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Kamran, Tayyab and Plymen, Roger

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K-THEORY AND THE CONNECTION INDEX

TAYYAB KAMRAN AND ROGER PLYMEN

Abstract. Let $\mathcal{G}$ denote a split simply connected almost simple p-adic group. The classical example is the special linear group $\mathrm{SL}(n)$. We study the K-theory of the unramified unitary principal series of $\mathcal{G}$ and prove that the rank of K_0 is the connection index $f(\mathcal{G})$. We relate this result to a recent refinement of the Baum-Connes conjecture, and show explicitly how generators of K_0 contribute to the K-theory of the Iwahori C^*-algebra $\mathfrak{I}(\mathcal{G})$.

1. Introduction

Let $\mathcal{G}$ denote a split simply connected almost simple p-adic group. The classical example is the special linear group $\mathrm{SL}(n)$. We study the K-theory of the unramified unitary principal series of $\mathcal{G}$ and prove that the rank of K_0 is the connection index $f(\mathcal{G})$.

In the spirit of noncommutative geometry, we construct a noncommutative C^*-algebra, the *spherical* C^*-algebra $\mathfrak{S}(\mathcal{G})$. The primitive ideal spectrum $\mathrm{Prim}\,\mathfrak{S}(\mathcal{G})$ of $\mathfrak{S}(\mathcal{G})$ can be identified with the irreducible representations in the unramified unitary principal series of $\mathcal{G}$. The C^*-algebra $\mathfrak{S}(\mathcal{G})$ is a direct summand of the Iwahori C^*-algebra $\mathfrak{I}(\mathcal{G})$, and so contributes to the K-theory of $\mathfrak{I}(\mathcal{G})$.

We relate this result with the recent conjecture in [4, §7]: this is a version, adapted to the K-theory of C^*-algebras, of the geometric conjecture developed in [2],[3],[4],[5].

Quite specifically, let F be a local nonarchimedean field of characteristic 0, let $\mathcal{G}$ be the group of F-rational points in a split, almost simple, simply connected, semisimple linear algebraic group defined over F, for example $\mathrm{SL}(n)$. Let $\mathcal{T}$ denote a maximal split torus of $\mathcal{G}$. Let $\mathcal{G}^\vee$, $\mathcal{T}^\vee$ denote the Langlands dual groups, and let G, T denote maximal compact subgroups:

$$G \subset \mathcal{G}^\vee, \quad T \subset \mathcal{T}^\vee.$$

Then G is a compact connected Lie group, of adjoint type, with maximal torus T.

Let $\pi_1(G)$ denote the fundamental group of G. The order of $\pi_1(G)$ is the *connection index*. The connection index is a numerical invariant

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attached to $\mathcal{G}$ denoted $f(\mathcal{G})$. The notation f is due to Bourbaki [10, VI, p.240].

The fundamental groups are listed in [10, Plates I–X, p.265–292]. They are

$$\mathbb{Z}/(n+1)\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/4\mathbb{Z}\ (n\ odd), \quad \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\ (n\ even)$$

$$\mathbb{Z}/3\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z}, \quad 0, \quad 0, \quad 0$$

for $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$, respectively.

The primitive ideal spectrum Prim $\mathfrak{S}(\mathcal{G})$ can be identified with the unramified unitary principal series of $\mathcal{G}$. In this identification, each element in Prim $\mathfrak{S}(\mathcal{G})$ is an irreducible constituent of a representation induced, by parabolic induction, from an unramified unitary character of $\mathcal{T}$.

When we compare the R-group computations in [13] with the above list, we see that *the maximal R-groups are isomorphic to the fundamental groups.* In this article, we explain this fact, via a geometrical approach.

The L-packet in the unramified unitary principal series of $\mathcal{G}$ with the maximal number of irreducible constituents has $f(\mathcal{G})$ constituents.

Theorem 1.1. *Let $\mathfrak{S}(\mathcal{G})$ denote the spherical C^*-algebra and let $f(\mathcal{G})$ be the connection index of $\mathcal{G}$. Then $K_0\,\mathfrak{S}(\mathcal{G})$ is a free abelian group on $f(\mathcal{G})$ generators, and $K_1\,\mathfrak{S}(\mathcal{G}) = 0$.*

In Theorem (6.1) we show explicitly how generators of $K_0\,\mathfrak{S}(\mathcal{G})$ contribute to the K-theory of the Iwahori C^*-algebra $\mathfrak{I}(\mathcal{G})$.

