# Commuting Involution Graphs of Certain Finite Simple Classical Groups 

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# COMMUTING INVOLUTION GRAPHS 

## OF CERTAIN FINITE SIMPLE CLASSICAL GROUPS

A thesis submitted to the University of Manchester for the degree of Doctor of Philosophy
in the Faculty of Engineering and Physical Sciences

## Alistaire Duncan Fraser Everett

## Contents

Abstract ..... 4
Declaration ..... 5
Copyright Statement ..... 6
Acknowledgements ..... 8
Index of Notation ..... 9
1 Introduction ..... 13
2 Background ..... 20
2.1 Classical Groups ..... 20
2.2 Commuting Involution Graphs ..... 31
2.3 Useful Results ..... 37
2.4 Final Remarks ..... 38
3 4-Dim. Symplectic groups over $G F(q), q$ even ..... 39
3.1 The Structure of $\mathcal{C}\left(G, X_{i}\right), i=1,3$ ..... 42
3.2 The Structure of $\mathcal{C}\left(G, X_{2}\right)$ ..... 49
4 4-Dim. Symplectic groups over $G F(q), q$ odd ..... 55
4.1 The Structure of $\mathcal{C}\left(G, Y_{1}\right)$ ..... 56
4.2 The Structure of $\mathcal{C}\left(G, Y_{2}\right)$ ..... 62
5 3-Dimensional Unitary Groups ..... 94
6 4-Dim. Unitary groups over $G F(q), q$ even ..... 111
6.1 The Structure of $\mathcal{C}\left(G, Z_{1}\right)$ ..... 112
6.2 The Structure of $\mathcal{C}\left(G, Z_{2}\right)$ ..... 115
7 Group Extensions and Affine Groups ..... 132
7.1 2-dimensional Projective General Linear Groups ..... 132
7.2 Affine Orthogonal Groups ..... 139
8 Prelude to Future Work ..... 146
8.1 Projective Symplectic Groups of Arbitrary Dimension ..... 146
8.2 4-Dimensional Projective Special Unitary Groups over Fields of Odd Characteristic ..... 147
8.3 Rank 2 Twisted Exceptional Groups of Lie Type ..... 149
Bibliography ..... 151

## The University of Manchester

Alistaire Duncan Fraser Everett<br>Doctor of Philosophy

Commuting Involution Graphs of Certain Finite Simple Classical Groups June 14, 2011

For a group $G$ and $X$ a subset of $G$, the commuting graph of $G$ on $X$, denoted by $\mathcal{C}(G, X)$, is the graph whose vertex set is $X$ with $x, y \in X$ joined by an edge if $x \neq y$ and $x$ and $y$ commute. If the elements in $X$ are involutions, then $\mathcal{C}(G, X)$ is called a commuting involution graph. This thesis studies $\mathcal{C}(G, X)$ when $G$ is either a 4 -dimensional projective symplectic group; a 3-dimensional unitary group; 4-dimensional unitary group over a field of characteristic 2; a 2-dimensional projective general linear group; or a 4-dimensional affine orthogonal group, and $X$ a $G$-conjugacy class of involutions. We determine the diameters and structure of the discs of these graphs.

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"Be nice to your kids... they'll choose your nursing home."
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## Index of Notation

## General Notation

| Symbol | Page | Symbol | Page |
| :--- | :--- | :--- | :--- |
| $(\cdot, \cdot)($ form $)$ | 21 | $P G L$ | 22 |
| $a_{i j},\left(a_{i j}\right)$ | 20 | $P G U$ | 26 |
| Aff $(G), A G$ | 30 | $P G O$ | 29 |
| $\mathcal{C}(-,-)$ | 31 | $P S L$ | 22 |
| $d(-,-)$ | 31 | $P S O$ | 29 |
| Diam | 31 | $P \Omega$ | 29 |
| $\Delta_{i}(-)$ | 31 | $P S p$ | 24 |
| $\Delta_{i}^{j}(-)$ | 35,45 | $P S U$ | 26 |
| $g_{i j}$ | 20 | $S L$ | 20 |
| $G$ | 38 | $S p$ | 24 |
| $G F(q)$ | 20 | $S U$ | 26 |
| $G L$ | 21 | $S O^{\varepsilon}$ | 28 |
| $G O^{\varepsilon}$ | 28 | $U^{\perp}$ | 21 |
| $G U$ | 26 | $U_{n}(q)$ | 26 |
| $H$ | 38 | $V$ | 38 |
| $J$ | 21 | $X_{i}$ | 40 |
| $L_{n}(q)$ | 22 | $Y_{i}$ | 56 |
| $O_{n}^{\varepsilon}(q)$ | 29 |  | 96,112 |
| $\Omega_{n}^{\varepsilon}$ | 28 | 20 |  |
| $p$ | 28 |  |  |

Notation for Chapter 3

| Symbol | Page | Symbol | Page |
| :--- | :--- | :--- | :--- |
| $\mathcal{C}(\Delta)$ | 50 | $t_{i}$ | 40 |
| $\Delta$ | 50 | $U_{i}(x)$ | 49 |
| $\Delta_{j}^{\mathcal{C}}(-)$ | 50 | $V(\mathcal{C}(\Delta))$ | 50 |
| $d^{\mathcal{C}}$ | 50 | $V(x)$ | 40 |
| $L$ | 41 | $V(\mathcal{Z})$ | 52 |
| $P_{i}$ | 49 | $y_{\alpha}$ | 46 |
| $Q_{i}$ | 40 | $Z_{R}, Z_{S}, Z_{T}$ | 49 |
| $S$ | 39 | $\mathcal{Z}$ | 52 |

Notation for Chapter 4

| Symbol | Page | Symbol | Page |
| :--- | :--- | :--- | :--- |
| $d^{-}$ | 66 | $G^{0}$ | 68 |
| $d^{+}$ | 67 | $G^{\tau}$ | 89 |
| $d^{0}$ | 69 | $\Gamma_{i}(-)$ | 79 |
| $d^{L}$ | 69 | $J_{0}$ | 55 |
| $\delta$ | 62 | $K_{y}$ | 86 |
| $\Delta_{i}^{-}(-)$ | 66 | $L$ | 68 |
| $\Delta_{i}^{+}(-)$ | 67 | $\widehat{L}$ | 89 |
| $\Delta_{i}^{0}(-)$ | 69 | $L_{1}, L_{2}$ | 66 |
| $\Delta_{i}^{L}(-)$ | 69 | $L_{t}, L_{y}$ | 63,86 |
| $\Delta_{2}^{K}(-), \Delta_{2}^{C_{G}(U)}(-)$ | 79 | $Q$ | 56 |
| $E, E^{\perp}$ | 57 | $\phi$ | 68 |
| $g_{Q}, g_{L}$ | 69 | $\rho$ | 78 |
| $G^{-}$ | 66 | $s$ | 55 |
| $G^{+}$ | 66 |  |  |

Notation for Chapter 4 (continued)

| Symbol | Page | Symbol | Page |
| :--- | :--- | :--- | :--- |
| $\Sigma, \Sigma_{\beta}$ | 57 | $\mathcal{U}_{i}(-)$ | 79 |
| $t, t_{L}$ | 62,69 | $v_{i}$ | 55 |
| $t^{\tau}$ | 89 | $\mathcal{W}_{i}(-)$ | 79 |
| $U_{\beta}$ | 57 | $X$ | 56 |
| $\mathcal{U}_{i}$ | 64 | $Y^{+}$ | 66 |
| $\mathcal{U}_{i}^{\varepsilon}$ | 66 | $Y^{0}$ | 69 |

Notation for Chapter 5

| Symbol | Page | Symbol | Page |
| :--- | :--- | :--- | :--- |
| $\bar{a}, \overline{\left(a_{i j}\right)}$ | 94 | $t$ | 96 |
| $\Delta_{2}^{\alpha}(-)$ | 100 | $x$ | 98 |
| $N_{1}, N_{2}$ | 95 | $y$ | 104 |
| $S, \widehat{S}$ | 94 | $z_{\gamma}$ | 104 |

Notation for Chapter 6

| Symbol | Page | Symbol | Page |
| :--- | :--- | :--- | :--- |
| $\bar{a}, \overline{\left(a_{i j}\right)}$ | 94 | $P_{x}$ | 118 |
| $C_{x}$ | 118 | $Q$ | 112 |
| $J_{0}$ | 111 | $Q_{i}$ | 113 |
| $L$ | 116 | RREF | 119 |
| $\mathcal{M}$ | 119 | $\rho$ | 119 |
| $\mathcal{M}_{i}$ | 120 | $S$ | 112 |
| $N_{2}$ | 95 | $S_{t}$ | 116 |
| $N_{x}$ | 118 | $t_{i}$ | 111 |
| $P$ | 115 | $U, U_{\beta}, U_{\alpha, \beta}, U_{\alpha, \beta, \gamma}$ | 120 |
| $P_{1}$ | 116 | $\mathcal{U}, \mathcal{U}_{\text {iso }}$ | 119 |

Notation for Chapter 7

| Symbol | Page | Symbol | Page |
| :--- | :--- | :--- | :--- |
| $d^{L}$ | 139 | $\varphi_{x},\left.\varphi_{x}\right\|_{U}$ | 140 |
| $\delta$ | 133 | $t_{\alpha}$ | 135 |
| $\Delta_{i}^{L}(-)$ | 139 | $X$ | 132 |
| $g_{V}, g_{L}$ | 139 | $X_{L}$ | 139 |
| $L$ | 139 |  |  |

## Chapter 1

## Introduction

One powerful method for investigating the structure of a group is by studying its action on a graph. In the study of finite simple groups from the 1950s, the method of embedding a group into the automorphism group of a graph has been used with many successful results. Recent methods within this realm of study have still shown to be beneficial. For $G$ a group and $X$ a subset of $G$, the commuting graph of $G$ on $X, \mathcal{C}(G, X)$, is the graph whose vertex set is $X$ with vertices $x, y \in X$ joined whenever $x \neq y$ and $x y=y x$. In essence commuting graphs first appeared in the seminal paper of Brauer and Fowler [17], famous for giving a proof that for a given isomorphism type of an involution centraliser, only finitely many non-abelian simple groups can contain it, up to isomorphism. The commuting graphs considered in [17] had $X=G \backslash\{1\}$ - such graphs have played an important role in recent work related to the Margulis-Platanov conjecture (see [35]). The complement of this type of commuting graph, called a non-commuting graph, appeared in [33] where B.H. Neumann solved a problem posed by Erdös. Moreover, a conjecture of Abdollahi, Akbari and Maimani states that if $G$ is a finite simple group and $M$ is a finite group with trivial centre, and the non-commuting graphs of $G$ and $M$ are isomorphic as graphs, then $G$ and $M$ are isomorphic as groups. This conjecture has been shown to be true in a variety of cases, in particular those where a conjecture of J . Thompson also holds (see [1], [23] and [21]). Various kinds of commuting graph have been deployed in the study of finite groups, particularly the non-abelian simple groups. For example, a
computer-free uniqueness proof of the Lyons simple group by Aschbacher and Segev [9] employed a commuting graph where the vertices consisted of the 3-central subgroups of order 3 .

A commuting involution graph is a specific kind of commuting graph of $G$, where the vertex set is a conjugacy class of involutions. Commuting involution graphs first arose in Fischer's work during his investigation into the 3-transposition groups (this work remains largely unpublished [25], [26]). Here, the vertices of the commuting involution graph were conjugate involutions such that the product of any two had order at most 3. This graph led, in part, to the construction of the three sporadic simple groups of Fischer; $F i_{22}, F i_{23}$ and $F i_{24}^{\prime}$. The construction and uniqueness of these groups are detailed in [8]. Shortly after, Aschbacher [7] found a condition on a commuting involution graph of a finite group, to guarantee the existence of a strongly embedded subgroup.

The detailed study of commuting involution graphs came to prominence with the work of Bates, Bundy, Hart (nèe Perkins) and Rowley; in particular, the diameters and disc sizes were determined. For $G$ a symmetric group, or more generally a finite Coxeter group; a projective special linear group; or a sporadic simple group, and $X$ a conjugacy class of involutions of $G$, the structure of $\mathcal{C}(G, X)$ has been investigated at length by this quartet ([11], [13], [14], and [15]). The commuting involution graphs of Affine Coxeter groups have also been studied in Perkins [34]. Further work on commuting graphs of the symmetric groups were explored in [12] and [18]. A different flavour of commuting graph has been examined in Akbari, Mohammadian, Radjavi, Raja [3] and Iranmanesh, Jafarzadeh [29]. There, for a group $G$, the vertex set is $G \backslash Z(G)$ with two distinct elements joined if they commute. Recently there has been work on commuting graphs for rings (see, for example, [2] or [4]).

This thesis presents a sequel of sorts to the research of Bates, Bundy, Hart and Rowley, in particular the commuting involution graphs of special linear groups [15]. Here, we present analogous results for the diameter and disc sizes of $\mathcal{C}(G, X)$ when $G$ is a finite 4 -dimensional projective symplectic group; a finite 3-dimensional unitary
group; or a finite 4-dimensional unitary group over a field of characteristic 2. Moreover, we investigate the structure of $\mathcal{C}(G, X)$ when $G$ is an affine orthogonal group, or a projective general linear group.

In Chapter 2, we give a brief overview of the finite classical groups, which we will be primarily working with. This chapter will be elementary, but fundamental in laying the foundations of what is to come. A review of the current research on commuting involution graphs will also be undertaken. Notation and general conventions will be set in stone in this chapter.

Chapter 3 explores the structure of the 4 -dimensional projective symplectic groups $H=S p_{4}(q) \cong P S p_{4}(q)=G$ when $q=2^{a}$ for some natural number $a$. There are three conjugacy classes of involutions in $G$ denoted by

$$
\begin{aligned}
& X_{1}=\left\{x \in G \mid x^{2}=1, \operatorname{dim} C_{V}(x)=3\right\} ; \\
& X_{2}=\left\{x \in G \mid x^{2}=1, \operatorname{dim} C_{V}(x)=2, \operatorname{dim} V(x)=3\right\} ; \text { and } \\
& X_{3}=\left\{x \in G \mid x^{2}=1, \operatorname{dim} C_{V}(x)=2, V(x)=V\right\} .
\end{aligned}
$$

where $V(x)=\left\{v \in V \mid\left(v, v^{x}\right)=0\right\}$. This chapter focusses on the proofs of Theorems 1.1 and 1.2.

Theorem 1.1. The commuting involution graph $\mathcal{C}\left(G, X_{i}\right)$, for $i=1,3$ is connected of diameter 2, with disc sizes

$$
\begin{aligned}
& \left|\Delta_{1}(t)\right|=q^{3}-2 ; \text { and } \\
& \left|\Delta_{2}(t)\right|=q^{3}(q-1) .
\end{aligned}
$$

Theorem 1.2. The commuting involution graph $\mathcal{C}\left(G, X_{2}\right)$ is connected of diameter 4, with disc sizes

$$
\begin{aligned}
& \left|\Delta_{1}(t)\right|=q^{2}(2 q-3) ; \\
& \left|\Delta_{2}(t)\right|=2 q^{2}(q-1)^{2} ; \\
& \left|\Delta_{3}(t)\right|=2 q^{3}(q-1)^{2} ; \text { and } \\
& \left|\Delta_{4}(t)\right|=q^{4}(q-1)^{2} .
\end{aligned}
$$

The general collapsed adjacency diagrams for $\mathcal{C}\left(G, X_{i}\right), i=1,3$, are presented in Figure 3.1.

Chapter 4 retains the family of classical groups but changes the field to that of odd characteristic. A brief examination of the commuting involution graphs of $H=S p_{4}(q)$ is given, before the study of the commuting involution graphs of $G=H / Z(H) \cong$ $P S p_{4}(q)$ is undertaken. There are two classes of involutions in $G$, denoted by $Y_{1}$ and $Y_{2}$. We denote $Y_{1}$ to be the conjugacy class of involutions whose elements are the images of an involution in $H$, and $Y_{2}$ to be the conjugacy class of involutions whose elements are the image of an element of $H$ of order 4 which square to the non-trivial element of $Z(H)$. The following two theorems are proved in this chapter.

Theorem 1.3. The commuting involution graph $\mathcal{C}\left(G, Y_{1}\right)$ is connected of diameter 2, with disc sizes

$$
\begin{aligned}
& \left|\Delta_{1}(t)\right|=\frac{1}{2} q\left(q^{2}-1\right) ; \text { and } \\
& \left|\Delta_{2}(t)\right|=\frac{1}{2}\left(q^{4}-q^{3}+q^{2}+q-2\right) .
\end{aligned}
$$

Theorem 1.4. (i) If $q \equiv 3(\bmod 4)$ then $\mathcal{C}\left(G, Y_{2}\right)$ is connected of diameter 3. Furthermore,

$$
\begin{aligned}
&\left|\Delta_{1}(t)\right|=\frac{1}{2} q\left(q^{2}+2 q-1\right) \\
&\left|\Delta_{2}(t)\right|=\frac{1}{16}(q+1)\left(3 q^{5}-2 q^{4}+8 q^{3}-30 q^{2}+13 q-8\right) ; \text { and } \\
&\left|\Delta_{3}(t)\right|=\frac{1}{16}(q-1)\left(5 q^{5}-4 q^{4}-2 q^{3}+4 q^{2}+5 q+5\right)
\end{aligned}
$$

(ii) If $q \equiv 1(\bmod 4)$ then $\mathcal{C}\left(G, Y_{2}\right)$ is connected of diameter 3. Furthermore,

$$
\begin{aligned}
& \left|\Delta_{1}(t)\right|=\frac{1}{2} q\left(q^{2}+1\right) \\
& \left|\Delta_{2}(t)\right|=\frac{1}{16}(q-1)\left(3 q^{5}-6 q^{4}+32 q^{3}-10 q^{2}-27 q-8\right) ; \text { and } \\
& \left|\Delta_{3}(t)\right|=\frac{1}{16}(q-1)\left(5 q^{5}+22 q^{4}-8 q^{3}+34 q^{2}+51 q+24\right)
\end{aligned}
$$

It is interesting to note that the proof of Theorem 1.4 is highly complex and a different viewpoint was needed to take on this task. The reason for this is that for $\mathcal{C}\left(G, X_{i}\right),(i=1,2,3)$ and $\mathcal{C}\left(G, Y_{1}\right)$ the graph can be studied effectively by working in
$H=S p_{4}(q)$ and looking at certain configurations in the natural symplectic module $V$, involving $C_{V}(x)$ for various $x \in X\left(X=X_{i}, i=1,2,3\right.$ or $\left.X Z(H) / Z(H)=Y_{1}\right)$. The key point being that, in these four cases for $x \in X, C_{V}(x)$ is a non-trivial subspace of $V$ whereas, for $x$ of order 4 and squaring into $Z(H), C_{V}(x)$ is trivial. If we change tack and look at $G$ acting on the projective symplectic space things are not much better. When $q \equiv 3(\bmod 4)$ elements of $Y_{2}$ fix no projective points, while in the case $q \equiv 1(\bmod 4)$ they fix $2 q+2$ projective points. However, even in the latter case, the fixed projective points didn't appear to be of much assistance. It is the isomorphism $P S p_{4}(q) \cong O_{5}(q)$ that comes to our rescue. If now $V$ is the 5 -dimensional orthogonal module and $x \in Y_{2}$, then $\operatorname{dim} C_{V}(x)=3$. Even so, probing $\mathcal{C}\left(G, Y_{2}\right)$ turns out to be a lengthy process. Fix $t \in Y_{2}$. Then by Lemma 4.7, $Y_{2} \subseteq \bigcup_{U \in \mathcal{U}_{1}} C_{G}(U)$ where $\mathcal{U}_{1}$ is the set of all 1-subspaces of $C_{V}(t)$ and as a result, by Lemma 4.8, $\mathcal{C}\left(G, Y_{2}\right)$ may be viewed as the union of commuting involution graphs for various subgroups of $G$. Up to isomorphism there are three of these commuting involution graphs (called $\mathcal{C}\left(G^{-}, Y^{-}\right), \mathcal{C}\left(G^{+}, Y^{+}\right)$and $\mathcal{C}\left(G^{0}, Y^{0}\right)$ in Chapter 4). After studying these three commuting involution graphs in Theorems 4.10, 4.12 and 4.18 it follows immediately (Theorem 4.19) that $\mathcal{C}\left(G, Y_{2}\right)$ is connected and has diameter at most 3. Using the sizes of the discs in $\mathcal{C}\left(G^{-}, Y^{-}\right), \mathcal{C}\left(G^{+}, Y^{+}\right)$and $\mathcal{C}\left(G^{0}, Y^{0}\right)$ we then complete the proof of Theorem 1.4. This "patching together" of the discs is quite complicated - for example we must confront such issues as $t$ and $x$ in $Y_{2}$ being of distance 3 in each of the commuting involution subgraphs which contain both $t$ and $x$, yet they have distance 2 in $\mathcal{C}\left(G, Y_{2}\right)$ (see Lemmas 4.33 to 4.38).

Chapter 5 investigates a different family of classical groups, namely the 3-dimensional unitary groups. We set $H=S U_{3}(q)$ and $G=H / Z(H) \cong U_{3}(q)$. We begin with a short review of the commuting involution graphs for $q$ even, before the much greater task where $q$ is odd is tackled. It should be noted that the commuting involution graphs for $S U_{3}(q)$ and $U_{3}(q)$ are isomorphic, due to $Z(H)$ being either trivial or of order 3. For ease, we work explicitly in $H$. There is only one conjugacy class of involutions in $H$, which is denoted by $Z_{0}$. Theorem 1.5 is the central focus of this chapter.

Theorem 1.5. (i) Let $q$ be even. The commuting involution graph $\mathcal{C}\left(G, t^{G}\right)$ for an involution $t \in G$ is disconnected, and consists of $q^{3}+1$ cliques on $q-1$ vertices.
(ii) Let $q$ be odd. The commuting involution graph $\mathcal{C}\left(H, Z_{0}\right)$ is connected of diameter 3, with disc sizes

$$
\begin{aligned}
& \left|\Delta_{1}(t)\right|=q(q-1) \\
& \left|\Delta_{2}(t)\right|=q(q-2)\left(q^{2}-1\right) ; \text { and } \\
& \left|\Delta_{3}(t)\right|=(q+1)\left(q^{2}-1\right)
\end{aligned}
$$

The general collapsed adjacency diagrams for arbitrary odd $q$ are constructed at the end of the chapter, with the third disc differing in orbit structure depending on whether $q \equiv 5(\bmod 6)$ or not. These can be found in Figures 5.1 and 5.2 respectively. Chapter 6 raises the dimension and we look at the 4 -dimensional unitary groups over fields of characteristic 2 . We set $H=S U_{4}(q) \cong U_{4}(q)=G$ and its two conjugacy classes by $Z_{1}$ and $Z_{2}$, where

$$
\begin{aligned}
& Z_{1}=\left\{x \in G \mid x^{2}=1, \operatorname{dim} C_{V}(x)=3\right\} ; \text { and } \\
& Z_{2}=\left\{x \in G \mid x^{2}=1, \operatorname{dim} C_{V}(x)=2\right\} .
\end{aligned}
$$

This chapter concentrates on the proofs of Theorems 1.6 and 1.7.

Theorem 1.6. The commuting involution graph $\mathcal{C}\left(G, Z_{1}\right)$ is connected of diameter 2, with disc sizes

$$
\begin{aligned}
& \left|\Delta_{1}(t)\right|=q^{4}-q^{2}+q-2 ; \text { and } \\
& \left|\Delta_{2}(t)\right|=q^{5}(q-1)
\end{aligned}
$$

Theorem 1.7. The commuting involution graph $\mathcal{C}\left(G, Z_{2}\right)$ is connected of diameter 3, with disc sizes

$$
\begin{aligned}
& \left|\Delta_{1}(t)\right|=q(q-1)\left(2 q^{2}+q+1\right)-1 ; \\
& \left|\Delta_{2}(t)\right|=q^{3}(q-1)\left(q^{3}+2 q^{2}+q-1\right) ; \text { and } \\
& \left|\Delta_{3}(t)\right|=q^{4}(q-1)\left(q^{3}-q+1\right) .
\end{aligned}
$$

Chapter 7, in a change of scenery, looks at the non-simple groups $P G L_{2}(q)$ and $A O_{4}^{ \pm}(q)$. In Chapter 4, Theorem 4.18 determines the diameter of the commuting involution graph of $\mathrm{AO}_{3}(q)$. However, an alternative proof using the machinery developed to tackle the 4 -dimensional case is presented here. It will be shown that in both the 3 - and 4-dimensional cases, the diameter of the commuting involution graph does not differ from the non-affine cases. However, as in Theorem 4.18, what will be highlighted is that distance is not preserved as we move between the two. This chapter is devoted to the proofs of the following theorems.

Theorem 1.8. Let $G=P G L_{2}(q)$ and suppose $q \equiv \delta(\bmod 4), \delta= \pm 1, q \notin\{3,7,11\}$. Let $X$ be the conjugacy class of involutions of $G$ such that $X \cap G^{\prime}=\varnothing$. Then $\mathcal{C}(G, X)$ is connected of diameter 3 with disc sizes

$$
\begin{aligned}
\left|\Delta_{1}(t)\right| & =\frac{1}{2}(q+\delta) ; \\
\left|\Delta_{2}(t)\right| & =\frac{1}{4}(q-1)(q-1+2 \delta) ; \text { and } \\
\left|\Delta_{3}(t)\right| & =\frac{1}{4}(q-5 \delta)(q+\delta) .
\end{aligned}
$$

Theorem 1.9. Let $L \in\left\{O_{3}(q), O_{4}^{+}(q), O_{4}^{-}(q)\right\}$ for $q$ odd, and $G=V L=\operatorname{Aff}(L)$. Let $X$ be a conjugacy class of involutions of $G$ such that $X_{L}=V X / V$ is a non-trivial conjugacy class of involutions in $L$. Then $\operatorname{Diam} \mathcal{C}\left(L, X_{L}\right)=\operatorname{Diam} \mathcal{C}(G, X)=3$.

Finally, Chapter 8 outlines some future avenues stemming from the work undertaken in this thesis, proving some initial results that will sow the seeds of upcoming research. In particular, motivating results about arbitrary dimensional symplectic groups over fields of characteristic 2, 4-dimensional projective unitary groups over fields of odd characteristic, and twisted exceptional groups of Lie rank 2 will be presented.

## Chapter 2

## Background

To begin, we give a background "crash course" in classical groups and provide a literary review of the recent research into commuting involution graphs. We use standard group theoretical notation as in, for example, [27]. Group nomenclature is lifted from the Atlas [22]. Conventions and non-standard notation will be defined in situ and will carry through the thesis. Any entry omitted from a matrix should be interpreted as zero. The Galois field of $q=p^{a}$ elements for $p$ prime will be denoted $G F(q)$. For any matrix $g$, the $(i, j)^{\text {th }}$ entry will be denoted $g_{i j}$.

### 2.1 Classical Groups

We present some background information on the finite simple classical groups. A detailed description of these groups alongside in-depth background reading can be found in [38]. The orders of the finite simple classical groups can be deduced from the orders of the full isometry group, as given in [22].

Let $V$ be an $n$-dimensional vector space over a field $K$, with basis $\left\{e_{1}, \ldots, e_{n}\right\}$. Let $\sigma$ be a linear transformation of $V$ onto itself. Supposing $e_{i}^{\sigma}=\sum_{j=1}^{n} a_{i j} e_{j}$ for $a_{i j} \in K$, $\sigma$ can be represented as a matrix $\left(a_{i j}\right)$.

Consider a map $(\cdot, \cdot): V \times V \rightarrow K$. If the map satisfies

$$
\begin{aligned}
(\lambda u+\mu v, w) & =\lambda(u, w)+\mu(v, w) \\
\text { and }(u, \lambda v+\mu w) & =\lambda(u, v)+\mu(u, w)
\end{aligned}
$$

for any $\lambda, \mu \in K$ and any $u, v \in V$ then the map is called a bilinear form. Let $\tau$ be an automorphism of $K$. If the map is linear in the first argument and satisfies $(u, v)=(v, u)^{\tau}$ then the map is called a sesquilinear form. Assume from now on the map is either a bilinear or sesquilinear form. If, for a fixed $u \in V,(u, v)=0$ for all $v \in V$ implies $u=0$, then the form is non-degenerate. The Gram matrix of a form on $V$ (denoted in this thesis by $J)$ is the matrix $J=\left(a_{i j}\right)$ where $a_{i j}=\left(e_{i}, e_{j}\right)$. A nondegenerate form implies $J$ is non-singular. Suppose $\sigma$ is a linear transformation that preserves the form, so $(u, v)=\left(u^{\sigma}, v^{\sigma}\right)$ for all $u, v \in V$. Then the matrix representing $\sigma$, say $A=\left(a_{i j}\right)$, satisfies $A^{T} J A=J$.

Let $U \leq V$ and define $U^{\perp}=\{v \in V \mid(u, v)=0$, for all $u \in U\}$. Then

$$
\begin{equation*}
\operatorname{dim} U+\operatorname{dim} U^{\perp}=\operatorname{dim} V \tag{2.1}
\end{equation*}
$$

and if the form is non-degenerate on restriction to $U$, then the form is non-degenerate on restriction to $U^{\perp}$ also. Any vector $v \in V$ such that $(v, v)=0$ is called an isotropic (or singular) vector. Any subspace $U$ of $V$ is called isotropic if it contains an isotropic vector. If $(u, v)=0$ for all $u, v \in U$, then we say $U$ is totally isotropic (that is, the Gram matrix of the form is the zero matrix).

## Linear Groups

Let $V$ be as before and denote the set of all invertible linear transformations from $V$ onto itself by $G L(V)$. For any $\sigma \in G L(V), \sigma$ can be represented as an invertible matrix. This gives an isomorphism from $G L(V)$ onto $G L_{n}(K)$, the general linear group. The subgroup of $G L_{n}(K)$ consisting of matrices of determinant 1 is denoted $S L_{n}(K)$, the special linear group. The centre, $Z$, of $G L_{n}(K)$ is precisely the set of all scalar matrices $\lambda I_{n}$. The centre, $Z_{S}$, of $S L_{n}(K)$ comprises of all the scalar matrices $\lambda I_{n}$ such that $\lambda^{n}=1$. Clearly, $Z$ (respectively $Z_{S}$ ) is the kernel of the induced action of
$G L_{n}(K)$ (respectively $S L_{n}(K)$ ) on the projective space $\mathcal{P}(V)=\left\{\langle u\rangle \mid u \in V^{\#}\right\}$. The group that acts faithfully on $\mathcal{P}(V)$ is the quotient group $P G L_{n}(K) \cong G L_{n}(K) / Z$, called the projective general linear group. One may also form the quotient group $P S L_{n}(K) \cong S L_{n}(K) / Z_{S}$, called the projective special linear group, which is a subgroup of $P G L_{n}(K)$ of index at most 2 . When $K$ is a finite field of $q=p^{a}$ elements, we write $G L_{n}(K)=G L_{n}(q)($ respectively $S L, P G L$ and $P S L)$. The order of $G L_{n}(q)$ is

$$
\left|G L_{n}(q)\right|=q^{\frac{1}{2} n(n-1)} \prod_{i=1}^{n}\left(q^{i}-1\right) .
$$

With the exceptions of $P S L_{2}(2) \cong \operatorname{Sym}(3)$ and $P S L_{2}(3) \cong \operatorname{Alt}(4), P S L_{n}(K)$ is simple. We write $L_{n}(q)$ for $P S L_{n}(q)$, following AtLas [22] notation. We prepare an elementary result regarding $S L_{2}(q)$.

Proposition 2.1. Let $q=p^{a}$. Any two distinct Sylow $p$-subgroups of $G=S L_{2}(q)$ intersect trivially, and $\left|\operatorname{Syl}_{p}(G)\right|=q+1$.

Proof. One may prove this directly, but instead we follow the proof as given by Satz 8.1 of Huppert [28].

All Sylow $p$-subgroups of $G$ are elementary abelian of order $q=p^{a}$, and are all conjugate. Let

$$
P=\left\{\left.\left(\begin{array}{ll}
1 & 0 \\
k & 1
\end{array}\right) \right\rvert\, k \in G F(q)^{*}\right\}
$$

and any element $g \in P$ only fixes vectors of the form $(m, 0)$ and thus fixes a single point $p=\langle(1,0)\rangle$ of the projective line. Any element normalizing $P$ must also fix $p$ and so $N_{G}(P) \leq C_{G}(p)$. Since $P=C_{G}((1,0)) \unlhd C_{G}(p)$, we have $N_{G}(P)=C_{G}(p)$. Since $G$ acts transitively on the projective line, we must have $\left[G: N_{G}(P)\right]=q+1$, which is precisely the number of Sylow $p$-subgroups of $G$. For $g \in G$, the elements of $P^{g}$ fix a single point $p^{g}$. Let $h \in P \cap P^{g}$, so $h$ fixes both $p$ and $p^{g}$. Since $h$ is an element of order $p$ and thus only fixes one point of the projective line, $p=p^{g}$. Hence $g \in C_{G}(p)=N_{G}(P)$ and so $P=P^{g}$.

The following theorem relating to $L_{2}(q)$ for odd $q$ will assist our calculations in Chapter 4.

Theorem 2.2. Let $\langle\varepsilon\rangle=G F(q)^{*},\langle\sqrt{\varepsilon}\rangle=G F\left(q^{2}\right)^{*}$ and $s \in G F\left(q^{2}\right)^{*}$ be a primitive $(q+1)^{\text {th }}$ root of unity. Set $r, t \in \mathbb{C}$ to be primitive $\frac{1}{2}(q-1)$ and $\frac{1}{2}(q+1)$ roots of unity, respectively. When $q \equiv 1(\bmod 4)$, let $i$ be the unique element of $G F(q)$ which squares to -1 . The general character table of $L_{2}(q)$ is given in Table 2.1 for $q \equiv 3$ $(\bmod 4)$, and in Table 2.2 for $q \equiv 1(\bmod 4)$, where $x, y, z \in G F(q), x \notin\{ \pm 1,0\}$, $y \neq 0$ and $\varepsilon^{a}=x ; s^{b}=y+\sqrt{\varepsilon} z ; \varepsilon^{c}=i ;$ and $s^{d}=\sqrt{\varepsilon} z$. If $q \equiv 1(\bmod 4)$ then we place the additional restriction $x \neq i$.

| Rep. | $\left(\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right)$ | $\left(\begin{array}{cc}1 & \varepsilon \\ 0 & 1\end{array}\right)$ | $\left(\begin{array}{cc}1 & \varepsilon^{2} \\ 0 & 1\end{array}\right)$ | $\left(\begin{array}{cc}x & 0 \\ 0 & x^{-1}\end{array}\right)$ | $\left(\begin{array}{ll}y & \varepsilon z \\ z & y\end{array}\right)$ | $\left(\begin{array}{cc}0 & \varepsilon z \\ z & 0\end{array}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Size | 1 | $\frac{\left(q^{2}-1\right)}{2}$ | $\frac{\left(q^{2}-1\right)}{2}$ | $q(q+1)$ | $q(q-1)$ | $\frac{q(q-1)}{2}$ |
| No. of Cols | 1 | 1 | 1 | $\frac{(q-3)}{4}$ | $\frac{(q-3)}{4}$ | 1 |
| $\chi_{1}$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_{2}$ | $q$ | 0 | 0 | 1 | -1 | -1 |
| $\chi_{3,4}$ | $\frac{(q-1)}{2}$ | $\frac{(-1 \pm \sqrt{-q)}}{2}$ | $\frac{(-1 \mp \sqrt{-q)}}{2}$ | 0 | $(-1)^{b+1}$ | $(-1)^{d+1}$ |
| $\chi_{5, \ldots, 5+\frac{(q-3)}{4}}^{4+1}$ | 1 | 1 | $r^{a}+r^{-a}$ | 0 | 0 |  |
| $\chi_{6+\frac{(q-3)}{4}, \ldots, \frac{(q+5)}{2}}$ | $q-1$ | -1 | -1 | 0 | $-t^{b}-t^{-b}$ | $-t^{d}-t^{-d}$ |

Table 2.1: The general character table for $L_{2}(q)$ when $q \equiv 3(\bmod 4)$

| Rep. | $\left(\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right)$ | $\left(\begin{array}{cc}1 & \varepsilon \\ 0 & 1\end{array}\right)$ | $\left(\begin{array}{cc}1 & \varepsilon^{2} \\ 0 & 1\end{array}\right)$ | $\left(\begin{array}{cc}x & 0 \\ 0 & x^{-1}\end{array}\right)$ | $\left(\begin{array}{cc}i & 0 \\ 0 & -i\end{array}\right)$ | $\left(\begin{array}{cc}y & \varepsilon z \\ z & y\end{array}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Size | 1 | $\frac{\left(q^{2}-1\right)}{2}$ | $\frac{\left(q^{2}-1\right)}{2}$ | $q(q+1)$ | $\frac{q(q+1)}{2}$ | $q(q-1)$ |
| No. of Cols | 1 | 1 | 1 | $\frac{(q-5)}{4}$ | 1 | $\frac{(q-1)}{4}$ |
| $\chi_{1}$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_{2}$ | $q$ | 0 | 0 | 1 | 1 | -1 |
| $\chi_{3,4}$ | $\frac{(q+1)}{2}$ | $\frac{(1 \mp \sqrt{q})}{2}$ | $\frac{(1 \pm \sqrt{q})}{2}$ | $(-1)^{a}$ | $(-1)^{c}$ | 0 |
| $\chi_{5, \ldots, 5+\frac{(q-3)}{4}}^{4}$ | $q+1$ | 1 | 1 | $r^{a}+r^{-a}$ | $r^{c}+r^{-c}$ | 0 |
| $\chi_{6+\frac{(q-3)}{4}, \ldots, \frac{(q+5)}{2}}$ | $q-1$ | -1 | -1 | 0 | 0 | $-t^{b}-t^{-b}$ |

Table 2.2: The general character table for $L_{2}(q)$ when $q \equiv 1(\bmod 4)$

Proof. See [30].

