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Divisibility provides the ideal of non-units of the Steenrod algebra A over the Galois field $\mathbf{F}_2$ with the structure of a partially ordered set. This can be examined for features of lattice structure. For example, the Steenrod squares Sq^1 and Sq^2 have a least upper bound, namely, $\mathrm{Sq}^1 \vee \mathrm{Sq}^2 = \mathrm{Sq}^3\mathrm{Sq}^1$. We show that a subposet consisting of the top elements of certain of the Poincaré duality subalgebras of A forms a lattice which can be identified with the lattice of column 2-regular partitions.

The subalgebras, under containment, provide another version of the lattice. as do the collections of the principal ideals of the top elements and of their annihilators. As the Steenrod algebra is local and locally finite, these subalgebras and ideals are associated with finite and with infinite subgroups of its unit group, the sets of which also form isomorphic lattices.

These results are formulated and proved in Sect. 1. The lattice of column 2-regular partitions itself is discussed in Sect. 2, where features of relevance to the Steenrod algebra are elaborated. For example, there is an algorithm for retrieving each such top element from the lattice alone. The setting of this section is that of our lattice realised as the lattice of finite subsets of positive integers in an order derivable from the product lattice on the product of countable many copies of the integers.

Notation. Vectors of non-negative integers appear here in several contexts. We find it convenient to assume that they are of a uniform and therefore infinite dimension. Most of our vectors, e.g., of indices or as partitions, have only a finite number of non-zero entries and so we adopt the setting of the direct sum $\bigoplus_\omega \mathbf{N}$ of a countable number of copies of the additive monoid $\mathbf{N}$ of non-negative integers (the set of positive integers is denoted by $\mathbf{P}$).

Vectors of indices are generally denoted by bold Roman letters, e.g., $\mathbf{x} = (x_1, x_2, \cdots), x_i \geq 0$. Partitions, however, are written as Greek letters. For example, ν indicates here an (infinite) sequence of non-negative integers $\nu = (\nu_1, \nu_2, \cdots)$, which is assumed to be non-increasing. Its length $\ell(\nu)$ is thus the largest integer ℓ for which $\nu_\ell \neq 0$. We take $\mathbf{0} := (0, 0, \cdots)$ as a partition, the partition of 0; its length $\ell(\mathbf{0}) = 0$. Vectors with infinitely many non-zero entries also appear, e.g., $\mathbf{1} := (1, 1, \cdots)$. For a partition ν, the elements $2^\nu := (2^{\nu_1}, 2^{\nu_2}, \cdots) \in \prod_\omega \mathbf{P}$ and $2^\nu - \mathbf{1} \in \bigoplus_\omega \mathbf{N}$ are important here.

We say that p *divides* q *on the left* if there is r such that $q = pr$ and we write $p|q$ (with $p|_\ell q$ and $p|_r q$ used to distinguish between left and right divisibility when required).

Other notation generally follows the usage in [Wo98] for the Steenrod algebra and in [St97] for lattices. These references also serve as a source of definitions.

A partition μ is *column p-regular* for a prime p if $\mu_i - \mu_{i+1} < p$ for all i, $i \geq 1$. Note that, with the convention that partitions are infinite vectors, the last non-zero entry of a (non-zero) column 2-regular partition is a 1. Partitions which are column p-regular play a significant role in the modular representation theory of the symmetric and general linear groups [Gr80; JK81]. As the Steenrod algebra is also relevant to the representation theoretic study of these groups in characteristic p, it is not surprising that such partitions manifest themselves in A. Indeed, for $p = 2$, Walker and Wood [WW01] show that a column 2-regular partition gives rise to an element of A which is related to the irreducible module corresponding to the partition in the representation theory of the monoid $\mathrm{M}_n(\mathbf{F}_2)$ of all $n \times n$ matrices over $\mathbf{F}_2$.

The *Young diagram* of a partition μ is a left-justified array consisting of $\ell(\mu)$ rows of juxtaposed squares, of which the ith contains exactly μ_i squares, $1 \leq i \leq \ell(\mu)$. This pictorial description of partitions makes it easy to appreciate the fact that the set of all partitions forms a lattice under the partial ordering of containment of Young diagrams. Formally, $\mu \leq \nu$ if $\mu_i \leq \nu_i$ for all i, $i \geq 1$. Union and intersection of Young diagrams serve to define the lattice operations of join and meet respectively. In terms of partitions, $(\mu \vee \nu)_i = \max\{\mu_i, \nu_i\}$ and $(\mu \wedge \nu)_i = \min\{\mu_i, \nu_i\}$ for all i, $i \geq 1$. The set of column 2-regular partitions forms a sublattice Λ of the lattice of all partitions.