Intuitively, we can observe a deformation retraction of the spherical C^*-algebra $\mathfrak{S}(\mathcal{G})$ onto the L-packet with $f(\mathcal{G})$ constituents. This L-packet is tempered. When $\mathcal{G} = \mathrm{SL}(n, F)$, this L-packet is elliptic; see [11, Theorem 3.4]. Elliptic representations share with the discrete series the property that their Harish-Chandra characters are not identically zero on the regular elliptic set; and K_0 sees these elliptic representations as if they were $f(\mathcal{G})$ isolated points in the discrete series.

2. The alcoves in the Lie algebra of T

The proof depends on the distinction between the affine Weyl group W_a and the extended affine Weyl group W_a'. The quotient W_a'/W_a is a finite abelian group which dominates the discussion.

Our reference at this point is [10, IX, p.309–327]. Let $\mathfrak{t}$ denote the Lie algebra of T, and let $\exp : \mathfrak{t} \to T$ denote the exponential map. The kernel of exp is denoted $\Gamma(T)$. The inclusion $\iota : T \to G$ induces the homomorphism $\pi_1(\iota) : \pi_1(T) \to \pi_1(G)$. Now $f(G,T)$ will denote the composite of the canonical isomorphism from $\Gamma(T)$ to $\pi_1(T)$ and the homomorphism $\pi_1(\iota)$:

$$f(G,T) : \Gamma(T) \simeq \pi_1(T) \to \pi_1(G).$$

Denote by $N(G,T)$ the kernel of $f(G,T)$. We have a short exact sequence

$$0 \to N(G,T) \to \Gamma(T) \to \pi_1(G).$$

Denote by $N_G(T)$ the normalizer of T in G. Let W denote the Weyl group $N_G(T)/T$. The affine Weyl group is $W_a = N(G,T) \rtimes W$ and the extended affine Weyl group is $W'_a = \Gamma(T) \rtimes W$; the subgroup W_a of W'_a is normal.

If $\mathfrak{t} - \mathfrak{t}_r$ denotes the union of the singular hyperplanes in $\mathfrak{t}$, then the *alcoves* of $\mathfrak{t}$ are the connected component of $\mathfrak{t}_r$.

The group W_a operates simply-transitively on the set of alcoves. Let A be an an alcove. Then $\overline{A}$ is a fundamental domain for the operation of W_a on $\mathfrak{t}$.

Let H_A be the stabilizer of A in W'_a. Then H_A is a finite abelian group which can be identified naturally with $\pi_1(G)$, see [10, IX, p.326]. The extended affine Weyl group W'_a is the semi-direct product

$$W'_a = W_a \rtimes H_A.$$

View $\mathfrak{t}$ as an additive group, and form the Euclidean group $\mathfrak{t} \rtimes O(\mathfrak{t})$. We have $W'_a \subset \mathfrak{t} \rtimes O(\mathfrak{t})$ and so W'_a acts as affine transformations of $\mathfrak{t}$. Now H_A leaves $\overline{A}$ invariant, so H_A acts as affine transformations of $\overline{A}$. Let $v_0, v_1, \ldots, v_n$ be the vertices of the simplex $\overline{A}$. We will use barycentric coordinates, so that

$$x = \sum_{i=0}^{n} t_i v_i$$

with $x \in \overline{A}$. The barycentre x_0 has coordinates $t_j = 1,\ 0 \le j \le n$ and so is H_A-fixed. Then $\overline{A}$ is equivariantly contractible to x_0:

$$r_t(x) := t x_0 + (1-t)x \tag{1}$$

with $0 \le t \le 1$. This is an affine H_A-equivariant retract from $\overline{A}$ to x_0.

