	-0.02280
34	75.87235	0.08409	0.48079	-6.38006	-0.48052
36	75.57715	0.06590	0.34055	-5.01746	-0.37739
38	73.31207	0.03362	0.14305	-2.66066	-0.33881
40	67.80988	0.00549	0.01641	-0.75812	-0.40229
42	71.36358	0.00764	0.02272	-0.91311	-0.39056
44	75.84479	0.01303	0.03908	-1.32332	-0.37445
46	81.06885	0.02547	0.07701	-2.33501	-0.34686
48	82.96421	0.01551	0.03711	-1.51303	-0.26318
50	85.03356	0.00354	0.00533	-0.49752	-0.20220
52	86.66678	0	0	-0.19206	-0.19206

Week 0: The share price is 100 and the delta of the option is 0.624. Thus, in order for a delta-neutral portfolio the institution has to take a long position in 0.624 of a share. The total cost for the institution at this point is 0.624*100-14.231 = 48.193.

Week 2: The share loses in value and the price drops to 93.611. The delta of the option has also changed to 0.534 meaning that the institution will have to sell (0.624-0.534) = 0.09 of the share to maintain the new long share position and a delta-hedged portfolio. The new cost is the previous cost grown at the risk free rate minus 0.09 * new share price = 39.866.

Week 3-19: As we move on to the end of the life of the option, it becomes clear that the call option will not be exercised and the delta tends to zero.

Week 20: By week 20 the institution has a naked position. The call option closes out-of-the-money and it is not exercised and the total cost of hedging the option is 0.192.

Table 2: Call option closes in-the-money

week	S(t)	Delta	C(t)	Total cost of purchasing the shares	Total value of the portfolio
0	100	0.62425	14.23135	-48.19382	0
2	106.51083	0.70018	18.22772	-56.37453	-0.02482
4	107.08322	0.70597	18.30104	-57.10321	0.19395
6	108.75284	0.72414	19.16039	-59.18929	0.40354
8	107.27921	0.70743	17.76511	-57.51026	0.61773
10	104.44657	0.67218	15.46473	-53.93907	0.80348
12	117.85224	0.81645	25.15282	-71.04573	0.02121
14	111.38956	0.75531	19.71485	-64.37111	0.04769
16	113.83953	0.78311	21.23814	-67.66042	0.25098
18	107.17351	0.70597	15.89201	-59.52262	0.24662
20	98.993538	0.58138	10.22497	-47.30383	0.02427
22	91.48603	0.44038	6.015392	-34.49598	-0.22194
24	101.75657	0.62227	11.1072	-53.07064	-0.85717
26	103.06588	0.64291	11.52368	-55.30026	-0.56101
28	96.20008	0.50999	7.13381	-42.61936	-0.69171
30	101.77110	0.61636	9.84368	-53.52680	-0.64241
32	108.40456	0.73569	13.88097	-66.56596	-0.69421
34	113.37245	0.81524	17.28702	-75.71249	-0.57351
36	105.53189	0.69116	10.86799	-62.76436	-0.69285
38	115.66152	0.86411	18.32653	-82.88908	-1.27031
40	107.51513	0.74374	11.23534	-70.10625	-1.37814
42	109.49432	0.79625	12.20745	-75.99089	-1.01321
44	110.65179	0.83753	12.59813	-80.70543	-0.62858
46	114.68921	0.92682	15.6616	-91.10014	-0.46599
48	107.31511	0.82545	8.570884	-80.39687	-0.38494
50	114.50242	0.99096	14.71535	-99.50316	-0.75139
52	119.20930	1	19.20904	-100.77256	-0.77256

Alternatively, the call option might close in-the-money. Delta hedging technique ensures the institution that it has a fully covered position by week 20. In this scenario, the option is exercised but most of the losses are offset by the gains in the long share position. The total cost of hedging is now 0.772 and the institution gains from placing the hedge the first time.

2.2.9 Static hedging

Delta hedging is an example of dynamic hedging where the portfolio has to be rebalanced frequently in order to be kept hedged.

On the other hand, the institution can adopt a Static hedging strategy. Static hedging involves creating a delta hedged portfolio initially and then never re-adjust this position in the shares throughout the life of the option. As we examine this technique later on we show that it works more or less as a speculative strategy.

2.2.10 Relationship with the Black-Scholes-Merton analysis

Delta hedging is closely related to what Black, Scholes and Merton showed. As we have seen earlier in (2.2.14) it is possible to construct a portfolio in such a way that the random term disappears. In particular, the appropriate portfolio to be used, in the case of a call option, is

$$\Pi = -C + \frac{dC}{dS} S \tag{2.2.27}$$

Recalling (2.2.14) the portfolio becomes

$$\Pi = -C + \Delta S \tag{2.2.28}$$

Therefore, we can say that Black and Scholes valued the derivatives by maintaining a delta hedged portfolio and by using absence of arbitrage they deduced that the portfolio would evolve at the risk free interest rate [29].

Badagnani in one of his articles showed that, by taking discrete steps in the time and the stock price and assuming the Black-Scholes formula, delta-hedging does not lead to a risk free self financing portfolio [3]. However, as we shall see in this project, a delta-hedging strategy leads to a risk free portfolio.

2.3 Hedging strategies in an incomplete market

Incomplete markets are those in which perfect risk transfer is not possible. This means that some payoffs cannot be hedged by the marketed securities [46]. As a result, one cannot construct a perfect hedging portfolio that eliminates all the risk [50], and there is no replication scheme that gives a unique price. Alternatively, a range of prices can be calculated for the actual price of a contingent claim.

There are numerous phenomena that cause the incompleteness of markets. First, the marketed assets are not sufficient compared to the risks a trader wants to hedge, as the risks might involve jumps or volatility of asset prices, or variables that are not derived from the market prices [46]. Incompleteness is can also be caused by transaction costs, particular constraints on the portfolio or other market frictions [46] that we do not discuss in this note.

The above reasons create the need for cross-hedging; hedging a position in one asset by taking an offsetting position in another asset that is highly correlated with the first one; however, this technique offers a decent hedge as long as the asset prices move in the same direction [32].

There have been studied several approaches to the problem of pricing and hedging such options, as the Black-Scholes formula is not appropriate in this case.

El karoui and Quenez studied maximum and minimum prices using stochastic control methods [11], whereas Kramkov [38] and then Follmer and Kabanov [16] proved a supermartingale decomposition of the price process. However, their results have shown that selling an option based on a super-replication price that takes into consideration all the risks associated, often leads to very high option price that no investor would agree to pay [50].

Moreover, Eberlein and Jacob [10] considered this case in pure-jump models and concluded that optimal criteria have to be imposed, whereas Bellamy and Jeanblanc [5] attempted the same using Merton’s jump diffusion models and realised that such models lead to large option price ranges.

Since the former approach cannot be adopted, a trader has to write the option for a sensible price and try to find a partial hedging strategy that will, unfortunately, leave him exposed to some risk in the end [50].

We consider, as an example, the case where energy producers want to hedge their exposure due to unexpected weather changes. The risk associated in this case can be split in two components; the price risk and the volume risk. Thus, when developing a hedging strategy, both components have to be taken into consideration. The price risk can be hedged by energy derivative contracts, futures, swaps, options, in the electricity derivatives market and the volume risk using weather options, i.e. the 'underlying' asset is a weather index [29].

We, briefly, discuss the two major approaches to the problem, as these appear in Xu's report [50].

One is to choose a particular martingale measure for pricing according to some optimal criterion. Unlike pricing in complete markets, incompleteness leads to infinitely many martingale measures each of which produces a no-arbitrage price. An incomplete list of references is: Fritelli, derived the martingale measure that minimises the relative entropy and proceeded in characterising its density [21]. Follmer and Schweizer introduced a minimal martingale measure [17], and Miyahara the minimal entropy martingale measure [41] for the option price based on the assumption that the prices follow a geometric Levy process. These processes provided a 'decent' hedging; however, they are impracticable as problems emerge when constructing the option price process, for these have to be based on the stochastic calculus of the Levy process. Furthermore, Goll and Ruschendorf [23] introduced the minimal distance martingale measure and its relationship with the minimax measure with respect to utility functions, as well as Bellini and Fritelli the minimax measure [6]. Once again, these methods provided a 'decent' hedging; nonetheless, these are not easily constructed in practice, because, as it turned out, a sufficient condition for the existence of such a measure must be imposed.

The other approach is to base the pricing of the derivatives according to the utility. This means that the security is priced in such a way that the utility remains the same whether the optimal trading portfolio includes a marginal amount of the security or not [50]. Fritelli [20], Rouge and El karoui [12], Henderson [25], Hugonnier et al [28] and Henderson and Hobson [26] attempted this approach. Although, these methods provide a good hedging strategy, Xu [50] argued that "in practice, it is quite unusual for the trader to explicitly write down her utility function for derivative pricing".

In this project, we use the cross hedging technique to hedge a European call option on a non-traded asset. However, this will result in an unhedgeable residual risk, the basis risk [19].

Alfredo Ibanez has shown that a European style option can be decomposed in two components; a robust component which is priced by arbitrage and another component that depends on a risk orthogonal to the traded securities [30]. As a result, the first component represents the unique price of the hedging portfolio and the second component that depends only on the risk premium associated with the basis risk.

Furthermore, Wang *et al* hedge such a contingent claim, and extent the case to other portfolios, by adjusting the hedging portfolio to reflect a risk premium due to the basis risk. Particularly, their aim is to construct a best local hedge, a hedge in which the residual risk is orthogonal to the risk which is hedged [19].

We will follow these ideas to construct a hedging scheme which minimises the variance of the resulting portfolio values and, then, compare its performance with other hedging strategies, such as the delta hedging and the stop-loss strategy. We will attempt to price the option in a way that it reflects all the risks associated with the short position. This methodology gives rise to a linear PDE that can be evaluated to obtain the option premium.

After solving the PDE, we hedge the option and construct a profit/loss distribution for the portfolio. The resulting distribution can then be used to calculate the mean, variance, hedge performance, value at risk (VaR) and the conditional value at risk (CVaR).

2.3.1 The PDE - Minimal Variance approach

Following Wang *et al* we suppose that we can trade a correlated asset H that follows a stochastic process

$$dH = \mu' H(t)dt + \sigma' H(t)dW(t) \tag{2.3.1}$$

where $dW(t)$ is the increment of a Wiener process [19].

The properties of the Wiener processes, $dW(t)$ and $dZ(t)$ are:

$dW(t)dW(t) = dt$, $dZ(t)dZ(t) = dt$, $dW(t)dZ(t) = \rho dt$

where ρ denotes the correlation of $dW(t)$ with $dZ(t)$.

Construct the portfolio

$$\Pi = -V(t) + xH(t) + B(t) \tag{2.3.2}$$

where x is the number of units of H held in the portfolio and B is a risk free bond.

We set

$$B(0) = V(0) - xH(0) \tag{2.3.3}$$

so that the portfolio at the beginning is

$$\Pi(0) = 0 \tag{2.3.4}$$

The change in the portfolio value is given by

$$d\Pi = -\left[V_t + \mu S V_s + \frac{\sigma^2 S^2}{2} V_{ss}\right]dt - \sigma S V_s dZ + r(V - xH)dt + x(\mu' H dt + \sigma' H dW)$$

$$= -\left[V_t + \mu S V_s + \frac{\sigma^2 S^2}{2} V_{ss} + r(V - xH) - x\mu' H\right]dt - \sigma S V_s dZ + x\sigma' H dW \tag{2.3.5}$$

The variance of $d\Pi$ is given by

$$E^P[(x\sigma' H dW - \sigma S V_s dZ)^2] = [x^2(\sigma')^2 H^2 + \sigma^2 S^2 V_s^2 - 2\sigma S V_s x \sigma' H \rho]dt \tag{2.3.6}$$

where E^P is the expectation under the P probability measure.

We, then, choose x to minimise the equation (2.3.6)

$$x = \left(\frac{S\sigma}{H\sigma'}\rho\right)V_s \tag{2.3.7}$$

Substituting equation (2.3.7) into equation (2.3.5) we get

$$d\Pi = -\left[V_t + \mu S V_s + \frac{\sigma^2 S^2}{2} V_{ss} + rV + \left(\frac{rS\sigma\rho}{\sigma'}\right)V_s - \left(\frac{S\sigma\rho\mu'}{\sigma'}\right)V_s\right]dt - \sigma S V_s dZ + \sigma S V_s \rho dW \tag{2.3.8}$$

Defining

$$r' = \mu - (\mu' - r)\frac{\sigma\rho}{\sigma'} \tag{2.3.9}$$

in equation (2.3.8) gives

$$d\Pi = -\left[V_t - r' S V_s + \frac{\sigma^2 S^2}{2} V_{ss} - rV\right]dt + \sigma S V_s(\rho dW - dZ) \tag{2.3.10}$$

To eliminate any arbitrage opportunities we require $r' \to r$ as $|\rho| \to 1$. Substituting (2.3.7) into (2.3.6) we get

$$\text{var}(d\Pi) = (1-\rho^2)\sigma^2 V_s^2 S^2 dt \tag{2.3.11}$$

We note that $[\rho dW - dZ, dW] = 0$, we obtain $\text{cov}[d\Pi, dW] = 0$ indicating that the residual risk is orthogonal to the hedging instrument.

Define a new Brownian increment

$$dX(t) = \frac{1}{\sqrt{1-\rho^2}}[\rho dW(t) - dZ(t)] \tag{2.3.12}$$

with the property $dX(t)dX(t) = dt$. Thus, according to Wang *et al* [19], the equation can be written as

$$d\Pi = -\left[V_t - r' SV_s + \frac{\sigma^2 S^2}{2} V_{ss} - rV\right]dt + \sigma SV_s \sqrt{1-\rho^2}\, dX \tag{2.3.13}$$

By requiring that the portfolio is mean self-financing

$$E^P[d\Pi] = 0 \tag{2.3.14}$$

we obtain the linear PDE

$$V_t + r' SV_s - rV + \frac{\sigma^2 S^2}{2} V_{ss} = 0 \tag{2.3.15}$$

2.3.2 PDE Solution – Option Price formula

The solution to equation (2.3.15) is the required European call option price.

Thus, in order to solve it we define

$$r' = r - q \tag{2.3.16}$$

where q is a constant, $q \in \Re$.

Substitute (2.3.16) into (2.3.15) to get

$$V_t + (r-q)SV_s - rV + \frac{\sigma^2 S^2}{2} V_{ss} = 0 \tag{2.3.17}$$

We observe that (2.3.17) is similar to the Black-Scholes PDE that accounts for dividends; although, q should not be misinterpreted as a dividend yield, the solution for this PDE can be calculated in a similar manner using the corresponding Black-Scholes formulas. That is;

$$C(S,t) = S_t e^{-q(T-t)} N(d_1) - Ke^{-r(T-t)} N(d_2) \tag{2.3.18}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{2.3.19}$$

and

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r - q - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t} \tag{2.3.20}$$

2.3.3 Derivation of parameters

Recall by (2.3.9) that

$$r' = \mu - (\mu' - r)\frac{\sigma\rho}{\sigma'} \tag{2.3.21}$$

which implies

$$\frac{\mu - r}{\sigma} = \frac{\rho(\mu' - r)}{\sigma'} \tag{2.3.22}$$

if $|\rho| = 1$ and $r' = r$ we obtain

$$\mu' = r + (\mu - r)\frac{\sigma'}{\sigma} \tag{2.3.23}$$

and if $|\rho| = -1$ and $r' = r$ we obtain

$$\mu' = r - (\mu - r)\frac{\sigma'}{\sigma} \tag{2.3.24}$$

The above suggest that μ' can be intepreted as

$$\mu' = r + f(\rho)(\mu - r)\frac{\sigma'}{\sigma} \tag{2.3.25}$$

with $f(1) = 1$ and $f(-1) = -1$.

