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Some recent work in Fréchet geometry*

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Abstract

Some recent work in Fréchet geometry is briefly reviewed. In particular an earlier result on the structure of second tangent bundles in the finite dimensional case was extended to infinite dimensional Banach manifolds and Fréchet manifolds that could be represented as projective limits of Banach manifolds. This led to further results concerning the characterization of second tangent bundles and differential equations in the more general Fréchet structure needed for applications. A summary is given of recent results on hypercyclicity of operators on Fréchet spaces.
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1 Introduction

Dodson and Radivoiovici [21, 22] proved that in the case of a finite n-dimensional manifold M, a vector bundle structure on T^2M can be well defined if and only if M is endowed with a linear connection: T^2M becomes then and only then a vector bundle over M with structure group the general linear group $GL(2n;\mathbb{R})$. The manifolds M that admit linear connections are precisely the paracompact ones. Manifolds with connections form a full subcategory $Man\nabla$ of the category Man of smooth manifolds and smooth maps; the constructions in the above theorems [21] provide a functor $Man\nabla \longrightarrow VBun$ [22]. A linear connection is a splitting of TLM, which then induces splitting in the second jet bundle J^2M (called a *dissection* by Ambrose et al. [5]) and we get also a corresponding splitting in T^2L^2M.

Dodson and Galanis [16] extended the results to manifolds M modeled on an arbitrarily chosen Banach space $\mathbb{E}$. Using the Vilms [46] point of view for connections on infinite dimensional vector bundles and a new formalism, it was proved that T^2M can be thought of as a Banach vector bundle over M with structure group $GL(\mathbb{E} \times \mathbb{E})$ if and only if M admits a linear connection. The case of non-Banach Fréchet modeled manifolds was investigated [16] but there are intrinsic difficulties with Fréchet spaces. These include pathological general linear groups, which do not even admit reasonable topological group structures. However, every Fréchet space admits representation as a projective limit of Banach spaces and under certain conditions this can persist into manifold structures. By restriction to those Fréchet manifolds which can be obtained as projective limits of Banach manifolds [23], it is possible to endow T^2M with a vector bundle structure over M with structure group a new topological group, that in a generalized sense is of Lie type. This construction is equivalent to the existence on M of a specific type of linear connection characterized by a generalized set of Christoffel symbols. We outline the methodology and a range of results in subsequent sections but first we mention what makes the Fréchet case difficult but important.

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In a number of cases that have significance in global analysis and physical field theory, Banach space representations break down and we need Fréchet spaces, which have weaker requirements for their topology, see for example Hamilton [29] and Neeb [36], Steen and Seebach [43]. However, there is a price to pay for these weaker structural constraints: Fréchet spaces lack a general solvability theory of differential equations, even linear ones; also, the space of continuous linear mappings drops out of the category while the space of linear isomorphisms does not admit a reasonable Lie group structure. We shall see that these shortcomings can be worked round to a certain extent. The developments described in this short review will be elaborated in detail in the forthcoming monograph by Dodson, Galanis and Vassilliou [20].

1.1 Fréchet spaces

A *seminorm* on (eg for definiteness a real) vector space X is a real valued map $p : X \to \mathbb{R}$ such that

$$p(x) \geq 0, \tag{i}$$
$$p(x+y) \leq p(x) + p(y), \tag{ii}$$
$$p(\lambda x) = |\lambda|\, p(x), \tag{iii}$$

for every $x, y \in X$ and $\lambda \in \mathbb{R}$.

A family of seminorms $\Gamma = \{p_\alpha\}_{\alpha \in I}$ on X defines a unique topology $\mathcal{T}_\Gamma$ compatible with the vector structure of X. The neighborhood base of $\mathcal{T}_\Gamma$ is determined by the family

$$\mathcal{B}_\Gamma = \{S(\Delta, \varepsilon) : \varepsilon > 0 \text{ and } \Delta \text{ a finite subset of } \Gamma\},$$

where

$$S(\Delta, \varepsilon) = \{x \in \mathbb{F} : p(x) < \varepsilon,\ \forall\, p \in \Delta\}.$$

The topology $\mathcal{T}_\Gamma$ induced on X by p is the finest making all the seminorms continuous but it is not necessarily Hausdorff. In fact $(X, \mathcal{T}_\Gamma)$ is a locally convex topological vector space and the local convexity of a topology on X is its subordination to a family of seminorms. Hausdorffness requires the further property

$$x = 0 \Leftrightarrow p(x) = 0,\ \ \forall p \in \Gamma.$$

and in that case it is metrizable if and only if the family of seminorms is countable. Convergence of a sequence $(x_n)_{n \in \mathbb{N}}$ in X is dependent on all the seminorms of Γ

$$x_n \to x \Leftrightarrow\ p(x_n - x) \to 0,\ \forall p \in \Gamma.$$

Completeness is if and only if every sequence $(x_n)_{n \in \mathbb{N}}$ in X with

$$\lim_{n.m \to \infty} p(x_n - x_m) = 0;\ \ \forall\, p \in \Gamma,$$

converges in X.

Definition 1.1. *A* Fréchet space *is a topological vector space* $\mathbb{F}$ *that is locally convex, Hausdorff, metrizable and complete.*

So, every Banach space is a Fréchet space, with just one seminorm and that one is a norm. More interesting examples include the following:

- The space $\mathbb{R}^\infty = \prod\limits_{n \in \mathbb{N}} \mathbb{R}^n$, endowed with the cartesian topology, is a Fréchet space with corresponding family of seminorms

$$\{p_n(x_1, x_2, \ldots) = |x_1| + |x_2| + \ldots + |x_n|\}_{n \in \mathbb{N}}.$$