The remaining classical groups arise from subgroups of $G L_{n}(K)$ that preserve certain forms on $V$, or equivalently the matrices $A$ that satisfy the relation $A^{T} J A=J$ where $J$ is the Gram matrix corresponding to the form.

## Symplectic Groups

Let $V$ be as before and let $(\cdot, \cdot)$ be a non-degenerate bilinear form on $V$ that also satisfies $(u, v)=-(v, u)$ for all $u, v \in V$ (such a property is called alternating). The form $(\cdot, \cdot)$ is called a symplectic form and the Gram matrix is skew-symmetric. Moreover, every vector in $V$ is isotropic. For $e_{1} \in V$, there exists $e_{1}^{\prime} \in V$ such that $\left(e_{1}, e_{1}^{\prime}\right) \neq 0$ since the symplectic form is non-degenerate. Setting $f_{1}=\left(e_{1}, e_{1}^{\prime}\right)^{-1} e_{1}^{\prime}$, we have $\left(e_{1}, f_{1}\right)=1$. Hence $\left\langle e_{1}, f_{1}\right\rangle \cap\left\langle e_{1}, f_{1}\right\rangle^{\perp}=\varnothing$ and it follows from (2.1) that $V=\left\langle e_{1}, f_{1}\right\rangle \oplus\left\langle e_{1}, f_{1}\right\rangle^{\perp}$. Continuing inductively, we see that $V$ must have even dimension so, for clarity, we will write the dimension of $V$ as $2 n$. We write $S p_{2 n}(K)=\left\{A \in G L_{2 n}(K) \mid A^{T} J A=J\right\}$ where $J$ is the Gram matrix corresponding to the symplectic form. Clearly, $S p_{2 n}(K)$ contains all invertible linear transformations on $V$ preserving the symplectic form, represented as matrices. We say $\left\{e_{1}, f_{1}, e_{2}, f_{2}, \ldots, e_{n}, f_{n}\right\}$ is a hyperbolic basis for $V$ if the Gram matrix of the symplectic form is

where $J_{0}=\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$. We say $\left\{e_{1}, e_{2}, \ldots, e_{n} \mid f_{1}, f_{2}, \ldots, f_{n}\right\}$ is a symplectic basis for $V$ if the Gram matrix of the symplectic form is

$$
J=\left(\begin{array}{l|l} 
& I_{n} \\
\hline-I_{n} &
\end{array}\right) .
$$

In general, any invertible skew-symmetric matrix $J$ defines a symplectic form on $V$. The determinant of all matrices in $S p_{2 n}(K)$ is 1 , and the centre is $\left\langle-I_{2 n}\right\rangle$. The quotient of $S p_{2 n}(K)$ by its centre is denoted by $P S p_{2 n}(K)$. When $K$ is a finite field of $q$ elements, we write $S p_{2 n}(K)=S p_{2 n}(q)$ (respectively $P S p$ ). The order of $S p_{2 n}(q)$ is

$$
\left|S p_{2 n}(q)\right|=q^{n^{2}} \prod_{i=1}^{n}\left(q^{2 i}-1\right)
$$

For $n \leq 4$, with the exception of $P S p_{4}(2) \cong \operatorname{Sym}(6), P S p_{n}(K)$ is simple. We present a result concerning the 4 -dimensional symplectic groups over a field of characteristic 2.

Proposition 2.3. Let $G \cong S p_{4}(K)$ for $K$ a field of characteristic 2. Then there exists an outer automorphism of $G$ that interchanges two conjugacy classes of involutions.

Proof. The group $G$ arises from the Dynkin diagram of type $B_{2}=C_{2}$, of which a graph automorphism exists when $\operatorname{char}(K)=2$ (see, for example, pages 224-225 of [20]). This automorphism is an outer automorphism of $G$. Each node of the Dynkin diagram corresponds to a subgroup isomorphic to $S L_{2}(K)$ and since this automorphism is outer, these $S L_{2}(K)$-subgroups must be non-conjugate. Consider the following subgroups

$$
S_{1}=\left\{\left.\left(\begin{array}{l|l|l}
1 & & \\
\hline & A & \\
\hline & & 1
\end{array}\right) \right\rvert\, A \in S L_{2}(q)\right\} \quad \text { and } \quad S_{2}=\left\{\left.\left(\begin{array}{ll}
A & \\
\hline & A
\end{array}\right) \right\rvert\, A \in S L_{2}(q)\right\}
$$

of $G$. All involutions in $S L_{2}(K)$ are conjugate and so using Lemma 7.7 of [10] we see that $S_{1}$ and $S_{2}$ are non-conjugate $S L_{2}(K)$-subgroups. The conjugacy classes of involutions containing those from $S_{1}$ and $S_{2}$ respectively are interchanged by the outer automorphism of $G$.

## Unitary Groups

Let $L$ be a quadratic extension of $K$ and $\tau$ be an automorphism of $L$. Let $V$ be an $n$-dimensional vector space over $L$ and define $(\cdot, \cdot)$ to be a non-degenerate sesquilinear form on $V$ with respect to $\tau$. When $\tau$ is of order 2 , the form $(\cdot, \cdot)$ is called a unitary (or Hermitian) form. For a matrix $A=\left(a_{i j}\right)$ representing a linear transformation on $V$, define $\bar{A}=\left(a_{i j}^{\tau}\right)$. Let $J$ be the Gram matrix with respect to the unitary form, and we write

$$
\begin{equation*}
G U_{n}(K)=\left\{A \in G L_{n}(L) \mid \bar{A}^{T} J A=J\right\} \tag{2.2}
\end{equation*}
$$

comprising of all the invertible matrices preserving the unitary form, called the general unitary group. Let $S U_{n}(K)$ denote the subgroup of $G U_{n}(K)$ of matrices of determinant 1, called the special unitary group. As with the general linear group, the quotient of $G U_{n}(K)$ (respectively $S U_{n}(K)$ ) by its centre yields the group $P G U_{n}(K)$ (respectively $P S U_{n}(K)$ ). When $K$ is a finite field of $q$ elements (and thus $L$ a finite field of $q^{2}$ elements), we write $G U_{n}(K)=G U_{n}(q)$ (respectively $\left.S U, P G U, P S U\right)$. The order of $G U_{n}(q)$ is

$$
\left|G U_{n}(q)\right|=q^{\frac{1}{2} n(n-1)} \prod_{i=1}^{n}\left(q^{i}-(-1)^{i}\right)
$$

A word of caution, however, that notation differs within existing literature. For example, some authors use $G U_{n}(L)$ to denote the group of matrices with entries over L. In the spirit of the Atlas [22], we follow the "smallest field" convention and use the definition as given in (2.2). Moreover, we often write $P S U_{n}(q)=U_{n}(q)$. With the exception of $U_{3}(2) \cong 3^{2} . Q_{8}, U_{n}(K)$ is simple for $n \geq 3$.

We note the following lemma regarding involutions in $S U_{n}(q)$.

Lemma 2.4. Suppose $q$ is odd, and let $J=I_{n}$ be the Gram matrix defining a unitary form on $V$, an n-dimensional vector space. Let $G=S U_{n}(q)$. Then conjugacy classes of involutions in $G$ are represented by the diagonal matrices

$$
t_{i}=\left(\begin{array}{c|c|c}
-I_{i} & & \\
\hline & I_{n-2 i} & \\
\hline & & -I_{i}
\end{array}\right)
$$

for $i=1, \ldots,\left[\frac{n}{2}\right]$.
Proof. A result of Wall (Page 34, Case(A)(ii) of [40]) reveals that any two involutions in $G U_{n}(q)$ are conjugate if and only if they are conjugate in $G L_{n}(q)$. This naturally restricts to an analogous result concerning conjugate involutions in $S U_{n}(q)$ and $S L_{n}(q)$.

## Orthogonal Groups

Let $V$ be an $n$-dimensional vector space over $K$ and let $Q: V \rightarrow K$ be a map such that $Q(a v)=a^{2} Q(v)$ for $a \in K$ and $v \in V$. We call $Q$ a quadratic form, and define a bilinear form $(\cdot, \cdot)$ by $(u, v)=Q(u+v)-Q(u)-Q(v)$ for $u, v \in V$. If $\operatorname{char}(K)$ is odd, then $(\cdot, \cdot)$ is uniquely determined by $Q$ (and vice versa). When $\operatorname{char}(K)=2$, $(\cdot, \cdot)$ is an alternating form. We say $Q$ is non-degenerate if $Q(v) \neq 0$, for all $v \in V^{\perp}$. When $\operatorname{char}(K)$ is odd, this is equivalent to the bilinear form being non-degenerate - such a bilinear form is called orthogonal. We define $G O(V, Q)$ to be the set of invertible linear transformations that preserve the non-degenerate quadratic form $Q$, called the general orthogonal group. The theory of quadratic forms is vastly different when $\operatorname{char}(K)=2$ as opposed to when $\operatorname{char}(K)$ is odd. This thesis only deals with orthogonal groups over fields of odd characteristic, and so until further notice we assume $\operatorname{char}(K)$ to be odd.

We now assume $Q$ to be non-degenerate and utilise the orthogonal form, $(\cdot, \cdot)$, uniquely determined by $Q$. Hence, $G O(V, Q)$ can also be described as the set of invertible linear transformations which preserve the orthogonal form. A hyperbolic plane is the unique (up to isometry) 2-dimensional vector space equipped with an orthogonal form with Gram matrix $J_{0}=\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)$, and contains an isotropic vector. The vector space $V$ can be decomposed as an orthogonal sum,

$$
V=H_{1} \perp H_{2} \perp \ldots \perp H_{n} \perp W
$$

where each of the $H_{i}$ are hyperbolic planes, $W$ is not a hyperbolic plane and $\operatorname{dim}(W) \leq$ 2. If $n$ is odd, then $\operatorname{dim}(W)=1$ but if $n$ is even then $\operatorname{dim}(W)=0$ or 2 . We say $V$ is an orthogonal space of + -type if $\operatorname{dim}(W)=0$ and --type if $\operatorname{dim}(W)=2$. If $n$ is odd, then all general orthogonal groups that preserve an orthogonal form are isomorphic, and are denoted by $G O^{0}(V)$, or just $G O(V)$. When $n$ is even, there are two isomorphism classes of general orthogonal group that preserve an orthogonal form. These stem from whether $V$ is an orthogonal space of + - or --type. The general orthogonal groups that preserve these forms are either denoted $G O^{+}(V)$ or $G O^{-}(V)$,
with superscripts referring to the type of $V$.
Let $U=\langle u, v\rangle$ be a 2-dimensional orthogonal space over $K$ with respect to an orthogonal form defined by the Gram matrix $J_{0}=\left(\begin{array}{ll}1 & 0 \\ 0 & a\end{array}\right)$, for some $a \in K^{*}$. If there exists a vector $w=(\alpha, \beta) \in U$ such that $(w, w)=0$ then $\alpha^{2}+a \beta^{2}=0$ and so $\alpha^{2}=-a \beta^{2}$. This occurs if and only if $-a$ is a square in $K$.

Let $J$ be the Gram matrix with respect to an orthogonal form $(\cdot, \cdot)$ on $V$. Since the orthogonal form is symmetric, $J$ is a symmetric matrix. There always exists a basis of $V$ such that $J$ is a diagonal matrix (such a basis is called an orthogonal basis). For brevity, we assume $J$ to be diagonal. For any 2-dimensional vector space with Gram matrix $\left(\begin{array}{ll}a & 0 \\ 0 & a\end{array}\right)$ for some $a \in K^{*}$, an alternative basis can be found such that the Gram matrix is $\left(\begin{array}{ll}1 & 0 \\ 0 & b\end{array}\right)$ for some $b \in K^{*}$. Hence, up to a reordering of basis, we may assume

$$
J=\left(\begin{array}{c|c}
I_{n-k} & \\
& J_{1}
\end{array}\right)
$$

where $J_{1}$ is either the $1 \times 1$ matrix (1) if $n$ is odd and $k=1$, or $\left(\begin{array}{cc}1 & 0 \\ 0 & -\mu\end{array}\right)$ for some $\mu \in K^{*}$ when $n$ is even and $k=2$. If $n$ is odd, then $J=J^{0}=I_{2 n+1}$. If $n$ is even, then

$$
J=\left(\begin{array}{l|l}
I_{n-1} & \\
\hline & -\mu
\end{array}\right)
$$

for $\mu \in K^{*}$. If $\mu$ is square in $K^{*}$, then define $J=J^{+}$. If $\mu$ is non-square in $K^{*}$, then define $J=J^{-}$. Note that $J^{\varepsilon}$ determines the type of $V$ to be of $\varepsilon$-type and define $G O_{n}^{\varepsilon}(K)=\left\{A \in G L_{n}(K) \mid A^{T} J^{\varepsilon} A=J^{\varepsilon}\right\}$ for $\varepsilon \in\{+,-, 0\}$.

Any matrix in $G O_{n}^{\varepsilon}(K)$ has determinant either 1 or -1 . The subgroup of $G O_{n}^{\varepsilon}(K)$ consisting of matrices of determinant 1 is denoted by $S O_{n}^{\varepsilon}(K)$, called the special orthogonal group. Unlike the other families of classical groups, in general $S O_{n}^{\varepsilon}(K)$ may not be perfect. In fact, the derived subgroup of $S O_{n}^{\varepsilon}(K)$, denoted $\Omega_{n}^{\varepsilon}(K)$, has index at most 2. An alternative description for $\Omega_{n}^{\varepsilon}(K)$ is via the notion of reflections.

For $r, v \in V$ such that $(r, r) \neq 0$, define $R_{r}: V \rightarrow V$ by $R_{r}: v \mapsto v-2(v, r)(r, r)^{-1} v$. Clearly, $r^{R_{r}}=-r$ and any $s \in V$ such that $(r, s)=0$ is fixed by $R_{r}$. We say $R_{r}$ is a reflection in the vector $r$ and such a reflection preserves the orthogonal form, and therefore lies in $G O^{\varepsilon}(V)$. Suppose $g$ is decomposed as a product of reflections $R_{r_{1}} R_{r_{2}} \ldots R_{r_{t}}$ in the vectors $r_{1}, r_{2}, \ldots, r_{t}$. Since char $(K)$ is odd, $g \in \Omega_{n}^{\varepsilon}(K)$ if and only if the product $\left(r_{1}, r_{1}\right)\left(r_{2}, r_{2}\right) \ldots\left(r_{t}, r_{t}\right)$ is square in $K$. This product is often referred to as the spinor norm of $g$.

The quotient of $G O_{n}^{\varepsilon}(K)$ by its centre is denoted $P G O_{n}^{\varepsilon}(K)$ (respectively $S O$ and $\Omega$ ). When $K$ is a finite field of $q$ elements, we replace $K$ with $q$ as before. The order of $G O_{n}(q)$ for $n$ odd is twice that of $S p_{2 n}(q)$ (note that $q$ is odd). The order of $G O_{2 n}^{\varepsilon}(q)$, for $\varepsilon= \pm 1$ is

$$
\left|G O_{2 n}^{\varepsilon}(q)\right|=2 q^{n(n-1)}\left(q^{2 n}-\varepsilon\right) \prod_{i=1}^{n-1}\left(q^{2 i}-1\right)
$$

For $n \geq 7, P \Omega_{n}^{\varepsilon}(K)$ is simple.
We give a second cautionary word to the reader regarding the extensive, and almost contradictory, notation within existing literature. Dickson's notation is becoming obsolete (see page xiii of [22] for a brief dictionary), being replaced with the $\Omega$ notation introduced by Dieudonné. Moreover, some authors regard $O_{n}^{\varepsilon}(K)$ as the full orthogonal group. In the interest of consistency, we follow the Atlas [22] notation, in particular Artin's convention of "single letter for a simple group". Thus, $O_{n}^{\varepsilon}(K)=P \Omega_{n}^{\varepsilon}(K)$ will denote the simple orthogonal group. If $K=G F(q)$ then the simple orthogonal group will be denoted by $O_{n}^{\varepsilon}(q)$. We present an important lemma.

Lemma 2.5. Let $G$ be an orthogonal group acting on the orthogonal $G$-module $V$ over $\operatorname{GF}(q)$. Then $G$ acts on the 1-subspaces of $V$ in 3 orbits.

Proof. Let $u_{0}, u_{1}, u_{2} \in V$ be such that $\left(u_{0}, u_{0}\right)=0,\left(u_{1}, u_{1}\right)$ is non-square in $\operatorname{GF}(q)^{*}$ and $\left(u_{2}, u_{2}\right)$ is square in $G F(q)^{*}$. By definition of $G,(v, v)=\left(v^{g}, v^{g}\right)$ for all $g \in G$ and all $v \in V$. Hence, the set of all isotropic 1-subspaces of $V$ forms a $G$-orbit. For any $\lambda \in G F(q),\left(\lambda u_{1}, \lambda u_{1}\right)=\lambda^{2}\left(u_{1}, u_{1}\right)$ will also be non-square in $G F(q)^{*}$. Similarly, $\left(\lambda u_{2}, \lambda u_{2}\right)$ will be square in $G F(q)^{*}$. Hence, $\left\{\langle v\rangle \mid(v, v)\right.$ is square in $\left.G F(q)^{*}\right\}$ forms a $G$-orbit, as does $\left\{\langle v\rangle \mid(v, v)\right.$ is non-square in $\left.G F(q)^{*}\right\}$.

We end this section on a general note that any form on $V$ that is either symplectic, unitary or orthogonal, will be referred to as a classical form.

## Affine Linear Groups

Let $K$ be a finite field of arbitrary characteristic. Let $H=G L_{n}(q)$ and $G \leq H$. We call $G$ a linear group and $G$ acts on the natural module $V$, an $n$-dimensional vector space over $G F(q)$. We form the semidirect product $V \rtimes G=\mathrm{Aff}(G)$, the affine group of $G$. If $G$ is a classical group defined in the earlier sections, we denote the affine analogue with the prefix $A$. For example if $G=S p_{2 n}(q)$ then $\operatorname{Aff}(G)=A S p_{2 n}(q)$.

## Isomorphisms Between Classical Groups

We present a list of exceptional isomorphisms between classical groups that will be of use.

Proposition 2.6. (i) $S L_{2}(q) \cong S p_{2}(q) \cong S U_{2}(q)$.
(ii) $L_{2}(q) \cong O_{3}(q)$.
(iii) $C_{\frac{q \mp 1}{2}} \cong O_{2}^{ \pm}(q)$.
(iv) $S L_{2}(q) \circ S L_{2}(q) \cong O_{4}^{+}(q)$.
(v) $L_{2}\left(q^{2}\right) \cong O_{4}^{-}(q)$.
(vi) $\operatorname{PSp}_{4}(q) \cong O_{5}(q)$
(vii) $U_{4}(q) \cong O_{6}^{-}(q)$.

Proof. These isomorphisms are well-known and can be proved in different ways. For a proof geometrical in nature, see [38]. The results here are scattered throughout the book, since more theory is developed as the book progresses. For a more algebraic proof (or for a more collated result) the reader is referred to Proposition 2.9.1 of [32].

### 2.2 Commuting Involution Graphs

We give a review of the recent study into commuting involution graphs, starting with a background in graph theory to cement our conventions.

A graph $\Gamma$ with vertex set $\Omega$ is undirected without loops if $(x, y)$ is an edge of $\Gamma$ exactly when $(y, x)$ is an edge of $\Gamma$ for all $x, y \in \Omega$, but $(x, x)$ is never an edge of $\Gamma$ for any $x \in \Omega$. The standard distance metric $d$ on $\Gamma$ is defined by $d(x, y)=i$ if and only if the shortest path between vertices $x$ and $y$ has length $i$. If no such path exists between $x$ and $y$, then the distance is infinite. For $x \in \Omega$, define the $i^{\text {th }}$ disc from $x$ to be

$$
\Delta_{i}(x)=\{y \in \Omega \mid d(x, y)=i\} .
$$

If $\left|\Delta_{1}(x)\right|=\left|\Delta_{1}(y)\right|$ for all $x, y \in \Omega$, then the graph is regular. We call $\left|\Delta_{1}(x)\right|$ the valency of a regular graph. If $\Gamma_{0}$ is a connected regular graph, then the diameter of $\Gamma_{0}, \operatorname{Diam} \Gamma_{0}$, is the greatest such $i$ such that $\Delta_{i}(x) \neq \varnothing$ and $\Delta_{i+1}(x)=\varnothing$ for any $x \in \Omega$.

For the entirety of this thesis, we consider only regular, undirected graphs without loops. Let $G$ be a group and $X$ a subset of $G$. We form a graph with vertex set $X$, denoted $\mathcal{C}(G, X)$, such that any two distinct vertices of $X$ are joined if and only if they commute. In particular, $\Delta_{1}(x)=\{y \in X \mid x y=y x\}$. Such a graph is called a commuting graph of $G$ on $X$. When $X$ is specifically a $G$-conjugacy class of involutions, we call $\mathcal{C}(G, X)$ a commuting involution graph. Due to the transitive action of $G$ on $X$ by conjugation, it is clear that $\mathcal{C}(G, X)$ is a regular, undirected graph without loops.

The detailed study of commuting involution graphs came to the fore in the early 2000's, when Peter Rowley and three of his then PhD Students and post-doctoral researchers - Chris Bates, David Bundy and Sarah Hart (nèe Perkins) - published a number of results describing the diameter and disc sizes of these graphs for various groups (see [14], [13], [34], [15] and [11]). Exact conditions when certain graphs had certain properties were determined. In 2006, a paper detailing the structure of the
commuting involution graphs for most sporadic simple groups was published. The remaining cases were then tackled in the late 2000s by two more of Rowley's PhD students, Paul Taylor and Benjamin Wright (see [39] and [43] respectively). We present a condensed overview of the results, the details and proofs of which can be found in the cited works.

Theorem 2.7 (Bates, Bundy, Perkins, Rowley). Let $G=\operatorname{Sym}(n)$ and $X$ a $G$ conjugacy class of involutions. Then $\mathcal{C}(G, X)$ is either disconnected or connected of diameter at most 4, with equality in precisely three cases.

Proof. The proof and exact conditions for this result can be found in [14].

Theorem 2.8 (Bates, Bundy, Perkins, Rowley). Let $G$ be a finite Coxeter group and $X$ a conjugacy class of involutions in $G$.
(i) If $G$ is of type $B_{n}$ or $D_{n}$, then $\mathcal{C}(G, X)$ is either disconnected or connected of diameter at most 5, with equality in exactly one case.
(ii) If $G$ is of type $E_{6}$, then $\mathcal{C}(G, X)$ is connected of diameter at most 5 .
(iii) If $G$ is of type $E_{7}$ or $E_{8}$, then $\mathcal{C}(G, X)$ is connected of diameter at most 4.
(iv) If $G$ is of type $F_{4}, H_{3}$ or $H_{4}$, then either $\mathcal{C}(G, X)$ is disconnected or connected of diameter 2.
(v) If $G$ is of type $I_{n}$, then $\mathcal{C}(G, X)$ is disconnected.

Proof. This is a highly condensed version of the result - the full details and proofs can be found in [13].

A sequel to these results, a result on the commuting involution graphs of a class of infinite groups, followed soon after.

Theorem 2.9 (Perkins). Let $G$ be an affine Coxeter group of type $\tilde{A}_{n}$, and $X$ a conjugacy class of involutions of $G$. Then $\mathcal{C}(G, X)$ is disconnected or is connected of diameter at most 6 .

Proof. As with Theorems 2.7 and 2.8, this is a compact description of the full result. The reader is referred to [34] for full details and proofs.

The next collection of results relating to commuting involution graphs provides, what can only be described as, the keystone to the research undertaken in this thesis. Bates, Bundy, Hart and Rowley explore the structure of the commuting involution graphs of the special linear and projective special linear groups over various fields. Due to the high relevance of this paper to this thesis, we present all three results as given in [15].

Theorem 2.10 (Bates, Bundy, Perkins, Rowley). Suppose $G \cong L_{2}(q)$, the 2-dimensional projective special linear group over the finite field of $q$ elements, and $X$ the $G$ conjugacy class of involutions.
(i) If $q$ is even, then $\mathcal{C}(G, X)$ consists of $q+1$ cliques each with $q-1$ vertices.
(ii) If $q \equiv 3(\bmod 4)$, with $q>3$, then $\mathcal{C}(G, X)$ is connected and $\operatorname{Diam} \mathcal{C}(G, X)=3$. Furthermore,

$$
\begin{aligned}
\left|\Delta_{1}(t)\right| & =\frac{1}{2}(q+1) \\
\left|\Delta_{2}(t)\right| & =\frac{1}{4}(q+1)(q-3) ; \text { and } \\
\left|\Delta_{3}(t)\right| & =\frac{1}{4}(q+1)(q-3)
\end{aligned}
$$

(iii) If $q \equiv 1(\bmod 4)$, with $q>13$, then $\mathcal{C}(G, X)$ is connected and $\operatorname{Diam} \mathcal{C}(G, X)=3$. Furthermore

$$
\begin{aligned}
\left|\Delta_{1}(t)\right| & =\frac{1}{2}(q-1) ; \\
\left|\Delta_{2}(t)\right| & =\frac{1}{4}(q-1)(q-5) ; \text { and } \\
\left|\Delta_{3}(t)\right| & =\frac{1}{4}(q-1)(q+7) .
\end{aligned}
$$

Theorem 2.11 (Bates, Bundy, Perkins, Rowley). Suppose that $G \cong S L_{3}(q)$ and $X$ the $G$-conjugacy class of involutions. Then $\mathcal{C}(G, X)$ is connected with $\operatorname{Diam} \mathcal{C}(G, X)=$ 3 and the following hold.
(i) If $q$ is even, then

$$
\begin{aligned}
& \left|\Delta_{1}(t)\right|=2 q^{2}-q-2 \\
& \left|\Delta_{2}(t)\right|=2 q^{2}(q-1) ; \text { and } \\
& \left|\Delta_{3}(t)\right|=q^{3}(q-1)
\end{aligned}
$$

(ii) If $q$ is odd, then

$$
\begin{aligned}
& \left|\Delta_{1}(t)\right|=q(q+1) \\
& \left|\Delta_{2}(t)\right|=\left(q^{2}-1\right)\left(q^{2}+2\right) ; \text { and } \\
& \left|\Delta_{3}(t)\right|=(q+1)(q-1)^{2}
\end{aligned}
$$

Theorem 2.12 (Bates, Bundy, Perkins, Rowley). Let $K$ be a (possibly infinite) field of characteristic 2, and suppose that $G \cong S L_{n}(K)$ and $X$ a $G$-conjugacy class of involutions containing $t$. Also let $V$ denote the natural $n$-dimensional $K G$-module, and set $k=\operatorname{dim}_{K}[V, t]$.
(i) If $n \geq 4 k$, then $\operatorname{Diam} \mathcal{C}(G, X)=2$.
(ii) If $3 k \leq n<4 k$, then $\operatorname{Diam} \mathcal{C}(G, X) \leq 3$.
(iii) If $2 k<n<3 k$, or $k$ is even and $n=2 k$, then $\operatorname{Diam} \mathcal{C}(G, X) \leq 5$.
(iv) If $n=2 k$ and $k$ is odd, then $\operatorname{Diam} \mathcal{C}(G, X) \leq 6$.

This thesis follows in the footsteps of [15], but for $G$ a 4-dimensional projective symplectic group, a 3-dimensional unitary group or a 4 -dimensional unitary group over a field of characteristic 2 .

The most recent family of groups whose commuting involution graphs were studied were the sporadic simple groups. Here, the notation for the conjugacy classes of involutions follows the Atlas convention.

Theorem 2.13 (Bates, Bundy, Hart, Rowley; Rowley, Taylor; Rowley). Let $K$ be a sporadic simple group and $K \leq G \leq$ Aut (K). Let $X$ be a conjugacy class of involutions in $G$.
(i) For $(K, X)$ not equal to $\left(J_{4}, 2 B\right),\left(F i_{24}^{\prime}, 2 B\right),\left(F i_{24}^{\prime}, 2 D\right),(B, 2 C),(B, 2 D)$ or $(\mathbb{M}, 2 B)$, the diameter of $\mathcal{C}(G, X)$ is at most 4 , with equality in precisely four cases.
(ii) For $(K, X)$ equal to $\left(J_{4}, 2 B\right)$, $\left(F i_{24}^{\prime}, 2 B\right)$ or $\left(F i_{24}^{\prime}, 2 D\right)$, the diameter of $\mathcal{C}(G, X)$ is 3.
(iii) For $(K, X)$ equal to $(\mathbb{M}, 2 B)$ the diameter of $\mathcal{C}(G, X)$ is 3.

Proof. Part (i) is given in the paper of Bates, Bundy, Hart and Rowley [11]. Part (ii) is proved in Taylor [39]. Part (iii) is determined in an unpublished manuscript of Rowley [36].

Conjecture: For $(K, X)$ equal to $(B, 2 C)$, the diameter of $\mathcal{C}(G, X)$ is 3 .

Due to the complexity of this particular case, a considerable portion of Wright [43] is devoted to studying the $C_{G}(t)$-orbits of $\mathcal{C}(G, X)$ with a view to proving this conjecture. It should be noted that the case when $(K, X)$ equal to ( $B, 2 D$ ) has, at time of writing, not been attempted.

## Collapsed Adjacency Diagrams

Now we present an overview of collapsed adjacency diagrams. As is customary we use a circle to denote a $C_{G}(t)$-orbit, and within the circle we note the name of this orbit and its size. An arrowed line from orbit $\Delta_{i}^{j}(t)$ to $\Delta_{k}^{l}(t)$, labelled by $\lambda$ says that a vertex in $\Delta_{i}^{j}(t)$ is joined to $\lambda$ vertices in $\Delta_{k}^{l}(t)$. The absence of arrowed lines from $\Delta_{i}^{j}(t)$ to $\Delta_{k}^{l}(t)$ indicates that there are no edges between vertices in $\Delta_{i}^{j}(t)$ and $\Delta_{k}^{l}(t)$. The graphs we are about to describe have collections of $C_{G}(t)$-orbits which display similar properties. In order to describe this and also make our collapsed adjacency graph easier to read we introduce some further notation. A square as described in Figure 2.1 is telling us that there are $\mu=k+1 C_{G}(t)$-orbits, $\Delta_{i}^{j}(t), \Delta_{i}^{j+1}(t), \ldots, \Delta_{i}^{j+k}(t)$ each of size $m$. For each of these orbits $\Delta_{i}^{l}(t)$, a vertex in $\Delta_{i}^{l}(t)$ is joined $\beta$ vertices in $\Delta_{i}^{l}(t)$ and to $\gamma$ vertices in $\Delta_{i}^{l^{\prime}}(t)$ for each $l^{\prime} \neq l$, for $j \leq l^{\prime} \leq j+k$. Now Figure 2.2 indicates that a vertex in any of the $C_{G}(t)$-orbits $\Delta_{i}^{j}(t), \Delta_{i}^{j+1}(t), \ldots, \Delta_{i}^{j+k}(t)$, is joined to $b$ vertices in $\Delta_{r}^{s}(t)$ and a vertex in $\Delta_{r}^{s}(t)$ is joined to $a$ vertices in


Figure 2.1: A collection of orbits in a collapsed adjacency diagram


Figure 2.2: The interactions between orbit collections in a collapsed adjacency diagram
each of $\Delta_{i}^{j}(t), \Delta_{i}^{j+1}(t), \ldots, \Delta_{i}^{j+k}(t)$. Directed arrows between square boxes as above mean that a vertex in each of the orbits $\Delta_{i}^{j}(t), \Delta_{i}^{j+1}(t), \ldots, \Delta_{i}^{j+k}(t)$ joins to $c$ vertices in each of the orbits $\Delta_{i^{\prime}}^{j^{\prime}}(t), \Delta_{i^{\prime}}^{j^{\prime}+1}(t), \ldots, \Delta_{i^{\prime}}^{j^{\prime}+k^{\prime}}(t)$, and a vertex in each of the orbits $\Delta_{i^{\prime}}^{j^{\prime}}(t), \Delta_{i^{\prime}}^{j^{\prime}+1}(t), \ldots, \Delta_{i^{\prime}}^{j^{\prime}+k^{\prime}}(t)$ is joined to $d$ vertices in each of the orbits $\Delta_{i}^{j}(t), \Delta_{i}^{j+1}(t), \ldots, \Delta_{i}^{j+k}(t)$.

### 2.3 Useful Results

There are some basic results which, whilst elementary, are fundamental in our study of commuting involution graphs. These are presented below.

Proposition 2.14. Let $G$ be a finite group acting on a graph $\Gamma$ with vertex set $\Omega$, with valency $k$. Let $\alpha, \beta \in \Omega$ such that $\beta \in \Delta_{1}(\alpha)$ (equivalently, $\alpha \in \Delta_{1}(\beta)$ ). Denote $\alpha^{G}$ and $\beta^{G}$ the $G$-orbits containing $\alpha$ and $\beta$ respectively. Then

$$
\left|\alpha^{G}\right|\left|\Delta_{1}(\alpha) \cap \beta^{G}\right|=\left|\beta^{G}\right|\left|\Delta_{1}(\beta) \cap \alpha^{G}\right| .
$$

Proof. By definition, $\left|\Delta_{1}(\alpha) \cap \beta^{G}\right|$ is the number of edges between $\alpha$ and $\beta^{G}$. Hence by the action of $G$, there exists $\left|\alpha^{G}\right|\left|\Delta_{1}(\alpha) \cap \beta^{G}\right|$ edges between the orbits $\alpha^{G}$ and $\beta^{G}$. By interchanging $\alpha$ and $\beta$, we see this number must also be equal to $\left|\beta^{G}\right|\left|\Delta_{1}(\beta) \cap \alpha^{G}\right|$, proving the lemma.

Proposition 2.15. Let $G$ be a group, and $V$ a module for $G$. For $g \in G$, we have $C_{G}(g) \leq \operatorname{Stab}_{G} C_{V}(g)$.