Substituting (2.3.25) in (2.3.21) and using $f(\rho)=\rho$, for simplicity, we conclude that

$$r' = (1-\rho^2)\mu + \rho^2 r \tag{2.3.26}$$

We can now evaluate q by substituting (2.3.26) into (2.3.16), and obtain

$$q = (r-\mu)(1-\rho^2) \tag{2.3.27}$$

2.3.4 Risk loading

Our aim is to modify (2.3.18) to account for the basis risk. In particular, we want to add a term to (2.3.27) that will reflect all the associated risks to the short position.

For pricing in the incomplete market, we proceed according to Wang *et al* [19], and thus we use the actuarial standard deviation principle in infinitesimal time. This is

$$E^P[d\Pi] = \lambda\sqrt{\frac{\mathrm{var}[d\Pi]}{dt}}dt \tag{2.3.28}$$

where λ is the risk loading parameter, which has units of $(time)^{-\frac{1}{2}}$. This means that there should be a premium earned in the portfolio during any time interval $[t, t+dt]$ that is proportional to the standard deviation on the particular interval.

From (2.3.11) we have

$$\sqrt{\frac{\mathrm{var}[d\Pi]}{dt}} = \sigma S|V_S|\sqrt{1-\rho^2} \tag{2.3.29}$$

Combining (2.3.13, 2.3.28, 2.3.29) we get the non-linear PDE

$$V_t + r'SV_s + S|V_s|\lambda\sigma\sqrt{1-\rho^2} + \frac{\sigma^2 S^2}{2}V_{ss} - rV = 0 \tag{2.3.30}$$

Therefore, the PDE[3] for the short option position is given by:

$$V_t + [r' + \lambda\sigma\sqrt{1-\rho^2}\,\mathrm{sgn}(V_s)]SV_s + \frac{\sigma^2 S^2}{2}V_{ss} - rV = 0 \tag{2.3.31}$$

And, similarly, the PDE for the long option position is given by:

[3] As long as the S derivative does not change sign, the PDE will be linear. In fact, this is the case when considering a European Call option.

$$V_t + [r' - \lambda\sigma\sqrt{1-\rho^2}\,\mathrm{sgn}(V_s)]SV_s + \frac{\sigma^2 S^2}{2} V_{ss} - rV = 0 \tag{2.3.32}$$

Therefore, using similar arguments when deriving (2.3.16) and then (2.3.27) we conclude that

$$q = (r - \mu)(1 - \rho^2) - \lambda\sigma\sqrt{1-\rho^2}\,\mathrm{sgn}(V_s) \tag{2.3.33}$$

Then, the derived PDE can also be used for pricing the option when delta hedging and stop-loss techniques are applied.

3 Methodology

Monte-Carlo simulation is used to simulate paths that can be taken by the asset. This is a forward induction method; however its convergence is slow and, as a result, it makes it very difficult to determine the error terms. In particular, the convergence to the correct value will be at a rate $N^{-1/2}$, where N the number of sample paths. As the computation takes very long to compile for a large number of simulations, our results will be based on only two thousand simulations.

For each price path, the call price is calculated at expiry and the final value of the portfolio is obtained. In order to achieve this, we first generate random numbers, which are uniformly distributed on [0, 1], by using the Mersenne Twister random number generator according to Bedaux [4]. We, then, use the Box-Muller method to transform them in normally distributed numbers. In particular, if x_1 and x_2 are two uniformly distributed numbers, the Box-Muller method suggests that

$$\phi_1 = \cos(2\pi x_2)\sqrt{-2\log(x_1)} \tag{3.0.1}$$

and

$$\phi_2 = \sin(2\pi x_1)\sqrt{-2\log(x_2)} \tag{3.0.2}$$

are two normally distributed numbers [36].

We approximate the normal distribution by using the trapezium rule.

We define the constant *'steps'* to indicate the number of discrete time steps we will be looking at, within the life of the option. In other words, *'steps'* will account for the number of times the hedge is rebalanced. Then, we define

$$dt = \frac{T}{steps} \tag{3.0.3}$$

the time increment, so that, the time grid becomes

$$0, dt, 2dt, \ldots\ldots, T-2dt, T-dt, T \tag{3.0.4}$$

3.1 Hedging a contingent claim with the underlying asset

Table 3: Parameters Involved

K=100	Strike price
S(0)=100	Stock price is initially set equal to the strike price so that an at-the-money written call option is considered and thus, allowing for hedging is sensible.
r=0.05	Risk-free rate of interest
σ=0.3	Volatility is examined for a range of values 0.1<σ<0.6
μ=0.1	Mean rate of return does not appear in the equation when assuming a risk-neutral world, still, in real world, it is assumed to take either positive or negative values
T=1	Maturity time is set to 1 year
steps=252	We initially assume that a year has 252 trading days. Daily hedging implies 252 steps, weekly hedging implies 52 steps etc.

3.1.1 Delta hedging

We construct the portfolio

$$\Pi(t) = -V(t) + \Delta(t)S(t) + B(t) \tag{3.1.1}$$

where

$$\Delta(t) = N[d_1(T-t, S(t))] \tag{3.1.2}$$

is the number of units of S held in the portfolio, B is the risk-free bond and V the value of the call option according to the Black-Scholes formula (2.2.20).

Initially, $t = 0$, we set

$$B(0) = V(0) - \Delta(0)S(0) \tag{3.1.3}$$

so that the portfolio is

$$\Pi(0) = 0 \tag{3.1.4}$$

For each hedging time we calculate the share price by

$$S(t+dt) = S(t)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)dt + \sigma\phi_1 dt\right] \qquad (3.1.5)$$

the option price V from (2.2.20) and the delta (Δ) of the option from (2.2.29).

We update the value of the portfolio by buying $\Delta(t+dt) - \Delta(t)$ shares. This will then update the value of the bond to

$$B(t+dt) = B(t)e^{rdt} - S(t+dt)[\Delta(t+dt) - \Delta(t)] \qquad (3.1.6)$$

The procedure proceeds to maturity $t = T$, at which the obtained final value of the portfolio reflects any profits or losses incurred due to the hedging.

3.1.2 Static hedging

We follow an identical procedure to the delta hedging one, but only at initial time. After that, we do not update the portfolio at any time within the life of the option.

We construct the portfolio

$$\Pi(t) = -V(t) + \Delta(0)S(t) + B(t) \qquad (3.1.7)$$

where

$$\Delta(t) = N[d_1(T-t, S(t))] \qquad (3.1.8)$$

is the number of units of S held in the portfolio, B is the risk-free bond and V the value of the call option according to the Black-Scholes formula (2.2.20).

Initially, $t = 0$, we set

$$B(0) = V(0) - \Delta(0)S(0) \qquad (3.1.9)$$

so that the portfolio is

$$\Pi(0) = 0 \qquad (3.1.10)$$

For each hedging time we calculate the share price by

$$S(t+dt) = S(t)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)dt + \sigma\phi_1 dt\right] \qquad (3.1.11)$$

the option price V from (2.2.20) and the delta (Δ) of the option from (2.2.29).

We do not readjust the long share position at any time interval. Still, the value of the bond grows at the predetermined risk-free rate of interest

$$B(t+dt)=B(t)e^{rdt} \tag{3.1.12}$$

and, therefore, a new value of the portfolio is obtained.

The procedure proceeds to maturity $t=T$, at which the obtained final value of the portfolio reflects any profits or losses incurred due to the hedging.

3.1.3 Stop-loss hedging

We construct the portfolio

$$\Pi(t)=-V(t)+fS(t)+B(t) \tag{3.1.13}$$

where

$$f=\begin{cases}1 \ \ if \ \ S(t)\geq K\\ 0 \ \ if \ \ S(t)<K\end{cases} \tag{3.1.14}$$

indicates the amount of units of S held.

Initially, $t=0$, we set

$$B(0)=V(0)-S(0) \tag{3.1.15}$$

and $f=1$ so that the portfolio is

$$\Pi(0)=0 \tag{3.1.16}$$

For each hedging time the share and the option prices are calculated using the above mentioned formulas and the portfolio is updated by either maintaining the long share position or closing it out. The share price, that arises each time, indicates whether the long position in shares is maintained (*f=1*) or closed out (*f=0*).

The values of the bond and f change in the following manner.

If $S(t+dt) \geq K$

$$f = 1;$$

If $S(t) < K$ then $B(t+dt) = B(t)e^{rdt} - S(t+dt)$ else $B(t+dt) = B(t)e^{rdt}$.

whereas

If $S(t+dt) < K$

$$f = 0;$$

If $S(t) < K$ then $B(t+dt) = B(t)e^{rdt}$ else $B(t+dt) = B(t)e^{rdt} + S(t+dt)$.

The procedure continues until maturity $t = T$, at which the obtained final value of the portfolio reflects any profits or losses incurred due to the hedging.

3.2 Hedging a contingent claim with a correlated asset

We recall that in this case the asset S cannot be traded and we assume a real world. We recall by (2.3.1) that H is a traded asset highly correlated with S. Therefore, some additional parameters are involved.

Table 4: Additional Parameters Involved

H(0)=100	When S cannot be traded, the correlated asset H is initially set equal to S for simplicity
σ'=0.3	Volatility of H is examined for a range of values 0.1<σ'<0.6
μ'=0.1	The drift rate of H depends on the drift rate of S as indicated by (2.3.24)
ρ=0.5	The correlation between weather and oil, in real world, lies around 0.7, thus it is reasonable to consider a range of values 0.5<ρ<1.0
λ=0.5	The risk loading parameter examined is in the range 0.0<λ<1.5, so that the behaviour of the VaR can be investigated
q	$q = (r-\mu)(1-\rho^2) - \lambda\sigma\sqrt{1-\rho^2}$

3.2.1 Minimal Variance hedging

The asset price S is given by

$$S(t+dt) = S(t)\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma\phi_1 dt\right] \tag{3.2.1}$$

The price of the correlated asset H is given by

$$H(t+dt) = H(t)\exp\left[\left(\mu' - \frac{1}{2}(\sigma')^2\right)dt + (\sigma'\rho\phi_1\sqrt{dt}) + (\sigma'\phi_2\sqrt{dt})\sqrt{1-\rho^2}\right] \tag{3.2.2}$$

We construct the portfolio

$$\Pi(t) = -V(t) + x(t)H(t) + B(t) \tag{3.2.3}$$

where

$$x(t)=\left(\frac{S(t)\sigma}{H(t)\sigma'}\rho\right)N[d_1(T-t,S(t))] \qquad (3.2.4)$$

is the number of units of H held in the portfolio, B is the risk-free bond and V the value of the call option according to the derived formula in (2.3.17).

Initially $t=0$, we set

$$B(0)=V(0)-x(0)H(0) \qquad (3.2.5)$$

so that the portfolio is

$$\Pi(0)=0 \qquad (3.2.6)$$

We update the value of the portfolio by buying $x(t+dt)-x(t)$ of the asset H . This will then update the value of the bond to

$$B(t+dt)=B(t)e^{rdt}-H(t+dt)[x(t+dt)-x(t)] \qquad (3.2.7)$$

The procedure proceeds to maturity $t=T$, at which the obtained final value of the portfolio reflects any profits or losses incurred due to the hedging.

3.2.2 Delta hedging

The asset price S is given by

$$S(t+dt)=S(t)\exp\left[\left(\mu-\frac{1}{2}\sigma^2\right)dt+\sigma\phi_1 dt\right] \qquad (3.2.8)$$

The price of the correlated asset H is given by

$$H(t+dt)=H(t)\exp\left[\left(\mu'-\frac{1}{2}(\sigma')^2\right)dt+(\sigma'\rho\phi_1\sqrt{dt})+(\sigma'\phi_2\sqrt{dt})\sqrt{1-\rho^2}\right] \qquad (3.2.9)$$

We construct the portfolio

$$\Pi(t)=-V(t)+\Delta(t)H(t)+B(t) \qquad (3.2.10)$$

where

$$\Delta(t)=N[d_1(T-t,S(t))] \qquad (3.2.11)$$

is the number of units of H held in the portfolio, B is the risk-free bond and V the value of the call option according to the derived formula in (2.3.17).

Initially $t = 0$, we set

$$B(0) = V(0) - \Delta(0)H(0) \tag{3.2.12}$$

so that the portfolio is

$$\Pi(0) = 0 \tag{3.2.13}$$

We update the value of the portfolio by buying $x(t+dt) - x(t)$ of the asset H. This will then update the value of the bond to

$$B(t+dt) = B(t)e^{rdt} - H(t+dt)[\Delta(t+dt) - \Delta(t)] \tag{3.2.14}$$

The procedure continues until maturity $t = T$, at which the obtained final value of the portfolio reflects any profits or losses incurred due to the hedging.

Due to the structure of the code in the case of a perfect positive correlation between the two assets, the Delta and Minimal Variance give the same result. Discussion about what causes this follows in the next chapter.

3.2.3 Stop-loss hedging

The method used for this type of hedging is similar to what it has been used in the complete market case with the only difference that the traded asset is the correlated asset H and not S. In other words, the value of the call option still depends only on S; however, any readjustments to the portfolio are achieved using the new value of the correlated asset H which can be traded.

We construct the portfolio

$$\Pi(t) = -V(t) + fH(t) + B(t) \tag{3.2.15}$$

where

$$f = \begin{cases} 1 \ \ if \ \ S(t) \geq K \\ 0 \ \ if \ \ S(t) < K \end{cases} \tag{3.2.16}$$

indicates the amount of units of H held.

Initially, $t = 0$, we set

$$B(0) = V(0) - H(0) \qquad (3.2.17)$$

and $f = 1$ so that the portfolio is

$$\Pi(0) = 0 \qquad (3.2.18)$$

For each hedging time the asset and the option prices are calculated using the above mentioned formulas and the portfolio is updated by either maintaining the long position in H or closing it out. The asset price, that arises each time, indicates whether the long position in H is maintained (*f=1*) or closed out (*f=0*).

The values of the bond and f change in the following manner.

If $S(t+dt) \geq K$

$$f = 1;$$

If $S(t) < K$ then $B(t+dt) = B(t)e^{rdt} - H(t+dt)$ else $B(t+dt) = B(t)e^{rdt}$.

whereas

If $S(t+dt) < K$

$$f = 0;$$

If $S(t) < K$ then $B(t+dt) = B(t)e^{rdt}$ else $B(t+dt) = B(t)e^{rdt} + H(t+dt)$.

The procedure proceeds to maturity $t = T$, at which the obtained final value of the portfolio reflects any profits or losses incurred due to the hedging.

4 Results

4.1 Hedging with the underlying asset: Complete market case

We begin by looking at the results obtained in the Complete market case after considering the three hedging strategies Delta hedging, Static hedging and the Stop-Loss hedging. The results, unless otherwise stated, are based on the parameters as these appear in Table 3.