Metrizability can be established by putting

$$d(x,y) = \sum_i \frac{|x_i - y_i|}{2^i(1+|x_i - y_i|)}. \tag{1}$$

In $\mathbb{R}^\infty$ the completeness is inherited from that of each copy of the real line. For if $x = (x_i)$ is Cauchy in $\mathbb{R}^\infty$ then for each i, $(x_i^m), m \in \mathbb{N}$ is Cauchy in $\mathbb{R}$ and hence converges, to X_i say, and $(X_i) = X \in \mathbb{R}^\infty$ with $d(x_i, X_i) \to 0$ as $i \to \infty$. Separability arises from the countable dense subset of elements having finitely many rational components and the remainder zero; second countability comes from metrizability. Hausdorfness implies that a compact subset of a Fréchet space is closed; a closed subspace is a Fréchet space and a quotient by a closed subspace is a Fréchet space. In fact, $\mathbb{R}^\infty$ is a special case from a classification for Fréchet spaces [36]. For each seminorm $p_n = ||\ ||_n$ we can define the normed subspace $F_n = F/p_n^{-1}(0)$ by factoring out the null space of p_n. Then, the seminorm requirement (1.1) provides a linear injection into the product of normed spaces

$$p : F \to \prod_{n\in\mathbb{N}} F_n : f \mapsto (p_n(f))_{n\in\mathbb{N}} \tag{2}$$

and the completeness of F is equivalent to the closedness of $p(V)$ in the Banach product of the closures $\overline{F_n}$ and p extends to an embedding of F in this product. This embedding can be used to construct limiting processes for geometric structures of interest in Fréchet manifolds modelled on F.

- More generally, any *countable* cartesian product of Banach spaces $\mathbb{F} = \prod_{n\in\mathbb{N}} \mathbb{E}^n$ is a Fréchet space with topology defined by the seminorms $(q_n)_{n\in\mathbb{N}}$, given by

  $$q_n(x_1, x_2, ...) = \sum_{i=1}^{n} \|x_i\|_i \,,$$

  where $\|\cdot\|_i$ denotes the norm of the i-factor $\mathbb{E}^i$.

- The space of continuous functions $C^0(\mathbb{R},\mathbb{R})$ is a Fréchet space with seminorms $(p_n)_{n\in\mathbb{N}}$ defined by

  $$p_n(f) = \sup\left\{\, |f(x)|\,,\ x \in [-n,n]\right\}.$$

- The space of smooth functions $C^\infty(I,\mathbb{R})$, where I is a compact interval of $\mathbb{R}$, is a Fréchet space with seminorms defined by

  $$p_n(f) = \sum_{i=0}^{n} \sup\left\{\, \left|D^i f(x)\right|,\ x \in I\right\}.$$

- The space $C^\infty(M,V)$, of smooth sections of the vector bundle V over compact smooth Riemannian manifold M with covariant derivative ∇, is a Fréchet space with

  $$||f||_n = \sum_{i=0}^{n} sup_x|\nabla^i f(x)|, \quad \text{for } n \in \mathbb{N}. \tag{3}$$

- Fréchet spaces of sections arise naturally as configurations of a physical field. Then the moduli space, consisting of inequivalent configurations of the physical field, is the quotient of the infinite-dimensional configuration space $\mathcal{X}$ by the appropriate symmetry gauge group. Typically, $\mathcal{X}$ is modelled on a Fréchet space of smooth sections of a vector bundle over a closed manifold. For example, see Omori [37, 38].

2 Banach second tangent bundle

Let M be a $C^\infty-$manifold modeled on a Banach space $\mathbb{E}$ and $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in I}$ a corresponding atlas. The latter gives rise to an atlas $\{(\pi_M^{-1}(U_\alpha), \Psi_\alpha)\}_{\alpha\in I}$ of the tangent bundle TM of M with

$$\Psi_\alpha : \pi_M^{-1}(U_\alpha) \longrightarrow \psi_\alpha(U_\alpha)\times\mathbb{E} : [c,x] \longmapsto (\psi_\alpha(x), (\psi_\alpha\circ c)'(0)),$$

where $[c,x]$ stands for the equivalence class of a smooth curve c of M with $c(0) = x$ and

$$(\psi_\alpha\circ c)'(0) = [d(\psi_\alpha\circ c)(0)](1).$$

The corresponding trivializing system of $T(TM)$ is denoted by

$$\{(\pi_{TM}^{-1}(\pi_M^{-1}(U_\alpha)), \widetilde{\Psi}_\alpha)\}_{\alpha\in I}.$$

Adopting the formalism of Vilms [46], a connection on M is a vector bundle morphism:

$$D : T(TM) \longrightarrow TM$$

with the additional property that the mappings $\omega_\alpha : \psi_\alpha(U_\alpha)\times\mathbb{E} \to \mathcal{L}(\mathbb{E},\mathbb{E})$ defined by the local forms of D:

$$D_\alpha : \psi_\alpha(U_\alpha)\times\mathbb{E}\times\mathbb{E}\times\mathbb{E} \to \psi_\alpha(U_\alpha)\times\mathbb{E}$$

with $D_\alpha := \Psi_\alpha\circ D\circ(\widetilde{\Psi}_\alpha)^{-1}$, $\alpha\in I$, via the relation

$$D_\alpha(y,u,v,w) = (y, w + \omega_\alpha(y,u)\cdot v),$$

are smooth. Furthermore, D is a linear connection on M if and only if $\{\omega_\alpha\}_{\alpha\in I}$ are linear with respect to the second variable.

Such a connection D is fully characterized by the family of Christoffel symbols $\{\Gamma_\alpha\}_{\alpha\in I}$, which are smooth mappings

$$\Gamma_\alpha : \psi_\alpha(U_\alpha) \longrightarrow \mathcal{L}(\mathbb{E}, \mathcal{L}(\mathbb{E},\mathbb{E}))$$

defined by $\Gamma_\alpha(y)[u] = \omega_\alpha(y,u)$, $(y,u)\in\psi_\alpha(U_\alpha)\times\mathbb{E}$.