Lemma 2.1. *Let $t_0 = \exp x_0$. There is a canonical isomorphism*

$$W(t_0) \simeq H_A.$$

Proof. Let $w \in W$. We have $w \cdot t_0 = t_0$ if and only if there exists $\gamma \in \Gamma(T)$, uniquely determined by w, for which $\gamma(w(x_0)) = x_0$. But γw will fix x_0 if and only if γw stabilizes A, i.e. $\gamma w \in H_A$. This determines the isomorphism

$$W(t_0) \simeq H_A, \quad w \mapsto \gamma w$$

□

In the special case of SL(3), the vector space $\mathfrak{t}$ is the Euclidean plane $\mathbb{R}^2$. The singular hyperplanes tessellate $\mathbb{R}^2$ into equilateral triangles. The interior of each equilateral triangle is an alcove. Barycentric subdivision refines this tessellation into isosceles triangles. The extended affine Weyl group W'_a acts simply transitively on the set of these isosceles triangles, but the closure $\overline{\Delta}$ of one such triangle is not a fundamental domain for the action of W'_a. The corresponding quotient space is [10, IX.§5.2]:

$$\mathfrak{t}/W'_a \simeq \overline{A}/H_A.$$

The abelian group H_A is the cyclic group $\mathbb{Z}/3\mathbb{Z}$ which acts on $\overline{A}$ by rotation about the barycentre of $\overline{A}$ through $2\pi/3$.

3. The spherical C^*-algebra

We will focus on the C^*-summand $\mathfrak{S}(\mathcal{G})$ in the reduced C^*-algebra $C^*_r(\mathcal{G})$ which corresponds to the unramified unitary principal series, see [16]. The algebra $C^*_r(\mathcal{G})$ is defined as follows. We choose a left-invariant Haar measure on $\mathcal{G}$, and form a Hilbert space $L^2(\mathcal{G})$. The left regular representation λ of $L^1(\mathcal{G})$ on $L^2(\mathcal{G})$ is given by

$$(\lambda(f))(h) = f * h$$

where $f \in L^1(\mathcal{G})$, $h \in L^2(\mathcal{G})$ and $*$ denotes the convolution. The C^*-algebra generated by the image of λ is the reduced C^*-algebra $C^*_r(\mathcal{G})$. Let $\mathfrak{K}$ denote the C^*-algebra of compact operators on the standard Hilbert space H.

As in §1, the Langlands dual of $\mathcal{G}$ is the complex reductive group $\mathcal{G}^\vee$ with maximal torus $\mathcal{T}^\vee$. Let G be a maximal compact subgroup of $\mathcal{G}^\vee$, let T be the maximal compact subgroup of $\mathcal{T}^\vee$.

The group $\Psi(\mathcal{T})$ of unramified unitary characters of $\mathcal{T}$ is isomorphic to T. The *spherical* C^*-algebra is given by the fixed point algebra

$$\begin{aligned}\mathfrak{S}(\mathcal{G}) :&= C(T, \mathfrak{K})^W \\ &= \{f \in C(T, \mathfrak{K}) : f(wt) = \mathfrak{c}(w : t) \cdot f(t), w \in W\}\end{aligned}$$

where $\mathfrak{a}(w : t)$ are normalized intertwining operators, and

$$\mathfrak{c}(w : t) = \operatorname{Ad} \mathfrak{a}(w : t)$$

as in [16]. Then $\mathfrak{a} : W \to C(T, U(H))$ is a 1-cocycle:

$$\mathfrak{a}(w_2 w_1 : t) = \mathfrak{a}(w_2 : w_1 t)\mathfrak{a}(w_1 : t).$$

Theorem 3.1. *The group $K_0(\mathfrak{S}(\mathcal{G}))$ is free abelian on $f(\mathcal{G})$ generators, and $K_1 = 0$.*

Proof. We have the exponential map $\exp : \mathfrak{t} \to T$. We lift f from T to a periodic function F on $\mathfrak{t}$, and lift $\mathfrak{a}$ from a 1-cocycle $\mathfrak{a} : W \to C(T, U(H))$ to a 1-cocycle $\mathfrak{b} : W'_a \to C(\mathfrak{t}, U(H))$:

$$F(x) := f(\exp x), \quad \mathfrak{b}(w' : x) := \mathfrak{a}(w : \exp x)$$

with $w' = (\gamma, w)$. The semidirect product rule is

$$w'_1 w'_2 = (\gamma_1, w_1)(\gamma_2, w_2) = (\gamma_1 w_1(\gamma_2), w_1 w_2).$$