4.1.1 Three Hedging Schemes Results

Table 5: Three Hedging Strategies results

Strategy	Days Between Hedge Rebalancing	Mean	Standard Deviation	Hedge Performance	95% VaR	95% CVaR
Stop-Loss	1	0.62837	9.41642	0.66167	16.93423	23.34122
Static	1	-0.33334	9.49293	0.66704	17.44834	24.68128
Delta	1	0.01681	0.55304	0.03886	0.89650	1.27920

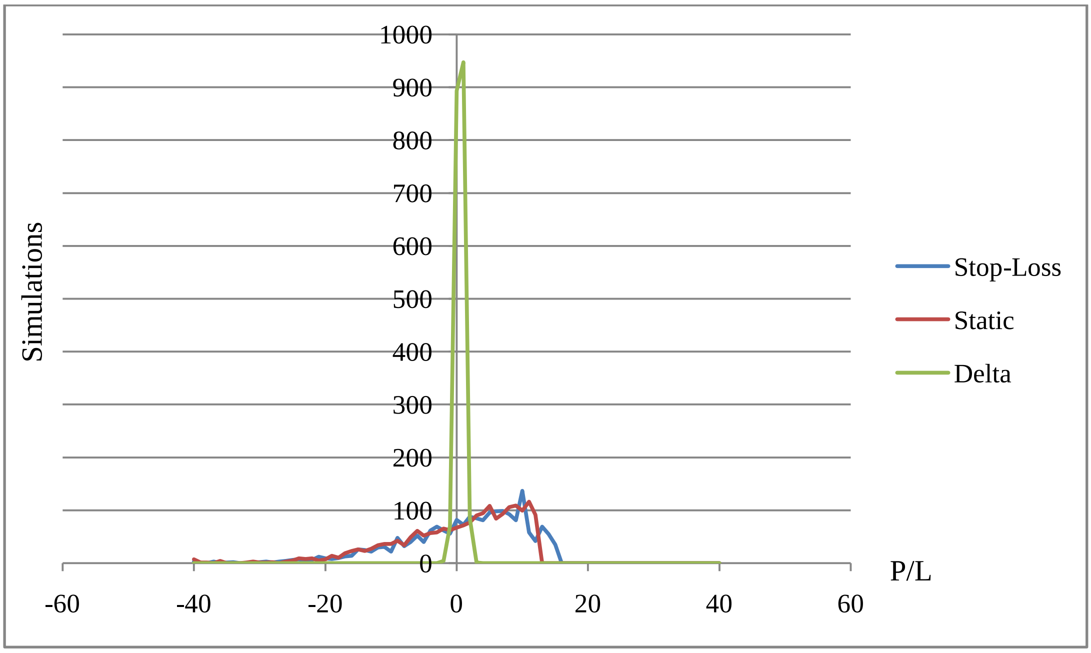

Figure 8: Profit/Loss Distributions

Table 3 along with Figure 8 presents the Profit/Loss Distributions and the statistics obtained after performing the three strategies with the hedge being rebalanced on a daily basis.

A stop-loss strategy might seem to work quite well in theory; however, a closer look at its key statistical features reveals that this is not the case. The standard deviation is very high and despite the fact that the hedge is rebalanced every trading day, its performance cannot drop more than 0.6.

The static hedging, on the other hand, maintains a delta hedged portfolio for just one day. After that, the short option position is exposed to any fluctuations of the share price. Indeed, looking at some of its statistical features, it does not work very well. The downside risk is huge and the standard deviation very high. The resulting portfolio's mean is -0.333 (negative), indicating that the cost of hedging the option is more expensive than its Black-Scholes price. This strategy can be considered as a gambling strategy since the portfolio is fully exposed to risks.

The delta hedging strategy manages to minimise the mean and standard deviation and looking at its hedge performance we observe that the hedge seems to work almost perfectly. Finite hedge rebalancing interval causes the performance not to be exactly perfect.

Clearly, a Delta hedging strategy is a huge improvement over the other two strategies; nevertheless, its immense effectiveness gives the false impression that the Static and Stop-Loss schemes do not work at all [see Figure 8].

Being based on the slope between the option change and the stock price change implies that the change in the long stock position made at time $t + dt$ is very small compared to the change made at time t. In other words, delta hedging does not involve changing from a fully naked position to a fully covered position every time the stock price crosses the strike price. Any profits or losses tend to vanish and the strategy leads to a risk free portfolio.

4.1.2 Delta Hedging Analysis

Since the Delta hedging proved to be more effective than the Static and the Stop-Loss Strategies, we proceed to investigate the efficiency of the Delta hedging scheme by examining it using varying hedge rebalancing intervals and several initial stock prices. Then, we assume a Real world drift and comparisons are made between this and the Risk-neutral world. Lastly, we use real historical data of Logica shares to illustrate the efficiency of the method in practice.

Table 6: Delta Hedging simulations

Days Between Hedge Rebalancing	Mean	Standard Deviation	Hedge Performance	95% VaR	95% CVaR
14	-0.04766	1.99763	0.14021	3.29265	4.61731
7	-0.01256	1.40325	0.09859	2.31706	3.16587
2	0.02709	0.74933	0.05252	1.19326	1.60500
1	0.01681	0.55304	0.03886	0.89650	1.27920

Table 6 shows the Delta hedging statistics obtained when the hedging interval varies from 1 to 14 trading days. As the hedge is monitored more frequently, the delta hedging strategy gets progressively better. The standard deviation drops down to 0.553 indicating that the portfolio return values are narrowly spread around the mean. We note that although the mean is close to zero at all cases, it is never exactly zero because of the finite rebalancing interval.

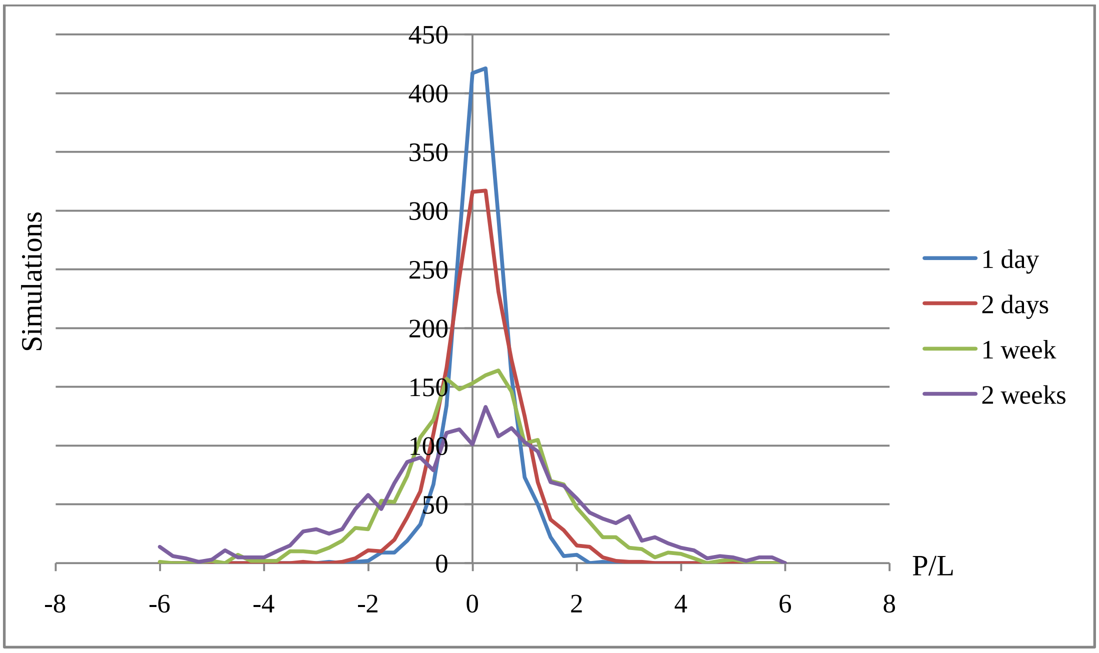

Figure 9: Profit/Loss Distributions

Figure 9 shows the resulting Profit/Loss Distributions obtained when the hedge rebalancing interval decreases. Improvement of the Profit/Loss distribution is spotted. We observe that any profits or losses tend to vanish, as the hedge position is readjusted more often. The curve becomes much smoother and its mean point goes very close to zero.

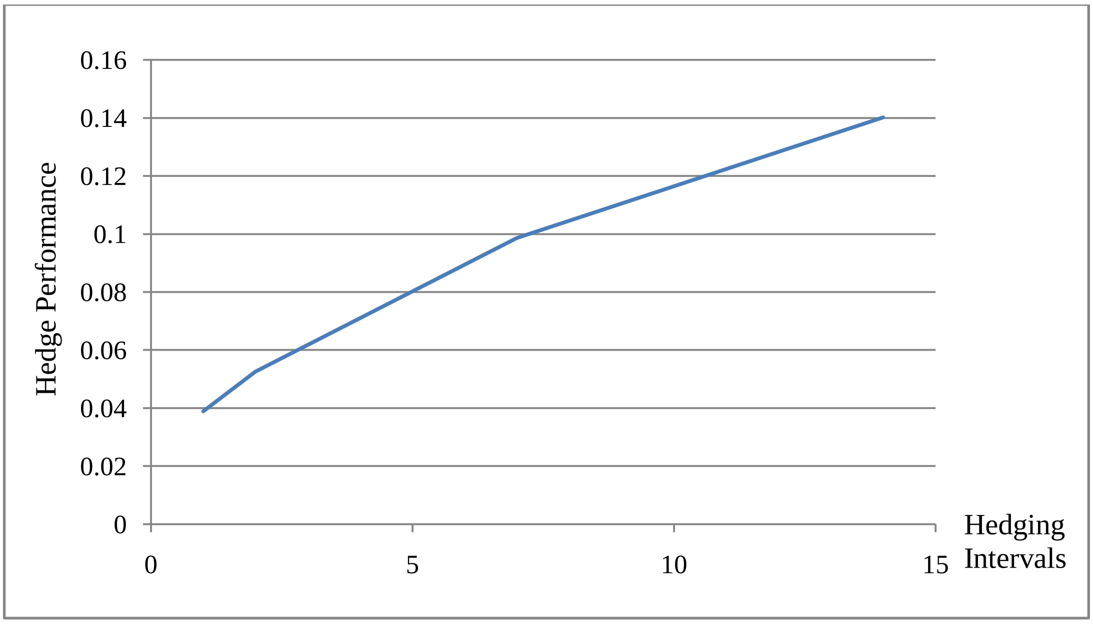

Figure 10: Hedge Performance with varying hedging intervals (in Days)

Figure 10 shows the relationship between the hedge performance measure and the days between hedge rebalancing. The hedge performance approaches zero[4] rapidly, verifying our expectations that the more frequently the hedge is rebalanced the better the strategy works and the costs of hedging are reduced.

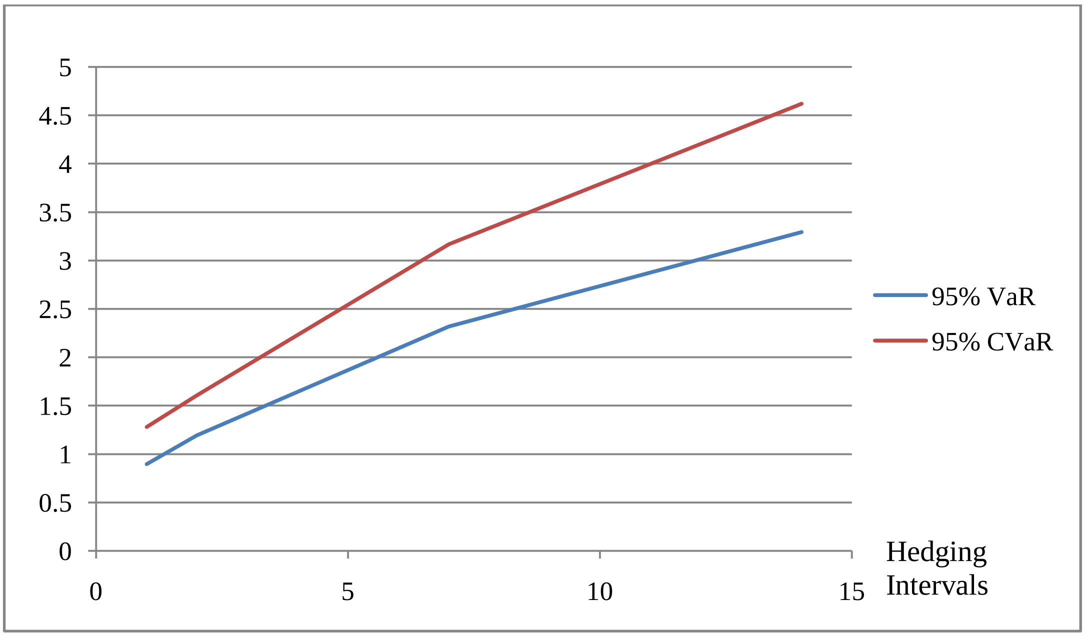

Figure 11: 95% VaR and CVaR with varying hedging intervals (in Days)

[4] Recall that a perfect hedge would have produced a hedge performance measure equal to zero.

Figure 11 shows the behaviour of the 95% VaR and CVaR as the hedging interval increases. as the days between hedge rebalancing decrease in number, convergence of the VaR and CVaR is noted. The difference between the two measures becomes bigger implying that when ‘things do get bad’, the institution will incur bigger losses.

Table 7: Hedging simulations with varying Spot prices

S(0)	Days Between Hedge Rebalancing	C(0)	Mean	Standard Deviation	Hedge Performance	95% VaR	95% CVaR
80	7	4.55322	0.01661	1.19257	0.27408	1.82889	2.82442
100	7	14.23135	-0.01256	1.40325	0.09860	2.31706	3.16587
120	7	28.88043	-0.00152	1.25064	0.04603	2.12751	2.91965

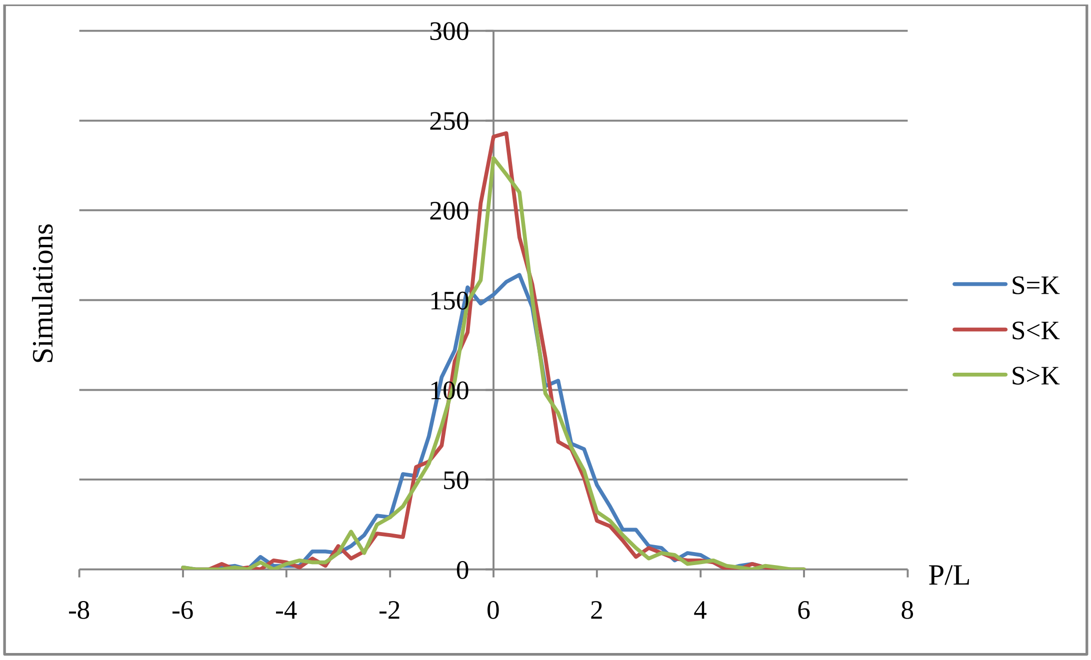

Figure 12: Profit/Loss Distributions with varying spot prices

Table 7 along with Figure 12 addresses the effectiveness of the strategy when the initial stock price is set below $[S(0)=80]$, equal $[S(0)=100]$ or above $[S(0)=120]$ the strike price.