1. The lattice in the Steenrod algebra

In the Steenrod algebra, the lattice Λ manifests itself in a variety of ways. It occurs as a lattice of elements, of subalgebras, of ideals, right and left, and of subgroups. The manifestation in terms of elements is the key to all the others. To each column 2-regular partition μ we associate an element t_μ defined as the Milnor basis element $\mathrm{Sq}(2^\mu - \mathbf{1})$. This notation mimics the use of $t_n := t_{(n+1,n,\cdots,2,1)}$ for the top element of the Poincaré duality algebra $A(n)$. For $n \geq 0$, $A(n)$ denotes the unital subalgebra of A generated by the Steenrod squares $\mathrm{Sq}^{2^i}, 0 \leq i \leq n$. We extend the notation here to the case $n = -1$ by defining t_{-1} as $t_{\mathbf{0}} = \mathrm{Sq}(\mathbf{0}) = 1$ and $A(-1)$ as $\mathbf{F}_2$. Each column 2-regular partition μ is associated with a subalgebra of A, denoted here as $A(\mu)$ and defined by a basis, namely, all Milnor basis elements $\mathrm{Sq}(\mathbf{s}), \mathbf{s} \leq 2^\mu - \mathbf{1}$. The fact that $A(\mu)$ is a Hopf subalgebra of A, and consequently a Poincaré duality algebra, whose top element is t_μ, is discussed below. In this notation, $A(n) = A((n{+}1, n, \cdots, 2, 1))$.

In terms of the Milnor basis, multiplication on the left by t_μ can be described explicitly.

1.1 Proposition. Let μ be a column 2-regular partition and let $\mathbf{s} \in \bigoplus_\omega \mathbf{N}$. Then

$$t_\mu \mathrm{Sq}(\mathbf{s}) = \begin{cases} \mathrm{Sq}((2^\mu - \mathbf{1}) + \mathbf{s}) & \text{if } 2^\mu \text{ divides } \mathbf{s}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. In calculating products of Milnor basis elements, we follow the (almost compatible) conventions of [Ma83, p. 228ff.; Wo98, p. 455ff.]. Let X be an allowable matrix for the product $\mathrm{Sq}(2^\mu - \mathbf{1})\mathrm{Sq}(\mathbf{s})$. For $k \geq 2$, we call the entries $\{x_{i,k-i} \mid 1 \leq i \leq k-1\}$ the *interior* entries of the kth diagonal of X.

Suppose that $\beta(X) = 1$. We show by induction on k "down from infinity" that all interior entries of X vanish. Assume then that the interior entries of all diagonals to the right of the kth vanish and that $k \geq 3$. Therefore $x_{k,0} = 2^{\mu_k} - 1$. As μ is column 2-regular, if $1 \leq i \leq k-1$, then $\mu_i \leq \mu_k + (k-i)$. As $0 \leq x_{i,0} = (2^{\mu_i} - 1) - \Sigma_{j \geq 1} 2^j x_{i,j}$, it follows that

$$x_{i,0} < 2^{\mu_k + (k-i)} - 2^{k-i} x_{i,k-i} = 2^{k-i}(2^{\mu_k} - x_{i,k-i}),$$

and hence $2^{\mu_k} > x_{i,k-i} \geq 0$. Unless $x_{i,k-i} = 0$, the multinomial coefficient corresponding to the kth diagonal is even and so $\beta(X) = 0$, contrary to hypothesis. Thus the interior entries of the kth diagonal vanish.

We have shown that the only allowable matrix which can contribute a term to the product is the *trivial* allowable matrix, i.e., that whose first column is $(*, 2^\mu - \mathbf{1})$ and whose first row is $(*, \mathbf{s})$. In this case the term in question is $\mathrm{Sq}((2^\mu - \mathbf{1}) + \mathbf{s})$. The coefficient β associated with this matrix is 1 if and only if, for all k, the multinomial coefficient $(2^{\mu_k} - 1, 0, \cdots, 0, s_k) = (2^{\mu_k} - 1, s_k)$ is odd. Suppose that $s_k \equiv r \bmod 2^{\mu_k}$, where $0 \leq r \leq 2^{\mu_k} - 1$. Then $(2^{\mu_k} - 1, s_k) \equiv (2^{\mu_k} - 1, r) \bmod 2$, even if $\mu_k = 0$. But this is 0 unless $r = 0$, i.e., unless 2^{μ_k} divides s_k. Consequently, $\beta = 1$ if and only if 2^μ divides $\mathbf{s}$. But this is the desired conclusion. □

We turn now to the manifestation of Λ in the structure of A. The ideals enter into the exposition through the elements t_μ. Left multiplication by t_μ is an endomorphism of A considered as right A-module. Its image $t_\mu A$ is a (principal) right ideal, and its kernel $t_\mu^\perp$, the right annihilator of t_μ, is also a right ideal. As non-zero ideals, these are infinite [Sa04].