The requirement that a connection is well defined on the common areas of charts of M, yields the Christoffel symbols satisfying the following compatibility condition:

$$\begin{aligned}\Gamma_\alpha(\sigma_{\alpha\beta}(y))(d\sigma_{\alpha\beta}(y)(u))[d(\sigma_{\alpha\beta}(y))(v)] + (d^2\sigma_{\alpha\beta}(y)(v))(u) = \\ = d\sigma_{\alpha\beta}(y)((\Gamma_\beta(y)(u))(v)),\end{aligned} \tag{4}$$

for all $(y,u,v)\in\psi_\alpha(U_\alpha\cap U_\beta)\times\mathbb{E}\times\mathbb{E}$, and d, d^2 stand for the first and the second differential respectively. Here by $\sigma_{\alpha\beta}$ we denote the diffeomorphisms $\psi_\alpha\circ\psi_\beta^{-1}$ of $\mathbb{E}$. For further details and the relevant proofs see [46].

Let M be a smooth manifold modeled on the Banach space $\mathbb{E}$ and $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in I}$ a corresponding atlas. For each $x\in M$ we define the following equivalence relation on $C_x = \{f : (-\varepsilon,\varepsilon)\to M \mid f$ smooth and $f(0) = x$, $\varepsilon > 0\}$:

$$f \approx_x g \Leftrightarrow f^{'}(0) = g'(0) \text{ and } f''(0) = g''(0), \tag{5}$$

where by $f^{'}$ and $f^{''}$ we denote the first and the second, respectively, derivatives of f:

$$\begin{aligned} f' &: (-\varepsilon,\varepsilon)\to TM : t\longmapsto [df(t)](1) \\ f'' &: (-\varepsilon,\varepsilon)\to T(TM) : t\longmapsto [df'(t)](1).\end{aligned}$$

The *tangent space of order two* of M at the point x is the quotient $T_x^2M = C_x/\approx_x$ and the *tangent bundle of order two* of M is the union of all tangent spaces of order 2: $T^2M := \bigcup_{x\in M} T_x^2M$. Of course, T_x^2M can be thought of as a topological vector space isomorphic to $\mathbb{E}\times\mathbb{E}$ via the bijection

$$T_x^2M \overset{\simeq}{\longleftrightarrow} \mathbb{E}\times\mathbb{E} : [f,x]_2 \longmapsto ((\psi_\alpha\circ f)'(0), (\psi_\alpha\circ f)''(0)),$$

where $[f,x]_2$ is the equivalence class of f with respect to $\approx_x$. However, this structure depends on the choice of the chart (U_α,ψ_α), hence a definition of a vector bundle structure on T^2M cannot be achieved by the use of the aforementioned bijections. The most convenient way to overcome this obstacle is to assume that the manifold M is endowed with the additional structure of a linear connection.

Theorem 2.1. *For every linear connection D on the manifold M, T^2M becomes a Banach vector bundle with structure group the general linear group $GL(\mathbb{E}\times\mathbb{E})$.*

Proof. Let $\pi_2 : T^2M \to M$ be the natural projection of T^2M to M with $\pi_2([f,x]_2) = x$ and $\{\Gamma_\alpha : \psi_\alpha(U_\alpha) \longrightarrow \mathcal{L}(\mathbb{E},\mathcal{L}(\mathbb{E},\mathbb{E}))\}_{a\in I}$ the Christoffel symbols of the connection D with respect to the covering $\{(U_a,\psi_a)\}_{a\in I}$ of M. Then, for each $\alpha\in I$, we define the mapping $\Phi_\alpha : \pi_2^{-1}(U_\alpha) \longrightarrow U_\alpha\times\mathbb{E}\times\mathbb{E}$ with

$$\Phi_\alpha([f,x]_2) = (x, (\psi_\alpha\circ f)'(0), (\psi_\alpha\circ f)''(0) + \Gamma_\alpha(\psi_\alpha(x))((\psi_\alpha\circ f)'(0))[(\psi_\alpha\circ f)'(0)]).$$

These are obviously well defined and injective mappings. They are also surjective since every element $(x,u,v)\in U_\alpha\times\mathbb{E}\times\mathbb{E}$ can be obtained through Φ_α as the image of the equivalence class of the smooth curve

$$f:\mathbb{R}\to\mathbb{E} : t\mapsto \psi_\alpha(x) + tu + \frac{t^2}{2}(v - \Gamma_\alpha(\psi_\alpha(x))(u)[u]),$$

appropriately restricted in order to take values in $\psi_\alpha(U_\alpha)$. On the other hand, the projection of each Φ_α to the first factor coincides with the natural projection π_2 : $pr_1\circ\Phi_\alpha = \pi_2$. Therefore, the trivializations $\{(U_\alpha,\Phi_\alpha)\}_{a\in I}$ define a fibre bundle structure on T^2M and we need now to focus on the behavior of the mappings Φ_α on areas of M that are covered by common domains of different charts. Indeed, if (U_α,ψ_α), (U_β,ψ_β) are two such charts, let $(\pi_2^{-1}(U_\alpha),\Phi_\alpha)$, $(\pi_2^{-1}(U_\beta),\Phi_\beta)$ be the corresponding trivializations of T^2M. Taking into account the compatibility condition (1) satisfied by the Christoffel symbols $\{\Gamma_\alpha\}$ we see that:

$$(\Phi_\alpha\circ\Phi_\beta^{-1})(x,u,v) = \Phi_\alpha([f,x]_2),$$

where $(\psi_\beta\circ f)'(0) = u$ and $(\psi_\beta\circ f)''(0) + \Gamma_\beta(\psi_\beta(x))(u)[u] = v$. As a result,