Proof. Let $h \in C_{G}(g)$ and $v \in C_{V}(g)$. Then $v^{h}=v^{g h}=v^{h g}$ and so $v^{h} \in C_{V}(g)$. Hence, $h \in \operatorname{Stab}_{G} C_{V}(g)$, so proving the result.

Proposition 2.16 (Witt's Lemma). Let $\left(V_{1}, \varphi_{1}\right)$ and $\left(V_{2}, \varphi_{2}\right)$ be vector spaces equipped with classical forms $\varphi_{i}$ on $V_{i}, i=1,2$. Let $W_{i} \leq V_{i}$ and assume there exists an isometry $\psi$ from $\left(W_{1}, \varphi_{1}\right)$ to $\left(W_{2}, \varphi_{2}\right)$. Then $\psi$ extends to an isometry from $\left(V_{1}, \varphi_{1}\right)$ to $\left(V_{2}, \varphi_{2}\right)$.

Proof. See Section 20 of [6].

We present a corollary to Witt's Lemma of vital importance to later results.

Corollary 2.17. Let $G$ be a classical group acting on the natural $G$-module $V$. Then $G$ acts transitively on the set of totally isotropic subspaces of $V$ of fixed dimension.

Proof. Let $W_{1}$ and $W_{2}$ be totally isotropic subspaces of $V$ with respect to the classical form $\varphi$ of the same dimension $k$. Clearly, $G$ induces isometries from $W_{1}$ to $W_{2}$ preserving $\varphi$. The result follows by Witt's Lemma.

### 2.4 Final Remarks

Each chapter from Chapter 3 to Chapter 6 deals with a different family of simple classical groups. We denote by $H$ the subgroup of $G L_{n}(q)$ of matrices with determinant 1 that preserve the given classical form, and $G$ will denote the image of $H$ obtained by factoring by its centre. In general $G^{\prime}$ will be a simple group, with $G=G^{\prime}$ if $G$ is a symplectic or a unitary group. We usually denote by $V$ the natural $G F(q) H$-module.

## Chapter 3

## 4-Dimensional Symplectic Groups

## over Fields of Characteristic 2

We start by considering the symplectic groups $H=S p_{4}(q)$ and $G=P S p_{4}(q) \cong$ $H / Z(H)$. In this chapter, we let $p=2$ and so $H=S p_{4}(q) \cong P S p_{4}(q)=G$. We set about proving Theorems 1.1 and 1.2, and determining the general collapsed adjacency diagram of $\mathcal{C}\left(G, X_{i}\right)$ for $i=1,3$. We denote by $V$ the symplectic $G F(q) G$-module with an associated symplectic form $(\cdot, \cdot)$ defined by the Gram matrix

$$
J=\left(\begin{array}{llll}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{array}\right),
$$

with respect to a suitable basis of $V$. We further define

$$
S=\left\{\left.\left(\begin{array}{cccc}
1 & a & b & c \\
0 & 1 & d & a d+b \\
0 & 0 & 1 & a \\
0 & 0 & 0 & 1
\end{array}\right) \right\rvert\, a, b, c, d \in G F(q)\right\}
$$

$Q_{1}=\left\{\left.\left(\begin{array}{cccc}1 & a & b & c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1\end{array}\right) \right\rvert\, a, b, c \in G F(q)\right\}$ and $Q_{2}=\left\{\left.\left(\begin{array}{cccc}1 & 0 & b & c \\ 0 & 1 & d & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{array}\right) \right\rvert\, b, c, d \in G F(q)\right\}$.
Lemma 3.1. (i) $S \in \operatorname{Syl}_{2} G$.
(ii) $S=Q_{1} Q_{2}$ with $Q_{1}^{\#} \cup Q_{2}^{\#}$ consisting of all the involutions of $S$.

Proof. It is straightforward to check that $S$ is a subgroup of $G$. Since $|G|=q^{4}\left(q^{2}-\right.$ 1) $\left(q^{4}-1\right)$ and $|S|=q^{4}$, we have part (i). If

$$
x=\left(\begin{array}{cccc}
1 & a & b & c \\
0 & 1 & d & a d+b \\
0 & 0 & 1 & a \\
0 & 0 & 0 & 1
\end{array}\right) \in S
$$

then $x^{2}=I_{4}$ if and only if $a=0$ or $d=0$, thus $x \in Q_{1} \cup Q_{2}$. Each $Q_{i}$ forms an elementary abelian group of order $q^{3}$, and an easy check shows that $Q_{1} Q_{2}=S$, and $Z(S)=Q_{1} \cap Q_{2}$, giving part (ii).

For any involution $x \in G$, note that $[V, x]^{\perp}=C_{V}(x)$ and $\operatorname{dim} V=\operatorname{dim}[V, x]+$ $\operatorname{dim} C_{V}(x)$. For an involution $x \in G$ we define $V(x)=\left\{v \in V \mid\left(v, v^{x}\right)=0\right\}$. As in Lemma 7.7 of [10], $G$ has three classes of involutions which may be described as

$$
\begin{aligned}
& X_{1}=\left\{x \in G \mid x^{2}=1, \operatorname{dim} C_{V}(x)=3\right\} ; \\
& X_{2}=\left\{x \in G \mid x^{2}=1, \operatorname{dim} C_{V}(x)=2, \operatorname{dim} V(x)=3\right\} ; \text { and } \\
& X_{3}=\left\{x \in G \mid x^{2}=1, \operatorname{dim} C_{V}(x)=2, V(x)=V\right\}
\end{aligned}
$$

The following three involutions are elements of $G$.

$$
t_{1}=\left(\begin{array}{llll}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right), t_{2}=\left(\begin{array}{llll}
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right), t_{3}=\left(\begin{array}{llll}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{array}\right) .
$$

Lemma 3.2. (i) For $i=1,2,3, t_{i} \in X_{i}$.
(ii) $C_{G}\left(t_{1}\right) \cong q^{3} S L_{2}(q)$ with $O_{2}\left(C_{G}\left(t_{1}\right)\right)=Q_{1}$ of order $q^{3}$.
(iii) $C_{G}\left(t_{2}\right)=S$.
(iv) $\left|X_{1}\right|=q^{4}-1$.
(v) $\left|X_{2}\right|=\left(q^{2}-1\right)\left(q^{4}-1\right)$.

Proof. Let $v=(\alpha, \beta, \gamma, \delta) \in V$. Then $v^{t_{1}}=(\alpha, \beta, \gamma, \alpha+\delta), v^{t_{2}}=(\alpha, \beta, \alpha+\gamma, \alpha+\beta+\delta)$ and $v^{t_{3}}=(\alpha, \alpha+\beta, \gamma, \gamma+\delta)$. Hence $\left[v, t_{1}\right]=(0,0,0, \alpha),\left[v, t_{2}\right]=(0,0, \alpha, \alpha+\beta)$ and $\left[v, t_{3}\right]=(0, \alpha, 0, \gamma)$. Consequently $\operatorname{dim}\left[V, t_{1}\right]=1$ and $\operatorname{dim}\left[V, t_{2}\right]=2=\operatorname{dim}\left[V, t_{3}\right]$. Thus $t_{1} \in X_{1}$. Now

$$
\left(v, v^{t_{2}}\right)=\alpha(\alpha+\beta+\delta)+\beta(\alpha+\gamma)+\gamma \beta+\delta \alpha=\alpha^{2}=0
$$

implies that $\alpha=0$ and so $\operatorname{dim} V\left(t_{3}\right)=3$. Therefore $t_{2} \in X_{2}$. Turning to $t_{3}$ we have that

$$
\left(v, v^{t_{3}}\right)=\alpha(\gamma+\delta)+\beta \gamma+\gamma(\alpha+\beta)+\delta \alpha=0
$$

implies that $V\left(t_{2}\right)=V$, as $v$ is an arbitrary vector of $V$. Hence $t_{3} \in X_{3}$, and we have (i).

By direct calculation we see that

Moreover

$$
S L_{2}(q) \cong L=\left\{\left.\left(\begin{array}{l|ll|l}
1 & & &  \tag{3.2}\\
& & & \\
& a & b & \\
& c & d & \\
& & d & \\
& & & 1
\end{array}\right) \right\rvert\, \begin{array}{l} 
\\
a, b, c, d \in G F(q) \\
a d+b c=1
\end{array}\right\} \leq C_{G}\left(t_{1}\right)
$$

with $Q_{1}$ a normal elementary abelian subgroup of $C_{G}\left(t_{1}\right)$ and $\left|Q_{1}\right|=q^{3}$. So $C_{G}\left(t_{1}\right)=$ $L Q_{1}$. Thus (ii) holds.

It is a routine calculation to show that $S \leq C_{G}\left(t_{2}\right)$. The involution $t_{2}$ satisfies the hypothesis of Lemma 7.11 (ii) of [10], the result of which shows that $C_{G}\left(t_{2}\right)$ is a 2group. Since $S \in \operatorname{Syl}_{2} G$ by Lemma 3.1(i), we have (iii).

From parts (ii) and (iii), $\left|C_{G}\left(t_{1}\right)\right|=q^{4}\left(q^{2}-1\right)$ and $\left|C_{G}\left(t_{2}\right)\right|=q^{4}$. Combining this with $|G|=q^{4}\left(q^{2}-1\right)\left(q^{4}-1\right)$ yields (iv) and (v).

### 3.1 The Structure of $\mathcal{C}\left(G, X_{i}\right), i=1,3$

As shown in Proposition 2.3, $G$ has an outer automorphism arising from the Dynkin diagram of type $C_{2}=B_{2}$. This outer automorphism interchanges the two involution conjugacy classes $X_{1}$ and $X_{3}$ and as a consequence $\mathcal{C}\left(G, X_{1}\right)$ and $\mathcal{C}\left(G, X_{3}\right)$ are isomorphic graphs. Thus we need only consider $\mathcal{C}\left(G, X_{1}\right)$.

Lemma 3.3. (i) Let $x \in \operatorname{Stab}_{G} C_{V}\left(t_{1}\right)$ be an involution. Then $x \in C_{G}\left(t_{1}\right)$.
(ii) $\mathcal{C}\left(G, X_{1}\right)$ is connected of diameter 2.

Proof. Combined, Proposition 4.1.19 of [32] and Lemma 3.2(ii) gives $\operatorname{Stab}_{G} C_{V}\left(t_{1}\right) \cong$ $C_{G}\left(t_{1}\right) \rtimes C_{(q-1)}$ and so $C_{G}\left(t_{1}\right) \unlhd \operatorname{Stab}_{G} C_{V}\left(t_{1}\right)$. Therefore any Sylow 2-subgroup of $\operatorname{Stab}_{G} C_{V}\left(t_{1}\right)$ is a Sylow 2-subgroup of $C_{G}\left(t_{1}\right)$ and, in particular, any involution stabilising $C_{V}\left(t_{1}\right)$ must lie in $C_{G}\left(t_{1}\right)$, proving (i).

Let $x \in X_{1}$ such that $x \notin C_{G}\left(t_{1}\right)$ (which exists since $\left\langle X_{1}\right\rangle$ is not abelian). If $C_{V}(x)=C_{V}\left(t_{1}\right)$ then $x \in \operatorname{Stab}_{G} C_{V}\left(t_{1}\right)$ and by (i), $x \in C_{G}\left(t_{1}\right)$, contradicting our choice of $x$. So $C_{V}(x) \neq C_{V}\left(t_{1}\right)$ and hence $C_{V}\left(\left\langle t_{1}, x\right\rangle\right) \nRightarrow C_{V}\left(t_{1}\right)$. Since $\operatorname{dim} C_{V}\left(t_{1}\right)=$ $\operatorname{dim} C_{V}(x)=3$, we necessarily have $\operatorname{dim}\left(C_{V}\left(\left\langle t_{1}, x\right\rangle\right)\right)=2$. Any 1-subspace of $V$ is isotropic by virtue of the symplectic form. Let $U$ be a 1-dimensional subspace of $C_{V}\left(\left\langle t_{1}, x\right\rangle\right) \leq V$. For any $y_{0} \in X_{1}, \operatorname{dim}\left[V, y_{0}\right]=1$ and hence isotropic. Therefore, there exists $y \in X_{1}$ such that $U=[V, y]$, since $G$ is transitive on the set of isotropic 1 -subspaces of $V$, by Witt's Lemma. Let $u \in U$ and so $u=v+v^{y}$ for some $v \in V$. Clearly

$$
u^{y}=\left(v+v^{y}\right)^{y}=v^{y}+v^{y^{2}}=v^{y}+v=u
$$

so $U^{y}=U$. Consider now $v+U \in V / U$. We then have

$$
(v+U)^{y}=v^{y}+U^{y}=v^{y}+\left(v^{y}+v\right)+U=v+U,
$$

thus $y$ fixes $V / U$ pointwise. Hence, $y$ stabilises any subspace $U \leq W \leq V$. In particular, $y$ stabilises $C_{V}\left(t_{1}\right)$ and $C_{V}(x)$, since $U \leq C_{V}\left(\left\langle t_{1}, x\right\rangle\right)$. Moreover, $y$ is an involution and so by (i), $y \in C_{G}\left(t_{1}\right) \cap C_{G}(x)$. Since $t_{1} \neq y \neq x$ we have $d\left(t_{1}, x\right)=2$. Moreover, $x$ is arbitrary and so $\mathcal{C}\left(G, X_{1}\right)$ is connected of diameter 2, so proving (ii).

Lemma 3.4. $\left|C_{G}\left(t_{1}\right) \cap X_{1}\right|=\left|\Delta_{1}\left(t_{1}\right)\right|=q^{3}-1$.
Proof. Let $s$ be an involution in $S$. Then, by Lemma 3.1(ii), $s \in Q_{1}^{\#} \cup Q_{2}^{\#}$. Let $v=(\alpha, \beta, \gamma, \delta)$ be a vector in $V$. Assume for the moment that $s \in Q_{1}$. Then

$$
s=\left(\begin{array}{llll}
1 & a & b & c \\
0 & 1 & 0 & b \\
0 & 0 & 1 & a \\
0 & 0 & 0 & 1
\end{array}\right)
$$

where $a, b, c \in G F(q)$. So $v^{s}=(\alpha, a \alpha+\beta, b \beta+\gamma, c \alpha+b \beta+a \gamma+\delta)$. Suppose that at least one of $a$ and $b$ is non-zero. If $v \in C_{V}(s)$, then we have $a \alpha=b \beta=c \alpha+b \beta+a \gamma=0$. If, say, $a \neq 0$ then this gives $\alpha=0$ and $b \beta+a \gamma=0$. Hence $\gamma=\lambda \beta$ for some $\lambda \in G F(q)$. Thus $\operatorname{dim} C_{V}(s)=2$, with the same conclusion if $b \neq 0$.

When $a=b=0$ we see that $\operatorname{dim} C_{V}(s)=3$. Therefore we conclude that

$$
\begin{equation*}
\left|Q_{1} \cap X_{1}\right|=q-1 . \tag{3.3}
\end{equation*}
$$

Now we suppose $s \in Q_{2} \backslash Q_{1}$. Then

$$
s=\left(\begin{array}{llll}
1 & 0 & a & b \\
0 & 1 & c & a \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

where $a, b, c \in G F(q)$ and $c \neq 0$. Here $v^{s}=(\alpha, \beta, a \alpha+c \beta+\gamma, b \alpha+a \beta+\delta)$ and so, if $v \in C_{V}(s), a \alpha+c \beta=b \alpha+a \beta=0$. Suppose that $a=0$ and $b \neq 0$. Then $c \beta=b \alpha=0$
which yields $\alpha=0=\beta$. Hence $\operatorname{dim} C_{V}(s)=2$. Likewise, when $a \neq 0$ and $b=0$ we get $\operatorname{dim} C_{V}(s)=2$. On the other hand, $a=0=b$ gives $\operatorname{dim} C_{V}(s)=3$.

Now consider the case when $a \neq 0 \neq b$ and $a^{2}+b c=0$. From $a \alpha+c \beta=0$ we obtain $\beta=a \alpha c^{-1}$ and so $0=b \alpha+a \beta=b \alpha+a^{2} c^{-1} \alpha=\left(b+a^{2} c^{-1}\right) \alpha$. Since $a^{2}+b c=0$, this equation holds for all $\alpha \in G F(q)$ and consequently $\operatorname{dim} C_{V}(s)=3$. Similar considerations show that $\operatorname{dim} C_{V}(s)=2$ when $a \neq 0 \neq b$ and $a^{2}+b c \neq 0$. So, to summarise, for $s \in Q_{2} \backslash Q_{1}, s \in X_{1}$ when either $a=0=b$ or $a \neq 0 \neq b$ and $a^{2}+b c=0$. For the former, there are $q-1$ such involutions (as $c \neq 0$ ). For the latter, there are $q-1$ choices for each of $b$ and $c$ and in each case $a$ is uniquely determined (as $G F(q)^{*}$ is cyclic of odd order), so giving $(q-1)^{2}$ involutions. Therefore

$$
\begin{equation*}
\left|\left(X_{1} \cap S\right) \backslash Q_{1}\right|=\left|X_{1} \cap\left(Q_{2} \backslash Q_{1}\right)\right|=q(q-1) \tag{3.4}
\end{equation*}
$$

Since any two distinct Sylow 2-subgroups of $S L_{2}(q)$ have trivial intersection and $S L_{2}(q)$ possesses $q+1$ Sylow 2-subgroups, Lemma 3.2(ii) together with (3.3) and (3.4) yields that

$$
\begin{aligned}
\left|C_{G}\left(t_{1}\right) \cap X_{1}\right| & =(q-1)+q(q-1)(q+1) \\
& =(q-1)\left(1+q^{2}+q\right)=q^{3}-1 .
\end{aligned}
$$

This proves Lemma 3.4.

As $\mathcal{C}\left(G, X_{1}\right)$ has diameter 2 , it is clear that $\left|\Delta_{2}\left(t_{1}\right)\right|=\left|X_{1}\right|-\left|\{t\} \cup \Delta_{1}\left(t_{1}\right)\right|$ and so

$$
\begin{aligned}
\left|\Delta_{2}\left(t_{1}\right)\right| & =\left(q^{4}-1\right)-\left(q^{3}-1\right) \\
& =q^{3}(q-1)
\end{aligned}
$$

This, together with Lemmas 3.3(ii) and 3.4, completes the proof of Theorem 1.1. We now set to constructing the collapsed adjacency diagram for $\mathcal{C}\left(G, X_{1}\right)$. Let $L$ be as
in (3.2) and so $C_{G}\left(t_{1}\right)=L Q_{1}$. If $g \in C_{G}(t)=L Q_{1}$, then by (3.1)
for $a, b, c, d, \beta, \gamma, \delta \in G F(q)$ and $a d+b c=1$. As is customary, we denote the $C_{G}\left(t_{1}\right)$ orbits by $\Delta_{i}^{j}\left(t_{1}\right)$ where $i$ denotes which disc the $C_{G}\left(t_{1}\right)$-orbit lies in, and $j$ indexes the orbits in ascending size.

Lemma 3.5. (i) $\Delta_{1}\left(t_{1}\right)$ is a union of $q-1 C_{G}\left(t_{1}\right)$-orbits, where $\left|\Delta_{1}^{i}\left(t_{1}\right)\right|=1$ for $i=1, \ldots, q-2$ and $\left|\Delta_{1}^{q-1}\left(t_{1}\right)\right|=q\left(q^{2}-1\right)$.
(ii) $\Delta_{1}^{i}\left(t_{1}\right) \subseteq Z\left(C_{G}\left(t_{1}\right)\right)$ for $i=1, \ldots, q-2$.
(iii) Let $x \in \Delta_{1}^{q-1}\left(t_{1}\right)$. Then $\left|\Delta_{1}^{q-1}\left(t_{1}\right) \cap C_{G}(x)\right|=q(q-1)$.

Proof. Recall that $Z\left(C_{G}\left(t_{1}\right)\right) \leq Z(S)=Q_{1} \cap Q_{2}$ and by Lemma 3.4, $X_{1} \cap Q_{1}=$ $X_{1} \cap Q_{1} \cap Q_{2}=X \cap Z\left(C_{G}\left(t_{1}\right)\right)$ with order $q-1$ and so $\left|\left(X \cap Z\left(C_{G}\left(t_{1}\right)\right)\right) \backslash\left\{t_{1}\right\}\right|=q-2$. This proves (ii). Let $x \in \Delta_{1}\left(t_{1}\right) \backslash Z\left(C_{G}\left(t_{1}\right)\right)$, so lies in a Sylow 2-subgroup of $C_{G}\left(t_{1}\right)$. From Lemma 3.4, $\left(\Delta_{1}\left(t_{1}\right) \cap S\right) \backslash Z\left(C_{G}\left(t_{1}\right)\right)=\Delta_{1}\left(t_{1}\right) \cap Q_{2}$. Hence $x$ must be $C_{G}\left(t_{1}\right)$ conjugate to an element in $X \cap Q_{2}$, by Sylow's Theorems. Let

$$
x_{1}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{3.5}\\
0 & 1 & a_{1} & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) \quad \text { and } \quad x_{2}=\left(\begin{array}{cccc}
1 & 0 & b_{1} & b_{2} \\
0 & 1 & b_{3} & b_{1} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

where $a_{1}, b_{i} \in G F(q)^{*}$ and $b_{1}^{2}+b_{2} b_{3}=0$, be such that $x$ is $C_{G}\left(t_{1}\right)$-conjugate to either $x_{1}$ or $x_{2}$. Hence $C_{G}\left(\left\langle t_{1}, x\right\rangle\right)^{g}=C_{G}\left(\left\langle t_{1}, x_{i}\right\rangle\right)$ for some $i=1,2$ and some $g \in C_{G}\left(t_{1}\right)$, so without loss of generality we may pick $x=x_{i}$ for $i=1$ or 2 . Since $x \in Q_{2}$ and $Q_{2}$ is abelian, $Q_{2} \leq C_{G}\left(\left\langle t_{1}, x\right\rangle\right)$. If $g \in C_{G}\left(t_{1}\right)$, then $x^{g}=x$ if and only if $g \in Q_{2}$. Hence $Q_{2}=C_{G}\left(\left\langle t_{1}, x\right\rangle\right)$ and so,

$$
\left|x^{C_{G}\left(t_{1}\right)}\right|=\frac{\left|C_{G}\left(t_{1}\right)\right|}{\left|C_{G}\left(\left\langle t_{1}, x\right\rangle\right)\right|}=\frac{q^{4}\left(q^{2}-1\right)}{q^{3}}=q\left(q^{2}-1\right) .
$$

However, $\left|\Delta_{1}\left(t_{1}\right) \backslash Z\left(C_{G}\left(t_{1}\right)\right)\right|=\left(q^{3}-2\right)-(q-2)=q\left(q^{2}-1\right)$ and hence $x^{C_{G}\left(t_{1}\right)}=$ $\Delta_{1}\left(t_{1}\right) \backslash Z\left(C_{G}\left(t_{1}\right)\right)$, proving (i). Therefore, $\Delta_{1}^{q-1}\left(t_{1}\right)=x^{C_{G}\left(t_{1}\right)}$.

Clearly for all $z \in X \cap Z\left(C_{G}\left(t_{1}\right)\right),[x, z]=1$ and for all $y \in C_{G}\left(t_{1}\right),[z, y]=1$. Recall that $d\left(x, x^{g}\right)=1$ if and only if $x^{g} \in C_{G}\left(\left\langle t_{1}, x\right\rangle\right)=Q_{2} \unlhd C_{G}\left(t_{1}\right)$. Since $Z\left(C_{G}\left(t_{1}\right)\right) \cap$ $\Delta_{1}^{q-1}\left(t_{1}\right)=\varnothing$, and Lemma 3.4 shows $\left|X \cap\left(Q_{2} \backslash Q_{1}\right)\right|=q(q-1)$, the result follows.

Lemma 3.6. $\Delta_{2}\left(t_{1}\right)$ is a union of $q-1 C_{G}\left(t_{1}\right)$-orbits and $\left|\Delta_{2}^{i}\left(t_{1}\right)\right|=q^{3}$ for $i=$ $1, \ldots, q-1$.

Proof. Let $g \in C_{G}\left(t_{1}\right)$ be as in (3.3), with $a, b, c, d, \beta, \gamma, \delta \in G F(q)$ and $a d+b c=1$. Define

$$
y_{\alpha}=\left(\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\alpha & 0 & 0 & 1
\end{array}\right)
$$

for $\alpha \in G F(q)^{*}$. An easy check shows that $y_{\alpha}^{2}=I_{4}$ and $\operatorname{dim}\left[V, y_{\alpha}\right]=1$, so $y_{\alpha} \in X_{1}$. Moreover, $y_{\alpha} \notin C_{G}\left(t_{1}\right)$ and by Lemma 3.3(ii), $y_{\alpha} \in \Delta_{2}\left(t_{1}\right)$. Direct calculation reveals,

$$
y_{\alpha}^{g}=\left(\begin{array}{cccc}
1+\alpha \delta & \alpha \beta \delta & \alpha \gamma \delta & \alpha \delta^{2} \\
\alpha \gamma & \alpha \beta \gamma+1 & \alpha \gamma^{2} & \alpha \gamma \delta \\
\alpha \beta & \alpha \beta^{2} & \alpha \beta \gamma+1 & \alpha \beta \delta \\
\alpha & \alpha \beta & \alpha \gamma & 1+\alpha \delta
\end{array}\right)
$$

and observe that $\left(y_{\alpha}^{g}\right)_{41}=\alpha$. Let $\alpha^{\prime} \in G F(q)^{*}$, so $y_{\alpha}^{g}=y_{\alpha^{\prime}}$ if and only if $\alpha=\alpha^{\prime}$. Hence, $y_{\alpha}^{C_{G}\left(t_{1}\right)}=y_{\alpha^{\prime}}^{C_{G}\left(t_{1}\right)}$ if and only if $\alpha=\alpha^{\prime}$. Therefore, there are at least $(q-1)$ distinct $C_{G}\left(t_{1}\right)$-orbits in $\Delta_{2}\left(t_{1}\right)$. If $y_{\alpha}^{g}=y_{\alpha}$, then $\beta=\gamma=\delta=0($ since $\alpha \neq 0)$, so $C_{G}\left(\left\langle t_{1}, y_{\alpha}\right\rangle\right)=L$. Therefore

$$
\left|y_{\alpha}^{C_{G}\left(t_{1}\right)}\right|=\frac{\left|C_{G}\left(t_{1}\right)\right|}{\left|C_{G}\left(\left\langle t_{1}, y_{\alpha}\right\rangle\right)\right|}=\frac{q^{4}\left(q^{2}-1\right)}{q\left(q^{2}-1\right)}=q^{3} .
$$

Hence for all $\alpha \in G F(q)^{*},\left|y_{\alpha}^{C_{G}\left(t_{1}\right)}\right|=q^{3}$. However, $\left|\Delta_{2}\left(t_{1}\right)\right|=q^{3}(q-1)$ by Theorem 1.1, so $\Delta_{2}\left(t_{1}\right)$ is a union of exactly $(q-1)$ orbits of length $q^{3}$.

Lemma 3.7. Let $x \in \Delta_{1}^{q-1}\left(t_{1}\right)$. Then $\left|C_{G}(x) \cap \Delta_{2}^{i}\left(t_{1}\right)\right|=q^{2}$ for $i=1, \ldots, q-1$.

Proof. Without loss of generality we may pick $x=x_{1}$ as in (3.5), and observe $y_{\alpha} \in$ $C_{G}(x) \cap \Delta_{2}^{j}\left(t_{1}\right)$, where $\alpha=\varepsilon^{j}$ and $\langle\varepsilon\rangle=G F(q)^{*}$. Clearly $d\left(x, y_{\alpha}\right)=d\left(x, y_{\alpha}^{g}\right)=1$ for some $g \in C_{G}\left(\left\langle t_{1}, x\right\rangle\right)=Q_{2}$ and since $C_{G}\left(\left\langle t_{1}, y_{\alpha}\right\rangle\right)=L$, we have $C_{G}\left(\left\langle t_{1}, x, y_{\alpha}\right\rangle\right)=$ $Q_{2} \cap L$ which has order $q$. Therefore there are $\frac{q^{3}}{q}=q^{2} C_{G}\left(t_{1}\right)$-conjugates, $y$, of $y_{\alpha}$ such that $[x, y]=1$. Since $\alpha$ is arbitrary, this holds for all $\Delta_{2}^{j}\left(t_{1}\right)$ and so $\left|C_{G}(x) \cap \Delta_{2}\left(t_{1}\right)\right| \geq$ $q^{2}(q-1)$. However, by Lemma 3.5(ii) and (iii), $\left|C_{G}\left(x_{i}\right) \cap\left(\Delta_{1}\left(t_{1}\right) \cup\left\{t_{1}\right\}\right)\right|=q(q-$ 1) $+(q-1)$ and we have

$$
\begin{aligned}
\left|C_{G}\left(x_{i}\right) \cap X_{1}\right| & \geq\left|C_{G}(x) \cap \Delta_{2}\left(t_{1}\right)\right|+\left|C_{G}\left(x_{i}\right) \cap\left(\Delta_{1}\left(t_{1}\right) \cup\left\{t_{1}\right\}\right)\right| \\
& \geq q^{2}(q-1)+q(q-1)+(q-1) \\
& =q^{3}-1 \\
& =\left|C_{G}\left(x_{i}\right) \cap X_{1}\right| .
\end{aligned}
$$

Hence we have equality and so $\left|C_{G}(x) \cap \Delta_{2}\left(t_{1}\right)\right|=q^{2}(q-1)$. This proves Lemma 3.7.

Lemma 3.8. Let $y \in \Delta_{2}^{i}\left(t_{1}\right)$ for some $i=1, \ldots, q-1$. Then

$$
\left|C_{G}(y) \cap \Delta_{2}^{j}\left(t_{1}\right)\right|= \begin{cases}q^{2}-1 & \text { if } i=j \\ q^{2} & \text { if } i \neq j\end{cases}
$$

Proof. Without loss of generality, we may choose $y=y_{\alpha}$, for some $\alpha \in G F(q)^{*}$. If $\alpha=\varepsilon^{j}$ where $\langle\varepsilon\rangle=G F(q)^{*}$ then we set $y_{\alpha}^{C_{G}\left(t_{1}\right)}=\Delta_{2}^{j}\left(t_{1}\right)$. Let

$$
h=\left(\begin{array}{cccc}
1 & \beta & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & \beta \\
0 & 0 & 0 & 1
\end{array}\right) \in C_{G}\left(t_{1}\right) \backslash C_{G}\left(y_{\alpha}\right),
$$

and a routine calculation yields

$$
y_{\alpha}^{h}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\alpha \beta & \alpha \beta^{2} & 1 & 0 \\
\alpha & \alpha \beta & 0 & 1
\end{array}\right),
$$

and $\left[y_{\alpha}, y_{\alpha}^{h}\right]=1$. Since $\left[y_{\alpha}, y_{\alpha}^{h g}\right]=1$ if and only if $g \in C_{G}\left(\left\langle t_{1}, y_{\alpha}\right\rangle\right)=L$, we have

$$
g=\left(\begin{array}{l|ll|l}
1 & & & \\
\hline & a & b & \\
& c & d & \\
\hline & & & 1
\end{array}\right),
$$

for $a d+b c=1$, and so

$$
y_{\alpha}^{h g}=\left(\begin{array}{ccc|c}
1 & & & \\
\hline b \alpha \beta & 1+a b \alpha \beta^{2} & b^{2} \alpha \beta^{2} & \\
a \alpha \beta & a^{2} \alpha \beta^{2} & 1+b a \alpha \beta^{2} & \\
\alpha & a \alpha \beta & b \alpha \beta & 1
\end{array}\right) \text {. }
$$

Now since $\left[y_{\alpha}, h\right] \neq 1$, we must have $\alpha \beta \neq 0$. However, $b \alpha \beta=0$ then implies that $b=0$, and $a \alpha \beta=\alpha \beta$ forces $a=1$. Also, since $a d+b c=1$, we get $d=1$. Hence, $C_{G}\left(\left\langle t, y_{\alpha}, y_{\alpha}^{h}\right\rangle\right)$ is isomorphic to a Sylow 2-subgroup of $L$ and hence $\left|C_{G}\left(\left\langle t, y_{\alpha}, y_{\alpha}^{h}\right\rangle\right)\right|=q$. Therefore, there exists $\frac{q\left(q^{2}-1\right)}{q}=q^{2}-1 C_{G}\left(t_{1}\right)$-conjugates, $y^{\prime}$, of $y_{\alpha}^{h}$ such that $\left[y_{\alpha}, y^{\prime}\right]=1$.

Letting $\alpha^{\prime}=\varepsilon^{k} \in G F(q)^{*} \backslash\{\alpha\}$ reveals $\left[y_{\alpha}, y_{\alpha^{\prime}}\right]=1$ and by Lemma 3.6, $C_{G}\left(\left\langle t_{1}, y_{\alpha}\right\rangle\right)=$ $C_{G}\left(\left\langle t, y_{\alpha^{\prime}}\right\rangle\right)=L$. By a completely analogous argument, we have that for an arbitrary $h^{\prime} \in C_{G}\left(t_{1}\right) \backslash C_{G}\left(y_{\alpha^{\prime}}\right)$, we have $\left|C_{G}\left(\left\langle t_{1}, y_{\alpha^{\prime}}, y_{\alpha^{\prime}}^{h^{\prime}}\right\rangle\right)\right|=q$ and there exists $q^{2}-1 C_{G}\left(t_{1}\right)$ conjugates, $y^{\prime \prime}$, of $y_{\alpha^{\prime}}^{h^{\prime}}$ such that $\left[y_{\alpha}, y^{\prime \prime}\right]=1$. Hence $\left|C_{G}\left(y_{\alpha}\right) \cap \Delta_{2}^{k}\left(t_{1}\right)\right|=q^{2}-1+1=q^{2}$. Since $\alpha^{\prime}$ was arbitrary, this occurs for every $C_{G}\left(t_{1}\right)$-orbit of $\Delta_{2}\left(t_{1}\right)$ not containing $y_{\alpha}$. So $\left|C_{G}\left(y_{\alpha}\right) \cap \Delta_{2}\left(t_{1}\right)\right| \geq q^{2}(q-1)$. Moreover, since $\left[x_{1}, y_{\alpha}\right]=1$ for $x_{1}$ as in (3.5), we have

$$
q\left(q^{2}-1\right) q^{2}=n q^{3}
$$

for some integer $n$, so clearly $n=q^{2}-1$. That is to say, there exists $q^{2}-1 C_{G}\left(t_{1}\right)-$ conjugates, $x$, of $x_{1}$ such that $\left[y_{\alpha}, x\right]=1$. Hence

$$
\begin{aligned}
\left|C_{G}\left(y_{\alpha}\right) \cap X_{1}\right| & \geq\left|C_{G}\left(y_{\alpha}\right) \cap \Delta_{2}\left(t_{1}\right)\right|+\left|C_{G}\left(y_{\alpha}\right) \cap \Delta_{1}^{q-1}\left(t_{1}\right)\right| \\
& \geq q^{2}(q-1)+q^{2}-1 \\
& =q^{3}-1 \\
& =\left|C_{G}\left(y_{\alpha}\right) \cap X_{1}\right| .
\end{aligned}
$$



Figure 3.1: The collapsed adjacency diagram for $\mathcal{C}\left(G, X_{i}\right)$, for $i=1,3$.
hence we have equality and so $\left|C_{G}\left(y_{\alpha}\right) \cap \Delta_{2}\left(t_{1}\right)\right|=q^{2}(q-1)$. Since $\mathcal{C}\left(G, X_{1}\right)$ is without loops, Lemma 3.8 holds.

Lemmas 3.5-3.8 determine the $C_{G}\left(t_{1}\right)$-orbit structure of $\mathcal{C}\left(G, X_{1}\right)$ and are summarized in a collapsed adjacency diagram as in Figure 3.1.

