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Multiple attractors in grazing-sliding bifurcations in an explicit example of a Filippov type flow

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Abstract

We present a constructed explicit example of a three-dimensional Filippov type flow where we show the birth of multiple attractors in grazing-sliding bifurcations. To the best of our knowledge, it is the first such an example of a Filippov type flow where grazing-sliding bifurcation is shown to trigger birth of multiple attractors, reported in the literature. Three qualitatively different scenarios are shown; namely, birth of period-two and period-three stable orbits with one sliding segment, chaotic attractor coexisting with stable period-three orbit characterised by a segment of sliding motion, and a coexistence of a period-three sliding orbit with two sliding segments and a limit cycle with no sliding segments. Our work reveals an important feature of the normal form map used to construct the Filippov flow that would produce the desired dynamics. Namely, due the fact that the normal form that we use is valid only locally around the grazing-sliding bifurcations, the scale of the variation of the bifurcation parameter past the grazing-sliding had to be carefully chosen to see the dynamics predicted by the map. In other words, sufficiently small neighbourhood where the normal form is valid, in the context of nonsmooth bifurcations, seems to mean different order of magnitude in the range of the bifurcation parameter variation than in the context of smooth bifurcations.

Keywords: grazing-sliding, multiple attractors, Filippov flows, explicit example
2000 MSC: 05.45.Ac, 05.45 -a, 02.30.Oz

1. Introduction

The dynamics of systems characterised by the continuous and discrete evolution has recently been given much research attention, see for instance [1, 2, 3, 4, 5]. This interest is motivated by the use of digital devices (microprocessors), which communicate with sensors that operate in continuous time, in many industries, e.g. car industry, telecommunications or aerospace. Also, on the macroscopic level, there are many physical processes which can be modelled using a combination of continuous and discrete dynamics. To give a few examples: genetic regulatory networks [6] have recently been modelled using switched vector fields, similarly power electronic converters [7, 8], mechanical systems friction, [9] or modern control systems [10].

Such systems have been shown to undergo topological transitions (bifurcations) which are triggered solely due to the combination of both – the continuous and discrete dynamics. In particular, systems characterised by the presence of manifolds in phase space where continuous time evolution undergoes a sudden jump, either in state variables or its derivatives, have been shown to exhibit so called discontinuity induced bifurcations (DIBs for short). These manifolds where switchings take place are termed in the literature as switching manifolds or discontinuity sets. A DIB occurs if, for instance, a limit cycle exhibits a grazing contact with a discontinuity set. It has been shown that such an instance may cause a sudden onset of chaotic dynamics [11]. Another intriguing scenario, observed solely in systems characterised by the presence of discontinuity manifolds, is the possibility of the birth of multiple attractor triggered by a DIB [12, 2]. However, so far, such scenario has been reported to occur only in piecewise smooth maps. In the current paper, for the first time, we present an example of a Filippov type flow, that is a flow generated by the vector field which is discontinuous across the switching surface, where grazing-sliding bifurcation leads to the onset of multiple attractors. We believe that the multiple attractors found in our model example may occur in many systems of relevance to applications, and perhaps they can be utilised in developing control strategies; depending on the application certain type of an attractor may have a more desirable characteristics than another one, and by appropriate choice of system parameters we may force our system on the desired attractor. Of course, these are speculations, and their validity can only be verified by further research.

The rest of the paper is outlined as follows. In Sec. 2 the methodology used to construct the explicit examples is shown. In the following Sec. 3, the derivation of the linearised return map about the grazing-sliding bifurcation having sufficient freedom to allow for the presence of complex Floquet multipliers characterising the grazing orbit, in the absence of an interaction with the switching surface, is presented. The method of determining the sliding vector field is shown in Sec. 4, and Sec. 5 is then devoted to presenting different cases of multistability according to our recent finding presented in [13]. Finally, Sec. 6 concludes the paper.

2. A Model System

The starting point, in the construction of a three-dimensional Filippov system with grazing and sliding for which all the relevant features (return map, switching surface, the coefficient of the sliding vector field) can be calculated explicitly, is a simple underlying flow with a periodic orbit. This flow needs to be chosen so that the (linearized) return map about the periodic orbit is easy to calculate analytically with sufficient freedom to be able to match any required linearization.