$$\begin{gathered}
(\Phi_\alpha\circ\Phi_\beta^{-1})(x,u,v) = \\
((\psi_\alpha\circ\psi_\beta^{-1})(\psi_\beta(x)), d(\psi_\alpha\circ\psi_\beta^{-1}\circ\psi_\beta\circ f)(0)(1), d^2(\psi_\alpha\circ\psi_\beta^{-1}\circ\psi_\beta\circ f)(0)(1,1)) + \\
\Gamma_\alpha((\psi_\alpha\circ\psi_\beta^{-1})(\psi_\beta(x)))(d(\psi_\alpha\circ\psi_\beta^{-1}\circ\psi_\beta\circ f)(0)(1))[d(\psi_\alpha\circ\psi_\beta^{-1}\circ\psi_\beta\circ f)(0)(1)] = \\
(\sigma_{\alpha\beta}(\psi_\beta(x)), d\sigma_{\alpha\beta}(\psi_\beta(x))(u), d\sigma_{\alpha\beta}(\psi_\beta(x))(d^2(\psi_\beta\circ f)(0)(1,1)) \\
+ d^2\sigma_{\alpha\beta}(\psi_\beta(x))(u)[u] + \Gamma_\alpha(\sigma_{\alpha\beta}(\psi_\beta(x)))(d\sigma_{\alpha\beta}(\psi_\beta(x))(u))[d\sigma_{\alpha\beta}(\psi_\beta(x))(u)]) = \\
(\sigma_{\alpha\beta}(\psi_\beta(x)), d\sigma_{\alpha\beta}(\psi_\beta(x))(u), d\sigma_{\alpha\beta}(\psi_\beta(x))(d^2(\psi_\beta\circ f)(0)(1,1) + \Gamma_\beta(\psi_\beta(x))(u)[u]) = \\
= (\sigma_{\alpha\beta}(\psi_\beta(x)), d\sigma_{\alpha\beta}(\psi_\beta(x))(u), d\sigma_{\alpha\beta}(\psi_\beta(x))(v)),
\end{gathered}$$

where by $\sigma_{\alpha\beta}$ we denote again the diffeomorphisms $\psi_\alpha\circ\psi_\beta^{-1}$. Therefore, the restrictions to the fibres

$$\Phi_{\alpha,x}\circ\Phi_{\beta,x}^{-1} : \mathbb{E}\times\mathbb{E}\to\mathbb{E}\times\mathbb{E} : (u,v) \longmapsto (\Phi_\alpha\circ\Phi_\beta^{-1})|_{\pi_2^{-1}(x)}(u,v)$$

are linear isomorphisms and the mappings:

$$T_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathcal{L}(\mathbb{E} \times \mathbb{E}, \mathbb{E} \times \mathbb{E}) : x \longmapsto \Phi_{\alpha,x} \circ \Phi_{\beta,x}^{-1}$$

are smooth since $T_{\alpha\beta} = (d\sigma_{\alpha\beta} \circ \psi_\beta) \times (d\sigma_{\alpha\beta} \circ \psi_\beta)$ holds for each $\alpha, \beta \in I$.

As a result, T^2M is a vector bundle over M with fibres of type $\mathbb{E} \times \mathbb{E}$ and structure group $GL(\mathbb{E} \times \mathbb{E})$. Moreover, T^2M is isomorphic to $TM \times TM$ since both bundles are characterized by the same cocycle $\{(d\sigma_{\alpha\beta} \circ \psi_\beta) \times (d\sigma_{\alpha\beta} \circ \psi_\beta)\}_{\alpha,\beta\in I}$ of transition functions. □

The converse of the theorem was proved also in [16]. These results coincide in the finite dimensional case with the earlier result since the corresponding transition functions are identical (see [21] Corollary 2).

The finite dimensional results [21, 22] on the *frame bundle of order two*

$$L^2(M) := \underset{x\in M}{\cup} \mathcal{L}is(\mathbb{E} \times \mathbb{E}, T_x^2 M),$$

were extended also to the Banach manifold M by Dodson and Galanis [17]:

Theorem 2.2. *Every linear connection ∇ of the second order tangent bundle T^2M corresponds bijectively to a connection ω of $L^2(M)$.*

3 Fréchet second tangent bundle

Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be two *Hausdorff locally convex topological vector spaces*, and let U be an open subset of $\mathbb{F}_1$. A continuous map $f : U \to \mathbb{F}_2$ is called *differentiable at* $x \in U$ if there exists a continuous linear map $Df(x) : \mathbb{F}_1 \to \mathbb{F}_2$ such that

$$R(t, v) := \begin{cases} \frac{1}{t}\left(f(x + tv) - f(x) - Df(x)(tv)\right), & t \neq 0 \\ 0, & t = 0 \end{cases}$$

is continuous at every $(0, v) \in \mathbb{R} \times \mathbb{F}_1$. The map f will be said to be *differentiable* if it is differentiable at every $x \in U$. We call $Df(x)$ the *differential* (or *derivative*) *of* f *at* x. As in classical (Fréchet) differentiation, $Df(x)$ is uniquely determined, see Leslie [33] and [34] for more details.

A map $f : U \to \mathbb{F}_2$, as before, is called C^1*-differentiable* if it is differentiable at every point $x \in U$, and the *(total) differential* or *(total) derivative*

$$Df : U \times \mathbb{F}_1 \to \mathbb{F}_2 : (x, v) \mapsto Df(x)(v)$$

is continuous.

This total differential Df does not involve the space of continuous linear maps $\mathcal{L}(\mathbb{F}_1, \mathbb{F}_2)$, thus avoiding the possibility of dropping out of the working category when $\mathbb{F}_1$ and $\mathbb{F}_2$ are Fréchet spaces. The notion of C^n-differentiability ($n \geq 2$) can be defined by induction and C^∞-differentiability follows.

Using the methodology of Galanis and Vassiliou [24, 45] for tangent and frame bundles, a vector bundle structure was obtained on the second order tangent bundles for those Frèchet manifolds which can be obtained as projective limits of Banach manifolds [16]. Let M be a smooth manifold modeled on the Frèchet space $\mathbb{F}$. Taking into account that the latter *always* can be realized as a projective limit of Banach spaces $\{\mathbb{E}^i; \rho^{ji}\}_{i,j\in\mathbb{N}}$ (i.e. $\mathbb{F} \cong \varprojlim \mathbb{E}^i$) we assume that the manifold itself is obtained as the limit of a projective system of Banach modeled manifolds $\{M^i;\varphi^{ji}\}_{i,j\in\mathbb{N}}$. Then, it was proved [16] that the second order tangent bundles $\{T^2M^i\}_{i\in\mathbb{N}}$ form also a projective system with limit (set-theoretically) isomorphic to T^2M. We define a vector bundle structure on T^2M by means of a certain type of linear connection on M. The problems concerning the structure group of this bundle are overcame by the replacement of the pathological $GL(\mathbb{F}\times\mathbb{F})$ by the new topological (and in a generalized sense smooth Lie) group:

$$\mathcal{H}^0(\mathbb{F}\times\mathbb{F}) := \{(l^i)_{i\in\mathbb{N}} \in \prod_{i=1}^{\infty} GL(\mathbb{E}^i\times\mathbb{E}^i) : \varprojlim l^i \text{ exists}\}.$$