### 3.2 The Structure of $\mathcal{C}\left(G, X_{2}\right)$

Before moving on to prove Theorem 1.2 we need additional preparatory material. If $W$ is a subspace of $V$, we recall that $\operatorname{dim} W+\operatorname{dim} W^{\perp}=\operatorname{dim} V=4$. By Lemma 3.2(i),(iii) we see that $C_{V}\left(C_{G}\left(t_{2}\right)\right)=\{(0,0,0, \alpha) \mid \alpha \in G F(q)\}$ is 1-dimensional. For $x \in X_{2}$ set $U_{1}(x)=C_{V}\left(C_{G}(x)\right)$ and $U_{2}(x)=C_{V}(x) . \quad$ So $\operatorname{dim} U_{1}(x)=1$ and $\operatorname{dim} U_{2}(x)=2$ (with the subscripts acting as a reminder). We denote the stabilizer in $G$ of $U_{1}\left(t_{2}\right)$, respectively $U_{2}\left(t_{2}\right)$, by $P_{1}$, respectively $P_{2}$. Then $P_{i} \cong q^{3} S L_{2}(q)(q-1)$ for $i=1,2$. Also $Q_{i}=O_{2}\left(P_{i}\right)$ with $C_{P_{i}}\left(Q_{i}\right)=Q_{i}$ for $i=1,2$.

We start analyzing $\mathcal{C}\left(G, X_{2}\right)$ by determining $\Delta_{1}\left(t_{2}\right)$. For $x \in X_{2}$ we let $Z_{C_{G}(x)}$ denote $Z\left(C_{G}(x)\right) \cap X_{2}$.

Lemma 3.9. (i) $X$ is a disjoint union of all $Z_{R}$ for all $R \in \operatorname{Syl}_{2} G$.
(ii) Let $R, T \in \operatorname{Syl}_{2} G$ be such that there exists $r_{0} \in Z_{R}, s_{0} \in Z_{T}$ such that $r_{0} s_{0}=s_{0} r_{0}$.

Then $\left[Z_{R}, Z_{S}\right]=1$.
Proof. Clearly $X_{2}=\bigcup_{R \in \mathrm{Syl}_{2} G} Z_{R}$ by Lemma 3.2(iii). If $Z_{R} \cap Z_{T} \neq \varnothing$ for $R, T \in \operatorname{Syl}_{2} G$, then we have some $x \in Z(R) \cap Z(T) \cap X_{2}$ whence, using Lemma 3.2(iii), $R=C_{G}(x)=$ $T$. So (i) holds.

Since $x y=y x, y \in C_{G}(x)=R$. Hence $Z(R) \leq C_{G}(y)=T$ and so $\left[Z_{R}, Z_{T}\right]=1$, giving (ii).

Let $\Delta$ be the building for $G$. Since $p=2$, every Borel subgroup is the normaliser of a Sylow 2-subgroup. Moreover, since $\operatorname{dim} V=4$ (thus a maximal flag of isotropic subspaces has length 2), there is only one conjugacy class of parabolic subgroups that properly contain a Borel subgroup. Clearly, $N_{G}(S)$ is a chamber of $\Delta$, and $N_{G}(S) \leq P_{i}$ for both $i=1,2$. Let $R \in \operatorname{Syl}_{2} G$, so $N_{G}(R)$ is adjacent to $N_{G}(S)$ if and only if $N_{G}(R) \leq P_{i}$ for some $i=1,2$. Let $\mathcal{C}(\Delta)$ denote the chamber graph, with the set of chambers $V(\mathcal{C}(\Delta))=\left\{N_{G}(R) \mid R \in \operatorname{Syl}_{2} G\right\}$ as its vertex set with an edge between two chambers if and only if they are adjacent. Equivalently, if $B=N_{G}(R)$ then the vertices of the chamber graph can be represented as cosets of $B$, such that two chambers $B g_{1}$ and $B g_{2}$ are adjacent if and only if there exists $g \in G$ such that $B g_{1} \subset P_{i} g$ and $B g_{2} \subset P_{i} g$ for $g_{i} \in G$. We use $d^{\mathcal{C}}$ to denote the standard distance metric in $\mathcal{C}(\Delta)$ and for a chamber $B$ put $\Delta_{j}^{\mathcal{C}}(B)=\left\{D \in \mathcal{C}(\Delta) \mid d^{\mathcal{C}}(B, D)=j\right\}$.

Lemma 3.10. $\mathcal{C}(\Delta)$ has diameter 4 with disc sizes

$$
\begin{aligned}
& \left|\Delta_{1}^{\mathcal{C}}(B)\right|=2 q ; \\
& \left|\Delta_{2}^{\mathcal{C}}(B)\right|=2 q^{2} ; \\
& \left|\Delta_{3}^{\mathcal{C}}(B)\right|=2 q^{3} ; \quad \text { and } \\
& \left|\Delta_{4}^{\mathcal{C}}(B)\right|=q^{4},
\end{aligned}
$$

for a chamber $B$ of $\Delta$.
Proof. Without loss of generality, set $B=N_{G}(S)$. Recall $G$ arises from the Dynkin diagram of type $C_{2}=B_{2}$ and so the Weyl group, $W=\left\langle w_{1}, w_{2}\right\rangle$, is dihedral of order 8, and hence the girth of the apartment is 8 . That is to say, the longest convex


Figure 3.2: An apartment of $\mathcal{C}(\Delta)$.


Figure 3.3: An impossible quadrangle within $\mathcal{C}(\Delta)$.
circuit in $\mathcal{C}(\Delta)$ has length 8 . Without loss of generality, we can set the $w_{i}$ to be the reflections in the walls contained in $B$, or equivalently automorphisms of $N_{G}\left(Q_{i}\right)$ that interchange $B$ with a $N_{G}\left(Q_{i}\right)$-conjugate of $B$. Since $G$ is a disjoint union of the double cosets of $(B, B)$ with each element of the Weyl group representing a different double coset, an apartment of $\mathcal{C}(\Delta)$ can be represented by Figure 3.2. Since the building is of rank 2 , there cannot be any convex circuits of length less than 8 , unless all chambers in the circuit all intersect in the same wall. Indeed if, for example, the quadrangle described in Figure 3.3 exists then $w_{2}=w_{1} w_{2} w_{1}$, contradicting the structure of $W$. All other cases are similarly shown. Suppose a quadrangle such as one described in Figure 3.4 exists for some $b, b^{\prime} \in B, i, j=1,2, i \neq j$. So $B$, $B w_{i}$ and $B w_{i} b$ all intersect in a common wall, or equivalently $B, B w_{i}$ and $B w_{i} b$ lie in the same parabolic subgroup, $P_{k}, k=1,2$. Similarly, $B w_{i}, B w_{i} b$ and $B w_{j} w_{i} b^{\prime}$ all lie in another distinct parabolic subgroup $P_{l}, l=1,2, l \neq k$. However, both $P_{k}$


Figure 3.4: Another impossible quadrangle within $\mathcal{C}(\Delta)$.
and $P_{l}$ are minimal parabolic subgroups and both contain $B w_{i}$ and $B w_{i} b$. Hence $P_{k} \cap P_{l}=P_{k}=P_{l}$ or $P_{k} \cap P_{l}=B w_{i}=B w_{i} b$, both of which cannot occur. Hence, no quadrangles containing non-adjacent chambers exist in the chamber graph.

Recall that $P_{k} \cong q^{3} S L_{2}(q)(q-1)$ and so $\left|\operatorname{Syl}_{2} P_{k}\right|=\left|\operatorname{Syl}_{2} S L_{2}(q)\right|$ since $q$ is even. Also, since any two Sylow 2-subgroups of $S L_{2}(q)$ intersect trivially, any two Sylow 2-subgroups of $P_{k}$ intersect in $Q_{k}$. There are $q+1$ Sylow 2-subgroups in $S L_{2}(q)$, one being $S=Q_{1} Q_{2}$. Hence, $\operatorname{Syl}_{2} P_{1} \cap \operatorname{Syl}_{2} P_{2}=\{S\}$ and so in each parabolic subgroup containing $B$, there exist $q$ (non-trivial) conjugates of $B$, thus the first disc of the chamber graph has order $2 q$.

If a circuit contains vertices that don't all intersect in a common wall, then the circuit must necessarily be of length 8 . This means that for $B_{1}, B_{2} \in \Delta_{1}^{\mathcal{C}}(B)$, we have $\Delta_{2}^{\mathcal{C}}(B) \cap \Delta_{1}^{\mathcal{C}}\left(B_{1}\right) \cap \Delta_{1}^{\mathcal{C}}\left(B_{2}\right)=\varnothing$. Hence for each $B_{i} \in \Delta_{1}^{\mathcal{C}}(B)$, there exist $q$ chambers in $\Delta_{2}^{\mathcal{C}}(B) \cap \Delta_{1}^{\mathcal{C}}\left(B_{i}\right)$ which are not contained in any other $\Delta_{2}^{\mathcal{C}}(B) \cap \Delta_{1}^{\mathcal{C}}\left(B_{j}\right), i \neq j$. Thus, $\left|\Delta_{2}^{\mathcal{C}}(B)\right|=q\left|\Delta_{1}^{\mathcal{C}}(B)\right|=2 q^{2}$. An analogous argument shows that $\left|\Delta_{3}^{\mathcal{C}}(B)\right|=2 q^{3}$.
Observe that $S$ acts simply-transitively on the set of chambers opposite a given chamber. That is to say, for $C_{1}, C_{2}$ opposite chambers of $B=N_{G}(S)$, there exists a unique $s \in S$ such that $C_{1}^{s}=C_{2}$. Hence there are $|S|=q^{4}$ opposite chambers of $B$ and thus $\left|\Delta_{4}^{\mathcal{C}}(B)\right|=q^{4}$. This proves Lemma 3.10.

The collapsed adjacency diagram for $\mathcal{C}(\Delta)$ is as described in Figure 3.5.
We now introduce a graph $\mathcal{Z}$ whose vertex set is $V(\mathcal{Z})=\left\{Z_{R} \mid R \in \operatorname{Syl}_{2} G\right\}$ with


Figure 3.5: The collapsed adjacency diagram for $\mathcal{C}(\Delta)$.
$Z_{R}, Z_{T} \in V(\mathcal{Z})$ joined if $Z_{R} \neq Z_{T}$ and $\left[Z_{R}, Z_{T}\right]=1$.

Lemma 3.11. The graphs $\mathcal{Z}$ and $\mathcal{C}(\Delta)$ are isomorphic.

Proof. Define $\varphi: V(\mathcal{Z}) \rightarrow V(\mathcal{C}(\Delta))$ by $\varphi: Z_{R} \mapsto N_{G}(R) \quad\left(R \in \operatorname{Syl}_{2} G\right)$. If $\varphi\left(Z_{R}\right)=$ $\varphi\left(Z_{T}\right)$ for $R, T \in \operatorname{Syl}_{2} G$, then $N_{G}(R)=N_{G}(T)$. Therefore $R=T$, and so $Z_{R}=Z_{T}$. Thus $\varphi$ is a bijection between $V(\mathcal{Z})$ and $V(\mathcal{C}(\Delta))$. Suppose $N_{G}(R)$ and $N_{G}(T)$ are distinct, adjacent chambers in $\mathcal{C}(\Delta)$. Without loss of generality we may assume $T=S$. Then $N_{G}(R), N_{G}(S) \leq P_{i}$ for $i \in\{1,2\}$. The structure of $P_{i}$ then forces $Z(R), Z(S) \leq Q_{i}$. Since $Q_{i}$ is abelian, we deduce that $\left[Z_{R}, Z_{S}\right]=1$. So $Z_{R}$ and $Z_{S}$ are adjacent in $\mathcal{Z}$. Conversely, suppose $Z_{R}$ and $Z_{S}$ are adjacent in $\mathcal{Z}$. Then [ $\left.Z_{R}, Z_{S}\right]=1$ with, by Lemma 3.9(i), $Z_{R} \cap Z_{S}=\varnothing$. Hence $Z_{R} \subseteq S$ and so by Lemma 3.1(ii), $Z_{R} \subseteq Q_{1} \cup Q_{2}$. Now $Q_{1} \cap Q_{2} \cap X_{2}=Z_{S}$ and so we must have $Z_{R} \subseteq Q_{i}$ for $i \in\{1,2\}$. The structure of $P_{i}$ now gives $N_{G}(R) \leq P_{i}$ and therefore $N_{G}(R)$ and $N_{G}(S)$ are adjacent in $\mathcal{C}(\Delta)$, which proves the lemma.

## Proof of Theorem 1.2

Since for all $x_{1}, x_{2} \in X,\left[x_{1}, x_{2}\right]=1$ if and only if $\left[Z_{C_{G}\left(x_{1}\right)}, Z_{C_{G}\left(x_{2}\right)}\right]=1$ by Lemma 3.9(i), then for $i>1, d^{\mathcal{C}}\left(x_{1}, x_{2}\right)=i$ if and only if $d^{\mathcal{Z}}\left(Z_{C_{G}\left(x_{1}\right)}, Z_{C_{G}\left(x_{2}\right)}\right)=i\left(\right.$ where $d^{\mathcal{Z}}$ denotes the distance in $\mathcal{Z})$. Note that if $d^{\mathcal{C}}\left(x_{1}, x_{2}\right)=1$, then either $Z_{C_{G}\left(x_{1}\right)}=Z_{C_{G}\left(x_{2}\right)}$ or $d^{\mathcal{Z}}\left(Z_{C_{G}\left(x_{1}\right)}, Z_{C_{G}\left(x_{2}\right)}\right)=1$. Since $X_{2}$ is a disjoint union of the elements of $\mathcal{Z}$, then
$\mathcal{C}\left(G, X_{2}\right)$ is connected of diameter 4. Now

$$
\Delta_{1}(t)=\bigcup_{\substack{R \in \mathrm{Sl}_{2} G \\\left[Z_{S}, Z_{R}\right]=1}} Z_{R} \quad \text { and } \quad \Delta_{i}(t)=\bigcup_{\substack{R \in \mathrm{Syl}_{G} G \\ d^{Z}\left(Z_{S}, Z_{R}\right)=i}} Z_{R}, \quad i>1
$$

and so $\left|\Delta_{1}(t)\right|=\left|Z_{S}\right|+2 q\left|Z_{S}\right|-1$. From $\left|Z_{S}\right|=(q-1)^{2}$ we get $\left|\Delta_{1}(t)\right|=(q-$ $1)^{2}+2 q(q-1)^{2}-1=q^{2}(2 q-3)$. The remaining disc sizes are immediate from the structure of the chamber graph $\mathcal{C}(\Delta)$.

This completes the proof of Theorem 1.2.

## Chapter 4

## 4-Dimensional Symplectic Groups over Fields of Odd Characteristic

We now consider $p>2$ so $G F(q)$ is a field of odd characteristic. Let $H=S p_{4}(q)$ and $G=H / Z(H) \cong P S p_{4}(q)$. Let $V$ be the symplectic $G F(q) H$-module equipped with a symplectic form $(\cdot, \cdot)$. Let $\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$ be a hyperbolic basis for $V$ with $\left(v_{2}, v_{1}\right)=\left(v_{4}, v_{3}\right)=1$. Thus if $J$ is the Gram matrix of this form then

$$
J=\left(\begin{array}{rrrr}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{array}\right),
$$

and $J$ has two diagonal blocks $J_{0}$ where $J_{0}=\left(\begin{array}{rr}0 & -1 \\ 1 & 0\end{array}\right)$. We remark that $H$ has exactly 2 conjugacy classes of involutions, one of which is contained in the centre of $H$. Let $s=\left(\begin{array}{c|c}-I_{2} & \\ & \\ & I_{2}\end{array}\right)$ which is clearly a non-central involution in $H$. Letting $g=\left(\begin{array}{l|l}A & B \\ \hline C & D\end{array}\right)$ where $A, B, C$ and $D$ are $2 \times 2$ matrices over $G F(q)$, direct calculation reveals that $[g, s]=1$ if and only if $B=C=0$. Moreover, since

$$
\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)^{T}\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right)\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)=\left(\begin{array}{cc}
0 & b c-a d \\
a d-b c & 0
\end{array}\right)
$$

and $g^{T} J g=J$, we must have $A^{T} J_{0} A=D^{T} J_{0} D=J_{0}$ and so $\operatorname{det} A=\operatorname{det} D=1$. Hence,

$$
C_{H}(s)=\left\{\left.\left(\begin{array}{l|l}
A & \\
& \\
& B
\end{array}\right) \right\rvert\, A, B \in S L_{2}(q)\right\} \cong S L_{2}(q) \times S L_{2}(q) .
$$

Let $x=\left(\begin{array}{c|c}A & \\ & \\ & B\end{array}\right) \in C_{H}(s)$ for some $A, B \in S L_{2}(q)$. Then $x$ is an involution if and only if $A$ and $B$ are involutions in $S L_{2}(q)$. Since $\pm I_{2}$ are the only elements in $S L_{2}(q)$ that square to $I_{2}$, the only involutions in $C_{H}(s)$ are $-I_{4}, s$ and $-s$. As $-I_{4}$ is central in $H$ (and hence non-conjugate to $s$ ), we have $s^{H} \cap C_{H}(s)=\{ \pm s\}$. Clearly $\left[H: C_{H}(s)\right]=\left|s^{H}\right|=q^{2}\left(q^{2}+1\right)$, and so $\mathcal{C}\left(H, s^{H}\right)$ is disconnected and consists of $\frac{1}{2} q^{2}\left(q^{2}+1\right)$ cliques on 2 vertices.

We now turn our attention to the simple group $G$ which has two conjugacy classes of involutions (see, for example, Lemma 2.4 of [42]). We shall let $Y_{1}$ denote the $G$-conjugacy class whose elements are the images of an involution in $H$, and $Y_{2}$ to denote the $G$-conjugacy class whose elements are the image of an element of $H$ of order 4 which squares to the non-trivial element of $Z(H)$. The main focus of this chapter is to prove Theorems 1.3 and 1.4.

### 4.1 The Structure of $\mathcal{C}\left(G, Y_{1}\right)$

This section is devoted to the proof of Theorem 1.3. In order to investigate the disc structure of $\mathcal{C}\left(G, Y_{1}\right)$ it is advantageous for us to work in $H=S p_{4}(q)$ (and so $\bar{H}=H / Z(H) \cong G)$. As before, we assume that $\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$ is a hyperbolic basis for $V$ with $\left(v_{2}, v_{1}\right)=\left(v_{4}, v_{3}\right)=1$, with $J$ and $J_{0}$ defined as above. We have $\bar{s} \in Y_{1}$ where $s=\left(\begin{array}{c|c}-I_{2} & \\ & \\ & I_{2}\end{array}\right)$. Put $X=s^{H}$. Then $Y_{1}=\{\bar{x} \mid x \in X\}$. For $x \in X$, set $N_{x}=N_{H}(\langle x, Z(H)\rangle)$. Evidently, for $\overline{x_{1}}, \overline{x_{2}} \in Y_{1}\left(\right.$ where $\left.x_{1}, x_{2} \in X\right) \overline{x_{1}}$ and $\overline{x_{2}}$ commute if and only if $x_{1} \in N_{x_{2}}$ (or equivalently $x_{2} \in N_{x_{1}}$ ). Now $N_{s}$ consists of $g \in H$ for which $s^{g}=s$ or $s^{g}=-s$. Letting $g=\left(\begin{array}{l|l}A & B \\ \hline C & D\end{array}\right)$ where $A, B, C$ and $D$ are $2 \times 2$ matrices over $G F(q)$, direct calculation reveals that either $B=C=0$ or
$A=D=0$. Also, as $g \in H$, we must have $A^{T} J_{0} A=D^{T} J_{0} D=J_{0}$ and therefore

$$
\begin{aligned}
N_{s} & =\left\{\left(\begin{array}{l|l}
A & \\
\hline & B
\end{array}\right), \left.\left(\begin{array}{l|} 
\\
\hline B
\end{array}\right) \right\rvert\, A, B \in S L_{2}(q)\right\} \\
& \cong\left(S L_{2}(q) \times S L_{2}(q)\right): 2 .
\end{aligned}
$$

Lemma 4.1. $\left|\Delta_{1}(\bar{s})\right|=\frac{1}{2} q\left(q^{2}-1\right)$.
Proof. Let $g=\left(\begin{array}{c|c}A & \\ & \\ & B\end{array}\right) \in X \cap N_{s}$. Then $A$ and $B$ must be involutions in $S L_{2}(q)$, hence either $I_{2}$ or $-I_{2}$. Thus the elements of $X$ of this form are precisely $\{s,-s\}$.
On the other hand, if $h=\left(\begin{array}{l|l} & A \\ \hline B & \end{array}\right) \in X \cap N_{s}$ then $A B=B A=I_{2}$ and so $B=A^{-1}$.
Since $X=s^{H}$ consists of all the involutions in $H \backslash Z(H)$, we have

$$
X \cap N_{s}=\left\{\left.\left(\begin{array}{l|l} 
& A \\
\hline A^{-1} &
\end{array}\right) \right\rvert\, A \in S L_{2}(q)\right\} \cup\left\{s, s^{-1}\right\} .
$$

Under the natural homomorphism to $G, \bar{x}=\overline{-x}$ for $x \in X$, and so $\left|\Delta_{1}(\bar{s})\right|=\frac{1}{2}\left|S L_{2}(q)\right|=\frac{1}{2} q\left(q^{2}-1\right)$.

Recall that a 2-space $\left\{\nu_{1}, \nu_{2}\right\}$ is called hyperbolic if $\left(\nu_{i}, \nu_{i}\right)=0$ and $\left(\nu_{2}, \nu_{1}\right)=1$. Put $E=\left\langle v_{3}, v_{4}\right\rangle$. Then $E^{\perp}=\left\langle v_{1}, v_{2}\right\rangle$ and we note that $C_{V}(s)=E$. Furthermore we have that $\operatorname{Stab}_{H}\left(\left\{E, E^{\perp}\right\}\right)=N_{s}$. Put

$$
\Sigma=\left\{\left\{F, F^{\perp}\right\} \mid F \text { is a hyperbolic 2-subspace of } V\right\} .
$$

Now let $\beta \in G F(q)$ and set $U_{\beta}=\langle(1,0,1,0),(0, \beta, 0,-\beta-1)\rangle$. Then $U_{\beta}$ is a hyperbolic 2-subspace of $V$ and so $\left\{U_{\beta}, U_{\beta}^{\perp}\right\} \in \Sigma$. The $N_{s}$-orbit of $\left\{U_{\beta}, U_{\beta}^{\perp}\right\}$ will be denoted by $\Sigma_{\beta}$.

Lemma 4.2. Let $F$ be a hyperbolic 2-subspace of $V$ with $F \neq E$ or $E^{\perp}$. Then $\left\{F, F^{\perp}\right\} \in \Sigma_{\beta}$ for some $\beta \in G F(q)$. Moreover, for $\beta \in G F(q), \Sigma_{\beta}=\Sigma_{-\beta-1}$.

Proof. Since $F \neq E$ or $E^{\perp}$, we may find $w_{1} \in F$ with $w_{1}=\left(\alpha_{1}, \beta_{1}, \gamma_{1}, \delta_{1}\right)$ and $\left\{\alpha_{1}, \beta_{1}\right\} \neq\{0\} \neq\left\{\gamma_{1}, \delta_{1}\right\}$. Now $N_{s}$ contains two $S L_{2}(q)$-subgroups for which $\left\langle v_{1}, v_{2}\right\rangle$ and $\left\langle v_{3}, v_{4}\right\rangle$ are natural $G F(q) S L_{2}(q)$-modules. Because $S L_{2}(q)$ acts transitively on
the non-zero vectors of such modules, we may suppose $w_{1}=(1,0,1,0)$. Now choose $w_{2} \in F$ such that $\left(w_{1}, w_{2}\right)=1$ (and so $\left.\left\langle w_{1}, w_{2}\right\rangle=F\right)$. Then if $w_{2}=(\alpha, \beta, \gamma, \delta)$ we must have $\beta+\delta=-1$ and so $w_{2}=(\alpha, \beta, \gamma,-\beta-1)$. The matrices in $N_{s}$ fixing $w_{1}$ are

Let $g=\left(\begin{array}{cc|cc}1 & 0 & & \\ a_{1} & 1 & & \\ & & 1 & 0 \\ & & a_{2} & 1\end{array}\right)$ where $a_{1}, a_{2} \in G F(q)$. Then $w_{1}^{g}=w_{1}$.
We single out the cases $\beta=0$ and $\beta=-1$ for special attention. If, say, $\beta=0$, then $w_{2}=(\alpha, 0, \gamma,-1)$. Hence $w_{2}-\alpha w_{1}=(0,0, \gamma-\alpha,-1)$ and $F=\left\langle w_{1}, w_{2}-\alpha w_{1}\right\rangle$. Since $(0,0, \gamma-\alpha,-1) g=\left(0,0,(\gamma-\alpha)-a_{2},-1\right)$ and choosing $a_{2}=-\gamma+\alpha$, we obtain $F g=U_{0}$. For $\beta=-1$ a similar argument works (using $w_{2}-\gamma w_{1}$ instead of $\left.w_{2}-\alpha w_{1}\right)$. So we may assume that $\beta \neq 0,-1$. From

$$
w_{2} g=(\alpha, \beta, \gamma,-\beta-1)=\left(\alpha+\beta a_{1}, \beta, \gamma+(-\beta-1) a_{2},-\beta-1\right)
$$

by a suitable choice of $a_{1}$ and $a_{2}$, as $\beta \neq 0,-1$, we get $w_{2} g=(0, \beta, 0,-\beta-1)$, whence $F g=U_{\beta}$. Thus we have shown $\left\{F, F^{\perp}\right\} \in \Sigma_{\beta}$ for some $\beta \in G F(q)$. Finally, for $\beta \in G F(q), \Sigma_{\beta}=\Sigma_{-\beta-1}$ follows from

$$
(0, \beta, 0,-\beta-1)\left(\begin{array}{l|l}
I_{2} \\
I_{2} &
\end{array}\right)=(0,-\beta-1,0, \beta)
$$

Let $\phi: G F(q) \backslash\{-1\} \rightarrow G F(q)$ be defined by

$$
\phi(\lambda)=-\left(1+(\lambda+1)^{-2}\left(1-\lambda^{2}\right)\right)^{-1} \quad(\lambda \in G F(q)) .
$$

There is a possibility that this is not well-defined should $1+(\lambda+1)^{-2}\left(1-\lambda^{2}\right)=0$. This would then give $(\lambda+1)^{2}+\left(1-\lambda^{2}\right)=0$ from which we infer that $\lambda=-1$. So we conclude that $\phi$ is well-defined.

Lemma 4.3. $\phi$ is injective.

Proof. Suppose $\phi(\lambda)=\phi(\mu)$ for $\lambda, \mu \in G F(q) \backslash\{-1\}$ with $\lambda \neq \mu$. Hence

$$
\left(1+(\lambda+1)^{-2}\left(1-\lambda^{2}\right)\right)^{-1}=\left(1+(\mu+1)^{-2}\left(1-\mu^{2}\right)\right)^{-1} .
$$

Simplifying and using the fact that $q$ is odd gives

$$
\mu^{2}+\mu-\mu \lambda^{2}-\lambda^{2}-\lambda+\lambda \mu^{2}=0,
$$

and then

$$
(\mu+\lambda)(\mu-\lambda)+(\mu-\lambda)+\lambda \mu(\mu-\lambda)=0 .
$$

Hence $(\mu-\lambda)(\mu+\lambda+1+\lambda \mu)=0$. Since $\mu \neq \lambda$, we get $\mu+\lambda+1+\lambda \mu=0$ from which we deduce that either $\lambda=-1$ or $\mu=-1$, a contradiction. So the lemma holds.

Lemma 4.4. The diameter of $\mathcal{C}\left(G, Y_{1}\right)$ is 2.

Proof. Let $x \in X$ be such that $x \notin\{\bar{s}\} \cup \Delta_{1}(\bar{s})$. Now $\left\{C_{V}(x), C_{V}(x)^{\perp}\right\} \in \Sigma$ and $C_{V}(x) \neq E$ or $E^{\perp}$ (otherwise $x \in\{s,-s\}$ and then $\bar{x}=\bar{s}$ ). Hence $\left\{C_{V}(x), C_{V}(x)^{\perp}\right\} \in \Sigma_{\mu}$ for some $\mu \in G F(q)$ by Lemma 4.2. Let $y=\left(\begin{array}{c|c} & I_{2} \\ \hline I_{2} & \end{array}\right) \in X \cap N_{s}$. Then $\bar{y} \in \Delta_{1}(\bar{s})$. Our aim is to choose an $x_{\lambda} \in N_{y} \cap X$ (so $\left.\overline{x_{\lambda}} \in \Delta_{1}(\bar{y})\right)$ for which $\left\{C_{V}\left(x_{\lambda}\right), C_{V}\left(x_{\lambda}\right)^{\perp}\right\} \in \Sigma_{\mu}$. Since $\Sigma_{\mu}$ is an $N_{s}$-orbit, there exists $h \in N_{s}$ such that $\left\{C_{V}\left(x_{\lambda}\right), C_{V}\left(x_{\lambda}\right)^{\perp}\right\}^{h}=\left\{C_{V}(x), C_{V}(x)^{\perp}\right\}$. As a consequence either $x=x_{\lambda}^{h}$ or $-x_{\lambda}^{h}$ and therefore $\bar{x}=\bar{x}^{\bar{h}}$, whence $d(\bar{s}, \bar{x}) \leq 2$.
We first look at the case when $\mu=-2^{-1}$. Then $\mu=-\mu-1$ and hence

$$
U_{-2^{-1}}=\langle(1,0,1,0),(0,1,0,1)\rangle .
$$

Observing that $U_{-2^{-1}}=C_{V}(y)$, we see that for $\mu=-2^{-1}, \bar{x} \in \Delta_{1}(y)$, which we are not concerned with here. So we may assume $\mu \neq-2^{-1}$.
Let $x_{\lambda}=\left(\begin{array}{c|c}\lambda I_{2} & -B \\ \hline B & -\lambda I_{2}\end{array}\right)$ where $\lambda \in G F(q)^{*}$ be such that $B$ has zero trace and determinant $1-\lambda^{2}$. So $x_{\lambda} \in X \cap N_{y}$. We now move onto the case when $\mu=0$ (or
equivalently $\mu=-1$ ). Here we take $\lambda=1$ and $B=\left(\begin{array}{ll}2 & -2 \\ 2 & -2\end{array}\right)$, noting that $B$ satisfies the conditions to ensure that $\overline{x_{1}} \in \Delta_{1}(\bar{y})$. Let $v=(\alpha, \beta, \gamma, \delta) \in V$. Then $v \in C_{V}\left(x_{1}\right)$ precisely when

$$
\begin{aligned}
2 \gamma+2 \delta=0 ; & -2 \gamma-2 \delta=0 \\
-2 \alpha-2 \beta-\gamma=\gamma ; & 2 \alpha+2 \beta-\delta=\delta ;
\end{aligned}
$$

thus the only conditions we get are $\gamma=-\beta-\alpha$ and $\alpha+\beta=\delta$. Thus

$$
\begin{aligned}
C_{V}\left(x_{1}\right) & =\{(\alpha, \beta,-\alpha-\beta, \alpha+\beta)\} \\
& =\langle(1,0,-1,1),(0,1,-1,1)\rangle .
\end{aligned}
$$

It is straightforward to check that $\left\{C_{V}\left(x_{1}\right), C_{V}\left(x_{1}\right)^{\perp}\right\} \in \Sigma_{0}$. Therefore we may also assume that $\mu \neq 0,-1$. Choosing $B=\left(\begin{array}{cc}\lambda & \lambda^{-1} \\ -\lambda & -\lambda\end{array}\right)$ we see that the requisite conditions are satisfied. Take $v=(\alpha, \beta, \gamma, \delta) \in V$ and calculating $v^{x_{\lambda}}$ gives the relations

$$
\begin{aligned}
(\lambda-1) \alpha+\gamma \lambda-\delta \lambda=0 ; & (\lambda-1) \beta+\gamma \lambda^{-1}-\delta \lambda=0 \\
-\lambda \alpha+\lambda \beta-(\lambda+1) \gamma=0 ; & -\lambda^{-1} \alpha+\lambda \beta-(\lambda+1) \delta=0
\end{aligned}
$$

which, after rearranging gives

$$
\begin{array}{ll}
\alpha=\lambda(\lambda-1)^{-1}(\delta-\gamma) ; & \beta=\lambda(\lambda-1)^{-1} \delta-\lambda^{-1}(\lambda-1)^{-1} \gamma ; \\
\gamma=\lambda(\lambda+1)^{-1}(\beta-\alpha) ; & \delta=\lambda(\lambda+1)^{-1} \alpha-\lambda^{-1}(\lambda+1)^{-1} \alpha ;
\end{array}
$$

and note that the relations for $\gamma$ and $\delta$ are satisfied after substitution for $\alpha$ and $\beta$. Hence

$$
\begin{align*}
C_{V}\left(x_{\lambda}\right) & =\left\{\left(\alpha, \beta, \lambda(\lambda+1)^{-1}(\beta-\alpha), \lambda(\lambda+1)^{-1} \beta-\lambda^{-1}(\lambda+1)^{-1} \alpha\right)\right\} \\
& =\left\langle\left(1,0,-\lambda(\lambda+1)^{-1},-\lambda^{-1}(\lambda+1)^{-1}\right),\left(0,1, \lambda(\lambda+1)^{-1}, \lambda(\lambda+1)^{-1}\right)\right\rangle . \tag{4.1}
\end{align*}
$$

We want to determine which $N_{s}$-orbit, $\Sigma_{\beta}$, that $C_{V}\left(x_{\lambda}\right)$ lies in. Our representative, $U_{\beta}$, for $\Sigma_{\beta}$ has $w_{1}=(1,0,1,0)$ as one component of the hyperbolic pair, so we need
an element of $N_{s}$ to send the first generator in (4.1) to $w_{1}$. We need to find conditions on $C, D \in S L_{2}(q)$ such that

$$
\left(1,0,-\lambda(\lambda+1)^{-1},-\lambda^{-1}(\lambda+1)^{-1}\right)\left(\begin{array}{l|l}
C & \\
\hline & D
\end{array}\right)=(1,0,1,0)
$$

and so without loss of generality we can take $C=I_{2}$. This reduces to solving

$$
\left(-\lambda(\lambda+1)^{-1},-\lambda^{-1}(\lambda+1)^{-1}\right)\left(\begin{array}{ll}
d_{1} & d_{2} \\
d_{3} & d_{4}
\end{array}\right)=(1,0)
$$

and after multiplying out, we get that $d_{3}=-\left(d_{1}+1\right) \lambda^{2}-\lambda$ and $d_{4}=-d_{2} \lambda^{2}$. Since $D$ has determinant 1 , we find that $d_{2}=\lambda^{-1}(\lambda+1)^{-1}$ and so $d_{4}=-\lambda(\lambda+1)^{-1}$. Without loss of generality, by taking $d_{1}=1$ we have that

$$
D=\left(\begin{array}{cc}
1 & \lambda^{-1}(\lambda+1)^{-1} \\
-2 \lambda^{2}-\lambda & -\lambda(\lambda+1)^{-1}
\end{array}\right)
$$

and a quick check shows that the first generator in (4.1) is mapped to $w_{1}$. Using the same matrix, by multiplying on the right of the second generator in (4.1), we get

$$
\left(0,1, \lambda(\lambda+1)^{-1}, \lambda(\lambda+1)^{-1}\right)\left(\begin{array}{c|c}
I_{2} & \\
\hline & D
\end{array}\right)=\left(*, 1, *,(\lambda+1)^{-2}\left(1-\lambda^{2}\right)\right)=u^{\prime}
$$

and $\left\langle w_{1}, u^{\prime}\right\rangle$ is a hyperbolic 2 -subspace conjugate to some $U_{\beta}$. Recall that for a fixed $\beta \in G F(q), N_{s}$ is transitive on $\{(\alpha, \beta, \gamma,-\beta-1) \mid \alpha, \gamma \in G F(q)\}$. Hence, we need only find the hyperbolic pair representing such a conjugate of $U_{\beta}$, to determine $\beta$. This is found by requiring that for some $\beta \in G F(q)^{*},\left(\beta u^{\prime}, w_{1}\right)=1$, that is

$$
\beta \cdot 1=-1-\beta\left((\lambda+1)^{-2}\left(1-\lambda^{2}\right)\right) .
$$

By expanding, we get that $\beta=-\left(1+(\lambda+1)^{-2}\left(1-\lambda^{2}\right)\right)^{-1}$ and so $C_{V}\left(x_{\lambda}\right) \in \Sigma_{\beta}$. By Lemma 4.3, $\phi: \lambda \mapsto-\left(1+(\lambda+1)^{-2}\left(1-\lambda^{2}\right)\right)^{-1}$ is an injective map from $G F(q) \backslash\{-1\}$ into $G F(q)$. Since $\mu \neq-2^{-1}, \mu \neq-\mu-1$ and therefore there exists $\lambda \in G F(q) \backslash\{-1\}$ such that $\phi(\lambda)=\mu$ or $-\mu-1$. Bearing in mind that $U_{\mu}=U_{-\mu-1}$ by Lemma 4.2, we conclude that $\left\{C_{V}\left(x_{\lambda}\right), C_{V}\left(x_{\lambda}\right)^{\perp}\right\} \in \Sigma_{\mu}$. Consequently we have proved that the diameter of $\mathcal{C}\left(G, Y_{1}\right)$ is 2 .