The next observation, an immediate corollary of Prop. 1.1, shows that the right ideals associated with t_μ have bases consisting of Milnor basis elements.

1.2 Proposition. Let μ be a column 2-regular partition. Then $t_\mu A$ has basis consisting of all $\mathrm{Sq}(\mathbf{t})$ for which $2^\mu \mid \mathbf{t} + \mathbf{1}$ and $t_\mu^\perp$ has basis consisting of all $\mathrm{Sq}(\mathbf{t})$ for which $2^\mu \not\mid \mathbf{t}$.

We can now state and prove the main result. In order to simplify the statement, we omit the quantifier, that is, an expression $f(\mu)$ is to be understood as the set $\{f(\mu) \mid \mu \text{ column } 2-\text{regular}\}$.

1.3 Theorem. *The following posets are isomorphic to the lattice* Λ*:*

1. $(2^\mu, \mid)$	3. $(t_\mu, \mid_\ell)$	5. $(t_\mu A, \supseteq)$	7. $(t_\mu^\perp, \subseteq)$
2. $(A(\mu), \subseteq)$	4. $(t_\mu, \mid_r)$	6. $(At_\mu, \supseteq)$	8. $({}^\perp t_\mu, \subseteq)$.

Proof. The observation concerning the vectors 2^μ is immediate from the definition used here for divisibility of vectors, i.e., $\mathbf{v}|\mathbf{w}$ if $v_i|w_i$ for all i. The fact that $A(\mu) \subseteq A(\nu)$ if and only if $\mu \leq \nu$ is clear from the bases given for these subalgebras.

The assertions about left divisibility of the elements t_μ follow from Prop. 1.1. If $\mu \leq \nu$, then

$$t_\nu = \mathrm{Sq}(2^\nu - \mathbf{1}) = \mathrm{Sq}(2^\mu - \mathbf{1})\mathrm{Sq}(2^\nu - 2^\mu) = t_\mu \mathrm{Sq}(2^\nu - 2^\mu).$$

Conversely, if $t_\mu | t_\nu$, then there is an $\mathbf{s}$ such that $t_\nu = t_\mu \mathrm{Sq}(\mathbf{s})$; necessarily, $\mathbf{s} = 2^\nu - 2^\mu$ and 2^μ divides 2^ν.

For right divisibility we use the conjugate χ of A. As $A(\mu)$ is a Hopf subalgebra of A, $\chi(A(\mu)) = A(\mu)$. Furthermore, $\chi(t_\mu) = t_\mu$, and so $t_\nu = t_\mu a$ if and only if $t_\nu = \chi(a) t_\mu$.

The assertions concerning principal ideals are now immediate from general principles of ring theory. For example, if x divides y on the left in a unital ring R, then $xR \supseteq yR$, and conversely.

For the assertions concerning annihilators note that, as a general principle, if x divides y on the right, then $x^\perp \subseteq y^\perp$, but the converse need not hold. Use of the conjugate shows that $\chi(x^\perp) = {}^\perp\chi(x)$ for $x \in A$. Thus, to complete the proof it suffices to show that $t_\mu^\perp \subseteq t_\nu^\perp$ implies that $2^\mu \mid 2^\nu$.

If not then there is an index i for which $\nu_i < \mu_i$. Define the vector κ via

$$\kappa_j := \begin{cases} \max\{\mu_j, \nu_j\} & \text{if } j \neq i, \\ \nu_i & \text{if } j = i. \end{cases}$$

But then 2^ν divides 2^κ whereas 2^μ does not. Thus, by Prop. 1.2, $\mathrm{Sq}(2^\kappa) \in t_\mu^\perp$ while $\mathrm{Sq}(2^\kappa) \notin t_\nu^\perp$, which is the desired contradiction. □

Remark. Lattices 5-8 are not sublattices of the lattice of ideals of A. In general, $t_\mu A \vee t_\nu A$ is not $t_\mu A + t_\nu A$ although $t_\mu A \wedge t_\nu A = t_\mu A \cap t_\nu A$; in contrast, $t_\mu^\perp \vee t_\nu^\perp = t_\mu^\perp + t_\nu^\perp$ while $t_\mu^\perp \wedge t_\nu^\perp \neq t_\mu^\perp \cap t_\nu^\perp$ in general.