There are, no doubt, many ways to achieve this. We have chosen to start with a two-dimensional flow such that one axis is invariant, then to interpret the invariant axis as the $z-$axis and restrict to the invariant positive half-plane in the other variable (r). By arranging for the existence of a stable fixed point in $r > 0$ the simple expedient of rotating the entire half-plane by introducing a polar angle θ with $\dot{\theta} > 0$ produces a flow on $\mathbb{R}^3$ with a stable periodic orbit having fixed r and z coordinates. We start by considering the motion in the $(r, z)-$ plane, i.e.

$$\begin{aligned} \dot{r} &= -4kr + 2rz \\ \dot{z} &= 4\alpha^2 - r^2 - z^2. \end{aligned} \tag{1}$$

Fixed points of this equation have $r = 0$ or $z = 2k$ (from the $\dot{r}$ equation) and then if $r = 0$, $z = \pm 2\alpha$; whilst if $z = 2k$ then $r = 2\sqrt{\alpha^2 - k^2}$ provided

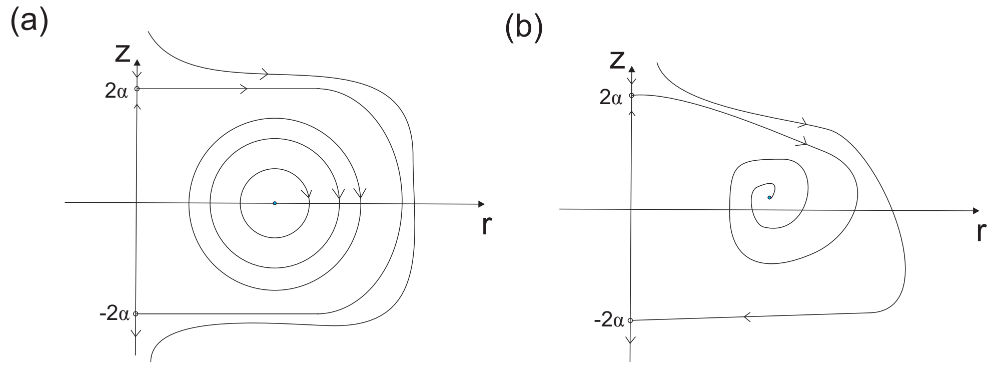

Figure 1: Schematic solutions of the two dimensional flow: (a) $k = 0$; (b) $k > 0$, k small.

$\alpha > k$. Fixed points in $r > 0$ correspond to periodic orbits of the full three-dimensional system, whilst fixed points with $r = 0$ are also fixed points of the full system. If $k = 0$ (1) is Hamiltonian, with

$$H(z,r) = 4\alpha^2 r - rz^2 - \frac{1}{3}r^3$$

which is how we chose this example, and the fixed point in $r > 0$ is stable if $k > 0$ and unstable if $k < 0$, and is a focus if $5k^2 < 4\alpha^2$ and a node if $5k^2 > 4\alpha^2 > 4k^2$. Sample phase portraits are shown in Figure 1.

Introducing an angle θ with $\dot{\theta} = \omega$, gives a flow in $\mathbb{R}^3$ which is given in cylindrical polar coordinates (r, θ, z) by

$$\begin{aligned} \dot{r} &= -4kr + 2rz \\ \dot{\theta} &= \omega \\ \dot{z} &= 4\alpha^2 - r^2 - z^2 \end{aligned} \tag{2}$$

or, in standard Cartesian coordinates,

$$\begin{aligned} \dot{x} &= -4kx - \omega y + 2xz \\ \dot{y} &= \omega x - 4ky + 2yz \\ \dot{z} &= 4\alpha^2 - x^2 - y^2 - z^2 \end{aligned} \tag{3}$$

This system has a periodic orbit if $\alpha^2 > k^2$ with $r = 2\sqrt{\alpha^2 - k^2}$, $z = 2k$ and $0 \leq \theta < 2\pi$. In the next section we analyze this periodic orbit, deriving an explicit formula for the linear part return map near the periodic orbit on the plane $y = 0$. We can then create a Filippov system by defining a flow on a half-space, e.g. $x < 2\sqrt{\alpha^2 - k^2} + \mu$.