From $|G|=\frac{q^{4}}{2}\left(q^{2}-1\right)\left(q^{4}-1\right)$ and $\left|C_{G}(\bar{s})\right|=q^{2}\left(q^{2}-1\right)^{2}$ we see that $\left|Y_{1}\right|=\frac{q^{2}}{2}\left(q^{2}+1\right)$. Using Lemmas 4.1 and 4.4 then gives

$$
\left|\Delta_{2}(\bar{s})\right|=\frac{1}{2}\left(q^{4}-q^{3}+q^{2}+q-2\right),
$$

which, combined, complete the proof of Theorem 1.3.

### 4.2 The Structure of $\mathcal{C}\left(G, Y_{2}\right)$

In this section we present a proof of Theorem 1.4. The uncovering of the disc structures of $\mathcal{C}\left(G, Y_{2}\right)$ will be a long haul. As discussed in Chapter 1, it will be advantageous for us to use the well known isomorphism that $P \operatorname{Sp}_{4}(q) \cong O_{5}(q)$ (see Corollary 12.32 of [38]). So we take $G=O_{5}(q)$ and from now on $V$ will denote the 5-dimensional orthogonal $G F(q)$-module for $G$. Thus the elements of $G$ are $5 \times 5$ orthogonal matrices with respect to the orthogonal form $(\cdot, \cdot)$ which have spinor norm a square in $G F(q)$. This section utilises an explicit representation of $G$ and its subgroups, in particular a subgroup isomorphic to $O_{3}(q)$. For ease of calculation, we assume that the Gram matrix with respect to $(\cdot, \cdot)$ is

$$
J=\left(\begin{array}{rr|rrr}
0 & 1 & & & \\
1 & 0 & & & \\
\hline & & 0 & 0 & 1 \\
& & 0 & -2 & 0 \\
& & 1 & 0 & 0
\end{array}\right) .
$$

Let

$$
t=\left(\begin{array}{l|rrr}
I_{2} & & & \\
\hline & 0 & 0 & 1 \\
0 & -1 & 0 \\
1 & 0 & 0
\end{array}\right)
$$

Then $t \in G$ and $Y_{2}=t^{G}$. Let $\delta= \pm 1$ where $q \equiv \delta(\bmod 4)$.

Lemma 4.5. (i) $\operatorname{dim}\left(C_{V}(t)\right)=3$.
(ii) $C_{V}(t)^{\perp}=[V, t]$ is a 2-subspace of $V$ of $\delta$-type.
(iii) $V=C_{V}(t) \perp C_{V}(t)^{\perp}$.

Proof. (i) is a consequence of direct calculation. Now

$$
C_{V}(t)^{\perp}=\left\{u \in V \mid(u, v)=0, \text { for all } v \in C_{V}(t)\right\}
$$

Then for any $v \in C_{V}(t)$, we have

$$
\left(v,-u+u^{t}\right)=-(v, u)+\left(v, u^{t}\right)=-(v, u)+\left(v^{t}, u\right)=-(v, u)+(v, u)=0
$$

and so $[V, t] \subseteq C_{V}(t)^{\perp}$. However, by dimensions, we must have $[V, t]=C_{V}(t)^{\perp}$. The restriction of the orthogonal form to $[V, t]$ has Gram matrix

$$
\left(\begin{array}{cc}
-2 & 0 \\
0 & -2
\end{array}\right)
$$

Hence $[V, t]$ contains an isotropic vector (and hence of +-type) if and only if -4 is a square in $G F(q)$. This occurs precisely when $q \equiv 1(\bmod 4)$. While $[V, t]$ is of --type if and only if $q \equiv-1(\bmod 4)$, proving (ii).

Since $\operatorname{dim} C_{V}(t)=3$ and $C_{V}(t) \cap[V, t]=0, \operatorname{dim}[V, t]=2$ and $V=C_{V}(t) \oplus[V, t]$. By (ii), $[V, t]=C_{V}(t)^{\perp}$ and so (iii) follows.

Put $L_{t}=C_{G}(t) \cap C_{G}([V, t])$.
Lemma 4.6. (i) Let $x \in Y_{2}$. Then $t=x$ if and only if $C_{V}(t)=C_{V}(x)$.
(ii) $C_{G}(t)=\operatorname{Stab}_{G}\left(C_{V}(t)\right) \cong\left(L_{2}(q) \times C_{\frac{q-\delta}{2}}\right) .2^{2}$.
(iii) $L_{t}$ acts faithfully on $C_{V}(t)$ and $L_{t} \cong L_{2}(q)$.

Proof. Suppose $C_{V}(x)=C_{V}(t)$. Then, using Lemma 4.5 (ii), $[V, x]=C_{V}(x)^{\perp}=C_{V}(t)^{\perp}=[V, t]$. Hence by Lemma 4.5(iii), $t x$ acts trivially on $V$ and thus $t x=1$. Therefore $t=x$ and (i) holds.
Plainly $C_{G}(t) \leq \operatorname{Stab}_{G}\left(C_{V}(t)\right)$, and if $g \in \operatorname{Stab}_{G}\left(C_{V}(t)\right)$, then $C_{V}(t)=C_{V}(t)^{g}=C_{V}\left(t^{g}\right)$. Since $t^{g} \in Y_{2}, t=t^{g}$ by part (i). So $g \in C_{G}(t)$ and thus $C_{G}(t)=\operatorname{Stab}_{G}\left(C_{V}(t)\right)$. That $\operatorname{Stab}_{G}\left(C_{V}(t)\right) \cong\left(L_{2}(q) \times C_{\frac{q-\delta}{2}}\right) .2^{2}$ can be read off from Proposition 4.1.6 of [32], giving (ii).

For any $g \in C_{G}(t)$, we have $[V, t]^{g}=C_{V}(t)^{\perp g}=C_{V}\left(t^{g}\right)^{\perp}=C_{V}(t)^{\perp}=[V, t]$ and so $C_{G}(t) \leq \operatorname{Stab}_{G}[V, t]$. If any element in $L_{t}$ acts trivially on $C_{V}(t)$, then it would act trivially on $V$ and thus be the identity. Hence $L_{t}$ acts faithfully on $C_{V}(t)$. Let $v \in C_{V}(t)$ and by Lemma 4.5(iii), we have $[V, t] \leq\langle v\rangle^{\perp}$. Hence $\langle v\rangle^{\perp}=[V, t] \oplus W$ where $W \leq C_{V}(t)$. But since $\operatorname{dim}\left(\langle v\rangle^{\perp}\right)=4$, we have $\operatorname{dim}(W)=2$ and so $C_{V}(t) \npreceq\langle v\rangle^{\perp}$. Therefore for all $u \in C_{V}(t),(v, u)=0$ if and only if $v=0$ and thus $(\cdot, \cdot)$ is non-degenerate on restriction to $C_{V}(t)$. Hence we have an embedding of $L_{t}$ into $G O\left(C_{V}(t)\right) \cong G O_{3}(q)$ since, by definition, $L_{t}$ fixes $[V, t]$ pointwise. Since $L_{t} \leq G$ and acts with determinant 1 on $[V, t]$, then it must act with determinant 1 on $C_{V}(t)$. In addition, as $L_{t}$ fixes [ $V, t$ ] pointwise, when the elements of $L_{t}$ are decomposed as products of refections, the vectors reflected will lie in $C_{V}(t)$. Since the spinor norm of the elements of $L_{t}$ are a square in $G F(q)$ and the vectors reflected lie in $C_{V}(t)$, then the spinor norm doesn't change on restriction to $C_{V}(t)$. Hence, $L_{t} \cong O_{3}(q) \cong L_{2}(q)$ proving (iii).

Let $\mathcal{U}_{i}$ denote the set of $i$-dimensional subspaces of $C_{V}(t), i=1,2$. In proving Theorem 1.4, our divide and conquer strategy is based on the following observation.

Lemma 4.7. $Y_{2} \subseteq \bigcup_{U \in \mathcal{U}_{1} \cup \mathcal{U}_{2}} C_{G}(U)$.
Proof. Let $x \in Y_{2} \backslash\{t\}$ and set $U=C_{V}(t) \cap C_{V}(x)$. By Lemmas 4.5(i) and 4.6(i), $U \in \mathcal{U}_{1} \cup \mathcal{U}_{2}$. Since $t, x \in C_{G}(U)$, we have Lemma 4.7.

The three cases we must chase down are presaged by our next result.

Lemma 4.8. (i) Let $U_{0}$ be an isotropic 1-subspace of $C_{V}(t)$. Then $C_{G}\left(U_{0}\right) \cong q^{3}: L_{2}(q)$.
(ii) Let $U_{\varepsilon}$ be a 1-subspace of $C_{V}(t)$, such that $U_{\varepsilon}^{\perp} \cap C_{V}(t)$ is a 2-space of $\varepsilon$-type $(\varepsilon= \pm 1)$. Then

$$
C_{G}\left(U_{\varepsilon}\right) \cong\left\{\begin{array}{lr}
S L_{2}(q) \circ S L_{2}(q) & \delta=\varepsilon \\
L_{2}\left(q^{2}\right) & \delta=-\varepsilon
\end{array}\right.
$$

Proof. Let $U_{0}$ be an isotropic 1-subspace of $C_{V}(t)$. From Proposition 4.1.20 of [32], we know that $\operatorname{Stab}_{G}\left(U_{0}\right) \cong A_{0}:\left(A_{1} \times A_{2}\right)\langle r\rangle$ where $A_{1}$ acts as scalars on $U_{0}, r$ a
reflection of $U_{0}$ and $A_{0} \cong q^{3}, A_{2} \cong L_{2}(q)$ fixing $U_{0}$ pointwise. Hence $C_{G}\left(U_{0}\right) \cong q^{3}: L_{2}(q)$, so proving (i). If $\delta=1$, then $[V, t]$ is a 2 -subspace of $V$ of + -type, and hence $U_{+}^{\perp}=\left(U_{+}^{\perp} \cap C_{V}(t)\right) \perp[V, t]$ is a 4-subspace of + -type. Similarly, $U_{\perp}^{\perp}=\left(U_{\perp}^{\perp} \cap C_{V}(t)\right) \perp[V, t]$ is a 4 -space of --type. If $\delta=-1$, then $[V, t]$ is a 2-subspace of $V$ of --type, and the results when $\delta=1$ interchange. Let $W_{+}$and $W_{-}$be 4 -subspaces of $V$ of + - and --type respectively, such that $W_{+}^{\perp}$ and $W_{-}^{\perp}$ are 1-subspaces of $C_{V}(t)$, observing that $\operatorname{Stab}_{G}\left(W_{ \pm}\right)=\operatorname{Stab}_{G}\left(W_{ \pm}^{\perp}\right)$. From Proposition 4.1.6 of [32], we have

$$
\operatorname{Stab}_{G}\left(W_{+}\right) \cong A_{+}\left\langle s_{+}\right\rangle \quad \text { and } \quad \operatorname{Stab}_{G}\left(W_{-}\right) \cong A_{-}\left\langle s_{-}\right\rangle
$$

where $A_{+} \cong S L_{2}(q) \circ S L_{2}(q)$ fixes $W_{+}^{\perp}$ pointwise, $A_{-} \cong L_{2}\left(q^{2}\right)$ fixes $W_{-}^{\perp}$ pointwise and $s_{+}, s_{-}$are reflections of $W_{+}^{\perp}$ and $W_{-}^{\perp}$ respectively. This proves (ii) and hence the lemma.

Lemma 4.9. (i) Let $U_{0}$ be a 2-subspace of $C_{V}(t)$ such that $U_{0}^{\perp} \cap C_{V}(t)$ is an isotropic 1-space. Then $C_{G}\left(U_{0}\right) \cong q^{2}: C_{\frac{q-\delta}{2}}$.
(ii) Let $U_{\varepsilon}$ be a 2-subspace of $C_{V}(t)$ of $\varepsilon$-type $(\varepsilon= \pm 1)$. Then $C_{G}\left(U_{\varepsilon}\right) \cong L_{2}(q)$.

Proof. (ii) is proved in a similar vein to Lemma 4.8. We can write $C_{V}(t)=V_{0} \perp\left(V_{1} \oplus V_{2}\right)$ where $V_{0}$ is a non-isotropic 1-space and $V_{1}$ and $V_{2}$ are isotropic 1-spaces such that $V_{1} \oplus V_{2}$ is a hyperbolic plane. Without loss of generality, we set $U_{0}=V_{0} \perp V_{1}$. So $C_{G}\left(U_{0}\right)=C_{G}\left(V_{0} \perp V_{1}\right)=C_{G}\left(V_{0}\right) \cap C_{G}\left(V_{1}\right)$. From Proposition 4.1.20 of [32], $C_{G}\left(V_{0}\right)=\Omega\left(\left(V_{1} \oplus V_{2}\right) \perp[V, t]\right)$ and $C_{G}\left(V_{1}\right)=R_{1}: \Omega\left(V_{0} \perp[V, t]\right)$ where $R_{1}$ is a $p$-group centralising the spaces $V_{1}$, $V_{1}^{\perp} / V_{1}$ and $V / V_{1}^{\perp}$. Using Proposition 4.1.6 of [32], we now have $C_{G}\left(V_{1}\right) \cap C_{G}\left(V_{0}\right) \cong R_{2}: O_{2}^{ \pm}(q)$ where $R_{2}$ is an elementary abelian group of order $q^{2}$. This proves (i) and hence Lemma 4.9.

Define the following subsets of $\mathcal{U}_{i}, i=1,2$.

$$
\begin{aligned}
& \mathcal{U}_{1}^{+}=\left\{U \in \mathcal{U}_{1} \mid C_{G}(U) \cong S L_{2}(q) \circ S L_{2}(q)\right\} \\
& \mathcal{U}_{1}^{-}=\left\{U \in \mathcal{U}_{1} \mid C_{G}(U) \cong L_{2}\left(q^{2}\right)\right\} \\
& \mathcal{U}_{1}^{0}=\left\{U \in \mathcal{U}_{1} \mid C_{G}(U) \cong q^{3}: L_{2}(q)\right\} \\
& \mathcal{U}_{2}^{+}=\left\{U \in \mathcal{U}_{2} \mid U \text { is of +-type }\right\} \\
& \mathcal{U}_{2}^{-}=\left\{U \in \mathcal{U}_{2} \mid U \text { is of --type }\right\} \\
& \mathcal{U}_{2}^{0}=\left\{U \in \mathcal{U}_{2} \mid C_{G}(U) \cong q^{2}: C_{\frac{q-\delta}{2}}\right\} .
\end{aligned}
$$

In the notation of Lemma 4.8, $\mathcal{U}_{1}^{+}$is the case $\delta=\varepsilon$ while $\mathcal{U}_{1}^{-}$is when $\delta=-\varepsilon$. Note by Lemmas 4.8 and 4.9 that $\mathcal{U}_{i}=\mathcal{U}_{i}^{0} \cup \mathcal{U}_{i}^{+} \cup \mathcal{U}_{i}^{-}, i=1,2$. We now study $C_{G}(U) \cap Y_{2}$ for $U \in \mathcal{U}_{1}$. By Lemma 4.8 there are three possibilities for the structure of $C_{G}(U)$. First we look at the case $U \in \mathcal{U}_{1}^{-}$, and set $G^{-}=C_{G}(U)$. Then $G^{-} \cong L_{2}\left(q^{2}\right)$ by definition of $\mathcal{U}_{1}^{-}$. Define $\Delta_{i}^{-}(t)=\left\{x \in G^{-} \cap Y_{2} \mid d^{-}(t, x)=i\right\}$ where $i \in \mathbb{N}$ and $d^{-}$is the distance metric on the commuting graph $\mathcal{C}\left(G^{-}, G^{-} \cap Y_{2}\right)$.

Theorem 4.10. If $q \neq 3$ then $\mathcal{C}\left(G^{-}, G^{-} \cap Y_{2}\right)$ is connected of diameter 3 with

$$
\begin{aligned}
\left|\Delta_{1}^{-}(t)\right| & =\frac{1}{2}\left(q^{2}-1\right) \\
\left|\Delta_{2}^{-}(t)\right| & =\frac{1}{4}\left(q^{2}-1\right)\left(q^{2}-5\right) ; \text { and } \\
\left|\Delta_{3}^{-}(t)\right| & =\frac{1}{4}\left(q^{2}-1\right)\left(q^{2}+7\right)
\end{aligned}
$$

Proof. Since $q^{2} \equiv 1(\bmod 4)$ and $q \neq 3$ implies $q^{2}>13$, this follows from Theorem 2.10 .

We move on to analyze $G^{+}=C_{G}(U)$ where $U \in \mathcal{U}_{1}^{+}$. Hence, by definition of $\mathcal{U}_{1}^{+}$, $G^{+} \cong L_{1} \circ L_{2}$ where $L_{1} \cong S L_{2}(q) \cong L_{2}$ (with the central product identifying $Z\left(L_{1}\right)$ and $\left.Z\left(L_{2}\right)\right)$. Set $Y^{+}=G^{+} \cap Y_{2}$. We begin by describing $Y^{+}$.

Lemma 4.11. $Y^{+}=\left\{x_{1} x_{2} \mid x_{i} \in L_{i}\right.$ and $x_{i}$ has order 4, $\left.i=1,2\right\}$.

Proof. Apart from the central involution $z$ of $G^{+}$, all other involutions of $G^{+}$are of the form $g_{1} g_{2}$ where $g_{i} \in L_{i}(i=1,2)$ has order 4 . Since all involutions in $L_{i} / Z\left(G^{+}\right)$
are conjugate, it quickly follows that $\left\{g_{1} g_{2} \mid g_{i} \in L_{i}\right.$ and $g_{i}$ has order $\left.4, i=1,2\right\}$ is a $G^{+}$-conjugacy class. Now $z$ acts as -1 on $U^{\perp}$ and thus $\operatorname{dim} C_{V}(z)=1$. Therefore $t \neq z$ whence, as $t \in G^{+}$, the lemma holds.

Let $d^{+}$denote the distance metric on the commuting graph $\mathcal{C}\left(G^{+}, Y^{+}\right)$and, for $i \in \mathbb{N}, \Delta_{i}^{+}(t)=\left\{x \in Y^{+} \mid d^{+}(t, x)=i\right\}$.

Theorem 4.12. Assume that $q \notin\{3,5,9,13\}$. Then $\mathcal{C}\left(G^{+}, Y^{+}\right)$is connected of diameter 3 with

$$
\begin{aligned}
\left|\Delta_{1}^{+}(t)\right| & =\frac{1}{2}(q-\delta)^{2}+1 \\
\left|\Delta_{2}^{+}(t)\right| & =\frac{1}{8}(q-\delta)^{3}(q-4-\delta)+(q-\delta)(q-2-\delta) ; \text { and } \\
\left|\Delta_{3}^{+}(t)\right| & =\frac{3}{8} q^{4}+\frac{1}{2}(1+3 \delta) q^{3}-\frac{1}{4}(7+6 \delta) q^{2}+\frac{7}{2}(1+\delta) q-\frac{1}{8}(29+20 \delta)
\end{aligned}
$$

Proof. Let $\overline{G^{+}}=G^{+} / Z\left(G^{+}\right)\left(=\overline{L_{1}} \times \overline{L_{2}}\right)$. Note that for $x_{1} x_{2} \in Y^{+}, x_{1}^{-1} x_{2}=x_{1} x_{2}^{-1}$ and $x_{1} x_{2}=x_{1}^{-1} x_{2}^{-1}$ and so the inverse image of $\overline{x_{1} x_{2}}$ contains two elements of $Y^{+}$. Let $d^{(i)}$ denote the distance metric on the commuting involution graph of $\overline{L_{i}}$ and $\Delta_{j}^{(i)}\left(\overline{x_{i}}\right)$ the $j^{\text {th }}$ disc of $\overline{x_{i}}$ in the commuting involution graph of $\overline{L_{i}}$. By Lemma 4.11, $t=t_{1} t_{2}$ where, for $i=1,2, t_{i} \in L_{i}$ has order 4 . Let $x=x_{1} x_{2} \in Y^{+}$with $x \neq t$. Then $t x=x t$ if and only if $t x$ has order 2 . So, bearing in mind that $Y^{+} \cup\{z\}$ (where $\langle z\rangle=Z\left(G^{+}\right)$) are all the involutions of $G^{+}$, we have that $t x=x t$ if and only if one of the following holds:- $x_{1}=t_{1}, x_{2}=t_{2}^{-1} ; x_{1}=t_{1}^{-1}, x_{2}=t_{2} ; \overline{x_{1}} \in \Delta_{1}^{(1)}\left(\overline{t_{1}}\right)$ and $\overline{x_{2}} \in \Delta_{1}^{(2)}\left(\overline{t_{2}}\right)$. Thus

$$
\begin{equation*}
\Delta_{1}^{+}(t)=\left\{x_{1} x_{2} \mid \overline{x_{i}} \in \Delta_{1}^{(i)}\left(\overline{t_{i}}\right), i=1,2\right\} \cup\left\{t_{1} t_{2}^{-1}\right\} \tag{4.2}
\end{equation*}
$$

Hence, using Theorem 2.10,

$$
\begin{equation*}
\left|\Delta_{1}^{+}(t)\right|=2\left(\frac{1}{2}(q-\delta)\right)^{2}+1=\frac{1}{2}(q-\delta)^{2}+1 \tag{4.3}
\end{equation*}
$$

Next we examine $\Delta_{2}^{+}(t)$. Let $x \in Y^{+}$. Assume that $x=x_{1} t_{2}$ or $x_{1} t_{2}^{-1}$ where $\overline{x_{1}} \in \Delta_{1}^{(1)}\left(\overline{t_{1}}\right)$. Then $x \in \Delta_{1}^{+}\left(t_{1} t_{2}^{-1}\right)$ (recall $t_{1} t_{2}^{-1}=t_{1}^{-1} t_{2}$ ) which implies, by (4.2), that $x \in \Delta_{2}^{+}(t)$. If $x=t_{1} x_{2}$ or $t_{1}^{-1} x_{2}$ where $\overline{x_{2}} \in \Delta_{1}^{(2)}\left(\overline{t_{2}}\right)$, we similarly get $x \in \Delta_{2}^{+}(t)$. Therefore

$$
\begin{equation*}
\left\{x_{1} x_{2} \mid \overline{x_{1}} \in \Delta_{1}^{(1)}\left(\overline{t_{1}}\right), \overline{x_{2}}=\overline{t_{2}}\right\} \cup\left\{x_{1} x_{2} \mid \overline{x_{2}} \in \Delta_{1}^{(2)}\left(\overline{t_{2}}\right), \overline{x_{1}}=\overline{t_{1}}\right\} \subseteq \Delta_{2}^{+}(t) \tag{4.4}
\end{equation*}
$$

Now suppose $x=x_{1} x_{2}$ where $\overline{x_{1}} \in \Delta_{2}^{(1)}\left(\overline{t_{1}}\right)$ and $\overline{x_{2}} \in \Delta_{1}^{(2)}\left(\overline{t_{2}}\right)$. So there exists $\overline{y_{1}} \in \overline{L_{1}}$ such that $\left(\overline{t_{1}}, \overline{y_{1}}, \overline{x_{1}}\right)$ is a path of length 2 in the commuting involution graph for $\overline{L_{1}}$. Then $\left(t=t_{1} t_{2}, y_{1} x_{2}^{-1}, x_{1} x_{2}=x\right)$ is a path of length 2 in $\mathcal{C}\left(G^{+}, Y^{+}\right)$. Thus, by (4.2), $x \in \Delta_{2}^{+}(t)$. If, on the other hand, $\overline{x_{1}} \in \Delta_{1}^{(1)}\left(\overline{t_{1}}\right)$ and $\overline{x_{2}} \in \Delta_{2}^{(2)}\left(\overline{t_{2}}\right)$ we obtain the same conclusion. Should we have $\overline{x_{1}} \in \Delta_{2}^{(1)}\left(\overline{t_{1}}\right)$ and $\overline{x_{2}} \in \Delta_{2}^{(2)}\left(\overline{t_{2}}\right)$, similar arguments also give $x \in \Delta_{2}^{+}(t)$. So

$$
\begin{array}{r}
\left\{x_{1} x_{2} \mid \overline{x_{1}} \in \Delta_{2}^{(1)}\left(\overline{t_{1}}\right), \overline{x_{2}} \in \Delta_{1}^{(2)}\left(\overline{t_{2}}\right)\right\} \cup\left\{x_{1} x_{2} \mid \overline{x_{1}} \in \Delta_{1}^{(1)}\left(\overline{t_{1}}\right), \overline{x_{2}} \in \Delta_{2}^{(2)}\left(\overline{t_{2}}\right)\right\} \\
\cup\left\{x_{1} x_{2} \mid \overline{x_{1}} \in \Delta_{2}^{(1)}\left(\overline{t_{1}}\right), \overline{x_{2}} \in \Delta_{2}^{(2)}\left(\overline{t_{2}}\right)\right\} \subseteq \Delta_{2}^{+}(t) . \tag{4.5}
\end{array}
$$

Since $x=x_{1} x_{2} \in \Delta_{2}^{+}(t)$ implies $d^{(i)}\left(\overline{t_{i}}, \overline{x_{i}}\right) \leq 2$ for $i=1,2, \Delta_{2}^{+}(t)$ is the union of the two sets in (4.4) and (4.5). Thus, employing Theorem 2.10,

$$
\begin{equation*}
\left|\Delta_{2}^{+}(t)\right|=\frac{1}{8}(q-\delta)^{3}(q-4-\delta)+(q-\delta)(q-2-\delta) \tag{4.6}
\end{equation*}
$$

Now, as $q \notin\{3,5,9,13\}$, by Theorem 2.10 the commuting involution graph for $\overline{L_{i}}$ is connected of diameter 3. Arguing as above we deduce that $\mathcal{C}\left(G^{+}, Y^{+}\right)$is also connected with diameter 3. Because $\left|Y^{+}\right|=2\left|{\overline{t_{1}}}_{\overline{L_{1}}}\right|\left|{\overline{t_{2}}}^{\overline{L_{2}}}\right|=\frac{1}{2} q^{2}(q+\delta)^{2}$, combining (4.3) and (4.6) we may determine $\left|\Delta_{3}^{+}(t)\right|$ to be as stated, so completing the proof of Theorem 4.12.

Finally we look at $C_{G}(U)$ where $U \in \mathcal{U}_{1}^{0}$. This will prove to be trickier than the other two cases. Put $G^{0}=C_{G}(U)$. So $G^{0} \cong q^{3}: L_{2}(q)$. We require an explicit description of $G^{0}$ which we now give. Let $Q=\{(\alpha, \beta, \gamma) \mid \alpha, \beta, \gamma \in G F(q)\}$ and

$$
L=\left\{\left.\left(\begin{array}{ccc}
a^{2} & 2 a b & b^{2} \\
a c & a d+b c & b d \\
c^{2} & 2 c d & d^{2}
\end{array}\right) \right\rvert\, \begin{array}{c}
a, b, c, d \in G F(q) \\
a d-b c=1
\end{array}\right\} .
$$

with $L$ acting on $Q$ by right multiplication. Then $Q \cong q^{3}$ and $L \cong L_{2}(q)$, with the latter isomorphism induced by the homomorphism $S L_{2}(q) \rightarrow L$ given by

$$
\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \mapsto\left(\begin{array}{ccc}
a^{2} & 2 a b & b^{2} \\
a c & a d+b c & b d \\
c^{2} & 2 c d & d^{2}
\end{array}\right) .
$$

Since $Q$ is the 3-dimensional $G F(q) L$-module (see the description on page 15 of [5]), $G^{0} \cong Q \rtimes L \cong A O_{3}(q)$. We will identify this semidirect product with $G^{0}$, writing $G^{0}=Q L$. Any $g \in G^{0}$ has a unique expression $g=g_{Q} g_{L}$ where $g_{Q} \in Q$ and $g_{L} \in L$ - in what follows we use such subscripts to describe this expression. Set $Y^{0}=G^{0} \cap Y_{2}$, let $d^{0}$ denote the distance metric and $\Delta_{i}^{0}(t)$ the $i^{\text {th }}$ disc of the commuting graph $\mathcal{C}\left(G^{0}, Y^{0}\right)$. In determining the discs of $\mathcal{C}\left(G^{0}, Y^{0}\right)$ we make use of the commuting involution graph of $L \cong L_{2}(q)$ (as given in Theorem 2.10). So we shall use $d^{L}$ to denote the distance metric on $\mathcal{C}\left(L, L \cap Y^{0}\right)$ and for $x \in L \cap Y^{0}$ and $i \in \mathbb{N}, \Delta_{i}^{L}(x)=\left\{y \in L \cap Y^{0} \mid d^{L}(x, y)=i\right\}$. The preimage in $S L_{2}(q)$ of an involution in $L_{2}(q)$ is an element of order 4 which squares to $-I_{2}$. These are all of the form $\left(\begin{array}{cc}a & b \\ c & -a\end{array}\right)$ and so the image in $L$ of such an element is given by

$$
\left(\begin{array}{cc}
a & b \\
c & -a
\end{array}\right) \mapsto\left(\begin{array}{ccc}
a^{2} & 2 a b & b^{2} \\
a c & b c-a^{2} & -a b \\
c^{2} & -2 a c & a^{2}
\end{array}\right) .
$$

Hence,

$$
L \cap Y^{0}=\left\{\left.\left(\begin{array}{ccc}
a^{2} & 2 a b & b^{2} \\
a c & b c-a^{2} & -a b \\
c^{2} & -2 a c & a^{2}
\end{array}\right) \right\rvert\, \begin{array}{c}
a, b, c \in G F(q) \\
a^{2}+b^{2}=-1
\end{array}\right\}
$$

and, as $G^{0}$ has one conjugacy class of involutions,

$$
Y^{0}=\left\{x_{Q} x_{L} \mid x_{L} \in L \cap Y^{0} \text { and } x_{L} \text { inverts } x_{Q}\right\}
$$

Without loss of generality, we take

$$
t=t_{L}=\left(\begin{array}{rrr}
0 & 0 & 1 \\
0 & -1 & 0 \\
1 & 0 & 0
\end{array}\right)
$$

and, up until Theorem 4.18, we will assume that $q \notin\{3,5,9,13\}$. Thus the diameter of $\mathcal{C}\left(L, L \cap Y^{0}\right)$ is 3 .

Lemma 4.13. (i) $Q t \cap Y^{0}=\{(\alpha, \beta,-\alpha) t \mid \alpha, \beta \in G F(q)\}$ and $\left|Q t \cap Y^{0}\right|=q^{2}$.
(ii) $Q t \cap \Delta_{1}^{0}(t)=\varnothing$.

Proof. Any $x_{Q} t \in Q t \cap Y^{0}$ has the property that $x_{Q}^{t}=x_{Q}^{-1}$. It is now a straightforward calculation to show both (i) and (ii).

Lemma 4.14. We have

$$
\Delta_{1}^{0}(t)=\left\{x \mid x_{Q}=(\alpha, 0, \alpha), x_{L}=\left(\begin{array}{ccc}
a^{2} & 2 a b & b^{2} \\
a b & b^{2}-a^{2} & -a b \\
b^{2} & -2 a b & a^{2}
\end{array}\right), a^{2}+b^{2}=-1\right\}
$$

and $\left|\Delta_{1}^{0}(t)\right|=\frac{1}{2} q(q-\delta)$.
Proof. Let $x, y \in Y^{0}$. If $[x, y]=1$ then clearly $\left[x_{L}, y_{L}\right]=1$. From [15] we have

$$
\Delta_{1}^{L}(t)=\left\{\left.\left(\begin{array}{ccc}
a^{2} & 2 a b & b^{2} \\
a b & b^{2}-a^{2} & -a b \\
b^{2} & -2 a b & a^{2}
\end{array}\right) \right\rvert\, a^{2}+b^{2}=-1\right\}
$$

If $x_{Q}=(\alpha, \beta, \gamma)$ and $x_{L} \in \Delta_{1}^{L}(t)$ then $[t, x]=1$ implies $\alpha=\gamma$ and $\beta=0$. Moreover, every $x=(\alpha, 0, \alpha) x_{L}$, where $x_{L} \in \Delta_{1}^{L}(t)$, is in $Y^{0}$. Hence, $\Delta_{1}^{0}(t)$ is as described above. By Theorem 2.10, for any involution $x_{L} \in L$ we have $\left|\Delta_{1}^{L}\left(x_{L}\right)\right|=\frac{1}{2}(q-\delta)$ and there are $q$ possible values that $\alpha$ can take for a fixed such $x_{L}$, proving the lemma.

Lemma 4.15. Let $x \in Y^{0}$ with $x_{L} \in \Delta_{1}^{L}(t)$. If $x \notin \Delta_{1}^{0}(t)$, then $x \in \Delta_{2}^{0}(t)$.
Proof. Suppose $x \in Y^{0}$ where $x_{Q}=(\alpha, \beta, \gamma)$ and

$$
x_{L}=\left(\begin{array}{ccc}
a^{2} & 2 a b & b^{2} \\
a b & b^{2}-a^{2} & -a b \\
b^{2} & -2 a b & a^{2}
\end{array}\right) .
$$

Then $x_{L}$ inverts $x_{Q}$ if and only if

$$
\begin{align*}
a^{2} \alpha+2 a b \beta+b^{2} \gamma & =-\alpha ; \\
a b \alpha+\left(b^{2}-a^{2}\right) \beta-a b \gamma & =-\beta ; \text { and }  \tag{4.7}\\
b^{2} \alpha-2 a b \beta+a^{2} \gamma & =-\gamma .
\end{align*}
$$

Suppose first that $\delta=-1$. Then, since -1 is not square in $G F(q)$, we must have $a, b \neq 0$. Rearranging the first equation gives $\alpha=2 a b^{-1} \beta+\gamma$ and (4.7) remains consistent. Note that when $\beta=0$, we have $\alpha=\gamma$ and so $x \in \Delta_{1}^{0}(t)$. So assume $\beta \neq 0$. Let $y \in \Delta_{1}^{0}(t)$ where $y_{Q}=\left(a b^{-1} \beta+\gamma, 0, a b^{-1} \beta+\gamma\right)$ and

$$
y_{L}=\left(\begin{array}{ccc}
b^{2} & -2 a b & a^{2} \\
-a b & a^{2}-b^{2} & a b \\
a^{2} & 2 a b & b^{2}
\end{array}\right) .
$$

It is a routine calculation to show that $[x, y]=1$, proving the lemma for $\delta=-1$. Now assume $\delta=1$. If $a, b \neq 0$ then the argument from the previous case still holds, so assume first that $a=0$, and hence $b$ is the unique element in $G F(q)$ that squares to -1 . Then (4.7) simplifies to $\alpha=\gamma$, and so $x_{Q}=(\alpha, \beta, \alpha)$. Let $z \in \Delta_{1}^{0}(t)$ where $z_{Q}=(\alpha, 0, \alpha)$ and

$$
z_{L}=\left(\begin{array}{rrr}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{array}\right)
$$

An easy calculation shows that $[x, z]=1$. Similarly, assuming $b=0$ then $a$ is the unique element of $G F(q)$ squaring to -1 and (4.7) simplifies to $\beta=0$. Then $x_{Q}=(\alpha, 0, \gamma)$ and if $w \in \Delta_{1}^{0}(t)$ where $w_{Q}=\left(2^{-1}(\alpha+\gamma), 0,2^{-1}(\alpha+\gamma)\right)$ and

$$
w_{L}=\left(\begin{array}{rrr}
0 & 0 & -1 \\
0 & -1 & 0 \\
-1 & 0 & 0
\end{array}\right)
$$

then an easy check shows that $[x, w]=1$, proving the lemma for $\delta=1$.