Note that $\mathfrak{b}$ is still a 1-cocycle:

$$\begin{aligned}\mathfrak{b}(w_2'w_1' : x) =&\mathfrak{a}(w_2w_1 : \exp x)\\ =&\mathfrak{a}(w_2 : w_1(\exp x)\mathfrak{a}(w_1 : \exp x)\\ =&\mathfrak{a}(w_2 : \exp w_1x)\mathfrak{b}(w_1' : x)\\ =&\mathfrak{a}(w_2 : \exp \gamma w_1x)\mathfrak{b}(w_1' : x)\\ =&\mathfrak{b}(w_2' : w_1'x)\mathfrak{b}(w_1' : x)\end{aligned}$$

Now we define

$$\mathfrak{d}(w : x) := Ad\,\mathfrak{b}(w : x), \qquad w \in W_a'$$

The fixed algebra $\mathfrak{S}(\mathcal{G})$ is as follows:

$$\{F \in C(\mathfrak{t}, \mathfrak{K}) : F(wx) = \mathfrak{d}(w : x) \cdot F(x), w \in W_a',\ x \in \mathfrak{t},\ F \text{ periodic}\}$$

Now F is determined by its restriction to $\overline{A}$. Upon restriction, we obtain

$$\mathfrak{S}(\mathcal{G}) \simeq \{f \in C(\overline{A}, \mathfrak{K}) : f(wx) = \mathfrak{d}(w : x) \cdot f(x), w \in H_A,\ x \in \overline{A}\}$$

We will write $H = H_A$. Let

$$\mathfrak{A} := C(\overline{A}, \mathfrak{K}), \qquad \mathfrak{B} := C(\overline{A}, \mathfrak{L}(V))$$

so that $\mathfrak{S}(\mathcal{G}) = \mathfrak{A}^H$.

We apply [15, Theorem 2.13]. We have to verify that all three conditions of this theorem are met. Let $R(t_0)$ denote the R-group attached to t_0. The calculations of Keys[13] show that $R(t_0) = W(t_0)$ and so condition (**) in [15] is met. We note that $\{\{\mathfrak{b}(w : x) : w \in H_A\} = \{\mathfrak{a}(w : t) : w \in W(t_0)\}$ and that $w \mapsto \mathfrak{a}(w : t_0)$ is a unitary representation of H_A. For this representation we have

$$\mathfrak{a}(w : t_0) = \rho_1(w)P_1 + \cdots + \rho_f(w)fP_f$$

where $\rho_1, \ldots, \rho_f$ are the characters of H_A, and $P_1, \ldots, P_f$ are the orthogonal projections onto the f irreducible subspaces $V_1, \ldots, V_f$ of the induced representation $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}\, t_0$. The representation $\mathfrak{a}(- : t_0)$ is quasi-equivalent to the regular representation of H_A, i.e. it contains the same characters (with different multiplicity), and so condition (***) is met. Let $W(t)$ denote the isotropy subgroup of $t \in T$. We have to compare the two following representations of representations of $W(t)$:

$$w \mapsto \mathfrak{a}(w : t_0)_{|W(t)}, \qquad w \mapsto \mathfrak{a}(w : t) \tag{2}$$

These two representations are quasi-equivalent, again by inspecting the results of Keys [13]. Since f can be non-prime only in cases A_n and D_n, these are the only two cases to be checked. In each subspace $V_1, \ldots, V_f$ choose an increasing sequence $e_1^n, \ldots, e_f^n$ of projections tending strongly to the identity operator on that subspace, and set $e_n = e_1^n + \cdots + e_f^n$. The compressions, with respect to the projections (e_n), of the two representations in (2) remain quasi-equivalent for each n: they each contain every character of $W(t)$. The condition (*) is now met.

The three conditions of [15, Theorem 2.13] are met. This yields a strong Morita equivalence

$$\mathfrak{A}^H \simeq_{Morita} C(\overline{A}) \rtimes H$$

We recall that

$$C(\overline{A}) \rtimes H \simeq C(\overline{A}, \operatorname{End} \mathbb{C}H)^H$$

where H acts via the regular representation on its group algebra $\mathbb{C}H$. Define

$$\mathfrak{M} := C(\overline{A}, \operatorname{End} \mathbb{C}H)^H, \qquad \mathfrak{N} := (\operatorname{End} \mathbb{C}H)^H$$