Precisely, $\mathcal{H}^0(\mathbb{F}\times\mathbb{F})$ is a topological group that is isomorphic to the projective limit of the Banach-Lie groups

$$\mathcal{H}^0_i(\mathbb{F}\times\mathbb{F}) := \{(l^1,l^2,...,l^i)_{i\in\mathbb{N}} \in \prod_{k=1}^{i} GL(\mathbb{E}^k\times\mathbb{E}^k) : \rho^{jk}\circ l^j = l^k\circ\rho^{jk} \;\; (k\le j\le i)\}.$$

Also, it can be considered as a generalized Lie group via its embedding in the topological vector space $\mathcal{L}(\mathbb{F}\times\mathbb{F})$.

Theorem 3.1. *If a Frèchet manifold $M = \varprojlim M^i$ is endowed with a linear connection D that can be realized also as a projective limit of connections $D = \varprojlim D^i$, then T^2M is a Fréchet vector bundle over M with structure group $\mathcal{H}^0(\mathbb{F}\times\mathbb{F})$.*

Proof. Following the terminology established above, we consider $\{(U_\alpha = \varprojlim U^i_\alpha, \psi_\alpha = \varprojlim \psi^i_\alpha)\}_{\alpha\in I}$ an atlas of M. Each linear connection D^i ($i\in\mathbb{N}$), which is naturally associated to a family of Christoffel symbols $\{\Gamma^i_\alpha : \psi^i_\alpha(U^i_\alpha)\to\mathcal{L}(\mathbb{E}^i,\mathcal{L}(\mathbb{E}^i,\mathbb{E}^i))\}_{\alpha\in I}$, ensures that T^2M^i is a vector bundle over M^i with fibres of type $\mathbb{E}^i$. This structure, as already presented in Theorem 2.1, is defined by the trivializations:

$$\Phi^i_\alpha : (\pi^i_2)^{-1}(U^i_\alpha) \longrightarrow U^i_\alpha\times\mathbb{E}^i\times\mathbb{E}^i,$$

with

$$\Phi^i_\alpha([f,x]^i_2) = (x, (\psi^i_\alpha\circ f)'(0), (\psi^i_\alpha\circ f)''(0) + \Gamma^i_\alpha(\psi^i_\alpha(x))((\psi^i_\alpha\circ f)'(0))[(\psi^i_\alpha\circ f)'(0)]);\;\; \alpha\in I.$$

The families of mappings $\{g^{ji}\}_{i,j\in\mathbb{N}}$, $\{\varphi^{ji}\}_{i,j\in\mathbb{N}}$, $\{\rho^{ji}\}_{i,j\in\mathbb{N}}$ are connecting morphisms of the projective systems $T^2M = \varprojlim(T^2M^i)$, $M = \varprojlim M^i$, $\mathbb{F} = \varprojlim \mathbb{E}^i$ respectively. These projections $\{\pi^i_2 : T^2M^i\to M^i\}_{i\in\mathbb{N}}$ satisfy

$$\varphi^{ji}\circ\pi^j_2 = \pi^i_2\circ g^{ji} \;\;\; (j\ge i)$$

and the trivializations $\{\Phi^i_\alpha\}_{i\in\mathbb{N}}$

$$(\varphi^{ji}\times\rho^{ji}\times\rho^{ji})\circ\Phi^j_\alpha = \Phi^i_\alpha\circ g^{ji} \;\;\; (j\ge i).$$

We obtain the surjection $\pi_2 = \varprojlim \pi^i_2 : T^2M \longrightarrow M$ and,

$$\Phi_\alpha = \varprojlim \Phi^i_\alpha : \pi_2^{-1}(U_\alpha) \longrightarrow U_\alpha\times\mathbb{F}\times\mathbb{F} \;\;\; (\alpha\in I)$$

is smooth, as a projective limit of smooth mappings, and its projection to the first factor coincides with π_2. The restriction to a fibre $\pi_2^{-1}(x)$ of Φ_α is a bijection since $\Phi_{\alpha,x} := pr_2\circ\Phi_\alpha|_{\pi_2^{-1}(x)} = \varprojlim(pr_2\circ\Phi^i_\alpha|_{(\pi^i_2)^{-1}(x)})$.

The corresponding transition functions $\{T_{\alpha\beta} = \Phi_{\alpha,x}\circ\Phi^{-1}_{\beta,x}\}_{\alpha,\beta\in I}$ can be considered as taking values in the generalized Lie group $\mathcal{H}^0(\mathbb{F}\times\mathbb{F})$, since $T_{\alpha\beta} = \epsilon\circ T^*_{\alpha\beta}$, where $\{T^*_{\alpha\beta}\}_{\alpha,\beta\in I}$ are the smooth mappings

$$T^*_{\alpha\beta} : U_\alpha\cap U_\beta \to \mathcal{H}^0(\mathbb{F}\times\mathbb{F}) : x \longmapsto (pr_2\circ\Phi^i_\alpha|_{(\pi^i_2)^{-1}(x)})_{i\in\mathbb{N}}$$

with ϵ the natural inclusion

$$\epsilon : \mathcal{H}^0(\mathbb{F}\times\mathbb{F}) \to \mathcal{L}(\mathbb{F}\times\mathbb{F}) : (l^i)_{i\in\mathbb{N}} \longmapsto \varprojlim l^i.$$