3. The linearized return map

Suppose that $5k^2 < 4\alpha^2$, so the periodic orbit of the full system exists and has period $\frac{2\pi}{\omega}$. Consider a small perturbation starting on the surface $\theta = 0$ at $t = 0$, and write

$$z = 2k + u, \quad r = 2\sqrt{\alpha^2 - k^2} + v.$$

Substituting into (1) and ignoring higher order terms gives the linearized system

$$\begin{aligned} \dot{u} &= -4ku - 4\sqrt{\alpha^2 - k^2}v \\ \dot{v} &= 4\sqrt{\alpha^2 - k^2}u \end{aligned} \tag{4}$$

or

$$\ddot{v} + 4k\dot{v} + 16(\alpha^2 - k^2)v = 0 \tag{5}$$

with characteristic equation $s^2 + 4ks + 16(\alpha^2 - k^2) = 0$ having roots $-2k \pm 2\sqrt{5k^2 - 4\alpha^2}$. Let $\Omega^2 = 4(4\alpha^2 - 5k^2)$, $\Omega > 0$, so the roots are $-2k \pm i\Omega$ and the solution of (5) is

$$v = e^{-2kt}\left(A\cos\Omega t + B\sin\Omega t\right) \tag{6}$$

and the second equation of (4) implies

$$u = \frac{e^{-2kt}}{4\sqrt{\alpha^2 - k^2}}\left((-2kA + B\Omega)\cos\Omega t + (-2kB - A\Omega)\sin\Omega t\right). \tag{7}$$

Now suppose that at $t = 0$ $(v, u) = (v_0, u_0)$, so in terms of the full three-dimensional system u_0 is the perturbation from the periodic orbit in the $z-$coordinate, and v the perturbation in the $x-$coordinate on the plane $y = 0$, then

$$A = v_0, \quad B = \frac{1}{\Omega}(4\sqrt{\alpha^2 - k^2}u_0 + 2kv_0)$$

and after a time of $\frac{2\pi}{\omega}$ the solutions return to the plane $y = 0$ with perturbations (v_1, u_1) where

$$\begin{pmatrix} v_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} v_0 \\ u_0 \end{pmatrix} \tag{8}$$

with, setting $E = \exp(-4\pi k/\omega)$,

$$\begin{aligned} a_{11} &= E(\cos\tfrac{2\pi\Omega}{\omega} + \tfrac{2k}{\Omega}\sin\tfrac{2\pi\Omega}{\omega}) \\ a_{12} &= \tfrac{4\sqrt{\alpha^2 - k^2}E}{\Omega}\sin\tfrac{2\pi\Omega}{\omega} \\ a_{21} &= \tfrac{E}{4\sqrt{\alpha^2 - k^2}}(-\Omega - \tfrac{4k^2}{\Omega})\sin\tfrac{2\pi\Omega}{\omega} \\ a_{22} &= E(\cos\tfrac{2\pi\Omega}{\omega} - \tfrac{2k}{\Omega}\sin\tfrac{2\pi\Omega}{\omega}). \end{aligned} \tag{9}$$

Note that consideration of the divergence of the vector field in (1) shows that the determinant of the matrix (9) is E^2, which can be verified directly from (9), so if we refer to this matrix as R then

$$\det\ R = E^2 = \exp(-8\pi k/\omega), \quad \mathrm{Tr}\ R = 2E\cos\frac{2\pi\Omega}{\omega} \tag{10}$$

and so, with $\omega = 1$, we can choose det R via k and then Tr R using α.

It is equally possible to treat the case of real eigenvalues, but the complex eigenvalue case is the one discussed in [13], so we stop the analysis here.

4. Constructing the sliding motion

In [13] the parameters for multistability are specified in coordinates in which the linear map near the periodic orbit takes the form

$$\begin{pmatrix} T & 1 \\ D & 0 \end{pmatrix} \tag{11}$$

and since we are working in a different set of coordinates we will need to take care that the coefficients are those corresponding to the correct choice of coordinates. For simplicity we will place the sliding bifurcation parameter μ in the switching surface H rather than in the differential equation and consider

$$\dot{\mathbf{x}}(t) = \begin{cases} F_1(\mathbf{x}(t)) & if \quad H(\mathbf{x}(t),\mu) > 0 \\ F_2(\mathbf{x}(t)) & if \quad H(\mathbf{x}(t),\mu) < 0, \end{cases} \tag{12}$$

where F_1, F_2 are sufficiently smooth vector functions, $F_1,\ F_2 : \mathbb{R}^3 \mapsto \mathbb{R}^3$ and $H(\mathbf{x}(t),\mu) : \mathbb{R}^3 \times \mathbb{R} \mapsto \mathbb{R}^3$ is some smooth scalar function depending on system states $\mathbf{x} \in \mathbb{R}^3$, and parameters $\mu \in \mathbb{R}$; $t \in \mathbb{R}$ is the time variable.