Spot and Call option prices are positively correlated. The mean is close to zero in all the three cases; however, in-the-money or out-of-the-money Call option portfolios have lower standard deviation than at-the-money Call options and face lower potential losses.

Figure 12 implies that an option that is either in-the-money or out-of-the-money at the beginning of the contract can be hedged rather easy as it can become apparent, at that point, whether the option is going to be exercised at expiry.

The Black-Scholes model assumes the absence of transaction costs and that the volatility is single and constant at all times. However, in real markets, the initial stock price has a significant impact on the implied volatilities. In particular, at-the-money options produce higher implied volatilities than in-the money or out-of-the money options. This behaviour results in the well known ‘volatility smile’ and, therefore, the results obtained by the Black-Scholes model do not reflect reality.

Furthermore, we investigate the performance of the Delta hedging scheme in both real and risk-neutral world. The obtained results are summarised in Table 8 and Figures 13 and 14.

Table 8: Hedging simulations in both Real and Risk-neutral world.

World	Days Between Hedge Rebalancing	Mean	Standard Deviation	Hedge Performance	95% VaR	95% CVaR
Real μ=0.1	14	-0.06250	1.97902	0.13906	3.34804	4.64677
Real μ=-0.1	14	-0.06002	1.96605	0.13815	3.24704	4.56002
Risk-neutral r=0.05	14	-0.04766	1.99763	0.14037	3.29267	4.61731

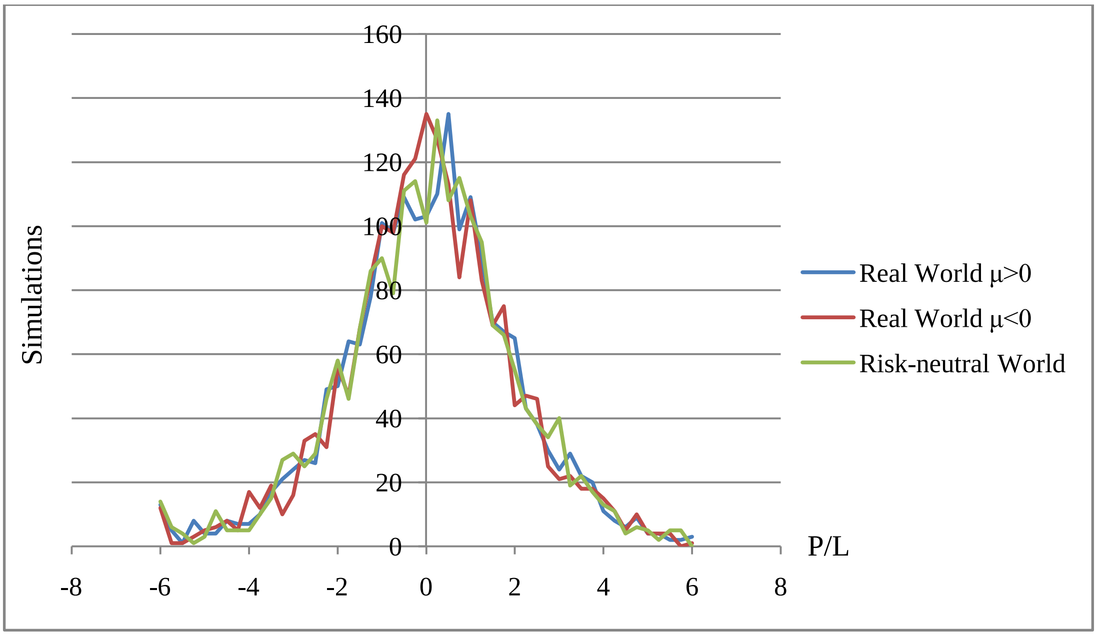

Figure 13: Profit/Loss Distributions in Real and Risk-neutral world

Figure 13 shows the Profit/Loss Distributions that arise in both the Risk-neutral world $(\mu = 0)$ and the Real world $(\mu = \pm 0.1)$.

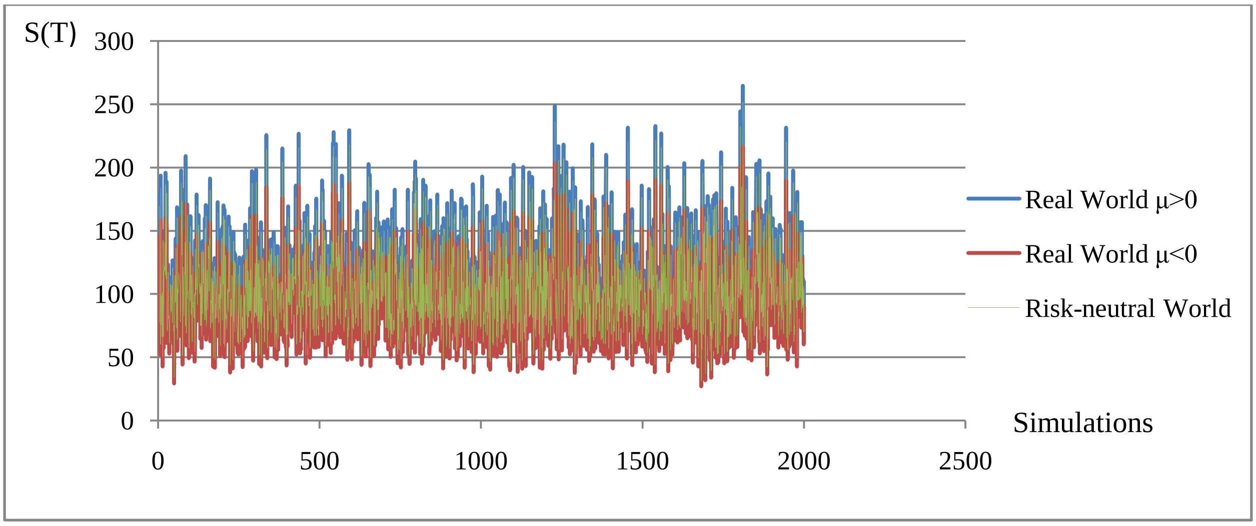

Figure 14: Stock price ranges at expiry in Real and Risk-neutral world

In the real world, the hedging scheme gives identical results for either choice of μ. Delta hedging in a Risk-neutral and a Real world seems to produce similar Profit/Loss distributions. This occurs due to the way the Black-Scholes equation is set up initially; it is constructed in such a way that the term μ is eliminated and never appears on the equation, thus it somehow takes care of the drift rate μ.

Figure 14 shows the stock price range at expiry in the Risk-neutral world $(\mu = 0)$ and the Real world $(\mu = \pm 0.1)$. We can, clearly, see that the stock price, at maturity, takes higher values when the mean rate of return is set to be positive than in the case where it is set to be negative. Intermediate values are produced when the hedging is considered to take place in a risk-neutral world.

4.1.3 Delta Hedging Example using Historical Data

Historical data of LOGICA share prices is used to examine the performance of the delta hedging method during a four-month period; Tuesday, 02/03/2010 to Wednesday, 02/07/2010.

We assume that a company has written a four-month European call option contract on 100 LOGICA shares on March the 3rd with a strike price $K = 120$. The data has been obtained from Livecharts.co.uk [40].The hedge rebalancing is set to be made every Tuesday.

Table 9: Delta hedging results using LOGICA shares

Date	LOG Share price	Delta	Long shares	Call price	B(t)	Portfolio
02/03/2010	120	0.57422	58	948.1673	-6011.83	0
09/03/2010	121.3	0.59651	60	994.1852	-6255.63	28.18915
16/03/2010	127.3	0.70194	71	1353.211	-7657.17	27.92224
23/03/2010	125.5	0.67292	68	1197.743	-7282.19	54.07087
30/03/2010	133.7	0.80461	81	1774.369	-9021.73	33.59914
06/04/2010	138.5	0.86740	87	2147.312	-9854.52	47.66683
13/04/2010	138.1	0.87035	88	2084.853	-9994.58	73.36991
20/04/2010	142.5	0.91839	92	2452.561	-10566.6	90.87896
28/04/2010	140	0.90584	91	2199.044	-10428.7	112.29873
05/05/2010	137.4	0.88942	89	1937.969	-10155.9	134.70479
12/05/2010	133.7	0.85336	86	1584.192	-9756.84	157.16646
19/05/2010	126.3	0.72104	73	958.844	-8116.88	144.17822
26/05/2010	118.9	0.50537	51	453.5701	-5502.69	107.64143
02/06/2010	126.2	0.74263	75	860.353	-8532.58	72.06664
09/06/2010	120.1	0.53785	54	410.1797	-6012.17	63.04682
16/06/2010	122	0.62343	63	456.7165	-7111.37	117.91702
23/06/2010	117	0.33825	34	136.4487	-3719.78	121.77365
30/06/2010	108.8	0.00154	1	0.155928	-130.116	-21.47169
02/07/2010	105	0	0	0	-25.1416	-25.14158

The option closes out-of-the-money and thus it is not exercised. The company gains in the short option position but loses in the long share position, resulting in a total cost of about 25. We divide the total cost of pricing and hedging the option with the theoretical call price and we obtain 0.0265. Thus, the hedging costs just about 2% of the call price. However, this is an example where the company, having placed a hedging strategy, is in a worse position than they would have been with no hedging.

4.2 Hedging with a correlated asset: Incomplete market case

In this section, the analysis consicles an Incomplete system of markets. The three hedging strategies, Delta hedging, Stop-Loss hedging and the Minimal Variance hedging, are demonstrated and the results are presented. The results, unless otherwise stated, are based on the parameters and the additional parameters as these appear in Table 3 and Table 4, respectively.

4.2.1 Three Hedging Schemes Results

Table 10: Delta Hedging results

λ	ρ	Mean	Standard Deviation	C(0)	Hedge Performance	95% VaR	95% CVaR
0	0.1	0.22498	12.70299	10.79186	1.17709	22.32044	28.19843
0	0.5	1.05291	9.37403	9.74208	0.96222	15.65492	19.52594
0	0.7	0.89881	7.19438	8.74346	0.82283	11.81312	14.73853
0	0.9	0.38928	4.09554	7.49686	0.54630	6.63764	8.37175
0	1	-0.02452	0.42491	6.80496	0.06844	0.72639	1.06988

Delta hedging results with $\lambda = 0$ and varying ρ, while the hedge is being rebalanced weekly.

Table 11: Stop-Loss Hedging results

λ	ρ	Mean	Standard Deviation	C(0)	Hedge Performance	95% VaR	95% CVaR
0	0.1	0.08702	12.59082	10.79186	1.16670	22.52190	28.36536
0	0.5	0.67098	9.52288	9.74208	0.97750	16.04310	20.16683
0	0.7	0.48197	7.59233	8.74346	0.86834	12.79252	15.62413
0	0.9	0.03098	4.98629	7.49686	0.66512	8.41221	10.02671
0	1	-0.30641	2.95209	6.80496	0.43381	5.61288	7.68007

Stop-Loss hedging results with $\lambda = 0$ and varying ρ, while the hedge is being rebalanced weekly.

Table 12: Minimal Variance Hedging results

λ	ρ	Mean	Standard Deviation	C(0)	Hedge Performance	95% VaR	95% CVaR
0	0.1	-0.35664	9.92932	10.79186	0.92008	18.59354	24.80927
0	0.5	-0.21489	8.26896	9.74208	0.84879	15.92202	20.20857
0	0.7	-0.11987	6.60652	8.74346	0.75560	12.39305	15.75615
0	0.9	-0.02873	3.89170	7.49686	0.51911	7.16513	8.84603
0	1	-0.01181	0.57475	6.80496	0.08446	0.94695	1.38944

Minimal variance hedging results with $\lambda = 0$ and varying ρ, while the hedge is being rebalanced weekly.

Tables 10, 11, 12 contain the statistics obtained after using each of the above mentioned strategies. Explicit comments and comparisons between the schemes can be seen in Figures 15, 16, 17 and 18.

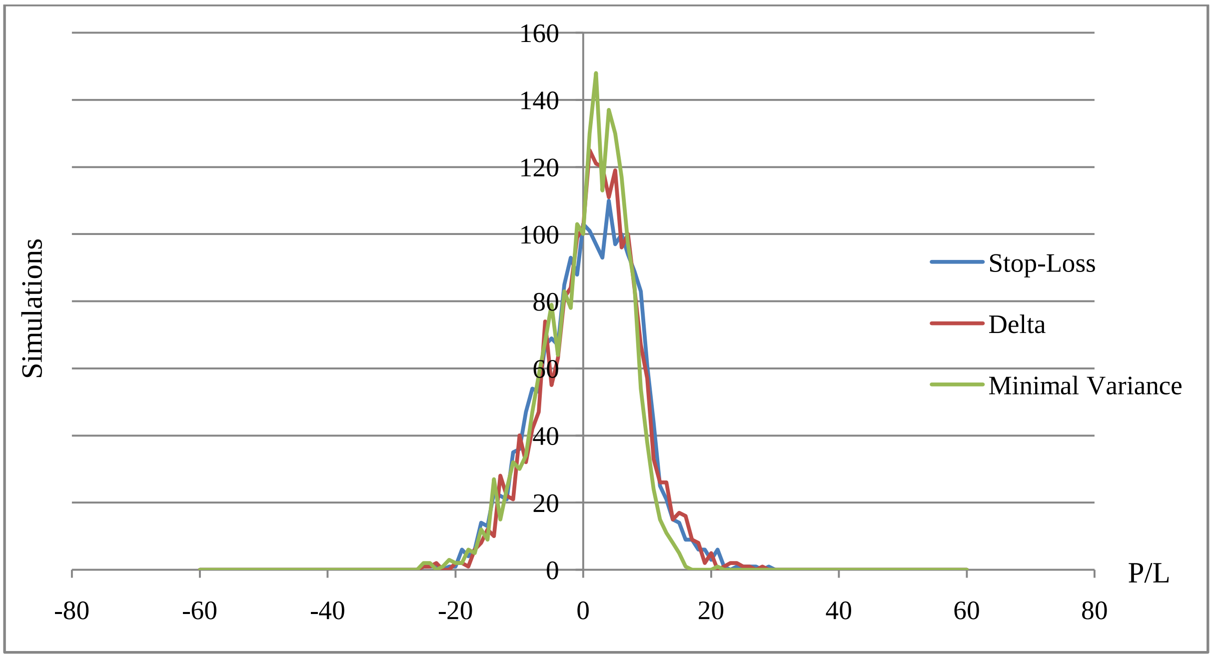

Figure 15: Profit/Loss Distributions with ρ=0.5

Figure 15 presents the Profit/Loss distributions obtained after performing the three strategies with the hedge being rebalanced in a weekly basis. Minimal Variance technique appears to produce the best Profit/Loss distribution and, therefore, it can be considered as the most effective strategy among them.

After comparing these results with the ones obtained in the complete market case, we deduce that the schemes are less effective. The imperfect correlation between the assets creates errors that are reflected in the final portfolio value. The three strategies, however, seem to produce very similar results whereas in the complete market case the delta scheme performed far better than the others.