Hence, T^2M admits a vector bundle structure over M with fibres of type $\mathbb{F}\times\mathbb{F}$ and structure group $\mathcal{H}^0(\mathbb{F}\times\mathbb{F})$. This bundle is isomorphic to $TM \times TM$ since they have identical transition functions:

$$T_{\alpha\beta}(x) = \Phi_{\alpha,x} \circ \Phi^{-1}_{\beta,x} = (d(\psi_a \circ \psi_\beta^{-1}) \circ \psi_\beta)(x) \times (d(\psi_a \circ \psi_\beta^{-1}) \circ \psi_\beta)(x)$$

□

Also, the converse is true:

Theorem 3.2. *If T^2M is an $\mathcal{H}^0(\mathbb{F}\times\mathbb{F})-$Frèchet vector bundle over M isomorphic to $TM\times TM$, then M admits a linear connection which can be realized as a projective limit of connections.*

4 Fréchet second frame bundle

Let $M = \varprojlim M^i$ be a manifold with connecting morphisms $\{\varphi^{ji} : M^j \to M^i\}_{i,j\in\mathbb{N}}$ and Fréchet space model the limit $\mathbb{F}$ of a projective system of Banach spaces $\{\mathbb{F}^i; \rho^{ji}\}_{i,j\in\mathbb{N}}$. Following the results obtained in [16], if M is endowed with a linear connection $D = \varprojlim D^i$, then T^2M admits a vector bundle structure over M with fibres of Fréchet type $\mathbb{F}\times\mathbb{F}$. Then T^2M becomes also a projective limit of manifolds via the identification $T^2M \simeq \varprojlim T^2M^i$.

Let

$$\mathcal{F}^2M^i = \bigcup_{x^i\in M^i} \{(h^k)_{k=1,\dots,i} : h^k \in \mathcal{L}is(\mathbb{F}^k\times\mathbb{F}^k, T^2_{\varphi^{ik}(x^i)}M^k) \text{ and}$$
$$g^{mk} \circ h^m = h^k \circ (\rho^{mk}\times\rho^{mk}),\ i \geq m \geq k\}.$$

We replace the pathological general linear group $GL(\mathbb{F})$ by

$$H_0(\mathbb{F}) := H_0(\mathbb{F},\mathbb{F}) = \{(l^i)_{i\in\mathbb{N}} \in \prod_{i=1}^{\infty} GL(\mathbb{F}^i) :\ \varprojlim l^i \text{ exists}\}.$$

The latter can be thought of also as a generalized Fréchet Lie group by being embedded in $H(\mathbb{F}) := H(\mathbb{F},\mathbb{F})$. Then [18],

Theorem 4.1. *$\mathcal{F}^2M^i$ is a principal fibre bundle over M^i with structure group the Banach Lie group $H^i_0(\mathbb{F}\times\mathbb{F}) := H^i_0(\mathbb{F}\times\mathbb{F},\mathbb{F}\times\mathbb{F})$.*
The limit $\varprojlim \mathcal{F}^2M^i$ is a Fréchet principal bundle over M with structure group $H_0(\mathbb{F}\times\mathbb{F})$.

We call the *generalized bundle of frames of order two* of the Fréchet manifold $M = \varprojlim M^i$ the principal bundle

$$\mathcal{F}^2(M) := \varprojlim \mathcal{F}^2M^i.$$

This is a natural generalization of the usual frame bundle and it follows

Theorem 4.2. *For the action of the group $H^0(\mathbb{F}\times\mathbb{F})$ on the right of the product $\mathcal{F}^2(M)\times(\mathbb{F}\times\mathbb{F})$:*

$$((h^i),(u^i,v^i))_{i\in\mathbb{N}} \cdot (g^i)_{i\in\mathbb{N}} = ((h^i\circ g^i),(g^i)^{-1}(u^i,v^i))_{i\in\mathbb{N}},$$

the quotient space $\mathcal{F}^2M\times(\mathbb{F}\times\mathbb{F})\diagup H^0(\mathbb{F}\times\mathbb{F})$ is isomorphic with T^2M.

Consider a connection of $\mathcal{F}^2(M)$ represented by the 1-form $\omega \in \Lambda^1(\mathcal{F}^2(M), \mathcal{L}(\mathbb{F}\times\mathbb{F}))$, with smooth atlas $\{(U_\alpha = \varprojlim U^i_\alpha, \psi_\alpha = \varprojlim \psi^i_\alpha)\}_{a\in I}$ of M, $\{(p^{-1}(U_\alpha), \Psi_\alpha)\}_{a\in I}$ trivializations of $\mathcal{F}^2(M)$ and $\{\omega_\alpha := s^*_\alpha \omega\}_{a\in I}$ the corresponding local forms of ω obtained as pull-backs with respect to the natural local sections $\{s_\alpha\}$ of $\{\Psi_\alpha\}$. Then a (unique) linear connection can be defined on T^2M by means of the Christoffel symbols

$$\Gamma_\alpha : \psi_\alpha(U_\alpha) \to \mathcal{L}(\mathbb{F}\times\mathbb{F}, \mathcal{L}(\mathbb{F}, \mathbb{F}\times\mathbb{F}))$$

with $([\Gamma_\alpha(y)](u))(v) = \omega_\alpha(\psi^{-1}_\alpha(y))(T_y\psi^{-1}_\alpha(v))(u)$, $(y,u,v) \in \psi_\alpha(U_\alpha)\times\mathbb{F}\times\mathbb{F}\times\mathbb{F}$.

However, in the framework of Fréchet bundles an arbitrary connection is not always easy to handle, since Fréchet manifolds and bundles lack a general theory of solvability for linear differential equations. Also, Christoffel symbols (in the case of vector bundles) or the local forms (in principal bundles) are affected in their representation of linear maps by the fact that continuous linear mappings of a Fréchet space do not remain in the same category. Galanis [24, 25] solved the problem for connections that can be obtained as projective limits and we obtain [18]

Theorem 4.3. *Let ∇ be a linear connection of the second order tangent bundle $T^2M = \varprojlim T^2M^i$ that can be represented as a projective limit of linear connections ∇^i on the (Banach modelled) factors. Then ∇ corresponds to a connection form ω of $\mathcal{F}^2M$ obtained also as a projective limit.*

Areas of application were outlined in [18].