and define the homomorphisms

$$f : \mathfrak{M} \to \mathfrak{N}, \quad \sigma \mapsto \sigma(x_0)$$

$$g : \mathfrak{N} \mapsto \mathfrak{M}, \quad Y \mapsto Y(-).$$

Then $(f \circ g) = id_{\mathfrak{N}}$ and $g \circ f$ sends the map σ to the constant map $\sigma(x_0)(-)$. With r_t as in Eqn.(1), we define

$$F_t(\sigma) = \sigma \circ r_t$$

then $F_1 = g \circ f$ and $F_0 = id_{\mathfrak{M}}$. So $g \circ f \sim_h id_{\mathfrak{M}}$. Therefore, $\mathfrak{M}$ and $\mathfrak{N}$ are homotopy equivalent C^*-algebras, and have the same K-theory. The fixed C^*-algebra $\mathfrak{M}$ is homotopy equivalent to its fibre $\mathfrak{N}$ over the fixed point x_0.

We therefore have

$$\mathfrak{S}(\mathcal{G}) = \mathfrak{A}^H \sim_{Morita} \mathfrak{M} \sim_h \mathfrak{N} \simeq \mathbb{C}^f$$

The C^*-algebra functors K_0 and K_1 are invariants of strong Morita equivalence and of homotopy type, whence

$$K_0\, \mathfrak{S}(\mathcal{G}) = \mathbb{Z}^f$$
$$K_1\, \mathfrak{S}(\mathcal{G}) = 0$$

where $f = f(\mathcal{G})$. □

Let $\mathcal{J}$ be a good maximal compact subgroup of $\mathcal{G}$. Choose left-invariant Haar measure μ on $\mathcal{G}$. Let

$$\begin{aligned} e_{\mathcal{J}}(x) &= \mu(\mathcal{J})^{-1} \text{ if } x \in \mathcal{J} \\ &= 0 \qquad\quad\ \text{ if } x \notin \mathcal{J} \end{aligned}$$

Then $e_{\mathcal{J}}$ is an idempotent in the reduced C^*-algebra $C_r^*(\mathcal{G})$. Let $\mathcal{J}_1, \ldots, \mathcal{J}_f$ be an enumeration of the good maximal compact subgroups of $\mathcal{G}$, one from each conjugacy class in $\mathcal{G}$. Let $e_{\mathcal{J}_1}, \ldots, e_{\mathcal{J}_f}$ be the corresponding idempotents. Their Fourier transforms are rank-one projections in $\mathfrak{K}(E_1), \ldots, \mathfrak{K}(E_f)$, and serve as generators for K_0.

4. The L-packet with $f(\mathcal{G})$ constituents

The H_A-fixed point $t_0 \in T$ determines an L-parameter:

$$\phi : W_F \times \mathrm{SL}(2, \mathbb{C}) \to G, \qquad (y\Phi_F^n, Y) \mapsto t_0^n$$

with W_F the Weil group of F, I_F the inertia subgroup of W_F, $y \in I_F$, Φ_F a geometric Frobenius in W_F, $Y \in \mathrm{SL}(2, \mathbb{C})$.

The L-parameter ϕ determines an L-packet $\Pi(t_0)$. The constituents of this L-packet have Kazhdan-Lusztig parameters

$$\{(t_0, 1, \rho) : \rho \in \mathbf{Irr}\, \pi_0\, \mathcal{Z}_G(t_0)\}$$

where $\mathcal{Z}_G(t_0)$ is the centralizer of t_0 in G. We have

$$\mathcal{Z}_G(t_0) = \mathcal{T}^\vee \cdot W(t_0)$$

and so

$$\pi_0\, \mathcal{Z}_G(t_0) = W(t_0).$$

By Lemma (2.1), we have $W(t_0) \simeq H_A$. It follows that the third Kazhdan-Lusztig parameter ρ is a character of the abelian group H_A. All such characters are allowed. Since H_A can be naturally identified with $\pi_1(G)$, the order of H_A is the connection index $f(\mathcal{G})$. As a consequence, the number of irreducible constituents in the L-packet $\Pi(t_0)$ is equal to the connection index $f(\mathcal{G})$.