To place the lattice Λ in context in A, several classes of Milnor basis element are relevant, namely, those $\mathrm{Sq}(\mathbf{r})$ for which:

I: $\mathbf{r} = 2^{\mathbf{u}} - \mathbf{1}$ for $\mathbf{u} \in \bigoplus_\omega \mathbf{N}$;
II: $\mathbf{r} = 2^\lambda - \mathbf{1}$ for a partition λ (non-increasing in our convention);
III: $\mathbf{r} = 2^\mu - \mathbf{1}$ for a column 2-regular partition μ;
IV: $\mathbf{r} = 2^{\mathbf{u}} - \mathbf{1}$ for $\mathbf{u} \in \bigoplus_\omega \mathbf{N}$, where $\mathbf{u}$ is such that, for all $i, j \geq 1$, $u_i \leq j + u_{i+j}$ or $u_j \leq u_{i+j}$;
V: $\mathbf{r}$ satisfies the condition: for all $i \geq 1$, $r_i \equiv -1 \bmod 2^{\omega(r_{i+1})}$, where $\omega(r)$ is the minimal non-negative exponent such that $2^{\omega(r)} > r$.

The condition defining class V is Monks' criterion for $\mathrm{Sq}(\mathbf{r})$ to be an admissible basis element (see [Mo98; CWW98]). That defining class IV is the criterion of Adams and Margolis for $\mathrm{Sq}(\mathbf{r})$ to be the top element of a Poincaré duality algebra, here denoted $A(\mathbf{r})$ (the earlier notation $A(\mu)$ can be viewed as an abbreviation for $A(2^\mu - \mathbf{1})$, an unambiguous notation). The Milnor basis elements of classes II and III mediate a role played by the Steenrod algebra in the representation theory of the full matrix algebra $\mathrm{M}_n(\mathbf{F}_2)$ (see [WW01, Thm. 5.1]). Class II is the intersection of classes I and V (its elements are called Milnor spikes in [WW01]). Class III, that which gives rise to Λ in A, is the intersection of classes II and IV and so also of classes IV and V.

Class IV is canonically determined within A in that Adams and Margolis showed that the corresponding algebras $A(\mathbf{r})$ are the only finite Hopf subalgebras of A. They also showed that each such $A(\mathbf{r})$ is a Poincaré duality algebra and that it has a basis consisting of all $\mathrm{Sq}(\mathbf{s}), \mathbf{s} \leq \mathbf{r}$ (see [AM74; Ma83, p. 233]). Whether class III is also canonical is an interesting issue.

2. The lattice of column 2-regular partitions

The lattice of column 2-regular partitions is one that admits a number of combinatorial interpretations. The Young diagram of a partition μ suggests another and related partition, namely, that corresponding to the columns of squares instead of the rows. This is the partition μ', the *conjugate* of μ. A partition is (*row*) *p-regular* for a prime p if its conjugate is column p-regular. Consequently, a partition is p-regular if none of its (non-zero) entries appears p times. It is clear from Young diagrams that the assignment $\mu \mapsto \mu'$ is an automorphism of order 2 of the lattice of partitions. Thus the collection of p-regular partitions forms a sublattice of the lattice of all partitions which is isomorphic to that of column p-regular partitions.

When $p = 2$, the conjugate of a column 2-regular partition admits a rather different interpretation. Let μ be column 2-regular and let $\lambda = \mu'$. Then, with $\ell = \ell(\lambda)$ and with $1 \leq i, j \leq \ell$, it is the case that $\lambda_i = \lambda_j$ if and only if $i = j$; that is, the partition λ has distinct parts and so can be viewed as a finite subset S of $\mathbf{P}$. Conversely, if S is such a subset written as $\{s_1 > s_2 > \cdots > s_\ell\}$, then the sequence λ defined by setting $\lambda_i = s_i$ if $1 \leq i \leq \ell = |S|$ and $\lambda_i = 0$ otherwise, is a partition whose conjugate μ is column 2-regular. Note that the numbers s_i indicate the positions at which the entries in the non-increasing sequence μ decrease by 1, i.e., $\mu_{s_i} = 1 + \mu_{1+s_i}$; indeed, $\mu_j = i$ if and only if $s_i \geq j > s_{i+1}$ (read s_0 as ∞). In the Steenrod algebra the lattice of 2-regular partitions has an association with the admissible basis. The results of the previous section given in terms of the Milnor basis have analogues in terms of the admissible basis.