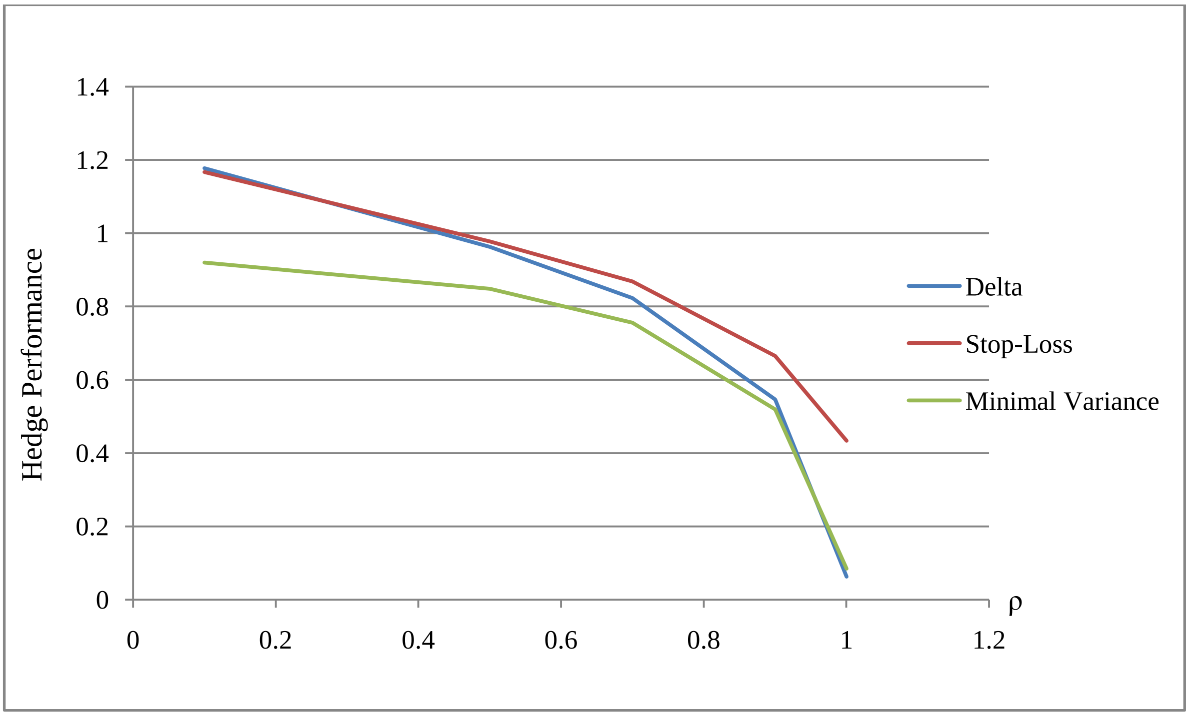

Figure 16: Hedge Performance with varying ρ

Figure 16 presents the improvement of the hedge performance of the strategies as the correlation between the assets increases.

Among the three trading strategies, Minimal Variance seems to work the best and Stop-Loss seems to work the worst. Even if perfect correlation is considered the hedge performance of the Stop-Loss scheme could not drop more than 0.4. Delta hedging gives intermediate values; low correlation case produces similar results to the stop-loss scheme, whereas high correlation produces similar results to the Minimal Variance scheme. Nonetheless, in the case of a perfect correlation between the assets, the Delta hedging performs almost the same as the Minimal Variance one (around 0.07), assuming that the two assets volatilities are equal. To realise this we recall the procedure used to set up the Minimal Variance scheme. By equation (2.3.7),

$$x = \left(\frac{S\sigma}{H\sigma'} \rho \right) V_s \tag{4.2.1}$$

we can see that when the correlation is perfect positive then $\rho = 1$ and if the volatilities of the two assets are equal then if the price of S changes by a certain amount, the price of H changes by exactly the same amount [see Equation (3.2.2)]. As a result, the term that appears in front of V_s cancels out and we are only left with the partial derivative that is, indeed, the Delta hedging strategy.

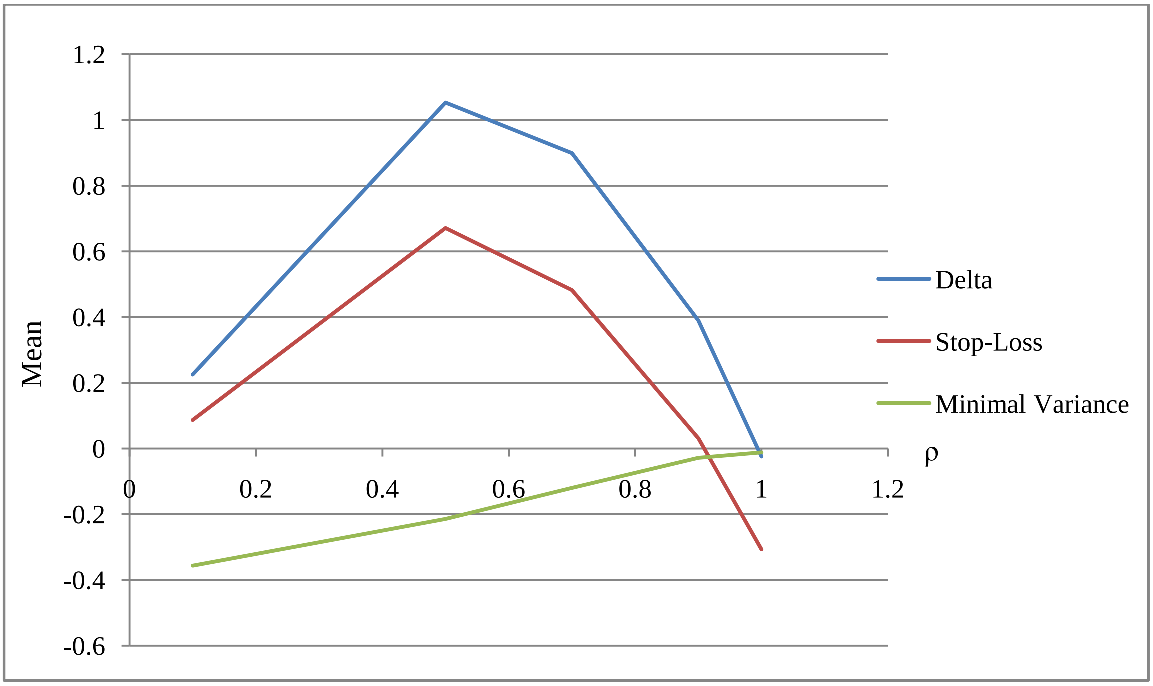

Figure 17: Mean with varying ρ

Figure 17 shows the behaviour of the mean as the correlation between the two assets increases. The absolute mean value approaches zero, as the correlation increases. There is a significant convergence in the case of the Minimal variance scheme. When there is a low correlation between the two assets, the mean values in the Delta and Stop-Loss case appear to be closer to zero than the one in the Minimal Variance case; however, as the assets become highly correlated, the latter approach forces the mean value to converge to zero rapidly.

We note that, although the mean is close to zero at all cases, it is never exactly zero because of the finite rebalancing interval.

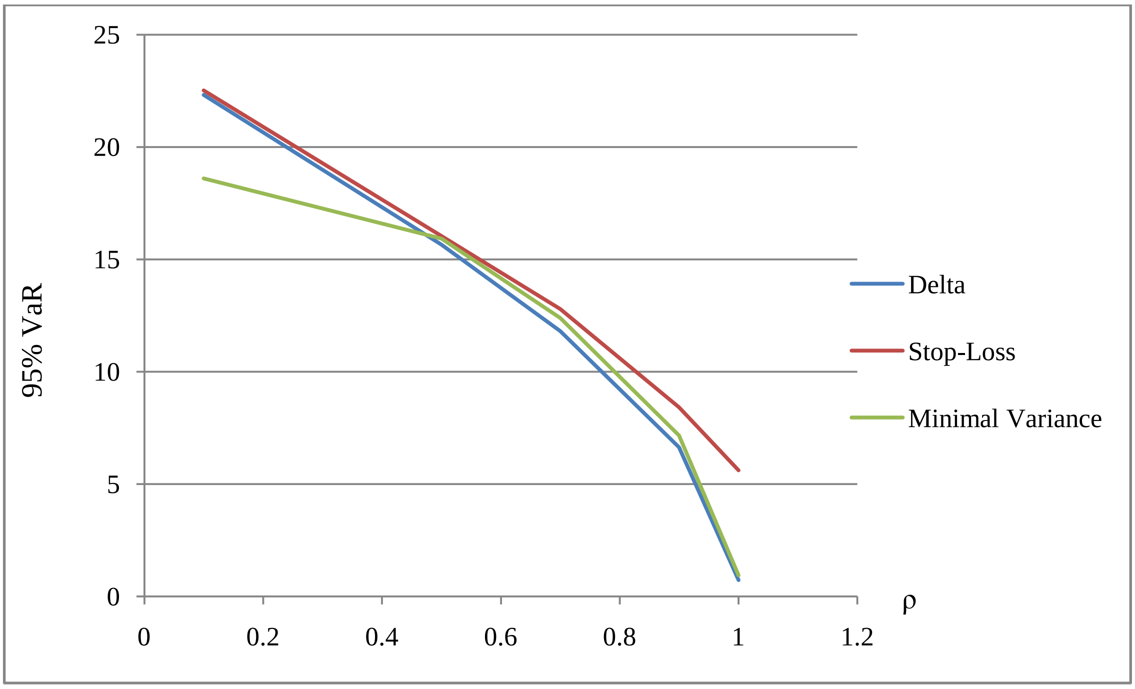

Figure 18: 95% VaR with varying ρ

Figure 18 shows the behaviour of the 95% VaR as the correlation changes from 0.1 to 1. As the correlation increases, the VaR decreases. This indicates that the loss a company can face is low when the hedge is based on highly correlated assets. Once again, the Minimal Variance scheme produces the best curve for all correlations; nonetheless, when perfect correlation occurs, the Delta hedging scheme gives identical results.

4.2.2 Minimal Variance Hedging Analysis

Since the Minimal Variance approach seems to have worked better than the Delta and the Stop-Loss Strategies, we proceed to investigate the effectiveness of the Minimal Variance approach by examining it using varying correlations, hedge rebalancing intervals, maturity times, interest rates, volatilities and drift rates. Then, a risk loading parameter is added to the equation and its effects are recorded.

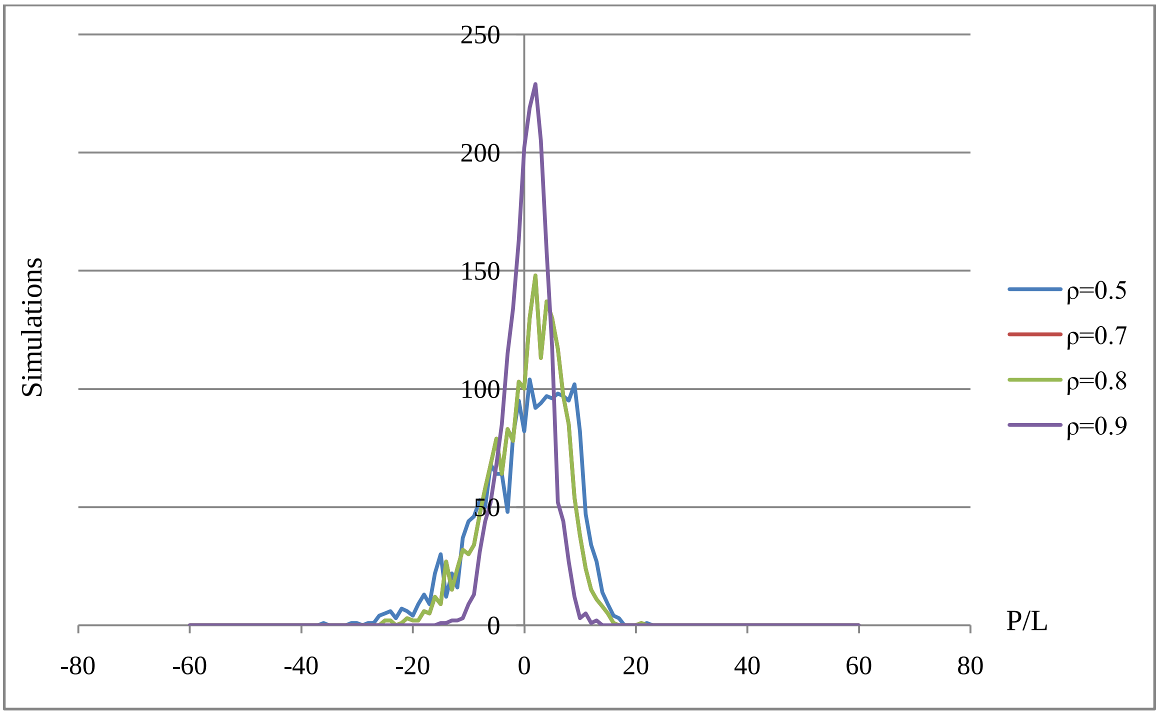

Figure 19: Profit/Loss Distributions with varying ρ

Figure 19 shows the resulting Profit/Loss distributions with varying ρ. As the correlation between the assets increases, the Profit/Loss distribution improves. However, imperfection is observed because, even if the correlation between the two assets is high, there is an amount of risk that cannot be hedged.

To realise this we recall the equation (2.3.13)

$$d\Pi = -\left[V_t - r'SV_s + \frac{\sigma^2 S^2}{2}V_{ss} - rV\right]dt + \sigma SV_s\sqrt{1-\rho^2}\,dX \qquad (4.2.2)$$

The standard deviation of the hedging error at time T is given by

$$\sqrt{1-\rho^2}\sqrt{T}\sigma \qquad (4.2.3)$$

as Ankirchner and Imkeller argued in [2].

Therefore, the first term of the formula, $\sqrt{1-\rho^2}$ plays a crucial role to the efficiency of the hedge.

Table 13: Causation of Basis Risk

ρ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99	1
$\sqrt{1-\rho^2}$	1	0.99499	0.979796	0.953939	0.91652	0.86603	0.8	0.71414	0.6	0.43589	0.14107	0

The second row of table 13 presents the percentage amount of standard deviation of the portfolio price that cannot be hedged, as correlation increases.

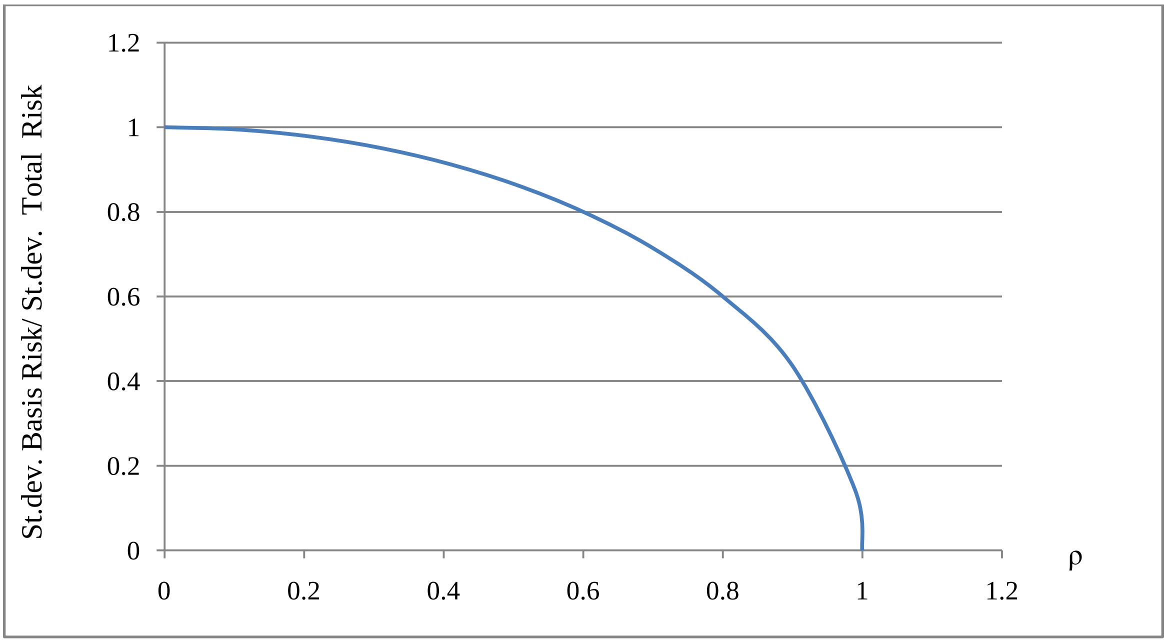

Figure 20: Percentage contribution of Basis Risk to the Total risk with varying ρ

Figure 20 presents the percentage contribution that the Basis Risk has to the Total risk when ρ varies. As the correlation decreases, the basis risk increases in a quadratic manner. Even if the two assets have an unrealistically high correlation, $\rho = 0.99$, there is still 14% of the standard deviation of the portfolio value that cannot be hedged.