5. The Baum-Connes correspondence: a refinement

Let $\mathcal{G}$ be a reductive p-adic group. The Baum-Connes correspondence is a definite isomorphism of abelian groups

$$K_j^{top}(\mathcal{G}) \simeq K_j\, C_r^*(\mathcal{G})$$

with $j = 0, 1$, see [14]. The left-hand-side, defined in terms of K-cycles, has never been directly computed for a noncommutative reductive p-adic group. A result of Higson-Nistor [12] and Schneider [17] allows us to replace the left-hand-side with the chamber homology groups. Chamber homology has been directly computed for only two noncommutative p-adic groups: SL(2), see [8] and GL(3), see [6]. In the case of GL(3), one can be sure that representative cycles in all the homology groups have been constructed only by checking with the right-hand-side. In other words, one always has to have an independent computation of the right-hand-side.

We now reflect on the conjecture in §7 of [4]. This is the geometric conjecture developed in [2],[3],[4],[5] adapted to the K-theory of C^*-algebras. We will focus on the Iwahori C^*-algebra $\mathfrak{I}(\mathcal{G}) \subset C_r^*(\mathcal{G})$. The primitive ideal spectrum of $\mathfrak{I}(\mathcal{G})$ can be identified with the irreducible tempered representations of $\mathcal{G}$ which admit nonzero Iwahori-fixed vectors.

In this special case, the conjecture in [4, §7] asserts that

$$K_j(\mathfrak{I}(\mathcal{G})) = K_W^j(T) \tag{3}$$

with $j = 0, 1$. Here $K_W^j(T)$ is the classical topological equivariant K-theory [1, §2.3] for the Weyl group W acting on the compact torus T.

We quote a recent theorem of Solleveld:

$$K_j(\mathfrak{I}(\mathcal{G})) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq K_j(C^*(W'_a)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

with $j = 0, 1$. This theorem is a special case of [18, Theorem 5.1.4] when $\Gamma = 1$. We have

$$\begin{aligned} C^*(W'_a) &= C^*(\Gamma(T) \rtimes W) \\ &\simeq C(T) \rtimes W \end{aligned}$$

by a standard Fourier transform. By the Green-Julg theorem [9, Theorem 11.7.1], we have

$$K_j(C(T) \rtimes W) \simeq K_W^j(T).$$

Therefore we have

$$K_j(\mathfrak{I}(\mathcal{G})) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq K_W^j(T) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{4}$$

so that Solleveld establishes Eqn.(3) *modulo torsion.*

Applying the equivariant Chern character for discrete groups [7] gives a map

$$ch_W : K_W^j(T) \to \bigoplus_l H^{j+2l}(T/\!/W; \mathbb{C}) \tag{5}$$

which becomes an isomorphism when $K_W^j(T)$ is tensored with $\mathbb{C}$. Hence the geometric conjecture at the level of C^*-algebra K-theory gives a much finer and more precise formula for $K_* C_r^*(\mathcal{G})$ than Baum-Connes alone provides; see [4].

In the formula (5) for the equivariant Chern character, $T/\!/W$ denotes the extended quotient of T by W. This is easily computed, as is clear from its definition, which we recall briefly. The quotient T/W is obtained by collapsing each orbit to a point and is again a compact Hausdorff space. For $t \in T$, $W(t)$ denotes the stabilizer group of t:

$$W(t) := \{w \in W : wt = t\}.$$

Denote by $c(W(t))$ the set of conjugacy classes of $W(t)$. The extended quotient, denoted $T/\!/W$, is constructed by replacing each orbit with $c(W(t))$ where t can be any point in the orbit. This construction is done as follows. First, set

$$\widetilde{T} := \{(w, t) \in W \times T : wt = t\}.$$

Then $\widetilde{T} \subset W \times T$. The group W acts on $\widetilde{T}$:

$$W \times \widetilde{T} \to \widetilde{T}, \qquad \alpha(w, t) = (\alpha w \alpha^{-1}, \alpha t)$$

with $(w,t) \in \widetilde{T}, \alpha \in W$. The extended quotient is defined by

$$T//W := \widetilde{T}/W.$$

Hence the extended quotient $T//W$ is the ordinary quotient for the action of W on $\widetilde{T}$.