In the interpretation in terms of partitions having distinct parts, Λ may be identified with the additive submonoid $\wp$ of $\bigoplus_\omega \mathbf{N}$ consisting of all sequences $(s_1, s_2, \cdots)$ in which $s_i \geq s_{i+1}$ for all $i \geq 1$ and either $s_i > s_{i+1}$ or $s_i = 0$. The subset $\wp$ is a locally finite sublattice of $\prod_\omega \mathbf{N}$ interpreted as the product lattice $\prod_\omega(\mathbf{N}, \leq)$. As lattice of column 2-regular partitions, Λ also has interpretations in product lattices. Via the vectors 2^μ, it is a sublattice of $\prod_\omega \mathbf{P}$ interpreted as the product lattice $\prod_\omega(\mathbf{P}, \leq)$. or, alternatively, as $\prod_\omega(\mathbf{P}, |)$, an interpretation more apt in our applications and that which is given in Theorem 1.3(1).

A generic construction on posets also gives rise to Λ. If $(P, \leq)$ is a poset and $\mathcal{P}^{\mathrm{fin}}(P)$ is the set of finite subsets of P, then, for $S, T \in \mathcal{P}^{\mathrm{fin}}(P)$, say that $S \leq T$ if there exists an injective mapping $\phi : S \longrightarrow T$ which is *non-decreasing* in the sense that $s \leq \phi(s)$ for all $s \in S$. The existence of such maps between subsets defines a poset structure on $\mathcal{P}^{\mathrm{fin}}(P)$, denoted $(\mathcal{P}^{\mathrm{fin}}(P), \leq)$. In the case $(\mathbf{P}, \leq)$ this construction produces a lattice $(\mathcal{P}^{\mathrm{fin}}(\mathbf{P}), \leq)$ which is precisely the finite subset version of Λ.

Additional conditions which give rise to interesting (weak) subposets may also be specified; for example:

- $\phi(s) = s$ for all $s \in S$;
- if $s' \leq s$ in S, then $\phi(s') \leq \phi(s)$ in T;
- $s' \leq s$ in S if and only if $\phi(s') \leq \phi(s)$ in T, and, if there are $s' \in S, t \in T$ such that $\phi(s') \leq t$, then there is $s \in S$ such that $t = \phi(s)$.

The first condition, containment, is the most familiar but makes no use of the poset structure on P. The second corresponds to monomorphisms on posets, i.e., order-preserving injections. The third is obtained from isomorphisms. Applied in the case of $\mathbf{P}$ the third condition provides another realisation of Λ in its interpretation as the lattice of 2-regular partitions.

The lattice Λ has many notable features. For example, it has trivial automorphism group. The proof of this assertion provides an (exceedingly slow) algorithm for reconstructing intrinsically the labels attached to its elements in, e.g., its interpretation as $(\mathcal{P}^{\mathrm{fin}}(\mathbf{P}), \leq)$. This can be accomplished by using the cover graph of Λ.

In $(\mathcal{P}^{\mathrm{fin}}(P), \leq)$, the criterion for a subset T to cover a subset S is that either $T = S \cup \{t\}$ for a minimal element t of P or there are elements $t \in T$ and $s \in S$ such that t covers s in P and $T - \{t\} = S - \{s\}$. Its interpretation for $(\mathcal{P}^{\mathrm{fin}}(\mathbf{P}), \leq)$ is given in the next lemma. It may also be obtained from the fact that covers in $(\wp, \leq)$ are determined as in a product of posets, namely, the entries in all components coincide except for one component in which one entry covers the other. The criterion has the consequence that $\mathcal{P}^{\mathrm{fin}}(\mathbf{P})$ is a graded lattice whose rank function ρ is given as $\rho(S) = \Sigma S$.