5 Connection choice

Dodson, Galanis and Vassiliou [19] studied the way in which the choice of connection influenced the structure of the second tangent bundle over Fréchet manifolds, since each connection determines one isomorphism of $T^2M \equiv TM\bigoplus TM$. They defined the second order differential T^2f of a smooth map $g : M \to N$ between two manifolds M and N. In contrast to the case of the first order differential Tg, the linearity of T^2g on the fibres ($T^2_xg : T^2_xM \to T^2_{g(x)}N$, $x \in M$) is not always ensured but they proved a number of results.

The connections ∇_M and ∇_N are called *g-conjugate* [44] (or *g-related*) if they commute with the differentials of g :

$$Tg \circ \nabla_M = \nabla_N \circ T(Tg). \tag{6}$$

Locally

$$\begin{gathered} DG(\phi_\alpha(x))(\Gamma^M_\alpha(\phi_\alpha(x))(u)(u)) = \\ \Gamma^N_\beta(G(\phi_\alpha(x)))(DG(\phi_\alpha(x))(u))(DG(\phi_\alpha(x))(u)) + D(DG)((\phi_\alpha(x))(u,u), \end{gathered} \tag{7}$$

for every $(x,u) \in U_\alpha \times \mathbb{E}$. For g-conjugate connections ∇_M and ∇_N the local expression of T^2_xg reduces to

$$(\Psi_{\beta,g(x)} \circ T^2_xg \circ \Phi^{-1}_{a,x})(u,v) = (DG(\phi_\alpha(x))(u), DG(\phi_\alpha(x))(v)). \tag{8}$$

Theorem 5.1. *Let T^2M, T^2N be the second order tangent bundles defined by the pairs (M, ∇_M), (N, ∇_N), and let $g : M \to N$ be a smooth map. If the connections ∇_M and ∇_N are g-conjugate, then the second order differential $T^2g : T^2M \to T^2N$ is a vector bundle morphism.*

Theorem 5.2. *Let ∇, ∇' be two linear connections on M. If g is a diffeomorphism of M such that ∇ and ∇' are g-conjugate, then the vector bundle structures on T^2M, induced by ∇ and ∇', are isomorphic.*

6 Differential equations

The importance of Fréchet manifolds arises from their ubiquity as quotient spaces of bundle sections and hence as environments for differential equations on such spaces. This context was addressed next in [1] and those authors provided a new way of representing and solving a wide class of evolutionary equations on Fréchet manifolds of sections.

First [1] considered a Banach manifold M, and defined an *integral curve* of ξ as a smooth map $\theta : J \to M$, defined on an open interval J of $\mathbb{R}$, if it satisfies the condition

$$T_t^2\theta(\partial_t) = \xi(\theta(t)). \tag{9}$$

Here ∂_t is the second order tangent vector of $T_t^2\mathbb{R}$ induced by a curve $c : \mathbb{R} \to \mathbb{R}$ with $c'(0) = 1, c''(0) = 1$. If M is simply a Banach space $\mathbb{E}$ with differential structure induced by the global chart $(\mathbb{E}, id_{\mathbb{E}})$, then the generalization is clear since the above condition reduces to the second derivative of θ:

$$T_t^2\theta(\partial_t) = \theta''(t) = D^2\theta(t)(1,1).$$

Then the following were proved [1].

Theorem 6.1. *Let ξ be a second order vector field on a manifold M modeled on Banach space $\mathbb{E}$. Then, the existence of an integral curve θ of ξ is equivalent to the solution of a system of second order differential equations on $\mathbb{E}$.*

Of course, these second order differential equations depend not only on the choice of the second order vector field but also the choice of the linear connection that underpins the vector bundle structure. In the case of a Banach manifold that is a Lie group, $M = (G, \gamma)$,

Theorem 6.2. *Let v be any vector of the second order tangent space of G over the unitary element. Then, a corresponding left invariant second order vector field ξ of G may be constructed. Also, every monoparametric subgroup $\beta : \mathbb{R} \to G$ is an integral curve of the second order left invariant vector field ξ^2 of G that corresponds to $\ddot{\beta}(0)$.*

Extending this to a Fréchet manifold M that is the projective limit of Banach manifolds [16], yielded the result:

Theorem 6.3. *Every second order vector field ξ on M obtained as projective limit of second order vector fields $\{\xi^i$ on $M^i\}_{i\in\mathbb{N}}$ admits locally a unique integral curve θ satisfying an initial condition of the form $\theta(0) = x$ and $T_t\theta(\partial_t) = y$, $x \in M$, $y \in T_{\theta(t)}M$, provided that the components ξ^i admit also integral curves of second order.*

7 Hypercyclicity

A continuous operator T on a topological vector space $\mathbb{E}$ is *cyclic* if for some $f \in \mathbb{E}$ the span of $\{T^n f, n \geq 0\}$ is dense in $\mathbb{E}$. Also, T is *hypercyclic* if, for some f, called a *hypercyclic vector*, $\{T^n f, n \geq 0\}$ is dense in $\mathbb{E}$, and *supercyclic* if the projective space orbit $\{\lambda T^n f, \lambda \in \mathbb{C}, n \geq 0\}$ is dense in $\mathbb{E}$. These properties are called *weakly hypercyclic*, *weakly supercyclic* respectively, if T has the property with respect to the weak topology. For example, the translation by a fixed nonzero $z \in \mathbb{C}$ is hypercyclic on the Fréchet space $\mathbb{H}(\mathbb{C})$ of entire functions, and so is the differentiation operator $f \mapsto f'$. Any power T^m of a hypercyclic linear operator is hypercyclic, Ansari [4]. Finite dimensional spaces do not admit hypercyclic operators, Kitai [32].