Lemma 4.16. We have $Q t \cap Y^{0} \subseteq\{t\} \cup \Delta_{2}^{0}(t) \cup \Delta_{3}^{0}(t)$. Moreover,

$$
\begin{aligned}
\left|Q t \cap \Delta_{2}^{0}(t)\right| & =\frac{1}{2}\left(q^{2}-(1+\delta) q+\delta\right) ; \text { and } \\
\left|Q t \cap \Delta_{3}^{0}(t)\right| & =\frac{1}{2}\left(q^{2}+(1+\delta) q-(2+\delta)\right)
\end{aligned}
$$

Proof. If $x \in Q t \cap Y^{0}$ and $x \neq t$ then $x_{Q}=(\alpha, \beta,-\alpha)$ and $x \notin \Delta_{1}^{0}(t)$ by Lemma
4.13. Let $y \in \Delta_{1}^{0}(t)$ where $y_{Q}=(\gamma, 0, \gamma)$ and

$$
y_{L}=\left(\begin{array}{ccc}
a^{2} & 2 a b & b^{2} \\
a b & b^{2}-a^{2} & -a b \\
b^{2} & -2 a b & a^{2}
\end{array}\right)
$$

with $a^{2}+b^{2}=-1$. Then $[x, y]=1$ if and only if $-a^{2} \alpha=a b \beta$ and $-b^{2} \beta=a b \alpha$. Assume first that $\delta=-1$. Since -1 is not square in $\operatorname{GF}(q)$, we have $a, b \neq 0$ and so $\alpha=-a^{-1} b \beta$. Hence if $y \in Q t$ is such that $y_{Q}=\left(-a^{-1} b \beta, \beta, a^{-1} b \beta\right)$, then $y \in \Delta_{2}^{0}(t)$. By looking at $\Delta_{1}^{L}(t)$, we see there are $q+1$ ordered pairs $(a, b)$ that satisfy $a^{2}+b^{2}=-1$. However, if $(a, b) \neq(c, d)$ where $a^{2}+b^{2}=c^{2}+d^{2}=-1$ and $a^{-1} b=c^{-1} d$, then an easy calculation shows that $(c, d)=(-a,-b)$. Hence there are $\frac{1}{2}(q+1)$ distinct values of $a^{-1} b$ satisfying the requisite conditions. If $\beta=0$, then $x=t$ and if $\beta \neq 0$, then there are $\frac{1}{2}\left(q^{2}-1\right)$ elements in $Q t \cap \Delta_{2}^{0}(t)$.

Assume now that $\delta=1$. If $a, b \neq 0$ then the arguments of the previous case still hold, with the exception that there are now $q-1$ ordered pairs $(a, b)$ that satisfy $a^{2}+b^{2}=-1$. However, as $a, b \neq 0$ we exclude the pairs $( \pm i, 0)$ and $(0, \pm i)$ where $i$ is the unique element of $G F(q)$ squaring to -1 . Hence there are $q-5$ ordered pairs $(a, b)$ satisfying $a^{2}+b^{2}=-1$, where $a, b \neq 0$ and thus $\frac{1}{2}(q-5)$ distinct values of $a^{-1} b$. Hence there are $\frac{1}{2}(q-5)(q-1)$ elements $z \in Q t \cap \Delta_{2}^{0}(t)$ such that $z_{Q}=\left(-a^{-1} b \beta, \beta, a^{-1} b \beta\right)$ where $\beta \neq 0$ (note that if $\beta=0$, then $z=t$ ). Suppose $a=0$, then $b \neq 0$ and so $\beta=0$. Hence $x_{Q}=(\alpha, 0,-\alpha)$ and all such $x$ lie in $\Delta_{2}^{0}(t)$ if $\alpha \neq 0$. Similarly, if $b=0$ then $a \neq 0$ and $x_{Q}=(0, \beta, 0)$ where $\beta \neq 0$ and all such $x$ lie in $\Delta_{2}^{0}(t)$. Therefore, $\left|Q t \cap \Delta_{2}^{0}(t)\right|=\frac{1}{2}(q-5)(q-1)+2(q-1)=\frac{1}{2}(q-1)^{2}$ as required.

Hence it suffices to show that these remaining involutions all lie in $\Delta_{3}^{0}(t)$. Let $w \in Q t$ be such that $w_{Q}=(\gamma, \varepsilon,-\gamma)$. Choose $s \in Y^{0}$ such that $s_{Q}=\left(a b \varepsilon-b^{2} \gamma, a b \gamma-a^{2} \varepsilon, b^{2} \gamma-a b \varepsilon\right)$ with $a b \gamma \neq a^{2} \varepsilon$ and

$$
s_{L}=\left(\begin{array}{ccc}
b^{2} & -2 a b & a^{2} \\
-a b & a^{2}-b^{2} & a b \\
a^{2} & 2 a b & b^{2}
\end{array}\right),
$$

with $a^{2}+b^{2}=-1$. It is an easy check to show that $s \in \Delta_{2}^{0}(t)$, and moreover $[w, s]=1$. This accounts for the remaining involutions in $Q t$, thus proving the lemma.

Lemma 4.17. Suppose $x \in Y^{0}$ with $x_{L} \in \Delta_{2}^{L}(t)$. Then $x \in \Delta_{2}^{0}(t)$.

Proof. It can be shown (see Remark 2.3 of [15], noting the result holds for any odd $q)$ that for fixed $a, b \in G F(q)$ such that $a^{2}+b^{2}=-1$,

$$
C_{L}\left(\left(\begin{array}{ccc}
a^{2} & 2 a b & b^{2} \\
a b & b^{2}-a^{2} & -a b \\
b^{2} & -2 a b & a^{2}
\end{array}\right)\right)=\left\{\left.\left(\begin{array}{ccc}
c^{2} & 2 c d & d^{2} \\
c e & d e-c^{2} & -c d \\
e^{2} & -2 c e & c^{2}
\end{array}\right) \right\rvert\, \begin{array}{c}
c^{2}+d e=-1 \\
b(e+d)=-2 a c
\end{array}\right\} .
$$

Let $y \in Y^{0}$ be such that $y_{Q}=(\alpha, \beta, \gamma)$ and

$$
y_{L}=\left(\begin{array}{ccc}
c^{2} & 2 c d & d^{2} \\
c e & d e-c^{2} & -c d \\
e^{2} & -2 c e & c^{2}
\end{array}\right) \in \Delta_{2}^{L}(t)
$$

So there exists $a, b \in G F(q)$ such that $a^{2}+b^{2}=-1$ and $b(e+d)=-2 a c$ with $d \neq e$. Since $y_{L}$ inverts $y_{Q}$, we have

$$
\begin{align*}
c^{2} \alpha+2 c d \beta+d^{2} \gamma & =-\alpha ; \\
c e \alpha+\left(d e-c^{2}\right) \beta-c d \gamma & =-\beta ; \text { and }  \tag{4.8}\\
e^{2} \alpha-2 c e \beta+c^{2} \gamma & =-\gamma .
\end{align*}
$$

Assume first that $\delta=-1$. Since -1 is not square in $G F(q)$, then $d, e \neq 0$ and any $a, b \in G F(q)$ such that $b(d+e)=-2 a c$ and $a^{2}+b^{2}=-1$ must also be non-zero. Moreover if $c=0$, then $d=-e^{-1}$ and $b\left(d-d^{-1}\right)=0$ implying that $d=-1$. But then $y_{L}=t \notin \Delta_{2}^{L}(t)$, so $c \neq 0$. The system (4.8) now simplifies to $\alpha=2 c e^{-1} \beta+d e^{-1} \gamma$. Let $x \in \Delta_{1}^{0}(t)$ be such that $x_{Q}=(\varepsilon, 0, \varepsilon)$ and

$$
x_{L}=\left(\begin{array}{ccc}
a^{2} & 2 a b & b^{2} \\
a b & b^{2}-a^{2} & -a b \\
b^{2} & -2 a b & a^{2}
\end{array}\right)
$$

where $\varepsilon=-a b c^{-1} e^{-1}\left(\gamma+(d-e)^{-1}\left(2 c+a^{-1} b e-a b^{-1} e-(a b)^{-1} e\right) \beta\right)$. Using the PolynomialAlgebra command in MAGMA [19] we verify that $[x, y]=1$ and so $y \in \Delta_{2}^{0}(t)$.

Assume now that $\delta=1$. Let $a, b \in G F(q)$ be such that $a^{2}+b^{2}=-1$ and $b(d+e)=-2 a c$. Suppose $c, d, e \neq 0$ and $d \neq-e$. Then $b(d+e)=-2 a c \neq 0$ and so $a, b \neq 0$. The argument for the case when $\delta=-1$ then holds. Suppose then $c, d, e \neq 0$ and $d=-e$. Then $b(d+e)=-2 a c=0$ and since $c \neq 0$ we must have $a=0$ and $b^{2}=-1$. The system (4.8) then becomes $\alpha=2 c e^{-1} \beta-\gamma$. If $x \in \Delta_{1}^{0}(t)$ is such that $x_{Q}=\left(-c^{-1} e^{-1} \beta, 0,-c^{-1} e^{-1} \beta\right)$ and

$$
x_{L}=\left(\begin{array}{rrr}
0 & 0 & -1 \\
0 & -1 & 0 \\
-1 & 0 & 0
\end{array}\right)
$$

then a routine check shows that $[x, y]=1$.
Now assume $c \neq 0$ and $d=0$. Since $y_{L} \in \Delta_{2}^{L}(t)$, we must have $e \neq 0$ and so $c^{2}=-1$. The system (4.8) becomes $\alpha=2 c e^{-1} \beta$ and using Magma [19] we deduce that if $x \in \Delta_{1}^{0}(t)$ where $x_{Q}=(\varepsilon, 0, \varepsilon)$,

$$
x_{L}=\left(\begin{array}{ccc}
a^{2} & 2 a b & b^{2} \\
a b & b^{2}-a^{2} & -a b \\
b^{2} & -2 a b & a^{2}
\end{array}\right)
$$

and $\varepsilon=\left(c e^{-1}\left(1-a^{2}\right)-a b\right) \beta-2^{-1} b^{2} \gamma$, then $[x, y]=1$. Similarly, if $c \neq 0$ and $e=0$, then $d \neq 0$ and $c^{2}=-1$. The system (4.8) becomes $\beta=2^{-1} c d \gamma$ and [19] will verify that if $x \in \Delta_{1}^{0}(t)$ where $x_{Q}=(\varepsilon, 0, \varepsilon)$,

$$
x_{L}=\left(\begin{array}{ccc}
a^{2} & 2 a b & b^{2} \\
a b & b^{2}-a^{2} & -a b \\
b^{2} & -2 a b & a^{2}
\end{array}\right)
$$

and $\varepsilon=2^{-1}\left(\gamma-b^{2} \alpha+a b c d \gamma-a^{2} \gamma\right)$, then $[x, y]=1$.
Finally, if $c=0$ then $d=-e^{-1}$ and so $a^{2}=-1$ and $b=0$ satisfies the required conditions. Note that if $d= \pm 1$ then $y_{L}=t$, so we may assume $d \neq \pm 1$. The system
(4.8) becomes $\alpha=d^{2} \gamma$, so if $x \in \Delta_{1}^{0}(t)$ where $x_{Q}=\left(2 d^{2} \gamma\left(1-d^{2}\right)^{-1}, 0,2 d^{2} \gamma\left(1-d^{2}\right)^{-1}\right)$ and

$$
x_{L}=\left(\begin{array}{rrr}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{array}\right)
$$

then a routine check again shows that $[x, y]=1$. Therefore, for all $y \in Y^{0}$ such that $y_{L} \in \Delta_{2}^{L}(t)$, there exists $x \in \Delta_{1}^{L}(t)$ such that $[x, y]=1$, so proving the lemma.

Theorem 4.18. If $q \notin\{3,5,9,13\}$, then $\mathcal{C}\left(G^{0}, Y^{0}\right)$ is connected of diameter 3, with disc sizes

$$
\begin{aligned}
& \left|\Delta_{1}^{0}(t)\right|=\frac{1}{2} q(q-\delta) \\
& \left|\Delta_{2}^{0}(t)\right|=\frac{1}{4}\left(q^{4}-(2 \delta+2) q^{3}+(1+2 \delta) q^{2}-2 q+2 \delta\right) ; \text { and } \\
& \left|\Delta_{3}^{0}(t)\right|=\frac{1}{4}\left(q^{4}+2(1+2 \delta) q^{3}-(3+2 \delta) q^{2}+2(1+\delta) q-2(2+\delta)\right)
\end{aligned}
$$

Proof. It is known that $\mathcal{C}\left(L, L \cap Y^{0}\right)$ has diameter 3. Hence, for any $h_{i} \in \Delta_{i}^{L}(t)$, there exists $h_{i \pm 1} \in \Delta_{i \pm 1}^{L}(t)$ that commutes with $h_{i}, i=1,2$. Therefore for any $x \in Y^{0}$ where $x_{L} \in \Delta_{i}^{L}(t)$, there exists $y \in Y^{0}$ with $y_{L} \in \Delta_{i \pm 1}^{L}(t)$ such that $[x, y]=1$. Since any $z \in Y^{0}$ where $z_{L} \in \Delta_{3}^{L}(t)$ must commute with some $w \in Y^{0}$ with $w_{L} \in \Delta_{2}^{L}(t)$ (which is contained in $\Delta_{2}^{0}(t)$ by Lemma 4.17), $z \in \Delta_{3}^{0}(t)$. This finally covers all possible involutions in $Y^{0}$ and so the diameter of $\mathcal{C}\left(G^{0}, Y^{0}\right)$ is 3 .

Now for each $x_{L} \in L \cap Y^{0},\left|Q x_{L} \cap Y^{0}\right|=q^{2}$ by Lemma 4.13, and therefore there are $\frac{1}{2} q^{2}(q-\delta)$ involutions $y \in Y^{0}$ such that $y_{L} \in \Delta_{1}^{L}(t)$. From Lemma 4.14, $\left|\Delta_{1}^{0}(t)\right|=\frac{1}{2} q(q-\delta)$. Therefore

$$
\left|\bigcup_{x_{L} \in \Delta_{1}^{L}(t)} Q x_{L} \cap \Delta_{2}^{0}(t)\right|=\frac{1}{2} q^{2}(q-\delta)-\frac{1}{2} q(q-\delta)=\frac{1}{2} q(q-1)(q-\delta) .
$$

There are $q^{2}\left|\Delta_{2}^{L}(t)\right|$ involutions $z \in Y^{0}$ such that $z_{L} \in \Delta_{2}^{L}(t)$, which is known to be $\frac{1}{4} q^{2}(q-\delta)(q-4-\delta)$ (see Theorem 2.10). Also, by Lemma 4.16,
$\left|Q t \cap \Delta_{2}^{0}(t)\right|=\frac{1}{2}\left(q^{2}-(1+\delta) q-\delta\right)$. Hence

$$
\begin{aligned}
\left|\Delta_{2}^{0}(t)\right| & =\left|Q t \cap \Delta_{2}^{0}(t)\right|+\left|\bigcup_{x_{L} \in \Delta_{1}^{L}(t)} Q x_{L} \cap \Delta_{2}^{0}(t)\right|+q^{2}\left|\Delta_{2}^{L}(t)\right| \\
& =\frac{1}{4}\left(q^{4}-(2 \delta+2) q^{3}+(1+2 \delta) q^{2}-2 q+2 \delta\right) .
\end{aligned}
$$

Finally, there are $\left|Y^{0}\right|=q^{2}\left|L \cap Y^{0}\right|=\frac{1}{2} q^{3}(q+\delta)$ involutions in $G^{0}$ and therefore

$$
\begin{aligned}
\left|\Delta_{3}^{0}(t)\right| & =\left|Y^{0}\right|-\left|\Delta_{2}^{0}(t)\right|-\left|\Delta_{1}^{0}(t)\right|-1 \\
& =\frac{1}{4}\left(q^{4}+2(1+2 \delta) q^{3}-(3+2 \delta) q^{2}+2(1+\delta) q-2(2+\delta)\right)
\end{aligned}
$$

which proves Theorem 4.18.

Theorem 4.19. $\mathcal{C}\left(G, Y_{2}\right)$ is connected of diameter at most 3.
Proof. For $q \leq 13$, this is easily checked using Magma [19], so assume $q>13$.
Combining Lemma 4.7 with Theorems 4.10, 4.12 and 4.18 yields the theorem.
We now focus on finding the disc sizes of $\mathcal{C}\left(G, Y_{2}\right)$. First, we need the following four lemmas.

Lemma 4.20. The sets $\mathcal{U}_{1}^{+}, \mathcal{U}_{1}^{-}$and $\mathcal{U}_{1}^{0}$ are single $C_{G}(t)$-orbits. Moreover,

$$
\begin{aligned}
\left|\mathcal{U}_{1}^{0}\right| & =q+1 ; \\
\left|\mathcal{U}_{1}^{+}\right| & =\frac{1}{2} q(q+\delta) ; \text { and } \\
\left|\mathcal{U}_{1}^{-}\right| & =\frac{1}{2} q(q-\delta)
\end{aligned}
$$

Proof. Since $C_{G}(t)$ acts orthogonally on $C_{V}(t)$, the first statement is immediate. Recall the Gram matrix $J$ for $V$ with respect to $(\cdot, \cdot)$ and the basis $\left\{v_{i}\right\}$. Observe that $C_{V}(t)=\{(\alpha, \beta, \gamma, 0, \gamma) \mid \alpha, \beta, \gamma \in G F(q)\}$ and so a basis for $C_{V}(t)$ is $\left\{v_{1}, v_{2}, v_{3}+v_{5}\right\}$. Let $v=(\alpha, \beta, \gamma, 0, \gamma)$ be a non-zero vector in $C_{V}(t)$ and so $(v, v)=2 \alpha \beta+2 \gamma^{2}$.
Suppose $v$ is isotropic, so $C_{G}(\langle v\rangle) \cong q^{3}: L_{2}(q)$ and $(v, v)=2 \alpha \beta+2 \gamma^{2}=0$. If $\gamma=0$, then $\alpha \beta=0$ and so either $\alpha=0$ or $\beta=0$ (but not both since $v \neq 0$ ). Hence there are $2(q-1)$ such vectors with $\gamma=0$. If $\gamma \neq 0$, then $\alpha=-\beta^{-1} \gamma^{2}$ and there are
$(q-1)^{2}$ such vectors satisfying this. Hence there are
$2(q-1)+(q-1)^{2}=(q-1)(q+1)$ non-zero isotropic vectors contained in $C_{V}(t)$ and thus $q+1$ isotropic 1-subspaces of $C_{V}(t)$.

Suppose now $v$ is $C_{G}(t)$-conjugate to $v_{3}+v_{5}$, which is non-isotropic. Note that $\left\langle v_{3}+v_{5}\right\rangle^{\perp} \cap C_{V}(t)$ is a 2-subspace of $V$ of + -type. If $\delta=1$, then by Lemma 4.5(ii), $\left\langle v_{3}+v_{5}\right\rangle^{\perp}$ is a 4 -subspace of $V$ of + -type and so $C_{G}\left(\left\langle v_{3}+v_{5}\right\rangle\right) \cong S L_{2}(q) \circ S L_{2}(q)$. While $\delta=-1$ gives that $\left\langle v_{3}+v_{5}\right\rangle^{\perp}$ is a 4 -subspace of $V$ of --type and so $C_{G}\left(\left\langle v_{3}+v_{5}\right\rangle\right) \cong L_{2}\left(q^{2}\right)$. A quick check shows that $\left(v_{3}+v_{5}, v_{3}+v_{5}\right)=2$ and so $(v, v)=2 \alpha \beta+2 \gamma^{2}=2 \lambda^{2}$ for some $\lambda \in G F(q)^{*}$. Thus, $\alpha \beta+\gamma^{2}=\lambda^{2}$ for some $\lambda \in G F(q)^{*}$. If $\gamma=0$, then $\alpha=\beta^{-1} \lambda^{2}$ and so there are $q-1$ such vectors that satisfy this. If $\gamma= \pm \lambda$, then $\alpha \beta=0$ and so for both values of $\gamma$, there are $2(q-1)+1$ vectors that satisfy this. Finally, if $\gamma \in G F(q) \backslash\{0, \lambda,-\lambda\}$, then $\alpha \beta=1-\gamma^{2} \neq 0$ and so $\alpha=\beta^{-1}\left(1-\gamma^{2}\right)$. There are $(q-1)(q-3)$ such vectors that satisfy this. Hence for any given $\lambda$, there exist $(q-1)+4(q-1)+2+(q-1)(q-3)=q(q+1)$ vectors that satisfy $\alpha \beta+\gamma^{2}=\lambda^{2}$. Since there are $\frac{1}{2}(q-1)$ squares in $G F(q)$, there are $q(q+1)(q-1)$ vectors that are $C_{G}(t)$-conjugate to $v_{3}+v_{5}$ and hence $\frac{1}{2} q(q+1) 1$-subspaces of $C_{V}(t)$ that are $C_{G}(t)$-conjugate to $\left\langle v_{3}+v_{5}\right\rangle$.

This leaves the remaining orbit. Recall there are $q^{2}+q+1$ subspaces of $C_{V}(t)$ of dimension 1 , and hence the size of the remaining orbit is
$q^{2}+q+1-(q+1)-\frac{1}{2} q(q+1)=\frac{1}{2} q(q-1)$, so proving the lemma.
Corollary 4.21. The sets $\mathcal{U}_{2}^{+}, \mathcal{U}_{2}^{-}$and $\mathcal{U}_{2}^{0}$ are single $C_{G}(t)$-orbits. Moreover,

$$
\begin{aligned}
\left|\mathcal{U}_{2}^{0}\right| & =q+1 ; \\
\left|\mathcal{U}_{2}^{+}\right| & =\frac{1}{2} q(q+1) ; \text { and } \\
\left|\mathcal{U}_{2}^{-}\right| & =\frac{1}{2} q(q-1)
\end{aligned}
$$

Proof. Since $C_{V}(t)$ is 3-dimensional, $U^{\perp} \cap C_{V}(t) \in \mathcal{U}_{1}$ for any $U \in \mathcal{U}_{2}$, and so the result is immediate by Lemma 4.20.

Lemma 4.22. Let $U, U^{\prime} \in \mathcal{U}_{2}$ be such that $U \neq U^{\prime}$. Then
$C_{G}(U) \cap C_{G}\left(U^{\prime}\right) \cap Y_{2}=\{t\}$.

Proof. Suppose $x \in C_{G}(U) \cap C_{G}\left(U^{\prime}\right) \cap Y_{2}$. Since $U \neq U^{\prime}$ and $x$ fixes each 2-subspace pointwise, $U+U^{\prime}=C_{V}(t)$ and so $x$ fixes $C_{V}(t)$ pointwise. That is to say, $C_{V}(x)=C_{V}(t)$ and so $t=x$ by Lemma 4.6(i).

Lemma 4.23. Let $U_{0} \in \mathcal{U}_{2}^{0}$, and suppose $G^{0}=Q L$ and $Y^{0}$ are as defined in the discussion prior to Lemma 4.13. Let $\rho: C_{G}\left(U_{0}^{\perp} \cap C_{V}(t)\right) \rightarrow G^{0}$ be an isomorphism such that

$$
t^{\rho}=\left(\begin{array}{rrr}
0 & 0 & 1 \\
0 & -1 & 0 \\
1 & 0 & 0
\end{array}\right)
$$

Then $C_{G}\left(U_{0}\right)$ is totally disconnected and $\left(C_{G}\left(U_{0}\right) \cap Y_{2}\right)^{\rho}=Q t \cap Y^{0}$.
Proof. Since $U_{0}^{\perp} \cap C_{V}(t)$ is isotropic, it must lie inside of $U_{0}$ and so $C_{G}\left(U_{0}\right) \leq C_{G}\left(U_{0}^{\perp} \cap C_{V}(t)\right)$. As $t$ fixes $U_{0}$ pointwise, $t^{\rho} \in\left(C_{G}\left(U_{0}\right)\right)^{\rho} \cong q^{2}: C_{\frac{q-\delta}{2}}$ by Lemma 4.9(i). The subgroup of $L$ with shape $C_{\frac{q-\delta}{2}}$ contains one single involution which must necessarily be $t^{\rho}$. For all $x \in Y^{0}$, we have $x_{L}^{2}=1$ and $x_{L}$ inverts $x_{Q}$, so $\left(C_{G}\left(U_{0}\right) \cap Y_{2}\right)^{\rho} \subseteq Q t \cap Y^{0}$. By comparing the orders of both sides, we get equality. By Lemma 4.13(ii) $C_{G}\left(U_{0}\right) \cap C_{G}(t) \cap Y_{2}=\{t\}$, hence $C_{G}\left(U_{0}\right)$ is totally disconnected.

Lemma 4.24. $\left|\Delta_{1}(t)\right|=\frac{1}{2} q\left(q^{2}+(1-\delta) q+\delta\right)$.
Proof. Clearly, $x \in \Delta_{1}(t)$ if and only if $x \in \Delta_{1}(t) \cap C_{G}(U)$ for $U=C_{V}(t) \cap C_{V}(x)$, so

$$
\Delta_{1}(t)=\bigcup_{U \in \mathcal{U}_{1} \cup \mathcal{U}_{2}}\left(\Delta_{1}(t) \cap C_{G}(U)\right) .
$$

If $W, W^{\prime} \in \mathcal{U}_{1}$ with $W \neq W^{\prime}$, then $W \oplus W^{\prime} \in \mathcal{U}_{2}$ and if $y \in C_{G}(W) \cap C_{G}\left(W^{\prime}\right)$ then $y \in C_{G}\left(W \oplus W^{\prime}\right)$ and hence $y \in C_{G}\left(W^{\prime \prime}\right)$ for any 1-subspace $W^{\prime \prime}$ of $W \oplus W^{\prime}$. Since there are $q+1$ subspaces of $W^{\prime \prime}$ of dimension 1 , any such $y$ will lie in exactly $q+1$ such $C_{G}(U)$ for $U \in \mathcal{U}_{1}$. Together with $C_{G}\left(W^{\prime \prime}\right)$ and Lemma 4.22,

$$
\left|\Delta_{1}(t)\right|=\sum_{U \in \mathcal{U}_{1}}\left|\Delta_{1}(t) \cap C_{G}(U)\right|-q \sum_{U \in \mathcal{U}_{2}}\left|\Delta_{1}\left(t_{1}\right) \cap C_{G}(U)\right| .
$$

Combining Lemmas 4.20, 4.23 and Corollary 4.21 with Theorems 4.10, 4.12, 4.18 and 2.10, we have

$$
\begin{aligned}
\left|\Delta_{1}(t)\right|= & \frac{1}{2} q(q+1)(q-\delta)+\frac{1}{2} q(q+\delta)\left[\frac{1}{2}(q-\delta)^{2}+1\right]+\frac{1}{4} q(q-\delta)\left(q^{2}-1\right) \\
& -\frac{1}{2} q(q-\delta)\left[\frac{1}{2} q(q+1)+\frac{1}{2} q(q-1)\right] \\
= & \frac{1}{2} q\left(q^{2}+(1-\delta) q+\delta\right)
\end{aligned}
$$

as required.

We now consider the second disc $\Delta_{2}(t)$. Here, we must be careful as elements that are distance 2 from $t$ in some subgroup $C_{G}(U)$ may not be distance 2 from $t$ in another subgroup $C_{G}\left(U^{\prime}\right)$. Moreover, there may be elements that are distance 3 from $t$ in every such subgroup centralizing an element of $\mathcal{U}_{1}$, but actually are distance 2 from $t$ in $G$. We introduce the following notation. Let $\Delta_{2}^{K}(t)$ be the second disc in the commuting involution graph $\mathcal{C}\left(K, K \cap Y_{2}\right)$ and

$$
\Gamma_{i}(K)=\left\{x \in \Delta_{2}^{K}(t) \mid \operatorname{dim} C_{V}(\langle t, x\rangle)=i\right\}
$$

for $K=C_{G}(U), U \in \mathcal{U}_{1} \cup \mathcal{U}_{2}$. Clearly, $\Delta_{2}(t)=\Gamma_{1}(G) \dot{\cup} \Gamma_{2}(G)$. A full list of cases with corresponding notation is found in Table 4.1. Also we use the following notation: for any $U \leq C_{V}(t)$, define $\mathcal{U}_{i}(U)$ to be the totality of $i$-dimensional subspaces of $U$ and $\mathcal{W}_{i}(U)$ to be the totality of $i$-dimensional subspaces of $C_{V}(t)$ containing $U$. Note that $\mathcal{U}_{i}=\mathcal{U}_{i}\left(C_{V}(t)\right)$.

Lemma 4.25. (i) If $W \in \mathcal{U}_{2}^{0}$, then $\left|\mathcal{U}_{1}^{0} \cap \mathcal{U}_{1}(W)\right|=1$ and $\left|\mathcal{U}_{1}^{+} \cap \mathcal{U}_{1}(W)\right|=q$.
(ii) If $W \in \mathcal{U}_{2}^{+}$, then $\left|\mathcal{U}_{1}^{0} \cap \mathcal{U}_{1}(W)\right|=2$ and $\left|\mathcal{U}_{1}^{+} \cap \mathcal{U}_{1}(W)\right|=\left|\mathcal{U}_{1}^{-} \cap \mathcal{U}_{1}(W)\right|=\frac{q-1}{2}$.
(iii) If $W \in \mathcal{U}_{2}^{-}$, then $\left|\mathcal{U}_{1}^{+} \cap \mathcal{U}_{1}(W)\right|=\left|\mathcal{U}_{1}^{-} \cap \mathcal{U}_{1}(W)\right|=\frac{q+1}{2}$.

Proof. Recall the Gram matrix $J$, with respect to the ordered basis $\left\{v_{i}\right\}$,
$i=1, \ldots, 5$. Suppose $W^{\perp} \cap C_{V}(t)=U_{0} \in \mathcal{U}_{1}^{0}$. Without loss of generality, choose $W=\left\langle v_{1}, v_{3}+v_{5}\right\rangle$. Clearly $\left\langle v_{1}\right\rangle \in \mathcal{U}_{1}^{0}$, and $\left\langle v_{3}+v_{5}\right\rangle^{\perp} \cap C_{V}(t) \in \mathcal{U}_{2}^{+}$. Since

$$
\left(v_{1}+\lambda\left(v_{3}+v_{5}\right), v_{1}+\lambda\left(v_{3}+v_{5}\right)\right)=\lambda^{2}\left(v_{3}+v_{5}\right)
$$

Case

Table 4.1: List of cases in $\Delta_{2}(t)$.
$v_{1}+\lambda\left(v_{3}+v_{5}\right)$ lies in the same $C_{G}(t)$-orbit as $v_{3}+v_{5}$ and so
$\left\langle v_{1}+\lambda\left(v_{3}+v_{5}\right)\right\rangle^{\perp} \cap C_{V}(t) \in \mathcal{U}_{2}^{+}$, proving (i).
Suppose now $W \in \mathcal{U}_{2}^{+}$. Without loss of generality, choose $W=\left\langle v_{1}, v_{2}\right\rangle$. Clearly $\left\langle v_{1}\right\rangle,\left\langle v_{2}\right\rangle \in \mathcal{U}_{1}^{0}$. Let $U_{\lambda}=v_{1}+\lambda v_{2}$ for $\lambda \neq 0$ and note that $\left(v_{1}+\lambda v_{2}, v_{1}+\lambda v_{2}\right)=2 \lambda=\mu \neq 0$. Since the type of $U_{\lambda}^{\perp}$ is determined by whether $\mu$ is a square or a non-square in $G F(q)$, and there are $\frac{q-1}{2}$ of each, it is clear that there exist $\frac{q-1}{2}$ such $U_{\lambda}$ for which $U_{\lambda}^{\perp}$ is of +-type, and similarly for --type, proving (ii). Finally suppose $W \in \mathcal{U}_{2}^{-}$, so for all $v \in W,(v, v) \neq 0$. The simple orthogonal group on $W$ is cyclic of order $\frac{q+1}{2}$ and acts on the 1 -subspaces of $W$ in exactly two orbits with representatives $\left\langle u_{1}\right\rangle$ and $\left\langle u_{2}\right\rangle$ where $\left(u_{1}, u_{1}\right)$ is a square and $\left(u_{2}, u_{2}\right)$ is a non-square in $G F(q)$. Since $\left|\mathcal{U}_{1}(W)\right|=q+1$, both orbits must be of size $\frac{q+1}{2}$. This proves (iii) and hence the lemma follows.

Corollary 4.26. Let $U \in \mathcal{U}_{1}$. Then,
(i) $\left|\mathcal{W}_{2}(U)\right|=q+1$.
(ii) If $U \in \mathcal{U}_{1}^{0}$, then $\left|\mathcal{U}_{2}^{0} \cap \mathcal{W}_{2}(U)\right|=1$ and $\left|\mathcal{U}_{2}^{+} \cap \mathcal{W}_{2}(U)\right|=q$.
(iii) If $U \in \mathcal{U}_{1}^{\delta}$, then $\left|\mathcal{U}_{2}^{0} \cap \mathcal{W}_{2}(U)\right|=2$ and $\left|\mathcal{U}_{2}^{+} \cap \mathcal{W}_{2}(U)\right|=\left|\mathcal{U}_{2}^{-} \cap \mathcal{W}_{2}(U)\right|=\frac{q-1}{2}$.
(iv) If $U \in \mathcal{U}_{2}^{-\delta}$, then $\left|\mathcal{U}_{2}^{+} \cap \mathcal{W}_{2}(U)\right|=\left|\mathcal{U}_{2}^{-} \cap \mathcal{W}_{2}(U)\right|=\frac{q+1}{2}$.

Proof. Let $U \leq W \leq C_{V}(t)$. Then $W^{\perp} \cap C_{V}(t) \leq U^{\perp} \cap C_{V}(t) \leq C_{V}(t)$. The result follows from Lemma 4.25.