2.1 Lemma. *Let T and S be finite subsets of $\mathbf{P}$. Then T covers S if and only if there is $h \geq 0$ such that $T - \{h+1\} = S - \{h\}$, $h+1 \in T$ and either $h \in S$ or $h = 0$.*

For use in the proof of the automorphism result, we describe the conditions above in more detail. Each non-empty finite subset T of $\mathbf{P}$ has a unique minimal expression as a disjoint union of intervals, written here as decreasing intervals to match the earlier partition convention, i.e., $[x, y] := \{n \in \mathbf{P} \mid x \geq n \geq y\}$. Thus, for $T = \{t_1 > \cdots > t_\ell\}$, we write $T = \cup_{1 \leq j \leq m} I_j, I_j = I_j(T) = [t_{i_{j-1}+1}, t_{i_j}]$ with $i_0 := 0 < \cdots < i_m = \ell$. By minimality, the only $t \in T$ for which $t - 1 \notin T$ are the t_{i_j} so that T covers precisely m subsets.

We denote by $\partial_j T$ the subset obtained by replacing t_{i_j} by $t_{i_j} - 1$ or by omitting t_{i_j} if equal to 1, in accordance with the previous lemma. We write the canonical decomposition of $\partial_j T$ as $\partial_j T = \cup_k I_{j,k}(T)$, where $I_{j,k}(T) := I_k(\partial_j T)$. Clearly, $I_{j,k}(T) = I_k(T)$ if $1 \leq k \leq j-1$. If $j < m$ and $I_j(T)$ is a singleton, i.e., $i_{j-1}+1 = i_j$, then

$$I_{j,j}(T) = [t_{i_j}-1, t_{i_j}-1] \text{ and, if } j+1 \leq k \leq m, I_{j,k}(T) = I_k(T)$$

if $t_{i_j}-2 > t_{i_j+1}$ while

$$I_{j,j}(T) = [t_{i_j}-1, t_{i_{j+1}}] \text{ and, if } j+1 \leq k \leq m-1, I_{j,k}(T) = I_{k+1}(T)$$

if $t_{i_j}-2 = t_{i_j+1}$. If $j = m$ and $I_m(T)$ is a singleton, then

$$I_{m,m}(T) = [t_\ell - 1, t_\ell - 1]$$

if $t_\ell > 1$ and is not present if $t_\ell = 1$ (for $m = 1$, consider $\emptyset$ as the decomposition of itself).

In the case in which $I_j(T)$ is not a singleton, if $j < m$, then

$$I_{j,j}(T) = [t_{i_{j-1}+1}, t_{i_j}+1], I_{j,j+1}(T) = [t_{i_j}-1, t_{i_j}-1]$$

and, if $j+2 \leq k \leq m+1$, $I_{j,k}(T) = I_{k-1}(T)$ if $t_{i_j}-2 > t_{i_j+1}$ while

$$I_{j,j}(T) = [t_{i_{j-1}+1}, t_{i_j}+1], I_{j,j+1}(T) = [t_{i_j}-1, t_{i_{j+1}}]$$

and, if $j+2 \leq k \leq m$, $I_{j,k}(T) = I_k(T)$ if $t_{i_j}-2 = t_{i_j+1}$. If $j = m$ and $I_m(T)$ is not a singleton, then

$$I_{m,m}(T) = [t_{i_{m-1}+1}, t_\ell + 1] \text{ and } I_{m,m+1}(T) = [t_\ell - 1, t_\ell - 1]$$

if $t_\ell > 1$ while

$$I_{m,m}(T) = [t_{i_{m-1}+1}, 2]$$

if $t_\ell = 1$.

2.2 Notation. The collection of subsets covered by T is denoted by cov^T. Thus, if $T \neq \emptyset$, $|\mathrm{cov}^T| = m$, the number of intervals in the canonical decomposition of T, and

$$\mathrm{cov}^T = \{\partial_j T \mid 1 \leq j \leq m\}.$$

The collection of subsets which cover T is denoted by cov_T. It is again the case that $|\mathrm{cov}_T| = m$ (with the convention concerning the decomposition of $\emptyset$).

2.3 Theorem. *The lattice of column 2-regular partitions has trivial automorphism group.*

Proof. Let $\phi \in \mathrm{Aut}(\mathcal{P}^{\mathrm{fin}}(\mathbf{P}), \leq)$. Let T be a finite subset of $\mathbf{P}$ and let $U = \phi(T)$. We show that $U = T$ by induction on the layers of Λ, i.e., on $\rho(T)$. In general, each layer is determined by the preceding layer in the sense that each element is determined by the elements which it covers. This fails, however, in precisely one case: $\mathrm{cov}^{\{3\}} = \{\{2\}\} = \mathrm{cov}^{\{2,1\}}$. For this reason, individual arguments must be made for each element of the first four layers. We assume then that $\rho(T) \geq 4$.