Table 14: Hedging simulations with varying hedging intervals

ρ	λ	Days Between Hedge Rebalancing	Mean	Standard Deviation	C(0)	Hedge Performance	95% VaR	95% CVaR
0.5	0	1	0.08180	8.04356	9.74208	0.82565	14.84266	19.26027
0.5	0	2	-0.09451	7.97080	9.74208	0.81818	14.91460	19.75606
0.5	0	7	-0.21489	8.26896	9.74208	0.84879	15.92202	20.20857
0.5	0	14	-0.00103	8.11262	9.74208	0.83274	14.92144	18.95544

Table 14 consists of the results obtained when the days between hedge rebalancing vary from one to fourteen trading days. The correlation seems to have a major impact on the hedge performance that overshadows the impact that the hedge rebalancing interval has. However, a slow convergence is observed in the standard deviation and VaR.

Table 15: Hedging results with varying maturity time

ρ	λ	T	Mean	Standard Deviation	C(0)	Hedge Performance	95% VaR	95% CVaR
0.5	0	1	-0.21489	8.26896	9.74208	0.84879	15.92202	20.20857
0.5	0	0.5	-0.13781	5.14827	5.51450	0.93359	10.14655	12.93159
0.5	0	0.25	-0.08837	3.30440	3.27114	1.01017	6.64362	8.52906

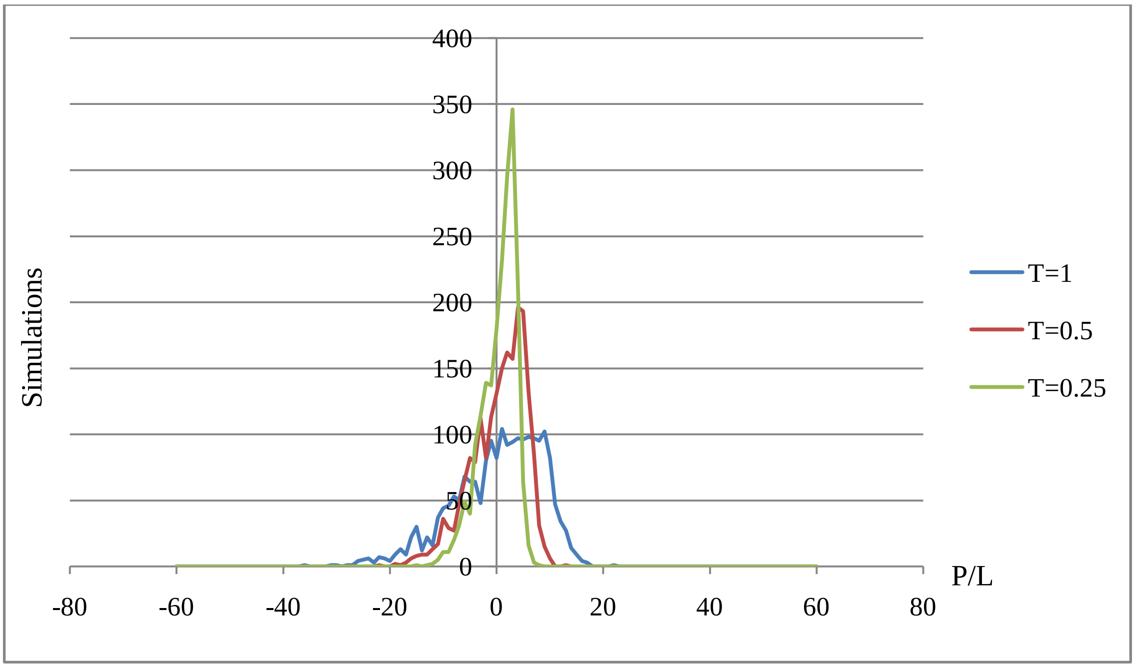

Figure 21: Profit/Loss distributions with varying maturity time

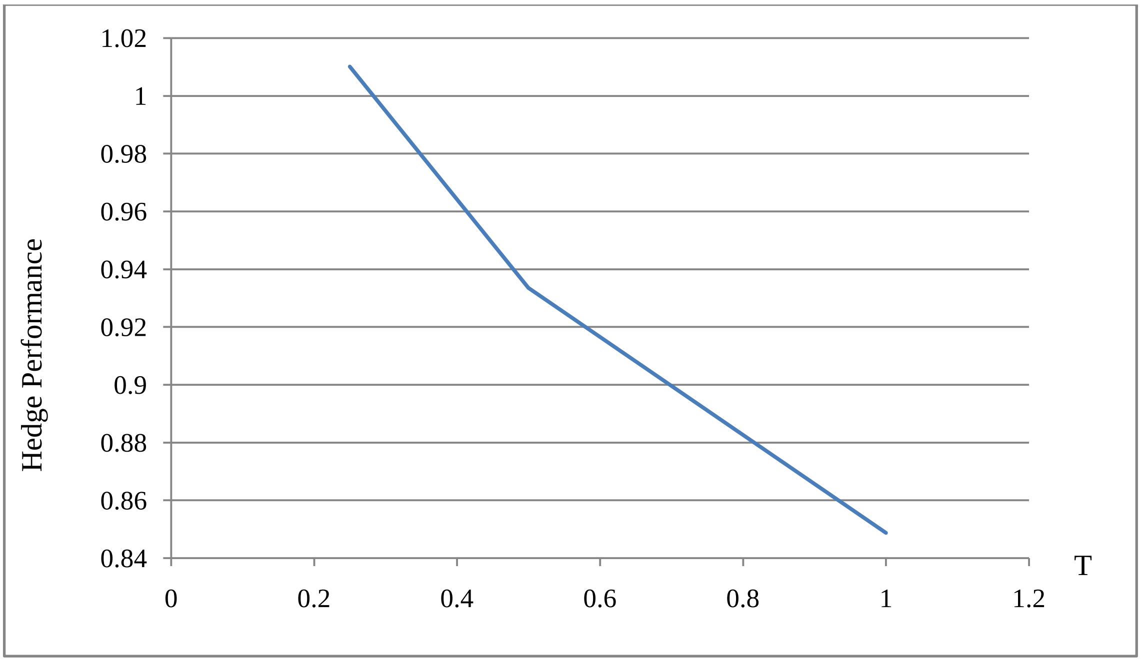

Figure 22: Hedge Performance with varying maturity date

Table 15 along with Figures 21 and 22 present the results obtained after considering several maturity dates. Narrower distributions are obtained when considering short-term contracts, due to that the cost of pricing and hedging the option is cheaper. To realise which maturity time gives the best results, one should only consider the hedge performances and not the rest of the statistical features.

Looking at Figure 22, one can see that the scheme performs better with long date option contracts. Assuming a weekly hedging interval, by getting a position in a long-term contract, a trader is able to readjust the portfolio more times than he should have done in the case of a short-term contract. This means that any unexpected losses in the portfolio would have enough time to be recovered before the end of the life of the contract.

Table 16: Hedging simulations with varying interest rates

ρ	λ	r	Mean	Standard Deviation	C(0)	Hedge Performance	95% VaR	95% CVaR
0.5	0	0.02	-0.21685	8.16526	9.38924	0.86964	15.71829	20.04063
0.5	0	0.05	-0.21489	8.26896	9.74208	0.84879	15.92201	20.20857
0.5	0	0.08	-0.21327	8.37056	10.08526	0.82998	15.98523	20.37178

Table 16 shows the behaviour of the strategy when assuming several interest rates values. Interest rates have insignificant impact on the hedge performance and the other statistical features of the Minimal Variance scheme. This happens because the change incurred in the bond value due to the risk-free rate is very small in relation to the changes that the asset prices face at each time interval.

Table 17: Hedging simulations with varying volatilities

λ	σ	σ'	Mean	Standard Deviation	C(0)	Hedge Performance	95% VaR	95% CVaR
0	0.6	0.6	0.36426	25.32933	26.14575	0.96877	45.29257	70.54721
0	0.6	0.1	-0.03939	24.80858	26.14575	0.94886	45.65060	71.47229
0	0.1	0.6	0.06737	4.01818	7.49686	0.53598	7.14933	8.96577
0	0.1	0.1	-0.02873	3.89170	7.49686	0.51911	7.16513	8.84603

In Table 17 one can see the results obtained after considering different combinations of the asset's volatilities. Clearly, the volatility of the asset H has little impact on the hedge performance and the other statistical features. Any significant results are obtained when the volatility of the original asset changes. In fact, this should be the case because the call option formula in (2.3.18) and the V_s in (2.3.7) depend only on the characteristics of the asset S.

Our analysis proceeds in examining the effectiveness of the hedging scheme when the drift rate of the non-traded asset S varies.

Table 18: Hedging simulations with varying drift rate μ

ρ	λ	μ	μ'	Mean	Standard Deviation	C(0)	Hedge Performance	95% VaR	95% CVaR
0.5	0	0.1	0.075	-0.21489	8.26896	9.742079	0.84879	15.92202	20.20857
0.5	0	0.05	0.05	-0.18155	7.01261	6.804957	1.03052	14.04273	18.46353
0.5	0	0	0.025	-0.12437	5.59430	4.44514	1.25852	12.08676	16.26631
0.5	0	-0.1	-0.025	-0.06025	2.83842	1.49183	1.90265	5.95774	10.16746
0.5	0	-0.2	-0.075	-0.02773	1.00520	0.34367	2.92490	0.80811	3.15746

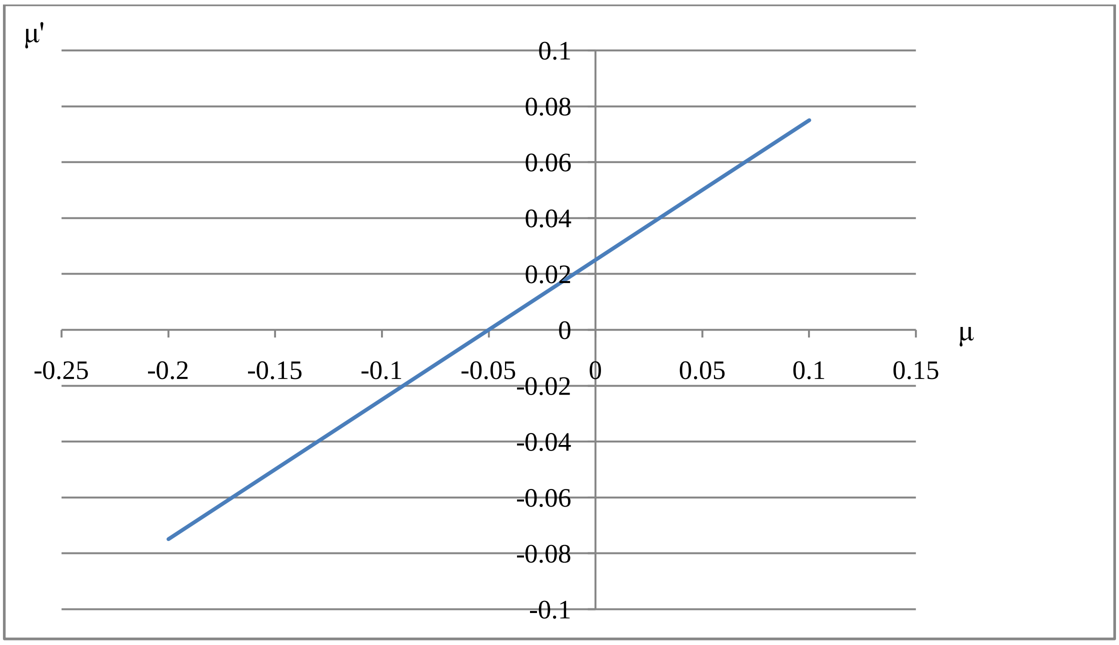

Figure 23: Relationship between μ and μ'

Figure 23 shows the relationship between the drift rates of the two assets S and H, which is positive and linear. This means that, as the drift rate of the non-tradable S increases, the gap between the two drift rates widens.

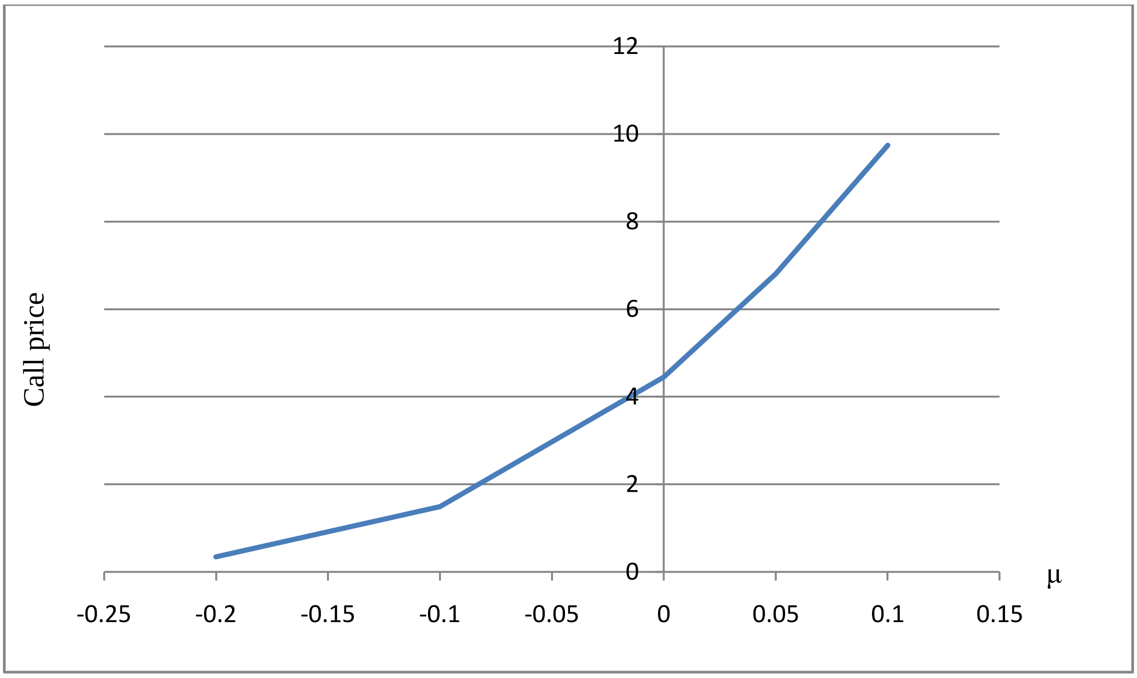

Figure 24: Relationship between the Call price and μ

Figure 24 shows the behaviour of the option price as the drift rate of the non-traded asset increases.