Lemma 4.27. Let $U \in \mathcal{U}_{1}^{0}$ and $W \in \mathcal{U}_{2}^{+} \cap \mathcal{W}_{2}(U)$. If $x \in Y_{2} \cap C_{G}(W)$ is such that $d^{C_{G}(W)}(t, x)=3$, then $d^{C_{G}(U)}(t, x)=3$. Moreover,

$$
\left|\Gamma_{1}\left(C_{G}(U)\right)\right|= \begin{cases}\frac{1}{4} q(q-3)(q-1)^{2} & q \equiv 1 \quad(\bmod 4) \\ \frac{1}{4} q(q-1)^{2}(q+1) & q \equiv-1 \quad(\bmod 4)\end{cases}
$$

Proof. Recall that $C_{G}(U)=Q L \cong G^{0}$ where $G^{0}$ is defined as in the discussion prior to Lemma 4.13. By conjugacy, we may assume $L=C_{G}(W)$. Now $C_{G}(U) \cap C_{G}(t)=Q_{0} C_{L}(t) \cong q: \operatorname{Dih}(q-\delta)$ where $Q_{0} \leq Q$ is elementary abelian of order $q$. Let $x \in Q_{0} C_{L}(t) \cap Y_{2}$, so $x_{L}^{2}=1$ and $x_{L}$ inverts $x_{Q}$. Clearly,
$x_{L}^{x_{Q}}=x_{L} x_{Q}^{2} \notin L$ since $Q_{0}$ is of odd order. Hence, $C_{L}(t)$ is self-normalizing in $Q_{0} C_{L}(t)$ and thus there are $q$ distinct conjugates of $C_{L}(t)$ in $Q_{0} C_{L}(t)$. Let $g \in Q_{0} C_{L}(t) \backslash C_{L}(t)$, so $C_{L}(t)^{g} \neq C_{L}(t)$. Now $\left[C_{L}(t), t\right]=\left[C_{L}(t)^{g}, t\right]=1$ and so $\left\langle C_{L}(t), C_{L}(t)^{g}\right\rangle$ centralizes $t$. If $C_{L}(t), C_{L}(t)^{g} \leq L^{h}$ for some $h \in Q L$, then $\left\langle C_{L}(t), C_{L}(t)^{g}\right\rangle \leq L^{h}$. However, $C_{L}(t) \supsetneqq\left\langle C_{L}(t), C_{L}(t)^{g}\right\rangle \leq C_{L}(t)$, a contradiction. Hence every conjugate of $C_{L}(t)$ lies in a different conjugate of $L$ and so there are $q$ distinct $Q_{0} C_{L}(t)$-conjugates of $L$. Therefore, $\mathcal{U}_{2}^{+} \cap \mathcal{W}_{2}(U)$ is contained in the same $C_{G}(U) \cap C_{G}(t)$-orbit, and $\left|\mathcal{U}_{2}^{+} \cap \mathcal{W}_{2}(U)\right|=q$ by Corollary 4.26. There are exactly $q$ +-type 2-subspaces of $C_{V}(t)$ containing $U$, all of which lie in the same $C_{G}(U) \cap C_{G}(t)$-orbit.

Let $x \in C_{G}(W) \cap Y_{2}$ be such that $d^{C_{G}(W)}(t, x)=3$. Suppose $W^{g} \in \mathcal{U}_{2}^{+} \cap \mathcal{W}_{2}(U)$ for some $g \in C_{G}(U) \cap C_{G}(t), W \neq W^{g}$. If $d^{C_{G}(U)}(t, x)=2$ then $d^{C_{G}(U)}\left(t^{g}, x^{g}\right)=d^{C_{G}(U)}\left(t, x^{g}\right)=2$, and $d^{C_{G}(W)}(t, x)=d^{C_{G}\left(W^{\prime}\right)}\left(t, x^{g}\right)=3$. Hence it suffices to prove the lemma for $C_{G}(W)$. By Theorem 4.18, any involution distance 3 away from $t$ in $L$ is necessarily distance 3 away from $t$ in $C_{G}(U)$, proving the first statement.

Let $W_{0} \in \mathcal{U}_{2}^{0} \cap \mathcal{W}_{2}(U)$, so $C_{G}\left(W_{0}\right) \cong q^{2}: C_{\frac{q-\delta}{2}}$. By Lemma 4.23, $\Delta_{2}^{C_{G}(U)}(t) \cap C_{G}\left(W_{0}\right)=Q t \cap \Delta_{2}^{C_{G}(U)}(t)$. Let $W_{i}, i=1, \ldots, q$ be the subspaces in $\mathcal{U}_{2}^{+} \cap \mathcal{W}_{2}(U)$. From Lemma 4.22, $C_{G}\left(W_{i}\right) \cap C_{G}\left(W_{j}\right) \cap Y_{2}=\{t\}$ if and only if $i=j$. Using Corollary 4.26(i) with Theorem 2.10, we have

$$
\begin{equation*}
\left|\bigcup_{i=1}^{q} \Delta_{2}^{C_{G}\left(W_{i}\right)}(t)\right|=\frac{1}{4} q(q-\delta)(q-4-\delta) \tag{4.9}
\end{equation*}
$$

Combining Lemma 4.16 with (4.9),

$$
\begin{align*}
\left|\Gamma_{2}\left(C_{G}(U)\right)\right| & =\left|\bigcup_{i=1}^{q} \Delta_{2}^{C_{G}\left(W_{i}\right)}(t)\right|+\left|\Delta_{2}^{C_{G}(U)}(t) \cap C_{G}\left(W_{0}\right)\right| \\
& =\frac{1}{4}\left(q^{3}-2(1+\delta) q^{2}+(2 \delta-1) q+2 \delta\right) . \tag{4.10}
\end{align*}
$$

Together, (4.10) and Theorem 4.18 give

$$
\begin{aligned}
\left|\Gamma_{1}\left(C_{G}(U)\right)\right| & =\left|\Delta_{2}^{C_{G}(U)}(t)\right|-\left|\Gamma_{2}\left(C_{G}(U)\right)\right| \\
& =\frac{1}{4} q\left(q^{3}-(2 \delta+3) q^{2}+(4 \delta+3) q-2 \delta-1\right),
\end{aligned}
$$

as required.

Lemma 4.28. Let $t, x \in L_{2}(q)$. Then $d^{L_{2}(q)}(t, x) \leq 2$ if and only if the order of $t x$ divides $\frac{1}{2}(q-\delta)$.

Proof. See Lemma 2.11 of [15].
Lemma 4.29. Let $U \in \mathcal{U}_{1}^{+}$, and $W \in\left(\mathcal{U}_{2}^{+} \cup \mathcal{U}_{2}^{-}\right) \cap \mathcal{W}_{2}(U)$.
(i) If $\delta=1$ and $W_{0} \in \mathcal{U}_{2}^{0} \cap \mathcal{W}_{2}(U)$, then $Y_{2} \cap C_{G}\left(W_{0}\right) \backslash\{t\} \subseteq \Delta_{3}^{C_{G}(U)}(t)$.
(ii) If $x \in Y_{2} \cap C_{G}(W)$ is such that $d^{C_{G}(W)}(t, x)=3$, then $d^{C_{G}(U)}(t, x)=3$ and

$$
\left|\Gamma_{1}\left(C_{G}(U)\right)\right|= \begin{cases}\frac{1}{8}(q-1)(q-3)\left(q^{2}-6 q+13\right) & q \equiv 1 \quad(\bmod 4) \\ \frac{1}{8}\left(q^{2}-1\right)\left(q^{2}-2 q+5\right) & q \equiv-1 \quad(\bmod 4)\end{cases}
$$

Proof. Recall that $C_{G}(U) \cong G^{+} \cong L_{1} \circ L_{2}$ for $L_{1} \cong S L_{2}(q) \cong L_{2}$. Suppose $y \in C_{G}(W)$ is such that $d^{C_{G}(W)}(t, y)=3$. Since $C_{G}(W) \cong L_{2}(q)$ is simple, then $y=g g^{\varphi}$ for some $g \in L_{1}$ and $\varphi$ an isomorphism from $L_{1}$ onto $L_{2}$. Since $t \in C_{G}(W)$, write $t=s s^{\varphi}$ for some $s \in L_{1}$. Then $d^{L_{1}}(s, g)=3$, so $d^{C_{G}(U)}(t, y)=3$ by Theorem 4.12, and thus

$$
\begin{equation*}
\Delta_{3}^{C_{G}(W)}(t) \subseteq \Delta_{3}^{C_{G}(U)}(t) \quad \text { for all } W \in\left(\mathcal{U}_{2}^{+} \cup \mathcal{U}_{2}^{-}\right) \cap \mathcal{W}_{2}(U) \tag{4.11}
\end{equation*}
$$

If $\delta=-1$, then $\mathcal{U}_{2}^{0} \cap \mathcal{W}_{2}(U)=\varnothing$ by Corollary 4.26. If $\delta=1$, there exists $W_{0} \in \mathcal{U}_{2}^{0} \cap \mathcal{W}_{2}(U)$. Recall that $W_{0}^{\perp} \cap C_{V}(t) \in \mathcal{U}_{1}^{0}$ so $C_{G}\left(W_{0}\right) \leq C_{G}\left(W_{0}^{\perp} \cap C_{V}(t)\right) \cong G^{0}=Q L$. By Lemma 4.23, if $x \in C_{G}\left(W_{0}\right) \cap Y_{2}$ then $x=x_{Q} t$ and $x_{Q}$ is inverted by $t$ and has order $p$. Since $x_{Q}$ also lies in $C_{G}(U)$, we can write $x_{Q}=h h^{\varphi}$ for some $h \in L_{1}$. Now $x_{Q}^{-1}=h^{-1} h^{-1 \varphi}$ and so $x_{Q}^{t}=x_{Q}^{s s^{\varphi}}=h^{s}\left(h^{\varphi}\right)^{s^{\varphi}}=h^{-1} h^{-1 \varphi}$. Therefore, $h^{s}=h^{-1}$ and $h^{\varphi s^{\varphi}}=h^{-1 \varphi}$. Moreover, $x=x_{Q} t=(h s)(h s)^{\varphi}$ where $h s \in L_{1}$ is an element of order 4 squaring to the non-trivial element of $Z\left(L_{1}\right)$, and $h=(h s) s$ has order $p$. By Lemma 4.28 and [15], $d^{L_{1}}(h s, s)=3$ and so $d^{C_{G}(U)}\left(t, x_{Q} t\right)=3$ by Theorem 4.12. Therefore,

$$
\begin{equation*}
C_{G}\left(W_{0}\right) \cap \Delta_{2}^{C_{G}(U)}(t)=\varnothing \quad \text { for all } W_{0} \in \mathcal{U}_{2}^{0} \cap \mathcal{W}_{2}(U) \tag{4.12}
\end{equation*}
$$

Hence combining (4.11) with Lemma 4.25, Theorem 2.10 and, if $\delta=1$, (4.12) we get

$$
\left|\bigcup_{U \leq W} \Delta_{2}^{C_{G}(W)}(t)\right|=\left|\Gamma_{2}\left(C_{G}(U)\right)\right|=\frac{1}{4}(q-\delta)^{2}(q-4-\delta)
$$

This, together with Theorem 4.12 yields

$$
\begin{aligned}
\left|\Gamma_{1}\left(C_{G}(U)\right)\right| & =\left|\Delta_{2}^{C_{G}(U)}(t)\right|-\left|\Gamma_{2}\left(C_{G}(U)\right)\right| \\
& =\frac{1}{8}(q-1)(q-1-2 \delta)\left(q^{2}-(4+2 \delta) q+9+4 \delta\right),
\end{aligned}
$$

which proves the lemma.

Lemma 4.30. Let $U \in \mathcal{U}_{1}^{-}$, and $W \in\left(\mathcal{U}_{2}^{+} \cup \mathcal{U}_{2}^{-}\right) \cap \mathcal{W}_{2}(U)$.
(i) If $\delta=-1$ and $W_{0} \in \mathcal{U}_{2}^{0} \cap \mathcal{W}_{2}(U)$, then $Y_{2} \cap C_{G}\left(W_{0}\right) \backslash\{t\} \subseteq \Delta_{3}^{C_{G}(U)}(t)$.
(ii) We have

$$
\left|\Gamma_{2}\left(C_{G}(U)\right) \backslash \bigcup_{W \in \mathcal{W}_{2}(U)} \Gamma_{2}\left(C_{G}(W)\right)\right|=\frac{1}{4}(q-2+\delta)\left(q^{2}-1\right)
$$

and $\left|\Gamma_{1}\left(C_{G}(U)\right)\right|=\frac{1}{4}(q-1)^{3}(q+1)$.
Proof. First assume $\delta=-1$, and consider $C_{G}\left(W_{0}\right)$. By Lemma 4.23, every involution in $C_{G}\left(W_{0}\right)$ can be written as $x t$ where $x$ has order $p$. But $(x t) t=x$ has order $p$, which does not divide $\frac{1}{2}\left(q^{2}-1\right)$, and hence $d^{C_{G}(U)}(x t, t)=3$. In other words, $Y_{2} \cap C_{G}\left(W_{0}\right) \backslash\{t\} \subseteq \Delta_{3}^{C_{G}(U)}(t)$, so proving (i).

Consider then $C_{G}(W) \cong L_{2}(q)$. We utilize the character table of $L_{2}(q)$ as given in Theorem 2.2. Recall that $L_{2}(q)$ contains one conjugacy class of involutions, and two conjugacy classes of elements of order $p$. The remaining conjugacy classes partition into two cases: those whose order divides $\frac{1}{2}(q-1)$ and those whose order divides $\frac{1}{2}(q+1)$. Let $C$ be a conjugacy class of elements in $C_{G}(W)$ and define $X_{C}=\left\{x \in Y_{2} \cap C_{G}(W) \mid t x \in C\right\}$. It is a well-known character theoretic result (see, for example, Theorem 4.2.12 of [27]) that

$$
\begin{equation*}
\left|X_{C}\right|=\frac{|C|}{\left|C_{C_{G}(W)}(t)\right|} \sum_{\chi \in \operatorname{Irr}\left(C_{G}(W)\right)} \frac{\chi(t x)|\chi(t)|^{2}}{\chi(1)} \tag{4.13}
\end{equation*}
$$

where $\operatorname{Irr}\left(C_{G}(W)\right)$ is the set of all irreducible characters of $C_{G}(W)$, and all $X_{C}$ are pairwise disjoint. Let $x \in Y_{2} \cap C_{G}(W)$. If the order of $t x$ divides $\frac{1}{2}\left(q^{2}-1\right)$ but not $\frac{1}{2}(q-\delta)$ then it must necessarily divide $\frac{1}{2}(q+\delta)$. Hence, if $C$ is a conjugacy class of elements of order dividing $\frac{q+\delta}{2}$, then any $y \in X_{C}$ has the property that $d^{C_{G}(W)}(t, y)=3$ but $d^{C_{G}(U)}(t, y)=2$, by Lemma 4.28. Recall that $\Gamma_{2}\left(C_{G}(U)\right) \backslash \bigcup_{W \in \mathcal{W}_{2}(U)}^{\bigcup_{2}} \Gamma_{2}\left(C_{G}(W)\right)$ is the set consisting of all such involutions. Therefore, it suffices to calculate the sizes of all such relevant $X_{C}$. We use $\mathcal{F}$ to denote the set of all conjugacy classes of elements with order dividing $\frac{q+\delta}{2}$.
By Theorem 2.2, we see that for any $C \in \mathcal{F},|C|=q(q-\delta)$ and
$\left|C_{C_{G}(W)}(t)\right|=(q-\delta)$. Hence (4.13) and Theorem 2.2 gives $\left|X_{C}\right|=q-\delta$. Now if $\delta=1$, then $|\mathcal{F}|=\frac{q-1}{4}$ by Theorem 2.2. If $\delta=-1$, then $|\mathcal{F}|=\frac{q-3}{4}$. Since $\left|\Delta_{3}^{C_{G}(W)}(t) \cap \Delta_{2}^{C_{G}(U)}\right|=\left|X_{C}\right||\mathcal{F}|$, and by Corollary 4.26, $\left|\mathcal{W}_{2}(U) \cap\left(\mathcal{U}_{2}^{+} \cup \mathcal{U}_{2}^{-}\right)\right|=q+\delta$, we obtain

$$
\begin{align*}
\left|\Gamma_{2}\left(C_{G}(U)\right) \backslash \bigcup_{W \in \mathcal{W}_{2}(U)} \Gamma_{2}\left(C_{G}(W)\right)\right| & =\left|\mathcal{W}_{2}(U) \cap\left(\mathcal{U}_{2}^{+} \cup \mathcal{U}_{2}^{-}\right)\right|\left|X_{C}\right||\mathcal{F}| \\
& =\left\{\begin{array}{lll}
\frac{1}{4}(q-1)\left(q^{2}-1\right) & q \equiv 1 & (\bmod 4) \\
\frac{1}{4}(q-3)\left(q^{2}-1\right) & q \equiv-1 & (\bmod 4)
\end{array}\right. \tag{4.14}
\end{align*}
$$

which proves the first part of (ii). We now prove the last part of (ii). Recall that

$$
\left|\bigcup_{W \in \mathcal{W}_{2}(U)} \Gamma_{2}\left(C_{G}(W)\right)\right|=(q+\delta)\left|\Delta_{2}^{C_{G}(W)}(t)\right|=\frac{1}{4}\left(q^{2}-1\right)(q-4-\delta)
$$

by Theorem 2.10 and Corollary 4.26. Together with (4.14), we have

$$
\begin{aligned}
\left|\Gamma_{2}\left(C_{G}(U)\right)\right| & =\left|\bigcup_{W \in \mathcal{W}_{2}(U)} \Gamma_{2}\left(C_{G}(W)\right)\right|+\left|\Gamma_{2}\left(C_{G}(U)\right) \backslash \bigcup_{W \in \mathcal{W}_{2}(U)} \Gamma_{2}\left(C_{G}(W)\right)\right| \\
& =\frac{1}{4}\left(q^{2}-1\right)(q-4-\delta)+\frac{1}{4}\left(q^{2}-1\right)(q-2+\delta) \\
& =\frac{1}{2}\left(q^{2}-1\right)(q-3)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left|\Gamma_{1}\left(C_{G}(U)\right)\right| & =\left|\Delta_{2}^{C_{G}(U)}(t)\right|-\left|\Gamma_{2}\left(C_{G}(U)\right)\right| \\
& =\frac{1}{4}(q-1)^{3}(q+1),
\end{aligned}
$$

and Lemma 4.30 holds.

## Lemma 4.31.

$$
\left|\bigcup_{U \in \mathcal{U}_{1}} \Gamma_{1}\left(C_{G}(U)\right)\right|=\left\{\begin{array}{lll}
\frac{1}{16} q\left(q^{2}-1\right)\left(3 q^{3}-11 q^{2}+21 q-29\right) & q \equiv 1 & (\bmod 4) \\
\frac{1}{16} q\left(q^{2}-1\right)(q-1)\left(3 q^{2}+2 q+7\right) & q \equiv-1 & (\bmod 4)
\end{array}\right.
$$

Proof. Since $\mathcal{U}_{1}=\mathcal{U}_{1}^{0} \dot{\cup} \mathcal{U}_{1}^{+} \dot{\cup} \mathcal{U}_{1}^{-}$, with each orbit size given in Lemma 4.20, the result follows immediately from Lemmas 4.27, 4.29 and 4.30.

Recall the list of cases in Table 4.1. The next lemma concerns Cases 2 and 3, in other words, $\bigcup_{U \in \mathcal{U}_{1}} \Gamma_{2}\left(C_{G}(U)\right)$.

Lemma 4.32.

$$
\left|\bigcup_{U \in \mathcal{U}_{1}} \Gamma_{2}\left(C_{G}(U)\right)\right|=\frac{1}{2}(q-\delta)\left(q^{3}-2 q^{2}-1\right)
$$

Proof. By Lemmas 4.16 and 4.23, for any $W_{0} \in \mathcal{U}_{2}^{0}$ we have $\left|\Delta_{2}^{C_{G}(U)}(t) \cap C_{G}\left(W_{0}\right)\right|=\frac{1}{2}(q-1)(q-\delta)$ for some $U \in \mathcal{U}_{1}\left(W_{0}\right)$. Additionally, for any $W \in\left(\mathcal{U}_{2}^{+} \dot{\cup} \mathcal{U}_{2}^{-}\right)$we have

$$
\begin{aligned}
\left|\Delta_{2}^{C_{G}(U)}(t) \cap C_{G}(W)\right| & =\left|\Delta_{2}^{C_{G}(W)}\right|+\left|\Delta_{3}^{C_{G}(W)}(t) \cap \Delta_{2}^{C_{G}(U)}(t)\right| \\
& =\frac{1}{2}(q-\delta)(q-3),
\end{aligned}
$$

for some $U \in \mathcal{U}_{1}(W)$, by Theorem 2.10 and Lemma 4.30. Since $\mathcal{U}_{2}=\mathcal{U}_{2}^{0} \dot{\cup} \mathcal{U}_{2}^{+} \dot{\cup} \mathcal{U}_{2}^{-}$, with the orbit sizes given in Corollary 4.21, this covers every involution in $\bigcup_{U \in \mathcal{U}_{1}} \Gamma_{2}\left(C_{G}(U)\right)$, and the lemma follows.

We now concern ourselves with the final two cases. These concern involutions that are distance 3 from $t$ in every $C_{G}(U)$ that they appear in, but actually are distance 2 from $t$ in $G$. Recall that for any involution $y \in Y_{2}, C_{G}(y)=\operatorname{Stab}_{G} C_{V}(y)=L_{y} K_{y}$ where $L_{y}=C_{G}(y) \cap C_{G}([V, y]) \cong L_{2}(q)$ and $\left|K_{y}\right|=2(q-\delta)$. Also note that $L_{y} \unlhd C_{G}(y)$ acts faithfully on $C_{V}(y)$, and $\operatorname{Syl}_{p} C_{G}(y)=\operatorname{Syl}_{p} L_{y}$. The following three lemmas concern Case 5.

Lemma 4.33. Let $W \in \mathcal{U}_{2}^{0} \cup \mathcal{U}_{2}^{-\delta}$ and $x \in C_{G}(W)$ be such that $d^{C_{G}(U)}(t, x)=3$ for all $U \in \mathcal{U}_{1}(W)$. Then $d(t, x)=3$.

Proof. If $W \in \mathcal{U}_{2}^{0}$, then any involution in $C_{G}(W)$ can be written as $x=x_{Q} t$ where $x_{Q}=x t$ has order $p$. If $W \in \mathcal{U}_{2}^{-\delta}$, then, from Lemma 4.30, any involution $x \in C_{G}(W)$ such that $t x$ has order dividing $\frac{1}{2}\left(q^{2}-1\right)$ must be distance 2 from $t$ in $C_{G}(U)$ for some $U \in \mathcal{U}_{1}(W)$. Hence, any $x$ satisfying the hypothesis must have the property that the order of $t x$ is $p$.

Let $W \in \mathcal{U}_{2}^{0} \cup \mathcal{U}_{2}^{-\delta}$ and suppose $d(t, x)=2$. Then there exists $y \in Y_{2}$ such that $t, x \in C_{G}(y)=L_{y} K_{y}$. Since $t x$ has order $p, t x \in L_{y}$ and so $t x \in C_{G}([V, y])$. As $L_{y}$ acts faithfully on $C_{V}(y)$, any element of order $p$ must fix a 1-subspace of $C_{V}(y)$, say $U_{y}$. Therefore, $t x \in C_{G}\left(U_{y} \oplus[V, y]\right)$. But $t x \in C_{G}(W+[V, y])$ and since $[V, y] \in \mathcal{U}_{2}^{\delta}$, we have $W \neq[V, y]$. Set $W+[V, y]=U_{y} \oplus[V, y]$.
Suppose $U_{y} \leq W$. Then $t, x, y \in C_{G}\left(U_{y}\right)$ and so $d^{C_{G}\left(U_{y}\right)}(t, x)=2$, contradicting our assumption. Hence $U_{y} \not \leq W$ and so $U_{y}=\left\langle u_{1}+u_{2}\right\rangle$ for $u_{1} \in W \backslash[V, y]$ and $u_{2} \in[V, y]$. Since $y \in C_{G}(y),\left(u_{1}+u_{2}\right)^{y}=u_{1}+u_{2}$. However, $\left(u_{1}+u_{2}\right)^{y}=u_{1}^{y}+u_{2}^{y}=u_{1}^{y}-u_{2}$ and so $u_{2}=-2^{-1} u_{1}+2^{-1} u_{1}^{y}$. Thus $u_{1}+u_{2}=2^{-1}\left(u_{1}+u_{1}^{y}\right)$ and so $U_{y}=\left\langle u_{1}+u_{1}^{y}\right\rangle$. Recall that $t, x \in C_{G}(y)$ and $u_{1} \in W \backslash[V, y]$, so $u_{1}^{t}=u_{1}^{x}=u_{1}$. Hence $u_{1}+u_{1}^{y}$ is centralised by both $t$ and $x$ and so $U_{y} \leq W=C_{V}(\langle t, x\rangle)$, a contradiction. Therefore, $d(t, x) \neq 2$ and the lemma holds.

Lemma 4.34. Let $W \in \mathcal{U}_{2}^{\delta}$. Then $\Delta_{3}^{C_{G}(W)}(t) \subseteq \Delta_{2}(t)$. In particular,

$$
\left|\Gamma_{2}(G) \backslash \bigcup_{U \in \mathcal{U}_{1}} \Gamma_{2}\left(C_{G}(U)\right)\right|=\left\{\begin{array}{lll}
q\left(q^{2}-1\right) & q \equiv 1 & (\bmod 4) \\
0 & q \equiv-1 & (\bmod 4) .
\end{array}\right.
$$

Proof. We deal first with the case when $\delta=-1$. From Lemma 4.30, the number of involutions distance 3 from $t$ in $C_{G}(W)$ that are actually distance 2 from $t$ in some $U \in \mathcal{U}_{1}(W)$ is $\frac{1}{4}(q+1)(q-3)=\left|\Delta_{3}^{C_{G}(W)}(t)\right|$. That is to say all elements in $\Delta_{3}^{C_{G}(W)}(t)$ are distance 2 from $t$ in $C_{G}(U)$ for some $U \in \mathcal{U}_{2}(W)$. This occurs for every such $W \in \mathcal{U}_{2}^{\delta}$ and so $\Gamma_{2}(G)=\bigcup_{U \in \mathcal{U}_{1}} \Gamma_{2}\left(C_{G}(U)\right)$.

Assume now that $\delta=1$. As before, any element $x$ in $\Gamma_{2}(G) \backslash \bigcup_{U \in \mathcal{U}_{1}} \Gamma_{2}\left(C_{G}(U)\right)$ must have the property that the order of $t x$ is $p$. Suppose $d(t, x)=2$, and so there exists $y \in Y_{2}$ such that $t, x \in C_{G}(y)$. If $W \neq[V, y]$ then the argument from Lemma 4.33 holds and results in a contradiction. So we must have $W=[V, y]$. Since $\operatorname{Stab}_{G} C_{V}(y)=\operatorname{Stab}_{G}[V, y]=C_{G}(y), C_{G}([V, y]) \leq C_{G}(y)$ and so any element in $C_{G}([V, y])=C_{G}(W)$ centralizes $y$. In particular, $\Delta_{3}^{C_{G}(W)}(t) \subseteq \Delta_{2}(t)$, establishing the first statement. By Lemma 4.30, the number of involutions distance 3 from $t$ in $C_{G}(W)$ that are actually distance 2 from $t$ in some $U \in \mathcal{U}_{1}(W)$ is $\frac{1}{4}(q-1)^{2}$. By Theorem 2.10, $\left|\Delta_{3}^{C_{G}(W)}(t)\right|=\frac{1}{4}(q-1)(q+7)$ and so by subtracting the two, there are $2(q-1)$ involutions in $\Delta_{3}^{C_{G}(W)}(t)$ that are distance 3 from $t$ in $C_{G}(U)$ for all $U \in \mathcal{U}_{1}(W)$, but are actually distance 2 from $t$ in $\mathcal{C}\left(G, Y_{2}\right)$. Since $\left|\mathcal{U}_{2}^{\delta}\right|=\frac{1}{2} q(q+\delta)$ by Corollary 4.21 , the lemma follows.

Finally we turn to Case $4, \Gamma_{1}(G) \backslash \bigcup_{U \in \mathcal{U}_{1}} \Gamma_{1}\left(C_{G}(U)\right)$.
Lemma 4.35. Let $U \in \mathcal{U}_{1}^{-} \cup \mathcal{U}_{1}^{0}$ and $x \in C_{G}(U)$ be such that $C_{V}(\langle t, x\rangle)=U$ and $d^{C_{G}(U)}(t, x)=3$. Then $d(t, x)=3$.

Proof. Assume first that $U \in \mathcal{U}_{1}^{-}$. By Lemma 4.28, $t x$ has order $p$ or divides $\frac{1}{2}\left(q^{2}+1\right)$. Suppose $d(t, x)=2$, then there exists $y \in Y_{2}$ such that $t, x \in C_{G}(y)$. Since $\frac{1}{2}\left(q^{2}+1\right)$ is coprime to $\left|C_{G}(y)\right|=q\left(q^{2}-1\right)(q-\delta), t x$ must have order $p$. Indeed, clearly $\frac{1}{2}\left(q^{2}+1\right)$ is coprime to both $q$ and $q^{2}-1$, and any factor dividing $q-\delta$ must divide $q^{2}-1$ and so $\frac{1}{2}\left(q^{2}+1\right)$ is coprime to $q-\delta$. Since $t x$ has order $p, t x \in L_{y}$. Assume now that $U \in \mathcal{U}_{1}^{0}$. Let $x$ be an involution in $C_{G}(U)=Q L \cong G^{0}$ as defined in the discussion prior to Lemma 4.13. Then $t x \in Q t x_{L}$ which has order $n$ dividing $\frac{1}{2}(q+\delta)$ in $Q L / L$. Therefore, $\left(Q t x_{L}\right)^{n} \in Q$ and so $(t x)^{n}$ has order $p$. Therefore, $t x$ has order dividing $\frac{1}{2} q(q+\delta)$. Suppose $d(t, x)=2$. Then there exists $y \in Y_{2}$ such that $t, x \in C_{G}(y)$. By the structure of $C_{G}(y) \cong\left(L_{2}(q) \times C_{\frac{q-\delta}{2}}\right): 2^{2}$, the order of $t x$ forces $t x \in L_{y}$.

We may now assume $U \in \mathcal{U}_{1}^{-} \cup \mathcal{U}_{1}^{0}$, so $t x \in L_{y}=C_{G}([V, y])$ and hence $t x \in C_{G}(U+[V, y])$. Suppose $U \not \leq[V, y]$, then $t x \in C_{G}(U \oplus[V, y])$ Also, $t x \in C_{G}\left(U_{y} \oplus[V, y]\right)$ for some $U_{y} \leq C_{V}(y)$. However, if $U=U_{y}$ then $t, x, y \in C_{G}(U)$
and $d^{C_{G}(U)}(t, x)=2$. While $U_{y} \neq U$ results in a contradiction using an analogous argument from Lemma 4.33. Hence $U \leq[V, y]$.

As $t, x \in C_{G}(y)=\operatorname{Stab}_{G}([V, y]), t x \in L_{y}=C_{G}([V, y])$ and $[V, y]=U \perp U^{\prime}$ where $U^{\prime}=U^{\perp} \cap[V, y]$. Then for $u \in[V, y]$ we have $u^{t x}=u$ and so $u^{t}=u^{x}$. In particular, if $u \in U^{\prime}$ then $u^{t}=u^{x}=-u$. Hence $[V, y]=U \perp([V, t] \cap[V, x])$. If $C_{V}(\langle t, y\rangle)$ is 1-dimensional, then $C_{V}(y)=C_{V}(\langle t, y\rangle) \perp[V, t]$ since $t$ stabilizes $C_{V}(y)$. However, then $[V, t] \oplus([V, t] \cap[V, x])$ is 3-dimensional, a contradiction. A similar argument holds for $C_{V}(\langle x, y\rangle)$. Therefore both $C_{V}(\langle t, y\rangle)$ and $C_{V}(\langle x, y\rangle)$ are 2-dimensional. But since $\operatorname{dim} C_{V}(y)=3$, this means $C_{V}(\langle t, y\rangle)$ and $C_{V}(\langle x, y\rangle)$ intersect non-trivially, that is $C_{V}(\langle t, x, y\rangle) \neq 0$, contradicting our assumption. Therefore, $d(t, x) \neq 2$, and consequently $d(t, x)=3$.

The final case when $U \in \mathcal{U}_{1}^{+}$is slightly trickier. Recall the definition of $Y_{1}$. For any $z \in Y_{1}$, we have $C_{G}(z) \cong\left(S L_{2}(q) \circ S L_{2}(q)\right): 2$ and $C_{V}(z)$ is 1-dimensional. We choose $z$ such that $t \in C_{G}(z)$ and $C_{V}(z)=U$, and return to work in the setting of $S p_{4}(q) /\left\langle-I_{4}\right\rangle=G^{\tau} \cong G$. We denote the image of any subgroup $K \leq G$ by $K^{\tau}$. Choose

$$
z=\left(\begin{array}{c|c}
-I_{2} &  \tag{4.15}\\
& I_{2}
\end{array}\right) \in G^{\tau}
$$

and note that $C_{G^{\tau}}(z) \cong C_{G}(U): 2$. Hence,

$$
C_{G}(U) \cong\left\{\left.\left(\begin{array}{l|l}
A & \\
\hline & B
\end{array}\right) \right\rvert\, A, B \in S L_{2}(q)\right\} /\left\langle-I_{4}\right\rangle=C_{G}(U)^{\tau}
$$

Let $t^{\tau}$ be the image of $t$ in $G^{\tau}$ and set $L \cong L_{2}(q)$ and $\widehat{L} \cong P G L_{2}(q)$. We start with a preliminary lemma concerning the commuting involution graph $\mathcal{C}(L, X)$ where $X$ is the sole conjugacy class of involutions in $L$.

Lemma 4.36. Let $x$ be an involution in L. Then $\Delta_{3}^{L}(x)$ splits into $\frac{1}{4}(q+2+5 \delta)$ $C_{L}(x)$-orbits of length $q-\delta$. Moreover, every $C_{L}(x)$-orbit in $\Delta_{3}^{L}(x)$ is $C_{\widehat{L}}(x)$-invariant.

Proof. Assume first that $\delta=-1$. Choose $x=\left(\begin{array}{rr}0 & -1 \\ 1 & 0\end{array}\right)$ and let $x_{\lambda}=\left(\begin{array}{cc}0 & \lambda \\ -\lambda^{-1} & 0\end{array}\right)$ for some $\lambda \in G F(q) \backslash\{ \pm 1\}$. There are two possibilities for an element of $C_{L}(x)$ :

$$
g_{1}=\left(\begin{array}{rr}
a_{1} & a_{2} \\
-a_{2} & a_{1}
\end{array}\right) \quad \text { and } \quad g_{2}=\left(\begin{array}{rr}
b_{1} & b_{2} \\
b_{2} & -b_{1}
\end{array}\right)
$$

By direct calculation, if $g_{1}^{-1} x_{\lambda} g_{1}=x_{\mu}$ for some $\lambda, \mu \in G F(q) \backslash\{ \pm 1\}$ then $\left(-\lambda^{-1}+\lambda\right) a_{1} a_{2}=0$. Note that since $\lambda \neq \pm 1$, we have $\lambda \neq \lambda^{-1}$. If $a_{1}=0$ then $a_{2}^{2}=1$, and so $\mu=\lambda^{-1}$. On the other hand, if $a_{2}=0$ then $a_{1}^{2}=1$ and so $\mu=\lambda$. Observe that in the case of $g_{2}$, neither $b_{1}$ or $b_{2}$ can be 0 and so $g_{2}^{-1} x_{\lambda} g_{2}=x_{\mu}$ requires $x y\left(\lambda-\lambda^{-1}\right)=0$, a contradiction. Hence for $\lambda, \mu \in G F(q) \backslash\{ \pm 1\}, x_{\lambda}$ and $x_{\mu}$ lie in different $C_{L}(x)$ orbits if and only if $\mu \notin\left\{\lambda, \lambda^{-1}\right\}$. Working modulo $\left\langle-I_{4}\right\rangle$, there are at least $\frac{1}{4}(q-3) C_{L}(x)$-orbits in $\Delta_{3}^{L}(x)$. However for any $\lambda \neq \pm 1$, $C_{L}\left(x, x_{\lambda}\right)=1$ and so each $C_{L}(x)$-orbit containing an $x_{\lambda}$ is of length $q+1$. But $\left|\Delta_{3}^{L}(x)\right|=\frac{1}{4}(q-3)(q+1)$ and so all involutions in $\Delta_{3}^{L}(x)$ are accounted for. Hence the first statement holds for $\delta=-1$, and each $C_{L}(x)$-orbit has representative $x_{\lambda}$ for some $\lambda \neq \pm 1$. Let

$$
e=\left(\begin{array}{rr}
1 & 0 \\
0 & -1
\end{array}\right) \in \widehat{L} \backslash L
$$

and note that $C_{\widehat{L}}(x)=\langle e\rangle C_{L}(x)$. An easy check shows $\left[e, x_{\lambda}\right]=1$ for all $\lambda \neq \pm 1$. Let $y \in \Delta_{3}^{L}(x)$, then $y=x_{\lambda}^{s}$ for some $s \in C_{L}(x)$. Let $g=e r \in C_{\widehat{L}}(x)$ for some $r \in C_{L}(x)$. Then $y^{g}=x_{\lambda}^{s^{e} r}$ and since $C_{L}(x) \unlhd C_{\widehat{L}}(x), s^{e} r \in C_{L}(x)$. That is, every $C_{L}(x)$-orbit in $\Delta_{3}^{L}(x)$ is $C_{\widehat{L}}(x)$-invariant.
Assume now that $\delta=1$. Choose $x=\left(\begin{array}{rr}i & 0 \\ 0 & -i\end{array}\right)$ where $i^{2}=-1$ and let $y=\left(\begin{array}{cc}\sigma & \mu \tau \\ \tau & \sigma\end{array}\right)$ for some $\sigma, \mu, \tau \in G F(q), \sigma \neq 0$ and $\mu$ either zero or a non-square in $G F(q)$. By [15], $y \in \Delta_{3}^{L}(x)$. There are two possibilities for an element of $C_{L}(x)$ :

$$
g_{1}=\left(\begin{array}{cc}
a^{-1} & 0 \\
0 & a
\end{array}\right) \quad \text { and } \quad g_{2}=\left(\begin{array}{cc}
0 & b \\
-b^{-1} & 0
\end{array}\right) .
$$

By direct calculation, if $g_{1}^{-1} x g_{1}=y$ then $a=1$. Note that $g_{2}^{-1} x g_{2} \neq y$ since $\pm b^{2} \neq \mu$ for any non-square $\mu$. Hence $C_{L}(\langle x, y\rangle)=1$. Since $y$ is arbitrary, each $C_{L}(x)$-orbit has length $q-1$. Now $\left|\Delta_{3}^{L}(x)\right|=\frac{1}{4}(q+7)(q-1)$ and so the first statement holds for $\delta=1$. Let

$$
e_{\nu}=\left(\begin{array}{ll}
0 & \nu \\
1 & 0
\end{array}\right) \in \widehat{L} \backslash L
$$

and observe that $C_{\widehat{L}}(x)=\left\langle e_{\nu}\right\rangle C_{L}(x)$ for any non-square $\nu$. It is easy to check that $y^{e_{\mu}}=y$. Let $g=e_{\mu} r \in C_{\widehat{L}}(x)$ for some $r \in C_{L}(x)$. Then $y^{g}=y^{e_{\mu} r}=y^{r}$ and since $y$ is arbitrary and $r \in C_{L}(x)$, every $C_{L}(x)$-orbit in $\Delta_{3}^{L}(x)$ is $C_{\widehat{L}}(x)$-invariant.