We will work with

$$H(\mathbf{x},\mu) = 2\sqrt{\alpha^2 - k^2} + \mu - x \tag{13}$$

and F_1 defined by (1) with the fixed parameters chosen below to satisfy the criteria of [13] for multistability. These choices imply that the periodic orbit analyzed in the previous section lies entirely in the region $H(\mathbf{x},\mu) > 0$ (where the Filippov flow is defined by F_1) provided $\mu > 0$. If $\mu = 0$ then the periodic orbit grazes H at the point $(2\sqrt{\alpha^2 - k^2}, 0, 2k)$, whilst if $\mu < 0$ then the periodic orbit intersects the switching surface transversally at two points (for small μ) and the flow near the periodic orbit will depend on the choice of F_2 in (12). The linear approximation of the PDM (see the appendix, and [5] for a detailed derivation) can be given by

$$PDM(X,Z) = \begin{cases} \mathbf{X} \quad \text{for} \quad X > 0 \\ \begin{pmatrix} 0 & 0 \\ \hat{C} & 1 \end{pmatrix} \begin{pmatrix} X \\ Z \end{pmatrix} \quad \text{for} \quad X \leq 0 \end{cases} \tag{14}$$

in terms of shifted local coordinates $\mathbf{X} = (X, Z)$ on $\{y = 0\}$. The constant $\hat{C}$ is a function of the vector field $F_2 = (F_2^x, F_2^y, F_2^z)^T$ and the jacobian matrix f_{ij} of F_1 evaluated at the grazing point:

$$\hat{C} = -\frac{F_2^z}{F_2^x} + \left(\frac{f_{11}}{f_{12}} + f_{12} \frac{F_2^z}{F_2^x} \right) F_1^z.$$

The choice of PDM in [13] to derive conditions for multistability is in coordinates in which the linear map takes the form given by (11) with the coefficient $\hat{C}$ replaced by C, with the relation between the coefficients given by

$$C = a_{22} + \hat{C} a_{12}.$$

5. The example and multistability

5.1. Case I: Two stable orbits

We now take a specific example of multistability from [13]: $T = 0.05$, $D = 0.31$ and $C = -3$. Using (10) we require

$$0.31 = E^2 = \exp(-8\pi k/\omega), \quad 0.005 = 2E \cos\left(\frac{2\pi\Omega}{\omega}\right).$$

By setting $\omega = 1$ we find $\alpha = 0.08$ and $k = 0.0466$. In our case $F_1^z = 0$ and since $F_2^x < 0$, if we let $F_2^x = -1$, we then have $F_2^z = -4.7192$. A stable period-three orbit with a sliding segment is depicted in Fig. 2 and a stable period-two orbit in Fig. 3.

5.2. Case II: A stable orbit and a chaotic attractor

The second case of multistability reported in [13] is the case of a chaotic attractor coexisting with a stable periodic orbit with sliding. This scenario was found for $T = 0.35$, $D = 0.3$ and $C = -3$. Fixing $\omega = 1$ gives $k = 0.04790$, $\alpha = 0.07297$ and $F_2^Z = -5.0727$ (for $F_2^x = -1$). A stable period-three orbit is depicted in Fig. 4 and a chaotic attractor in Fig. 5.

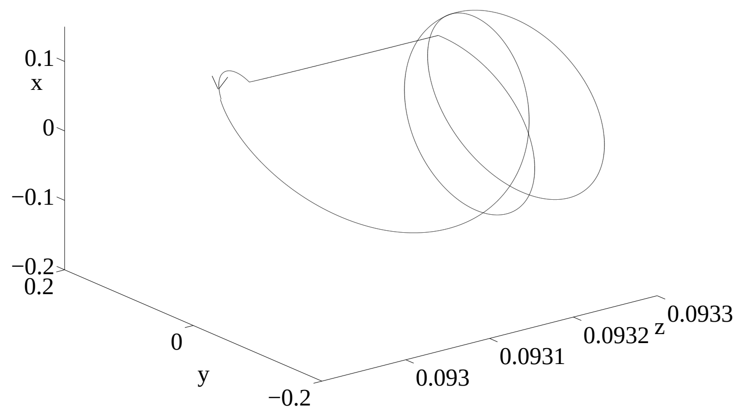

Figure 2: Stable period-3 orbit with sliding for $\omega = 1$, $k = 0.0466$ and $\alpha = 0.08$, with the initial condition $[x,\, y,\, z]^T = [0.036291,\, 0.124810,\, 0.093028]^T$ and $\mu = -0.000005$