By induction, ϕ fixes each of the elements of cov^T. As ϕ is an automorphism,

$$\text{cov}^U = \phi(\text{cov}^T) := \{\phi(S) \mid S \in \text{cov}^T\} = \text{cov}^T.$$

With $m = |\text{cov}^T|$, if $m = 1$, i.e., $T = [t_1, t_\ell]$, there are four patterns for cov^T depending on whether or not $t_1 = t_\ell$ and on whether or not $t_\ell = 1$. It is readily seen that T is determined by cov^T except for the case $T = \{2\}$.

Assume then that $m > 1$. Note first that t_1 is determined, i.e., $t_1 = u_1$, as

$$t_1 = \max \cup \text{cov}^T = \max \cup \text{cov}^U,$$

as can be seen from the description of the sets $\partial_j T$.

We next show that $I_1(T)$ is determined by cov^T. This interval is a singleton if and only if there is $X \in \text{cov}^T$ for which $t_1 \notin X$. As the same applies to U, $I_1(T) = I_1(U)$ in this case.

Suppose that $I_1(T)$ is not a singleton. Then

$$I_1(T) = \cup_{X \in \text{cov}^T} I_1(X)$$

so that again $I_1(T) = I_1(U)$. Moreover, there is a unique $X \in \text{cov}^T$ for which $I_1(X) \neq I_1(T)$, namely, $X = \partial_1 T = \partial_1 U$. The interval $I_2(X) = I_{1,2}(T) = I_{1,2}(U)$ is a singleton if and only if $t_{i_1} - 2 > t_{i_1+1}$ if and only if $u_{i_1} - 2 > u_{i_1+1}$ (note that i_1 is unambiguous as $t_{i_1} = t_1 - (i_1 - 1)$). The remaining intervals in the decomposition of X coincide with those of T and of U, that is, if $2 \leq j \leq m$, then $I_j(U) = I_{j+1}(X) = I_j(T)$. If $I_2(X)$ is not a singleton, then $I_2(X) = [t_{i_1} - 1, t_{i_2}]$ so that $I_2(U) = I_2(T)$; further, if $3 \leq j \leq m$, $I_j(U) = I_j(X) = I_j(T)$. In all cases we conclude that $U = \cup_j I_j(U) = \cup_j I_j(T) = T$ as required. $\square$

While Λ may admit no non-trivial automorphisms, it has infinitely many self-embeddings: for example, every non-decreasing injection from $\mathbf{P}$ to itself induces an embedding.

The sequences $\{n\}$ and $[n]$ are readily distinguished within $(\mathcal{P}^{\text{fin}}(\mathbf{P}), \leq)$. The subalgebras of A to which they correspond are also significant. The first corresponds to a chain of exterior algebras (see Sect. 5.2 of [Wo98]), the second to the series of subalgebras $A(n)$. In terms of partitions, for column 2-regular partitions having exactly ℓ parts, $\ell \geq 0$, the top elements of these subalgebras are the Milnor basis elements associated with the "lightest" partition, i.e., that μ for which $\Sigma\mu := \Sigma_{i \geq 1} \mu_i$ is minimal, and with the "heaviest", i.e., that for which $\Sigma\mu$ is maximal, namely, $\text{Sq}(1, 1, \cdots, 1, 1)$ in which the rightmost 1 appears in the ℓth coordinate, and $\text{Sq}(2^\ell - 1, 2^{\ell-1} - 1, \cdots, 3, 1)$. Features which distinguish these elements are set out in our next result, whose proof is straightforward. Recall that (non-empty) intervals are distinguished from other subsets by the fact that they cover precisely one element.

2.4 Proposition. *Let $T \in \mathcal{P}^{\mathrm{fin}}(\mathbf{P})$. Then* cov^T *consists of an interval if and only if there is $n \geq 1$ for which $T = \{n\}$ or $T = [n]$, while* cov_T *consists of an interval if and only if $T = \emptyset$ or there is $n \geq 1$ for which $T = \{n\}$.*

If the lexicographic order on $\mathcal{P}^{\mathrm{fin}}(\mathbf{P})$ is also brought into play, then each layer of $(\mathcal{P}^{\mathrm{fin}}(\mathbf{P}), \leq)$ is totally ordered. This can provide a systematic approach to drawing a Hasse diagram for Λ. Forming the set of maximal elements over all layers of positive rank, we obtain the set of subsets $\{n\}$, $n \geq 1$. The set of minimal elements contains not only the subsets $[n]$ but also the subsets in the the intervals $[\,[n], [n+1]\,]$, $n \geq 1$. Both sets of extreme elements are saturated chains in $(\mathcal{P}^{\mathrm{fin}}(\mathbf{P}), \leq)$; they then serve as the two "sides" of the Hasse diagram.