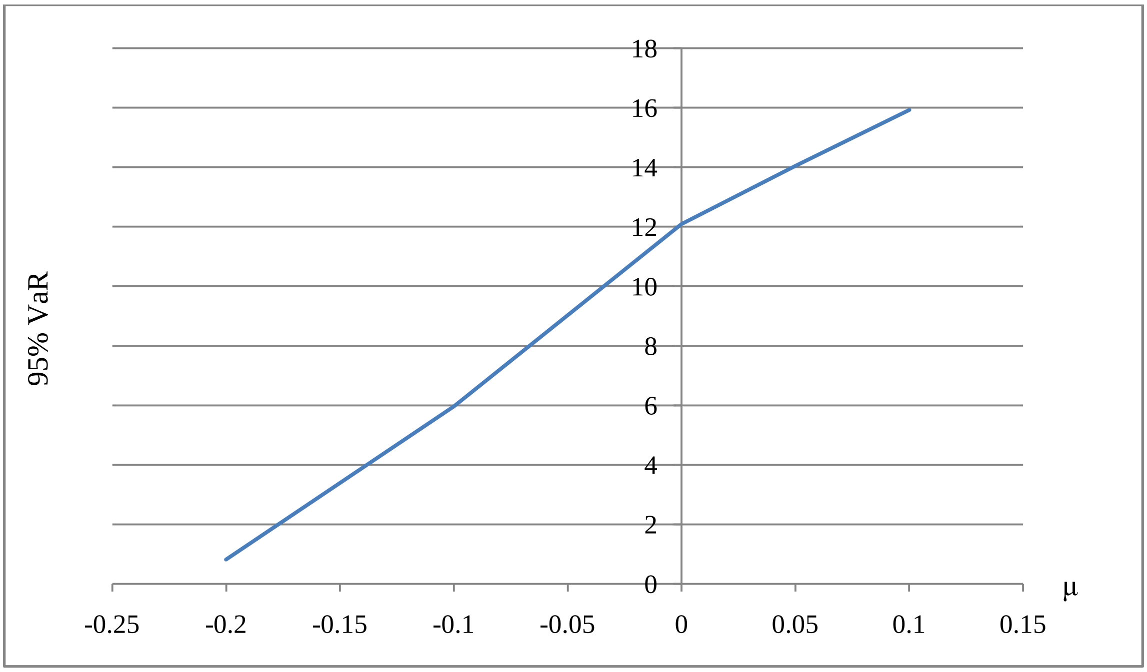

Figure 25: Relationship between the VaR and μ

Figure 25 shows the behaviour of the VaR as μ increases, which is positive and almost linear.

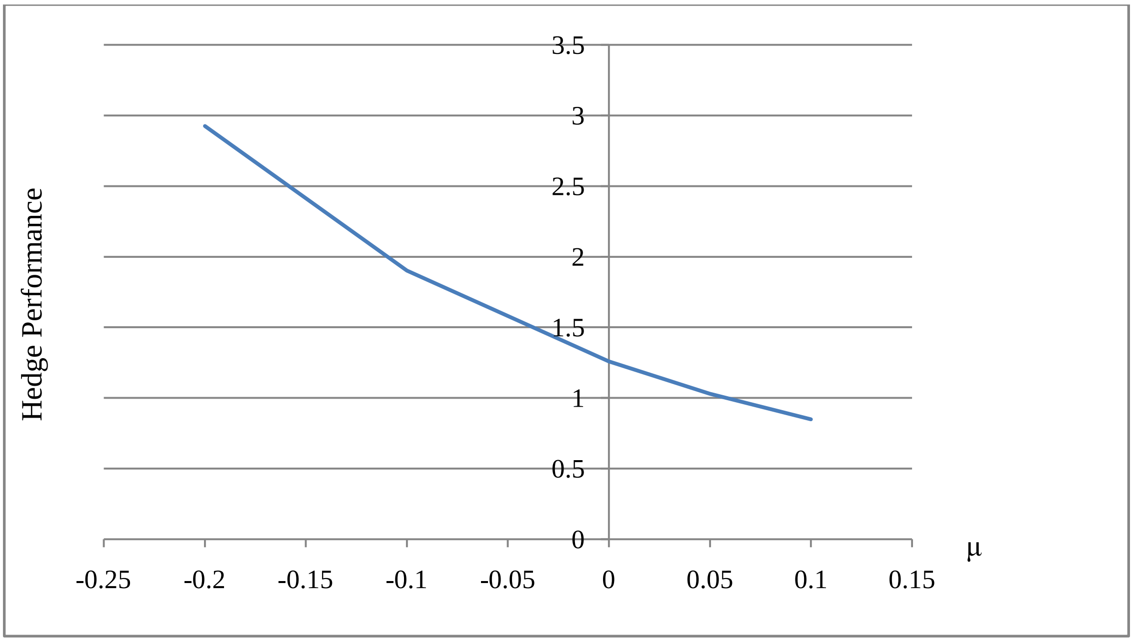

Figure 26: Hedge Performance with varying μ

Figure 26 shows the behaviour of the hedge performance as the drift rate of the non-tradable asset S increases.

As the drift rate increases, the theoretical call option price increases [see Figure 24] and, as a result, the VaR increases linearly [see Figure 25]; however, the hedge performance converges to zero [see Figure 26]. To understand this, we recall the meaning of the drift rate μ. The change of the random variable S over a given time interval dt is μdt, indicating that high drift rates result in high stock prices. On the one hand, such prices increase the potential losses (VaR) in the written option but, on the other hand, the call option premium gains in value so that the potential losses will be minimised.

The increase in the Call price along with the gap created between the two drift rates results in a hedging scheme that is relatively more effective.

4.2.3 Minimal Variance Hedging with Risk Loading

Table 19: Hedging results with varying ρ and λ

ρ	λ	Mean	Standard Deviation	C(0)	Hedge Performance	95% VaR	95% CVaR
0.5	-0.5	-3.85051	8.28803	6.40074	1.29485	19.46954	23.97190
0.5	0	-0.21489	8.26896	9.74208	0.84879	15.92201	20.20857
0.5	0.5	4.10161	8.28240	13.74673	0.60250	11.55155	15.81694
0.5	1	8.93690	8.31328	18.26170	0.45523	6.66593	10.94122
0.5	1.5	14.15564	8.35047	23.15594	0.36062	1.39693	5.69938
0.7	-0.5	-3.09031	6.62725	6.10318	1.08587	15.55993	18.94575
0.7	0	-0.11987	6.60652	8.74346	0.75560	12.39305	15.75610
0.7	0.5	3.33374	6.63825	11.86216	0.55962	8.84903	12.16151
0.7	1	7.17916	6.70181	15.37513	0.43589	4.99404	8.24656
0.7	1.5	11.32849	6.77987	19.19793	0.35316	0.86973	4.06088
0.9	-0.5	-1.83332	3.90932	5.95507	0.65647	9.20179	10.87138
0.9	0	-0.02873	3.89170	7.49686	0.51911	7.16513	8.84603
0.9	0.5	1.96566	3.92794	9.21908	0.42607	4.98828	6.70351
0.9	1	4.13056	4.00475	11.11528	0.36029	2.81267	4.45334
0.9	1.5	6.44439	4.10822	13.16593	0.31203	0.44479	2.09935

For different correlations examined, a range of λ values is considered and the results are listed in Table 19. The impact of the risk loading parameter to each of the portfolio's characteristics may be seen explicitly in Figures 29-32.

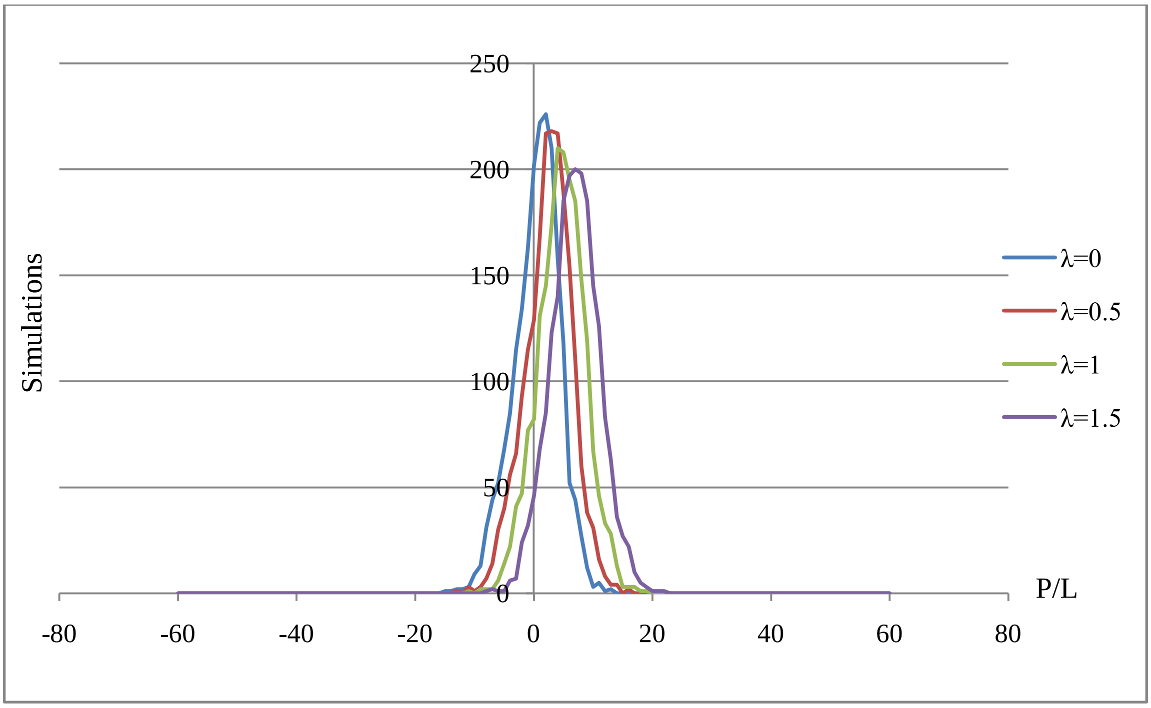

Figure 27: Profit/Loss distributions with varying λ and ρ=0.9

In Figure 27, the correlation is set to 0.9 and the Profit/Loss distributions are recorded, as the risk loading parameter increases. As λ increases, the Profit/Loss distribution of the portfolio gains in mean. The impact that the increasing risk loading parameter has on the distribution is the shift it to the right and a reduction of its peak point.

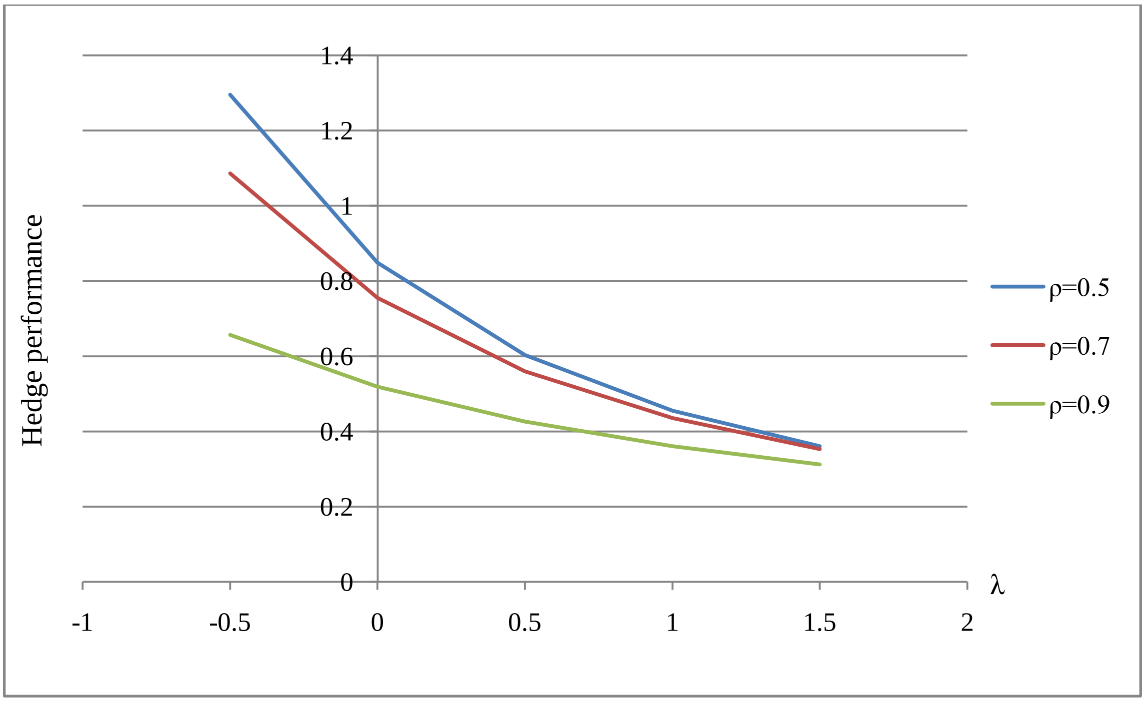

Figure 28: Hedge performances with varying λ

Figure 28 shows the impact of λ to the hedge performance. The hedge performance seems to improve as the value of the risk loading parameter λ increases. To understand why this is the case we need to set up the following diagrams.

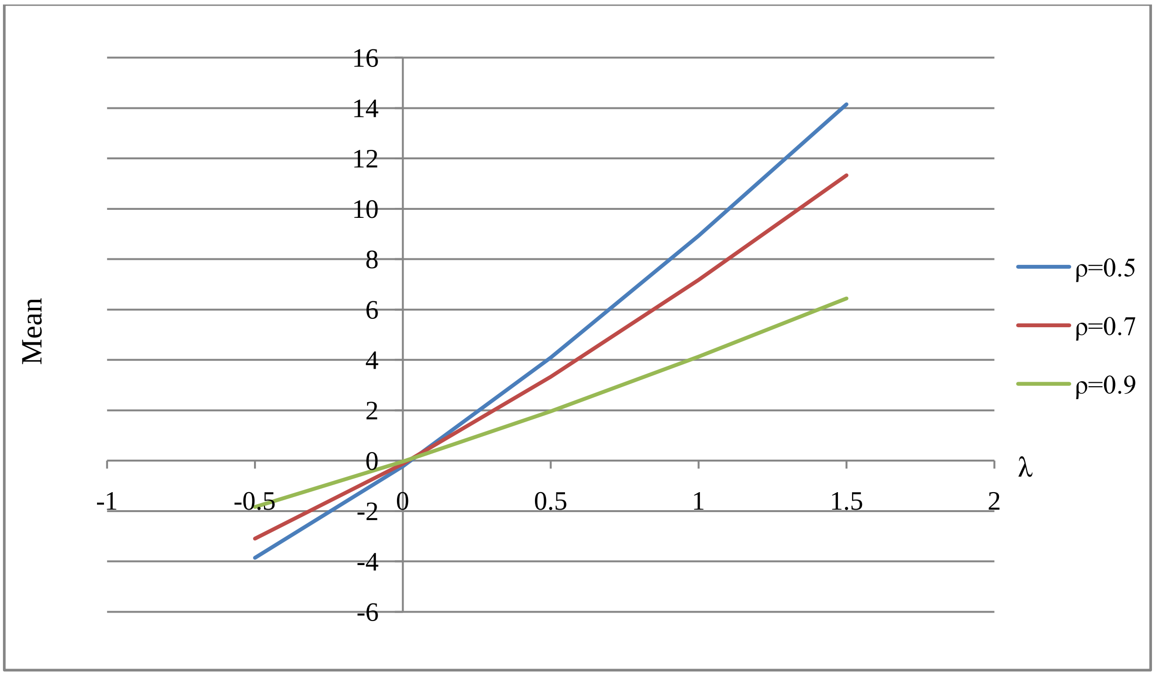

Figure 29: Mean results with varying λ

Figure 29 shows the behaviour of the mean as the risk loading parameter increases. The increasing λ causes the mean to increase in value. Interesting to note that the higher the correlation, the less the absolute mean value.