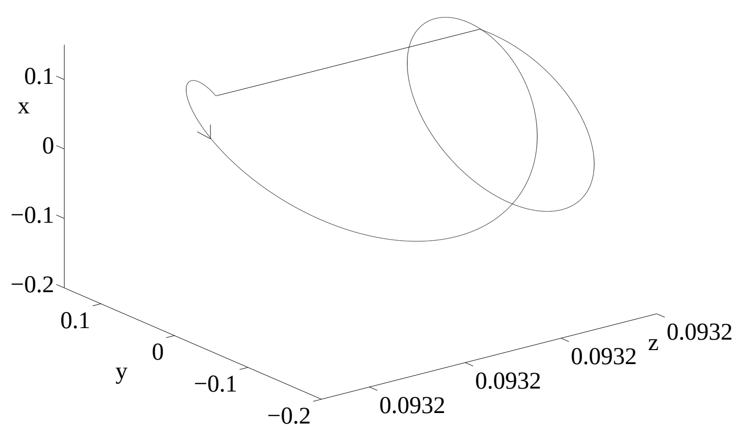

Figure 3: Stable period-2 orbit with sliding for $\omega = 1$, $k = 0.0466$ and $\alpha = 0.08$, with the initial condition $[x,\, y,\, z]^T = [-0.028117,\, -0.126974,\, 0.093209]^T$ and $\mu = -0.000005$

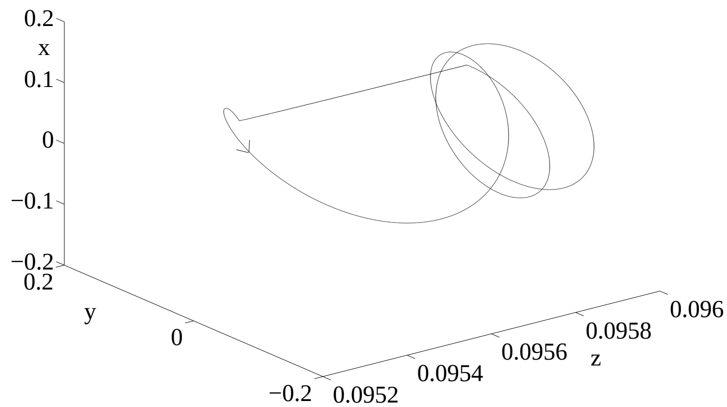

Figure 4: Stable period-3 orbit with sliding for $\omega = 1$, $k = 0.04790$, $\alpha = 0.07297$ and $\mu = -0.00005$, with the initial condition $[x,\, y,\, z]^T = [0.0259013,\, 0.1067204,\, 0.0959262]^T$

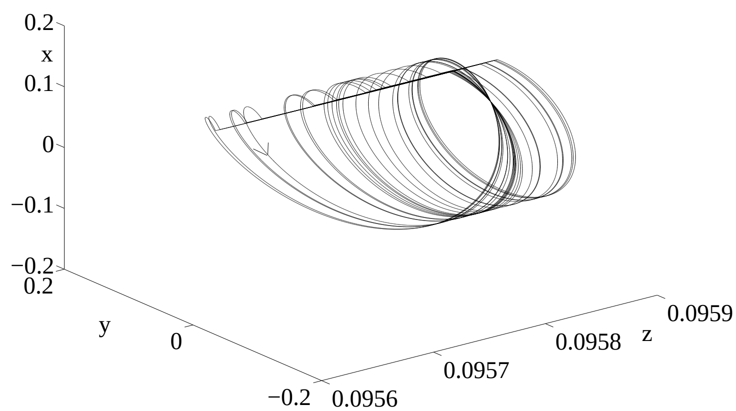

Figure 5: Chaotic attractor for $\omega = 1$, $k = 0.04790$, $\alpha = 0.07297$ and $\mu = -0.00005$. The initial condition $[x,\, y,\, z]^T = [-0.085976,\, -0.068551,\, 0.095775]^T$