Certain aspects of the elements of $\mathcal{P}^{\mathrm{fin}}(\mathbf{P})$ can be determined locally from the cover graph, as the next result illustrates. Its interpretation in the Steenrod algebra is that, in the Milnor basis expression for a top element in the lattice, whether the initial entries coincide can be determined locally.

2.5 Proposition. *Let T be a finite subset of* $\mathbf{P}$. *Whether $1 \in T$ can be determined from the ball of radius 2 centred on the node corresponding to T in the cover graph of Λ.*

Proof. (Sketch) For $i \geq 1$, let k^i denote the number of elements of cov^T whose canonical decompositions consist of i intervals. These numbers can thus be calculated by reference to nodes of the cover graph at distance ≤ 2 from the node corresponding to T. Let $k_i, i \geq 1$, denote the analogous numbers of elements obtainable from cov_T.

Suppose that the canonical decomposition of T has m intervals, i.e., $m = |\mathrm{cov}^T| = |\mathrm{cov}_T|$. Analysis of the decompositions of the elements of cov^T and of cov_T shows that

$$1 \in T \text{ if and only if } k_m + 2k_{m-1} = k^m + 2k^{m-1}.$$

(There is such a formula in the contrary case as well:

$$1 \notin T \text{ if and only if } k_m + 2k_{m-1} = k^m + 2k^{m-1} + 1.)$$

□

There is a way of looking at Λ which corresponds well to the subalgebras $A(n)$ of A. As $(\mathcal{P}^{\mathrm{fin}}(\mathbf{P}), \leq)$, it can be seen as the union of its sublattices $\mathcal{P}([n])$. The union can be viewed as proceeding by a form of doubling. For $n \geq 0$, $(\mathcal{P}([n+1]), \leq)$ consists of two copies of $(\mathcal{P}([n]), \leq)$, namely, $\mathcal{P}([n])$ itself and $\{\{n+1\} \cup S \mid S \in \mathcal{P}([n])\}$ with the induced lattice structure on each together with, to give only the covers, all relations $(\{n+1\} \cup S, \{n\} \cup S)$, where $S \in \mathcal{P}([n-1])$.

An alternative approach to some of this material is available by using trees. In the lattice $(\mathcal{P}^{\mathrm{fin}}(\mathbf{P}), \leq)$ there is a subposet which has the structure of an infinite binary tree and which signals significant aspects of A. A realisation of the infinite binary tree convenient for our purpose is the following. Its vertices are the non-zero vectors of $\bigoplus_\omega \mathbf{F}_2$, i.e., vectors of countable length which have only finitely many non-zero entries; its edges are the pairs $\{\mathbf{x}^-, \mathbf{x}\}$, where this ordering serves to define a poset structure, i.e., $\mathbf{x}^- \leq \mathbf{x}$ (here $\mathbf{x}^-$ denotes the vector $\mathbf{x}$ shifted to the left one place, i.e., $\mathbf{x}^- := (x_2, x_3, \cdots)$). There is a monomorphism from the tree to $(\mathcal{P}^{\mathrm{fin}}(\mathbf{P}), \leq)$ given by sending $\mathbf{x}$ to $\{i \mid x_i = 1\}$. Note that $(1, 0, 0, \cdots)$ is the root of the tree, that only the empty set is not in the image of the monomorphism and that a pair $\{\mathbf{x}^-, \mathbf{x}\}$ is sent to one of the form $\{S - 1, S\}$, where $S - 1 := \{s - 1 \mid s \in S\} - \{0\}$. The image of this tree in the lattice $(\mathcal{P}^{\mathrm{fin}}(\mathbf{P}), \leq)$ can be envisaged so that the path ascending from the root along one "side" of the tree has the subsets $\{\ell\}$ as its vertices corresponding to the sequence of elements $\mathrm{Sq}(1, 1, \cdots, 1)$, ℓ 1's, and the path along the other "side" has the subsets $[\ell]$ as its vertices corresponding to the sequence of elements $t_{\ell-1}$. The latter sequence of top elements is used in [Wo98] as the basis of the strapping technique in A. Each infinite path ascending from the root gives rise to an analogous sequence of elements of the Steenrod algebra. An analogue of strapping can be formulated in each case but apparently without the potency of Wood's usage.

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