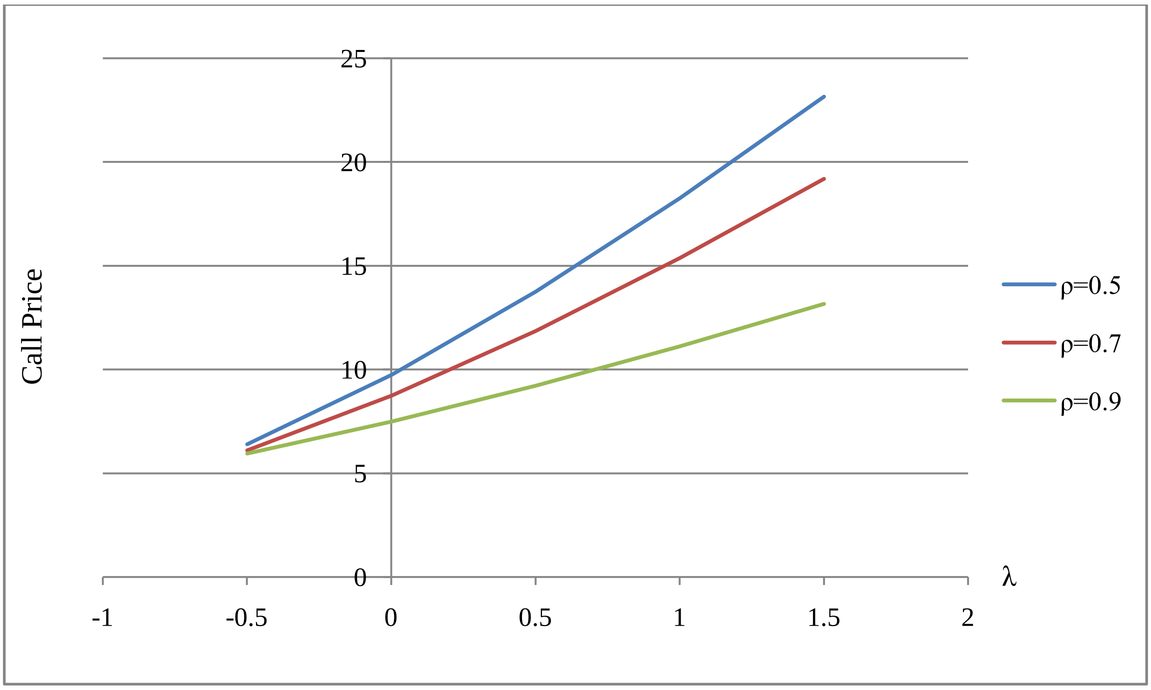

Figure 30: Relationship between the Call price and λ

Figure 30 shows the relationship between the call price and the risk loading parameter. Clearly, adding the parameter λ to account for the basis risk, the call price gains in value. Figure 30 shows that when using low correlated assets, such an action results in having the call price to raise reluctantly much.

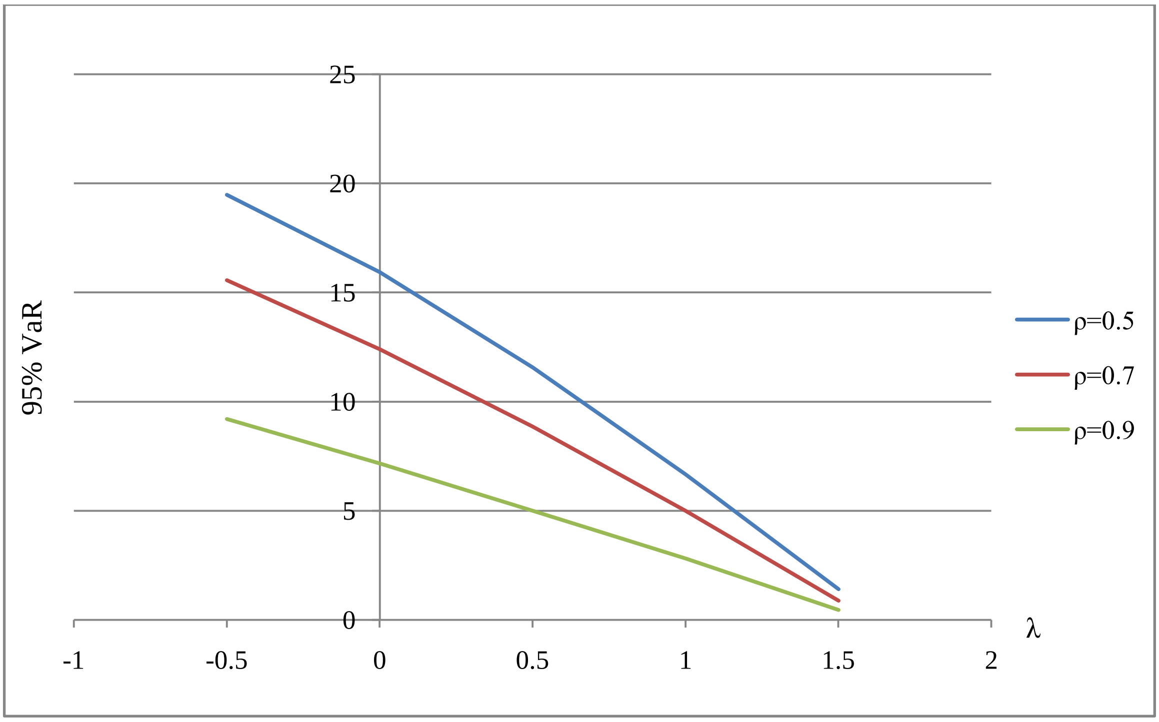

Figure 31: Relationship between 95% VaR and λ

Figure 31 shows the relationship between the 95% VaR and λ, which appears to be linear and negative, meaning that considering risk loading when writing the option, the replicating strategy is improved; however, as seen in Figure 29, this happens at a higher option cost.

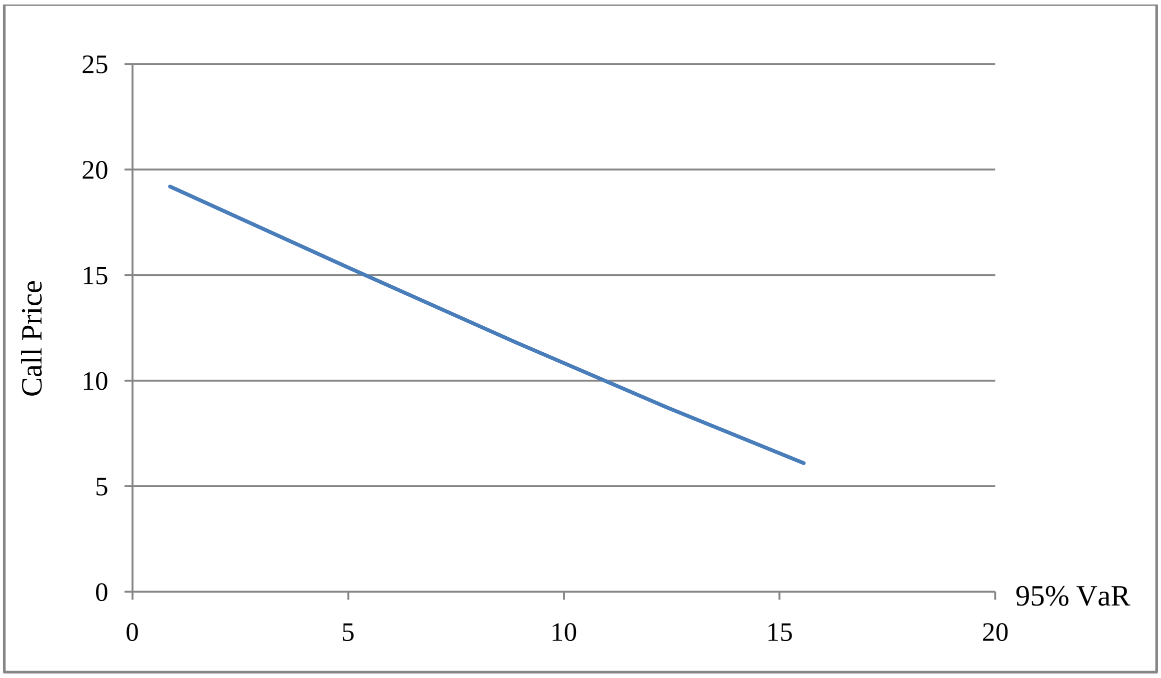

Figure 32: Relationship between the Call price and the 95% VaR with ρ=0.7 and varying λ

Figure 32 shows the relationship between the call price and the VaR which appears to be linear and negative. The relationship appears to be more a straight line than a curve. This is, of course, reasonable; a trader increases the option price by a certain amount so that his potential risks would reduce by that amount. This risk reduction is reflected in the VaR value.

Since the relationship between the call value and the VaR is negative and linear, it is left to the trader to examine and decide the value of λ that will result in a VaR he is willing to face in the case things do go wrong. In fact, we note that high correlation between the two assets reduces the call value and the VaR, thus the higher the correlation, the less the value of λ that the trader needs to consider adding.

5 Discussion

5.1 Conclusions-Recommendations

In this project we have examined several trading strategies in both complete and incomplete systems of markets.

5.1.1 Hedging with the underlying asset

The Static hedging offers almost no hedging at all. This happens due to the fact that the portfolio is delta neutral only at the first trading day. After that, the portfolio value moves according to the stock price movements and it can either end up positive or negative. Therefore, on the one hand, the strategy does not induce any transaction costs but, on the other hand, the possibility of achieving a perfect hedge in an over-the-counters market is small due to the limited liquid assets [24].

The Stop-Loss strategy also provides limited performance. Problems arise when considering such a strategy: if the stock price never crosses the strike price, the writer of the option had paid nothing, whereas if the stock price crosses the strike price many times, the hedging scheme becomes very expensive.

In practice, a trader that places a hedge, when the stock price is equal to the strike price, cannot know in which direction the stock price will move, either above or below the strike price. That is why an additional cost should be added to any purchases and sales made. Frequent hedging can reduce this additional cost; but the hedge performance cannot drop more than a certain level, around 0.6.

The Delta hedging technique is a huge improvement over the Static and the Stop-Loss strategies because it is based on the slope between the option change and the stock price change. Any profits or losses tend to vanish and the strategy leads to a risk free portfolio.

The call option premium depends on the current stock price relative to the strike price. Undoubtedly, an in-the-money option costs more than an out-of-the-money option. If the option was either out-of-the-money or in-the-money at the time it was first issued, the hedge could be easily set up, as it would become clear whether the option would be exercised at maturity, and the downside risks minimised. However, the implied volatility depends on the initial stock price and, as a result, the volatility smile arises; unfortunately, the Black-Scholes model is not able to capture this.

The current price is not the price that a trader can buy the stock for, but it is the price at which the stocks last changed hands from seller to buyer. Therefore, purchases and sales cannot be made at exactly this price [29]. One should take into consideration the bid-ask spread when placing any hedging scheme. The spread value, measured in percentage, will indicate whether it is appropriate to perform a purchase or a sale. For instance, large bid-ask spreads indicate that the stock is not very liquid and that if one buys a stock with a certain amount of bid-ask spread, he will have to make a profit on the stock of at least that amount just to break even. The more liquid the market, the less the bid-ask spread value and, therefore, it is more difficult and, consequently, less effective to place a hedging strategy with frequent rebalancing intervals in illiquid markets.

5.1.2 Hedging with a correlated asset

We remind the reader that in an incomplete markets system, there is no formula that can give us the exact call option value, thus any conclusions made are restricted in our findings using the derived PDE in (2.3.31).

The non-perfect correlation between the assets creates a basis risk. As the correlation approaches positive perfection, the basis risk decreases; yet, there is always a significant amount of the risk that cannot be hedged.

All of the examined strategies have produced a reasonable hedging.

Stop-Loss hedging provided a reasonable hedging. Increasing the correlation improved its performance; still, when S is very volatile, the presence of transaction costs results in a very costly scheme.

Delta hedging, on the other hand, works slightly better. Perfect correlation results in a very efficient hedging and a hedge performance of only 0.07. However, the standard deviation of the portfolio did not show any improvement compared to the Stop-Loss strategy.

Minimal Variance performed better than Delta and Stop-Loss hedging, verifying our expectations. Minimising the variance of the portfolio achieves low VaR and CVaR values with no additional cost. Perfect correlation leads to identical results between the Delta and the Minimal Variance only if the volatilities of the two assets are equal.

When focusing on the Minimal Variance, we obtained similar results to those of Wang *et al* obtained [19]. We have shown that, as the correlation increases, the standard deviation decreases and, consequently, VaR and CVaR improve dramatically. Nonetheless, perfect or close to perfect hedge cannot occur in any case due to the non-perfect correlation and the finite rebalancing interval.

Wang *et al* [19] showed that, by readjusting the portfolio at frequent times, convergence of the standard deviation of the Profit/Loss distribution occurs. Our results have shown that such an action leads to a very little and sometimes insignificant convergence. However, I believe that if the number of simulations used was larger, a reasonable convergence should have been realised.

Since the hedge is not perfect, we tried to price the risk using the actuarial standard deviation principle in infinitesimal time. Using a risk loading parameter λ we have achieved to slightly raise the call option value so that the loss incurred due to the basis risk will be offset by the excess value in the short call.

Finally, according to the Black-Scholes model, a risk-free portfolio can be achieved by dynamic hedging due to the absence of transaction costs. Nevertheless, in the real world where time is continuous, dynamic hedging implies infinite trading which, in turn, causes a large amount of transaction costs. Therefore, this project is restricted within the limits set by our choice of the price model (Black-Scholes). This limitation could only be overcome by choosing an alternative model that would take into consideration the presence of transaction costs. Related work has been done by Forsyth *et al* [18] who examined this assuming that asset price follows a jump diffusion process and demonstrated that a dynamic hedging can be sufficiently effective without incurring large transaction costs.

5.2 Further Work

We recall the non-linear PDE for the short position is given by

$$V_t + [r' + \lambda\sigma\sqrt{1-\rho^2}\,\mathrm{sgn}(V_s)]SV_s + \frac{\sigma^2 S^2}{2}V_{ss} - rV = 0 \tag{5.2.1}$$

and the non-linear PDE for the long position by

$$V_t + [r' - \lambda\sigma\sqrt{1-\rho^2}\,\mathrm{sgn}(V_s)]SV_s + \frac{\sigma^2 S^2}{2}V_{ss} - rV = 0 \tag{5.2.2}$$

By solving these two PDE’s we can extent our analysis to replicate more complicated payoffs such as bear call spreads and bull call spreads. Examination could be made to derive the appropriate value of λ that gives the most suitable call premium value for the written option, according to the resulting Value at Risk.

What is more, we can examine payoffs produced by positions in put options by deriving the corresponding non-linear PDE’s. In addition, analysis can be made to payoffs created by combinations in both European call and put options. The case of a straddle position has been examined by Wang *et al* [19], therefore one could, similarly, examine the case of butterfly or box spreads.

Furthermore, the analysis can be extended to other more complicated types of options such as Asian and American options, where optimal exercise boundaries have to be taken into consideration.

Lastly, the model can be extended to assume the presence of transaction costs and that the volatility is stochastic, a study that has also been undertaken by Gondzio *et al* [24]. This will enable us to price and hedge such contingent claims more accurately.

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