6. Equivariant line bundles over T

It follows from the C^*-Plancherel theorem [16] that the C^*-ideal $\mathfrak{S}(\mathcal{G})$ is a direct summand of the C^*-algebra $\mathfrak{I}(\mathcal{G})$. We therefore have

$$K_0\,\mathfrak{S}(\mathcal{G}) \hookrightarrow K_0\,\mathfrak{I}(\mathcal{G}). \tag{6}$$

When we combine (6) with (4), we obtain

$$K_0\,\mathfrak{S}(\mathcal{G}) \hookrightarrow K_W^0(T) \otimes \mathbb{C} \tag{7}$$

We will now make the map (7) explicit. We recall that the group $\psi(\mathcal{T})$ of unramified unitary characters of the maximal torus $\mathcal{T}$ is isomorphic to the compact torus T. The Hilbert space of the induced representation

$$\pi_t := \mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}\, t$$

will be denoted E_t with $t \in T$. We recall from §4 that

$$\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}\, t_0 = \Pi(t_0)$$

is an L-packet with f irreducible constituents.

The collection $\{E_t : t \in T\}$ forms a W-equivariant bundle of Hilbert spaces via the action of the intertwining operators:

$$\mathfrak{a}(w:t) : E_t \to E_{wt}$$

We will construct 1-dimensional sub-bundles of fixed vectors as follows. Let $\mathcal{J}$ be a good maximal compact subgroup of $\mathcal{G}$. Let $e_{\mathcal{J}}$ be the corresponding idempotent in $C_r^*(\mathcal{G})$. Let π be a unitary representation of $\mathcal{G}$ on the Hilbert space E. We define the linear operator

$$\pi(\phi) := \int_{\mathcal{G}} \phi(g)\pi(g)dg$$

whenever ϕ is a suitable test-function. When $\phi = e_{\mathcal{J}}$, the linear operator $\pi(e_{\mathcal{J}})$ is a projection onto the subspace $E^{\mathcal{J}}$ of $\mathcal{J}$-fixed vectors. As we vary π in the primitive ideal spectrum $\mathrm{Prim}\,\mathfrak{S}(\mathcal{G})$, we obtain a continuous field of rank 1 projections over the compact torus T. This idempotent projects onto the complex hermitian line bundle $E^{\mathcal{J}}$ over T of $\mathcal{J}$-fixed vectors. The fibre of $E^{\mathcal{J}}$ at the point t is given by

$$E_t^{\mathcal{J}} = \pi_t(e_{\mathcal{J}})E_t$$

This creates a W-equivariant line bundle over T. As we vary $\mathcal{J}$ among $\mathcal{J}_1, \ldots, \mathcal{J}_f$, we obtain f distinct W-equivariant line bundles over T. To show that they are distinct, we proceed as follows.

Let $\mathcal{J} = \mathcal{J}_k$. Then, for each $1 \leq k \leq f$, the fibre of $E^{\mathcal{J}}$ at the point $t_0 \in T$ is a 1-dimensional subspace of the irreducible subspace V_k of the L-packet $\Pi(t_0)$, by a result of Keys on class-1 representations [Theorem, §4][13]. The irreducible subspaces $V_1, \ldots, V_f$ are *inequivalent* representations of $\mathcal{G}$, and so the W-equivariant line bundles $E^{\mathcal{J}}$ with $1 \leq k \leq f$ are distinct. This establishes the next result.

Theorem 6.1. *The injective homomorphism (7) is given explicitly by*

$$K_0\,\mathfrak{S}(\mathcal{G}) \hookrightarrow K^0_W(T) \otimes \mathbb{C}, \qquad e_{\mathcal{J}} \mapsto E^{\mathcal{J}}$$

with $\mathcal{J} = \mathcal{J}_1, \ldots, \mathcal{J}_f$. The image of this map is a free abelian group of rank f.

For the exceptional groups $\mathcal{G} = E_8,\ F_4$ or G_2, the connection index $f = 1$, and so the map (6) is especially simple. There is no L-packet. The K-theory of the unramified unitary principal series of $\mathcal{G}$ is that of a point.

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T. Kamran: Centre for Advanced Mathematics and Physics, National University of Sciences and Technology, H-12, Islamabad, Pakistan
E-mail address: tkamran@camp.nust.edu.pk

R. Plymen: School of Mathematics, Manchester University, Manchester M13 9PL, England *and* School of Mathematics, Southampton University, SO17 1BJ, England
E-mail address: plymen@manchester.ac.uk, r.j.plymen@soton.ac.uk