More generally, a sequence of linear operators $\{T_n\}$ on a topological is called hypercyclic if, for some $f \in \mathbb{E}$, the set $\{T_n f, n \in \mathbb{N}\}$ is dense in $\mathbb{E}$; see Chen and Shaw [13] for a discussion of related properties. The sequence $\{T_n\}$ is said to satisfy the *Hypercyclicity Criterion* for an increasing sequence $\{n(k)\} \subset \mathbb{N}$ if there are dense subsets $X_0, Y_0 \subset \mathbb{E}$ satisfying:

$$(\forall f \in X_0) \quad T_{n(k)} f \to 0$$
$$(\forall g \in Y_0) \quad \text{there is a sequence } \{u(k)\} \subset \mathbb{E} \text{ such that } u(k) \to 0 \text{ and } T_{n(k)} u(k) \to 0.$$

Bes and Peris [11] proved that on a separable Fréchet space $\mathbb{F}$ a continuous linear operator T satisfies the Hypercyclicity Criterion if and only if $T \oplus T$ is hypercyclic on $\mathbb{F} \oplus \mathbb{F}$. Moreover, if T satisfies the Hypercyclicity Criterion then so does every power T^n for $n \in \mathbb{N}$.

The book by Bayart and Matheron [7] provides more details of the theory of hypercyclic operators. Bermúdez et al. [8] investigated hypercyclicity, topological mixing and chaotic maps on Banach spaces. Bernal and Grosse-Erdmann studied the existence of hypercyclic semigroups of continuous operators on a Banach space. Albanese et al. [3] considered cases when it is possible to extend Banach space results on C_0-semigroups of continuous linear operators to Fréchet spaces. Every operator norm continuous semigroup in a Banach space X has an infinitesimal generator belonging to the space of continuous linear operators on X; an example is given to show that this fails in a general Fréchet space. However, it does not fail for countable products of Banach spaces and quotients of such products; these are the Fréchet spaces that are quojections, the projective sequence consisting of surjections. Examples include the sequence space $\mathbb{C}^{\mathbb{N}}$ and the Fréchet space of continuous functions $C(X)$ with X a σ-compact completely regular topological space and compact open topology.

Grosse-Erdmann [28] related hypercyclicity to the topological universality concept, and showed that an operator T is hypercyclic on a separable Fréchet space $\mathbb{F}$ if it has the *topological transitivity property*: for every pair of nonempty open subsets $U, V \subseteq \mathbb{F}$ there is some $n \in \mathbb{N}$ such that $T^n(U) \bigcap V \neq \emptyset$. Chen and Shaw [13] related hypercyclicity to topological mixing, following Costakis and Sambarino [14] who showed that if T^n satisfies the Hypercyclicity Criterion then T is *topologically mixing* in the sense that: for every pair of nonempty open subsets $U, V \subseteq \mathbb{F}$ there is some $N \in \mathbb{N}$ such that $T^n(U) \bigcap V \neq \emptyset$ for all $n \geq N$. See also Bermúdez et al. [8] for further studies of hypercyclic and chaotic maps on Banach spaces in the context of topological mixing.

It was known that the direct sum of two hypercyclic operators need not be hypercyclic but recently De La Rosa and Read [15] showed that even the direct sum of a hypercyclic operator with itself $T \oplus T$ need not be hypercyclic. Bonet and Peris [12] showed that every separable infinite dimensional Fréchet space $\mathbb{F}$ supports a hypercyclic operator. Moreover, from Shkarin [42], there is a linear operator T such that the direct sum $T \oplus T \oplus ... \oplus T = T^{\oplus m}$ of m copies of T is a hypercyclic operator on $\mathbb{F}^m$ for each $m \in \mathbb{N}$. An m-tuple $(T, T, ..., T)$ is called *disjoint hypercyclic* if there exists $f \in \mathbb{F}$ such that $(T_1^n f, T_2^n f, ..., T_m^n f), n = 1, 2, ...$ is dense in $\mathbb{F}^m$. See Salas [41] and Bernal-González [9] for examples and recent results.

O'Regan and Xian [40] proved fixed point theorems for maps and multivalued maps between Fréchet spaces, using projective limits and the classical Banach theory. Further recent work on set valued maps between Fréchet spaces can be found in Galanis et al.[26, 27, 39] and Bakowska and Gabor [6].

Montes-Rodriguez et al. [35] studied the Volterra composition operators V_φ for φ a measurable self-map of $[0, 1]$ on functions $f \in L^p[0, 1], \; 1 \leq p \leq \infty$

$$(V_\varphi f)(x) = \int_0^\varphi (x) f(t) dt \tag{10}$$

These operators generalize the classical Volterra operator V which is the case when φ is the identity. V_φ is measurable, and compact on $L^p[0, 1]$.

Consider the Fréchet space $\mathbb{F} = C_0[0, 1)$, of continuous functions vanishing at zero with the topology of uniform convergence on compact subsets of $[0, 1)$. It was known that the action of V_φ on $C_0[0, 1)$

is hypercyclic when $\varphi(x) = x^b, b \in (0,1)$ [30]. This result has now been extended by Montes-Rodriguez et al. to give the following complete characterization.

Theorem 7.1. *[35] For $\varphi \in C_0[0,1)$ the following are equivalent*
(i) *φ is strictly increasing with $\varphi(x) > x$ for $x \in (0,1)$* **(ii)** *V_φ is weakly hypercyclic* **(iii)** *V_φ is hypercyclic.*

Karami et al [31] seem to obtain examples of hypercyclic operators on $H_{bc}(\mathbb{E})$, the space of bounded functions on compact subsets of Banach space $\mathbb{E}$. For example, when $\mathbb{E}$ has separable dual $\mathbb{E}^*$ then for nonzero $\alpha \in \mathbb{E}$, $T_\alpha : f(x) \mapsto f(x+\alpha)$ is hypercyclic.

Yousefi and Ahmadian [47] studied the case that T is a continuous linear operator on an infinite dimensional Hilbert space $\mathbb{H}$ and left multiplication is hypercyclic with respect to the strong operator topology. Then there is a Fréchet space $\mathbb{F}$ containing $\mathbb{H}$, $\mathbb{F}$ is the completion of $\mathbb{H}$, and for every nonzero vector $f \in \mathbb{H}$ the orbit $\{T^n f, n \geq\}$ meets any open subbase of $\mathbb{F}$.

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