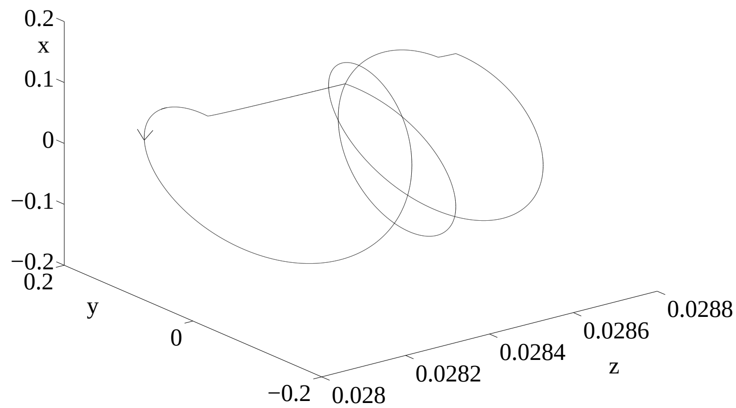

Figure 6: A stable period-3 orbit with two sliding segments for $\omega = 1$, $k = 0.01419$, $\alpha = 0.06679$ and $\mu = 5 \times 10^{-6}$. For the simulations the initial condition $[x, y, z]^T = [2\sqrt{\alpha^2 - k^2} + \mu + 10^{-4}, 10^{-5}, 2k]^T$ was used (we disregard the transient).

5.3. Case III: A stable orbit with sliding and a stable non-sliding orbit

The last case of multistability reported in [13] is that of a stable period-3 orbit coexisting with a stable non-sliding orbit. Representative parameter values at which these attractors where reported to exist are $T = -0.1$, $D = 0.7$, $C = -1.8$ with $\mu > 0$. Fixing $\omega = 1$ gives $k = 0.01419$, $\alpha = 0.06679$, and $F_2^Z = -1.97425287$ with $F_1^X = -1$. A period-three stable orbit detected for $\mu = 5 \times 10^{-6}$ is depicted in Fig. 6, and finally a stable non-sliding orbit is reported in Fig. 7.

6. Conclusions

In the paper we present, for the first time, an example of a explicit Filippov type flow where grazing-sliding bifurcation leads to the onset of multiple attractors. Three qualitatively different scenarios of multiple attractors in grazing-sliding bifurcations are shown. To find the parameter values for which these qualitatively different scenarios occur we use the analytical conditions reported in [13], where the classification of a one-dimensional normal form map for grazing-sliding bifurcations for three-dimensional Filippov type flows was conducted.

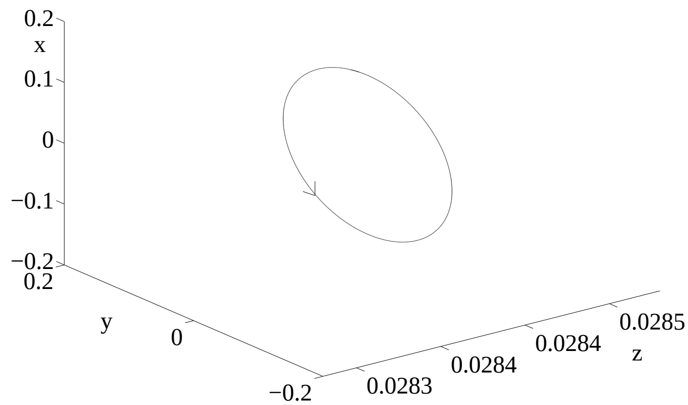

Figure 7: A stable periodic orbit for $\omega = 1$, $k = 0.01419$, $\alpha = 0.06679$ and $\mu = 5.e - 6$. The initial condition on the orbit $[x,\, y,\, z]^T = [0.129880,\, 0.013031,\, 0.028383]^T$

The theoretical predictions verified in this paper are based on the normal form for grazing sliding bifurcations [5]. This normal form is, by definition, local, in that it is derived to model solutions near the periodic orbit at grazing in phase space and near the bifurcation value in parameter space. As with almost all such local theories, the domain on which the normal form is defined is not specified by the analysis, which holds for a 'sufficiently small' neighbourhood of the bifurcation. One of the consequences of our analysis is that it is possible to make some comments about how small 'small' really is in this example.

The attentive reader will have noted that the examples are shown for μ of the order of 10^{-4} to 10^{-6}. For larger, but still small, values of μ we observe dynamics which is different from that predicted, though in some cases it is consistent with the expectations of the normal form, but *not* for the parameter values we have so carefully chosen. In other words, had we simply matched behaviour in the flow with behaviour in the normal form we would have found no contradiction for some examples, but this would have been due to the fact that we were matching to parameters other than those that really apply. It is only at the smaller values of μ used in the reported simulations that the precise predictions of the normal form analysis are observed.

This poses two questions. First, why is 'small' so small (in smooth systems one can frequently get away with 10^{-2} or even larger)? It may of course be bad luck, and induction from a single example is notoriously fool-hardy, but we could speculate that this is generally true for grazing bifurcations; the parameter $\sqrt{\mu}$ intervenes naturally here as the width of the sliding region, and small $\sqrt{\mu}$ implies very small μ. In other words, the corrections to the normal form are order $|x|^{\frac{3}{2}}$ rather than the standard x^2 terms for smooth systems – this is because the PDM is of the form

$$PDM(X, Z, q; \mu) = \begin{pmatrix} X \\ Z \end{pmatrix} + \beta(X, Z, q; \mu)q^2, \tag{15}$$

where $q = \sqrt{-H(x, \mu)}$, so we require q^2 to be small compared to q^3.

The second question is about how normal forms are used. If the aim is simply to make the numerical results more reasonable theoretically, then it might not be necessary to obtain a precise match up between parameters in the normal form and parameters in the flow, but if the aim is to use the normal form as a predictive tool then the examples here underline the importance of ensuring that parameters really are close enough in the two cases. Whether our somewhat pessimistic view of the domain of strict applicability of the normal form analysis in nonsmooth sliding bifurcations is justified remains to be seen.

Our finding of the explicit Filippov flow with multiple attractors points to a number of research directions and open problems. First of all, it would be interesting to determine if the number of possible attractors born in the grazing-sliding bifurcations in three-dimensional type flows is limited to two. Secondly, using the explicit example presented in the current paper it would be interesting to construct Filippov type flows where corner-collision bifurcations lead to the onset of multiple attractors. The question then arises if the possible pattern of emerging attractors is equivalent to the ones for grazing-sliding bifurcations. Finally, an interesting point would be the use the knowledge on the existence on multiple attractors in grazing-sliding bifurcations to use it as a control strategy. Suppose we have a Filippov system operating in some stable oscillatory state, now, can we use the knowledge on the existence of multiple attractors in discontinuity induced bifurcations to make our system evolve on some other attractor.

7. Appendix

Following [5] we can write the two-dimensional PDM, in the case of a three-dimensional Filippov type flow and using (X, Z) coordinates, as

$$PDM(X, Z, q; \mu) = \begin{pmatrix} X \\ Z \end{pmatrix} + \beta(X, Z, q; \mu)q^2, \tag{16}$$

where

$$q^2 + H(X, Z; \mu) = 0$$

with q being an independent variable that measures the penetration of trajectories below the switching surface for orbits sufficiently close to the grazing orbit. The coordinates (X, Z) are chosen such that the grazing incidence takes place at $(X, Z, q; \mu) = (0, 0, 0; 0)$. The linearisation of (16) at grazing gives

$$PDM_L(X, Z, q; \mu) = \begin{pmatrix} X \\ Z \end{pmatrix} - \beta(0, 0, 0; 0)\langle H_x, [X, Z]\rangle - \beta(0, 0, 0; 0)H_\mu \mu, \tag{17}$$

where

$$\beta(0, 0, 0\ 0) = \frac{1}{\langle H_x, (F_2 - F_1)\rangle} F_2 + \frac{\langle (H_x F_1)_x, F_2\rangle}{\langle (H_x F_1)_x, F_1\rangle} F_1$$

with the right-hand side evaluated at the grazing point $(0, 0, 0; 0)$, and $\langle , \rangle$ denoting the standard dot product. At grazing the X-component of the vector field F_1 is naught. Using the definition of the switching function given in Sec. 4, and ignoring the parameter dependence we arrive at the map (14) given in Sec. 4.

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