Lemma 4.37. $\left|\Delta_{3}^{C_{G}(U)}(t) \cap \Gamma_{1}(G)\right|=\frac{1}{4}(q-\delta)^{2}(q+2+5 \delta)$.
Proof. We first work in the setting of $G^{\tau}$. Choose

$$
t^{\tau}=\left(\begin{array}{rr|r}
0 & -1 & \\
1 & 0 & \\
\hline & & 0
\end{array} \begin{array}{r}
-1 \\
\end{array}\right.
$$

By direct calculation, it is easily seen that

$$
C_{G^{\tau}}\left(t^{\tau}\right) \subseteq\left\{\left.\left(\begin{array}{c|c}
A_{1} & A_{2} \\
\hline A_{3} & A_{4}
\end{array}\right) \right\rvert\, A_{i}^{-1} J_{0} A_{i}=J_{0} \quad\left(\bmod \left\langle-I_{4}\right\rangle\right)\right\}
$$

and any involution $y \in C_{G^{\tau}}\left(t^{\tau}\right)$ has the additional properties that

$$
\begin{array}{ll} 
& \operatorname{det} A_{1}+\operatorname{det} A_{3}=\operatorname{det} A_{2}+\operatorname{det} A_{4}=1  \tag{4.16}\\
\text { and } & A_{1}^{2}+A_{2} A_{3}=A_{3} A_{2}+A_{4}^{2}=-I_{2} .
\end{array}
$$

Recall that if $x \in C_{G}(U)^{\tau}$ then $x=\left(\begin{array}{c|l}A & \\ \hline & B\end{array}\right)$ for some $A, B \in S L_{2}(q)$ and by Theorem 4.12, $x \in \Delta_{3}^{C_{G}(U)^{\tau}}\left(t^{\tau}\right)$ if and only if $A, B$ are involutions in $L$ and either $d^{L}\left(A, J_{0}\right)=3$ or $d^{L}\left(B, J_{0}\right)=3$. So without loss of generality, set $A=B_{i}$ where $d^{L}\left(B_{i}, J_{0}\right)=i$ and choose $B \in \Delta_{3}^{L}\left(J_{0}\right)$.
If $x \in \Delta_{2}^{G^{\tau}}\left(t^{\tau}\right)$ then there exists $y=\left(\begin{array}{c|c}A_{1} & A_{2} \\ \hline A_{3} & A_{4}\end{array}\right) \in C_{G^{\tau}}\left(t^{\tau}\right)$ such that $y^{2}=1$ and $[x, y]=1$. Suppose $\operatorname{det} A_{2}=0$. Then $\operatorname{det} A_{4}=1$ by (4.16), and so $A_{4} \in C_{L}\left(J_{0}\right)$. As
$[x, y]=1,\left[A_{4}, B\right]=1$. However $C_{L}\left(\left\langle J_{0}, B\right\rangle\right)=1$, by Lemma 4.36 and so $A_{4}= \pm I_{2}$. But then $A_{3} A_{2}=-2 I_{2}$ by (4.16), which is impossible since $\operatorname{det} A_{2}=0$. An analogous argument holds for $\operatorname{det} A_{3}$. Hence $\operatorname{det} A_{2}, \operatorname{det} A_{3} \neq 0$. Since $[x, y]=1$, $B_{i} A_{2} B= \pm A_{2}$ and so $B_{i}$ and $B$ must be $C_{\widehat{L}}\left(J_{0}\right)$-conjugate. In other words, if $B_{i}$ and $B$ are not $C_{\widehat{L}}\left(J_{0}\right)$-conjugate, then $[x, y] \neq 1$. By Lemma 4.36, every $C_{L}\left(J_{0}\right)$-orbit is an $C_{\widehat{L}}\left(J_{0}\right)$-orbit and so if $[x, y]=1$ then $B_{i}$ and $B$ must be $C_{L}\left(J_{0}\right)$-conjugate. Assume then $B_{i}$ and $B$ are $C_{L}\left(J_{0}\right)$-conjugate and let $A \in C_{L}\left(J_{0}\right)$ be such that $B_{i}^{A}=B$. Hence if $y_{A}=\left(\begin{array}{l|l} & A \\ \hline-A^{-1} & \end{array}\right) \in C_{G^{\tau}}\left(t^{\tau}\right)$, then $\left[y_{A}, x\right]=1$ and so $d^{G^{\top}}\left(t^{\tau}, x\right)=2$. By Lemma 4.36, each $C_{L}\left(J_{0}\right)$-orbit of $\Delta_{3}^{L}\left(J_{0}\right)$ is of length $q-\delta$, and there are $\frac{1}{4}(q+2+5 \delta)$ such orbits. Moreover, for any involution $x_{0} \in C_{G}(U)^{\tau}$ conjugate to $t^{\tau}$ and $z$ as in (4.15), $z x_{0}$ is also an involution in $C_{G}(U)^{\tau}$ conjugate to $t^{\tau}$ which has not been accounted for. Therefore, the number of involutions in $\Delta_{3}^{C_{G}(U)^{\tau}}\left(t^{\tau}\right)$ that are actually distance 2 from $t^{\tau}$ in $G^{\tau}$ is $\frac{1}{2}(q-\delta)^{2}(q+2+5 \delta)$. We now return to the setting of $G$, and first assume that $\delta=-1$ and so by Corollary 4.26(i), $\left|\mathcal{W}_{2}(U)\right|=q+1$, and for every $W \in \mathcal{W}_{2}(U), C_{G}(W) \cong L_{2}(q)$. For each $W$, there exists $U_{W} \in \mathcal{U}_{1}^{+}$such that $C_{G}(W) \leq C_{G}\left(U_{W}\right) \cong L_{2}\left(q^{2}\right)$ by Lemma 4.25 , and $\Delta_{3}^{C_{G}(W)}(t) \subseteq \Delta_{2}^{C_{G}\left(U_{W}\right)}(t)$ by Lemma 4.34. Hence, there are $\frac{1}{4}(q+1)^{2}(q-3)$ involutions already counted (from Case 3) and the remaining involutions do not fix a 2-subspace of $C_{V}(t)$. Therefore

$$
\begin{aligned}
\left|\Delta_{3}^{C_{G}(U)}(t) \cap \Gamma_{1}(G)\right| & =\frac{1}{2}(q+1)^{2}(q-3)-\frac{1}{2}(q+1)^{2}(q-3) \\
& =\frac{1}{4}(q+1)^{2}(q-3),
\end{aligned}
$$

as required. Now assume that $\delta=1$ and so by Corollary 4.26. For each $W$, there exists $U_{W} \in \mathcal{U}_{1}^{-}$such that $C_{G}(W) \leq C_{G}\left(U_{W}\right) \cong L_{2}\left(q^{2}\right)$ by Lemma 4.25 and $\left|\Delta_{3}^{C_{G}(W)}(t) \cap \Delta_{2}^{C_{G}\left(U_{W}\right)}(t)\right|=\frac{1}{4}(q-1)^{2}$ by Lemma 4.30. Since $\left|\mathcal{W}_{2}(U) \cap\left(\mathcal{U}_{1}^{+} \cup \mathcal{U}_{1}^{-}\right)\right|=q-1$ by Corollary $4.26($ iii $)$, this accounts for $\frac{1}{4}(q-1)^{3}$ involutions. Suppose now $W_{0} \in \mathcal{W}_{2}(U) \cap \mathcal{U}_{2}^{0}$. By Lemma 4.25, there exists $U_{0} \in \mathcal{U}_{1}^{0}$ such that $C_{G}\left(W_{0}\right) \leq C_{G}\left(U_{0}\right)$. From Lemmas 4.16 and 4.23, $\left|C_{G}(W) \cap \Delta_{2}^{C_{G}\left(U_{0}\right)}(t)\right|=\frac{1}{2}(q-1)^{2}$. Since $\left|\mathcal{W}_{2}(U) \cap \mathcal{U}_{2}^{0}\right|=2$ by Corollary $4.26(\mathrm{iii})$,
this yields a further $(q-1)^{2}$ involutions. Finally, if $W \in \mathcal{U}_{2}^{+}$, then by Lemma 4.34, $\Delta_{3}^{C_{G}(W)}(t) \subseteq \Delta_{2}(t)$ and there are $2(q-1)$ involutions in $\Delta_{3}^{C_{G}(W)}(t)$ not already enumerated. Now $\left|\mathcal{U}_{2}^{+} \cap \mathcal{W}_{2}(U)\right|=\frac{1}{2}(q-1)$ by Corollary 4.26(iii), and this yields another $(q-1)^{2}$ involutions. Hence, there are $\frac{1}{4}(q-3)^{2}+2(q-1)^{2}=\frac{1}{4}(q-1)^{2}(q+7)$ involutions already counted (from Cases 3 and 5) and the remaining involutions do not fix a 2-subspace of $C_{V}(t)$. Consequently

$$
\begin{aligned}
\left|\Delta_{3}^{C_{G}(U)}(t) \cap \Gamma_{1}(G)\right| & =\frac{1}{2}(q-1)^{2}(q+7)-\frac{1}{2}(q-1)^{2}(q+7) \\
& =\frac{1}{4}(q-1)^{2}(q+7),
\end{aligned}
$$

as required.
Corollary 4.38. $\left|\Gamma_{1}(G) \backslash \bigcup_{U \in \mathcal{U}_{1}} \Gamma_{1}\left(C_{G}(U)\right)\right|=\frac{1}{8} q(q-\delta)\left(q^{2}-1\right)(q+2+5 \delta)$.
Proof. Since $\left|\mathcal{U}_{1}^{+}\right|=\frac{1}{2} q(q+\delta)$, the result holds by Lemmas 4.36 and 4.37.
Lemma 4.39. If $q \equiv 3(\bmod 4)$, then

$$
\begin{aligned}
& \left|\Delta_{2}(t)\right|=\frac{1}{16}(q+1)\left(3 q^{5}-2 q^{4}+8 q^{3}-30 q^{2}+13 q-8\right) ; \text { and } \\
& \left|\Delta_{3}(t)\right|=\frac{1}{16}(q-1)\left(5 q^{5}-4 q^{4}-2 q^{3}+4 q^{2}+5 q+5\right)
\end{aligned}
$$

If $q \equiv 1(\bmod 4)$, then

$$
\begin{aligned}
& \left|\Delta_{2}(t)\right|=\frac{1}{16}(q-1)\left(3 q^{5}-6 q^{4}+32 q^{3}-10 q^{2}-27 q-8\right) ; \text { and } \\
& \left|\Delta_{3}(t)\right|=\frac{1}{16}(q-1)\left(5 q^{5}+22 q^{4}-8 q^{3}+34 q^{2}+51 q+24\right)
\end{aligned}
$$

Proof. The cases listed in Table 4.1 are disjoint. Hence $\left|\Delta_{2}(t)\right|$ is determined by summing the values calculated in Lemmas 4.31, 4.32, 4.34 and 4.38. By Theorem 4.19, $\mathcal{C}\left(G, Y_{2}\right)$ has diameter 3 and so $\left|\Delta_{3}(t)\right|=\left|Y_{2}\right|-\left|\Delta_{1}(t)\right|-\left|\Delta_{2}(t)\right|$. Since $|G|=\frac{1}{2} q^{4}\left(q^{2}-1\right)\left(q^{4}-1\right)$ and $\left|C_{G}(t)\right|=q\left(q^{2}-1\right)(q-\delta),\left|Y_{2}\right|=\frac{1}{2} q^{3}(q+\delta)\left(q^{2}+1\right)$. Together with Lemma 4.24, this proves the lemma.

Together, Theorem 4.19 and Lemmas 4.24 and 4.39 complete the proof of Theorem 1.4.

## Chapter 5

## 3-Dimensional Unitary Groups

We now change our focus and consider a family of twisted groups of Lie type. We let $H=S U_{3}(q)$ and $G=H / Z(H) \cong U_{3}(q)$. For any $a \in G F\left(q^{2}\right)$, we write $\bar{a}=a^{q}$ and $\overline{\left(a_{i j}\right)}=\left(\overline{a_{i j}}\right)$. First assume $p=2$. We have $G \cong H$ and there is one single class of involutions (see, for example, Lemma 6.1 of [10]). Let $t$ be an involution in $G$, and define the unitary form preserved by the elements of $G$ by the Gram matrix

$$
J=\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{array}\right)
$$

Since $|G|=q^{3}\left(q^{3}+1\right)\left(q^{2}-1\right)$ and $\left|t^{G}\right|=(q-1)\left(q^{3}+1\right)$ (see Theorem 4 of $[24]$ ), $\left|C_{G}(t)\right|=q^{3}(q+1)$. By Proposition 4.1.18 of [32] (see also Lemma 6.2 of [10]), $O_{2}\left(C_{G}(t)\right) \cong q^{3}$ and $C_{G}(t) \cong q^{3}: C_{(q+1)}$. A Sylow 2-subgroup $T$ of $C_{G}(t)$ has order $q^{3}$ and so $T \in \operatorname{Syl}_{2}(G)$.

Let

$$
\widehat{S}=\left\{\left.\left(\begin{array}{lll}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{array}\right) \right\rvert\, a, b, c \in G F\left(q^{2}\right)\right\} \in \operatorname{Syl}_{2}\left(G L_{3}\left(q^{2}\right)\right)
$$

By a direct calculation, the set of all $s \in \widehat{S}$ such that $\bar{s}^{T} J s=J$ forms a group

$$
S=\left\{\left.\left(\begin{array}{ccc}
1 & a & b \\
0 & 1 & \bar{a} \\
0 & 0 & 1
\end{array}\right) \right\rvert\, a \bar{a}=b+\bar{b}\right\} \in \operatorname{Syl}_{2}(G)
$$

The set of involutions in $S$ form an elementary abelian group of order $q$. Indeed, a direct calculation reveals that the involutions in $S$ require $a=0$ and $b \in G F(q)$. Since $S \in \operatorname{Syl}_{2}(G)$ is normal in $C_{G}(t)$, all involutions of $C_{G}(t)$ lie in $S$ and thus must all commute with each other. There are $q-1$ involutions in $C_{G}(t)$ and for any $x \in C_{G}(t) \cap t^{G} \backslash\{t\}$ we have $C_{G}(x) \cap t^{G}=C_{G}(t) \cap t^{G}$. That is to say, the involutions in $C_{G}(t)$ form a clique in $\mathcal{C}\left(G, t^{G}\right)$. For any involution $y \notin C_{G}(t)$, $C_{G}(y) \cap C_{G}(t) \cap t^{G}=\varnothing$ and so the commuting involution graph is disconnected. This proves Theorem 1.5(i).

Remark: The above is an example of when the group contains a strongly embedded subgroup. In such cases, the commuting involution graph of such a group with respect to a conjugacy class of involutions will always be disconnected and consist of cliques. Finite simple groups with strongly embedded subgroups have already been established by Bender [16]. Other examples include $L_{2}(q)$ for $q$ even (see Theorem 2.10), and the simple Suzuki groups $\mathrm{Sz}\left(2^{2 m+1}\right)$ mentioned later, in Theorem 8.3.

We now assume $p>2$ and set $H=S U_{3}(q)$ and $G=H / Z(H) \cong U_{3}(q)$. Since $|Z(H)| \in\{1,3\}$, without loss of generality we work in the setting of $H$. We devote the remainder of this chapter to the proof of Theorem 1.5(ii). Define the unitary form preserved by the elements of $G$ by the Gram matrix $J=I_{3}$. We first prepare a lemma:

Lemma 5.1. The polynomials $f_{1}=x^{q+1}-\lambda$ and $f_{2}=x^{q}+x-\lambda$ split over $G F\left(q^{2}\right)$, for all $\lambda \in G F(q)$.

Proof. Let $N_{1}: G F\left(q^{2}\right)^{*} \rightarrow G F(q)^{*}$ and $N_{2}: G F\left(q^{2}\right) \rightarrow G F(q)$ be given by $N_{1}(x)=x^{q+1}$ for all $x \in G F\left(q^{2}\right)^{*}$ and $N_{2}(y)=y^{q}+y$ for all $y \in G F\left(q^{2}\right)$. It is easy to see that both $N_{1}$ and $N_{2}$ are group homomorphisms. Since $G F\left(q^{2}\right)^{*}$ is cyclic of order $q^{2}-1$, there exists a cyclic subgroup of order $q+1$ consisting of elements that are $(q+1)^{\text {th }}$ roots of unity. Hence this subgroup is precisely ker $N_{1}$ and which has order $q+1$. Therefore $\left|\operatorname{Im} N_{1}\right|=q-1=\left|G F(q)^{*}\right|$ and $N_{1}$ is surjective. Let
$\lambda \in G F(q)$, so there exists $y \in G F\left(q^{2}\right)$ such that $N_{1}(y)=\lambda$. Moreover if $\mu \in \operatorname{ker} N_{1}$, $N_{1}(\mu y)=N_{1}(y)$ and so there are $q+1$ values in $G F\left(q^{2}\right)$ satisfying $f_{1}$. Consider now $N_{2}$ and let $x \in G F(q)$. Then $2^{-1} x \in G F(q)$ and $N_{2}\left(2^{-1} x\right)=x$. So $N_{2}$ is surjective and so $\left|\operatorname{Im} N_{2}\right|=q$. Hence $\left|\operatorname{ker} N_{2}\right|=q$ and by a similar argument as above, there are $q$ values of $G F\left(q^{2}\right)$ satisfying $f_{2}$, so proving the lemma.

It will also be useful to note the conditions for a $2 \times 2$ matrix $A$ to be unitary. Let $A=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in G U_{2}(q)$ be such that $\bar{A}^{T} A=I_{2}$. Hence $\bar{A}^{T} A=\left(\begin{array}{ll}\bar{a} a+\bar{c} c & \bar{a} b+\bar{c} d \\ \bar{b} a+\bar{d} c & \bar{b} b+\bar{d} d\end{array}\right)$ and so

$$
\begin{gather*}
\bar{a} a+\bar{c} c=\bar{b} b+\bar{d} d=1 ; \text { and } \\
\bar{a} b+\bar{c} d=\bar{b} a+\bar{d} c=0 . \tag{5.1}
\end{gather*}
$$

Clearly $\overline{(\bar{b} a+\bar{d} c)}=\overline{\bar{b}} \bar{a}+\overline{\bar{d}} \bar{c}=\bar{a} b+\bar{c} d=0$. It is also easy to see that the determinant of a matrix in $G U_{2}(q)$ is always a $(q+1)^{\text {th }}$ root of unity. Indeed,

$$
\begin{aligned}
(a d-b c)^{q+1} & =\overline{(a d-b c)}(a d-b c) \\
& =\overline{(a d)} a d+\overline{(b c)} b c-\bar{a} b c \bar{d}-a \bar{b} \bar{c} d \\
& =\overline{(a d)} a d+\overline{(b c)} b c+\overline{(a b)} a b+\overline{(c d)} c d \\
& =(\bar{a} a+\bar{c} c)(\bar{b} b+\bar{d} d)=1
\end{aligned}
$$

as claimed. Let

$$
t=\left(\begin{array}{rrr}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{array}\right)
$$

which is an involution in $H$. Using Lemma 2.4, there is only one class of involutions in $H$, which we denote by $Z_{0}$.

Lemma 5.2. (i) $C_{H}(t) \cong G U_{2}(q)$
(ii) $\left|Z_{0}\right|=q^{2}\left(q^{2}-q+1\right)$
(iii) $\left|\Delta_{1}(t)\right|=q(q-1)$.
(iv) If $x_{0} \in \Delta_{1}(t)$, then $\left|\Delta_{1}(t) \cap \Delta_{1}\left(x_{0}\right)\right|=1$.

Proof. Clearly

$$
\left\{\left.\left(\begin{array}{c|cc}
D^{-1} & & \\
\hline & a & b \\
& c & d
\end{array}\right) \right\rvert\, \begin{array}{c}
a, b, c, d \in G F\left(q^{2}\right) \\
D=a d-b c \neq 0
\end{array}\right\}
$$

consists of all matrices in $S L_{3}\left(q^{2}\right)$ that centralise $t$. An easy calculation shows that any unitary matrix from the above set necessarily requires $a, b, c, d$ to satisfy (5.1).

Hence

$$
\left.C_{H}(t)=\left\{\left.\left(\begin{array}{lll}
D^{-1} & &  \tag{5.2}\\
& a & b \\
& c & d
\end{array}\right) \right\rvert\, \begin{array}{cc}
a & b \\
c & d
\end{array}\right) \in G U_{2}(q) \quad \begin{array}{cc}
D=a d-b c \neq 0
\end{array}\right\} \cong G U_{2}(q)
$$

proving (i).
Recall that $\left|G U_{2}(q)\right|=q(q+1)\left(q^{2}-1\right)$ and $|H|=q^{3}\left(q^{3}+1\right)\left(q^{2}-1\right)$. Hence

$$
\left|Z_{0}\right|=\frac{q^{3}\left(q^{3}+1\right)\left(q^{2}-1\right)}{q(q+1)\left(q^{2}-1\right)}=q^{2}\left(q^{2}-q+1\right)
$$

which proves (ii).
Let $x=\left(\begin{array}{l|l}\operatorname{det} A^{-1} & \\ & A\end{array}\right) \in C_{H}(t) \cap Z_{0}$. Using a result of Wall [40], there are two classes of involutions in $G U_{2}(q)$, represented by $-I_{2}$ and $\left(\begin{array}{cc}-1 & 0 \\ 0 & 1\end{array}\right)$. If $A=-I_{2}$, then $x=t$. Assume then that $A$ is the latter choice, so
$x=\left(\begin{array}{rrr}-1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1\end{array}\right) \in \Delta_{1}(t)$ and $\Delta_{1}(t)=x^{C_{H}(t)}$.
By a routine calculation as in part (i), it is easy to see that

$$
C_{H}(x)=\left\{\left.\left(\begin{array}{l|l}
A & \\
\hline & \operatorname{det} A^{-1}
\end{array}\right) \right\rvert\, A \in G U_{2}(q)\right\},
$$

and so

$$
C_{H}(\langle t, x\rangle)=\left\{\left.\left(\begin{array}{ccc}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & (a b)^{-1}
\end{array}\right) \right\rvert\, a, b \in G F\left(q^{2}\right), \bar{a} a=\bar{b} b=1\right\}
$$

with $\left|C_{H}(\langle t, x\rangle)\right|=(q+1)^{2}$. Hence $\left|\Delta_{1}(t)\right|=\frac{\left|C_{H}(t)\right|}{\left|C_{H}\langle\langle t, x\rangle)\right|}=q(q-1)$, proving (iii), while (iv) follows immediately from the structure of $C_{H}(\langle t, x\rangle)$.
Henceforth, we set $x=\left(\begin{array}{rrr}-1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1\end{array}\right) \in \Delta_{1}(t)$.
Lemma 5.3. (i) Let $g, h \in \Delta_{2}(t)$. If $g_{11} \neq h_{11}$, then $g$ and $h$ are not
$C_{H}(t)$-conjugate.
(ii) $\Delta_{2}(t) \cap \Delta_{1}(x)=\left\{\left.\left(\begin{array}{cc|c}a & b & \\ \bar{b} & -a & \\ \hline & & -1\end{array}\right) \right\rvert\, \bar{b} b=1-a^{2}, a \in G F(q) \backslash\{ \pm 1\}\right\}$.
(iii) For each $a \in G F(q) \backslash\{ \pm 1\}$, there are $q+1$ elements $g$ of $\Delta_{2}(t) \cap \Delta_{1}(x)$ such that $g_{11}=a$.

Proof. By an analogous method to that in Lemma 5.2(i), it is clear that

$$
\Delta_{1}(x)=\left\{\left.\left(\begin{array}{cc|c}
a & b & \\
c & -a & \\
\hline & & -1
\end{array}\right) \right\rvert\, a, b, c \in G F\left(q^{2}\right), a^{2}+b c=1\right\} .
$$

Let

$$
g=\left(\begin{array}{cc|c}
a & b & \\
c & -a & \\
\hline & & -1
\end{array}\right) \in \Delta_{1}(x)
$$

for $a, b, c \in G F\left(q^{2}\right)$, and $h \in C_{H}(t)$. Now $\left(h^{-1} g h\right)_{11}=h_{11}^{-1} a h_{11}=a$ and so any two $C_{H}(t)$-conjugate elements have the same top-left entry, so proving (i). If $b=0$ then $a^{2}+b c=a^{2}=1$ and so $a= \pm 1$. But then $\bar{a} a=1$ and thus $\bar{c} c=0$ implying $c=0$. Similarly, if $c=0$ then $b=0$. If $a= \pm 1$, then $1+b c=1$ and so $b c=0$. Hence, either $b=0$ or $c=0$ and therefore both are zero. However, $a=1$ implies $g=t$, and $a=-1$ implies $g \in \Delta_{1}(t)$. Therefore if $a= \pm 1$, then $g \notin \Delta_{2}(t)$. In particular, if $a \neq \pm 1$ then $g \in \Delta_{2}(t)$, since $d(t, x)=1$ and $[g, x]=1$. Suppose now $a \neq \pm 1$, so $b, c \neq 0$. Then by (5.1), we have $\bar{a} a+\bar{c} c=\bar{a} a+\bar{b} b=1$ and $\bar{a} b=a \bar{c}$. Therefore $\bar{a} a+\bar{c} c=a^{2} \bar{c} b^{-1}+\bar{c} c=1$ and so $a^{2} b^{-1}+c=\overline{c^{-1}}$. It follows that
$b \bar{c}^{-1}=a^{2}+b c=1$ and hence $b=\bar{c}$. However, this yields $\bar{a}=a$, implying $a \in G F(q) \backslash\{ \pm 1\}$, proving (ii).

By combining parts (i) and (ii), $\Delta_{1}(x) \cap \Delta_{2}(t)$ is partitioned into $C_{H}(\langle t, x\rangle)$-orbits, with the action of $C_{H}(\langle t, x\rangle)$ leaving the diagonal entries unchanged. Since $a \neq \pm 1$, $\bar{b} b \neq 0$ and $\bar{b} b-\left(1+a^{2}\right)=0$. By Lemma 5.1, there are $q+1$ solutions in $G F\left(q^{2}\right)$ to the equation $x^{q+1}=\lambda$ for any fixed $\lambda \in G F(q)$, so there are $q+1$ values of $b$ that satisfy this equation. Therefore $x$ is centralised by $q+1$ involutions sharing a common top-left entry, proving (iii).

Lemma 5.4. There are exactly $(q-2) C_{H}(t)$-orbits in $\Delta_{2}(t)$.
Proof. By Lemma 5.3(i), there are at least $(q-2) C_{H}(t)$-orbits in $\Delta_{2}(t)$. It suffices to prove that any two matrices commuting with $x$ that share a common top-left entry are $C_{H}(\langle t, x\rangle)$-conjugate. Let $g \in \Delta_{2}(t) \cap \Delta_{1}(x)$, and $a \in G F(q) \backslash\{ \pm 1\}$ be fixed such that $g_{11}=a$ and set $g_{12}=b$. By direct calculation, the diagonal entries of $g$ remain unchanged under conjugation by $C_{H}(\langle t, x\rangle)$. Let

$$
h=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & \beta^{-1}
\end{array}\right) \in C_{H}(\langle t, x\rangle)
$$

where $\bar{\beta} \beta=1$. Then

$$
h^{-1} g h=\left(\begin{array}{cc|c}
a & b \beta & \\
\beta^{-1} \bar{b} & -a & \\
\hline & & -1
\end{array}\right) .
$$

Clearly $b \beta$ takes $q+1$ different values for the $q+1$ different values of $\beta$. However, since there are only $q+1$ possible values for $b$, all such values are covered. That is to say, all matrices of the form

$$
\left(\begin{array}{cc|c}
a & b & \\
\bar{b} & -a & \\
\hline & & -1
\end{array}\right) \in \Delta_{2}(t) \cap \Delta_{1}(x), a \neq \pm 1, \bar{b} b=1-a^{2}
$$

lie in the same $C_{H}(\langle t, x\rangle)$ orbit, and thus are all $C_{H}(t)$-conjugate. Therefore, all involutions that centralise $x$ and share a common top-left entry are $C_{H}(t)$-conjugate and so the lemma follows.

Lemma 5.5. $\left|\Delta_{2}(t)\right|=q\left(q^{2}-1\right)(q-2)$.

Proof. Let

$$
g=\left(\begin{array}{c|cc}
-1 & & \\
\hline & a & b \\
\bar{b} & -a
\end{array}\right) \in \Delta_{1}(t) \text { and } h=\left(\begin{array}{cc|c}
\alpha & \beta & \\
\bar{\beta} & -\alpha & \\
\hline & & -1
\end{array}\right) \in \Delta_{2}(t) \cap \Delta_{1}(x)
$$

for $\alpha \neq \pm 1$ and $\bar{\beta} \beta=1-\alpha^{2}$ fixed. Then

$$
g h=\left(\begin{array}{ccc}
-\alpha & a \beta & b \beta \\
-\bar{\beta} & -a \alpha & -b \alpha \\
0 & -\bar{b} & a
\end{array}\right) \text { and } h g=\left(\begin{array}{ccc}
-\alpha & -\beta & 0 \\
a \bar{\beta} & -a \alpha & -b \\
0 & -\bar{b} \alpha & a
\end{array}\right)
$$

If $[g, h]=1$ then $a \bar{\beta}=-\bar{\beta}$ and $b \beta=0$ imply $a=-1$ and $b=0$, since $\beta \neq 0$.
Therefore, $g=x$ and thus $h$ commutes with a single element of $\Delta_{1}(t)$. Since $\Delta_{1}(t)$ is a single $C_{H}(t)$-orbit, and combining Lemmas 5.2(iii) and 5.3(iii), all $C_{H}(t)$-orbits in $\Delta_{2}(t)$ have length $q(q-1)(q+1)=q\left(q^{2}-1\right)$. Hence $\left|\Delta_{2}(t)\right|=q\left(q^{2}-1\right)(q-2)$, since $\Delta_{2}(t)$ is a partition of $C_{H}(t)$-orbits.

For each $\alpha \in G F(q) \backslash\{ \pm 1\}$, define $\Delta_{2}^{\alpha}(t)$ to be the $C_{H}(t)$-orbit in $\Delta_{2}(t)$ consisting of matrices with top-left entry $\alpha \in G F(q) \backslash\{ \pm 1\}$. By (5.2) and Lemma 5.3(iii), $\Delta_{2}^{\alpha}(t)$ can be written explicitly as

$$
\Delta_{2}^{\alpha}(t)=\left\{\left.\left(\begin{array}{ccc}
\alpha & a D \beta & b D \beta  \tag{5.3}\\
d \bar{\beta} D^{-2} & (-a d \alpha+b c) D^{-1} & b d D^{-1}(1-\alpha) \\
-c \bar{\beta} D^{-2} & a c D^{-1}(\alpha-1) & (b c \alpha-a d) D^{-1}
\end{array}\right) \right\rvert\, \begin{array}{cc}
a & b \\
c & d
\end{array}\right) \in G U_{2}(q) ~\left(\begin{array}{cc} 
& \\
D=a d-b c \\
\bar{\beta} \beta=1-\alpha^{2}
\end{array}\right\} .
$$

Clearly, $\Delta_{2}(t)=\bigcup_{\alpha \in G F(q) \backslash\{ \pm 1\}} \Delta_{2}^{\alpha}(t)$.

Lemma 5.6. Suppose

$$
g=\left(\begin{array}{cc|c}
\alpha & \beta & \\
\bar{\beta} & -\alpha & \\
\hline & & -1
\end{array}\right) \in \Delta_{2}^{\alpha}(t) \cap \Delta_{1}(x)
$$

and

$$
h=\left(\begin{array}{ccc}
\gamma & a D \delta & b D \delta \\
d \bar{\delta} D^{-2} & (-a d \gamma+b c) D^{-1} & b d D^{-1}(1-\gamma) \\
-c \bar{\delta} D^{-2} & a c D^{-1}(\gamma-1) & (b c \gamma-a d) D^{-1}
\end{array}\right) \in \Delta_{2}^{\gamma}(t)
$$

satisfy the conditions of (5.3). If $[g, h]=1$ then
(i) $d=a \bar{\beta} \beta^{-1} \overline{\delta^{-1}} \delta D^{3}$;
(ii) if $b, c \neq 0$, then $a=-(1+\alpha)(1-\gamma)^{-1} \overline{\beta^{-1} \delta} D^{-1}$ and
$b=2 D \beta^{-1}(1-\gamma)^{-1}(\beta \gamma-a \alpha \delta D) c^{-1} ;$ and
(iii) if $b=c=0$, then $\beta \gamma=a \alpha \delta D$.

Proof. Recall that since $\alpha, \gamma \neq \pm 1$, we have $\beta, \delta \neq 0$. Direct calculation shows that

$$
g h=\left(\begin{array}{ccc}
\alpha \gamma+\beta d \bar{\delta} D^{-2} & \alpha a D \delta+\beta D^{-1}(b c-a d \gamma) & \alpha b D \delta+\beta b d D^{-1}(1-\gamma) \\
\bar{\beta} \gamma-\alpha d \bar{\delta} D^{-2} & \bar{\beta} a D \delta-\alpha D^{-1}(b c-a d \gamma) & \bar{\beta} b D \delta-\alpha b d D^{-1}(1-\gamma) \\
c \bar{\delta} D^{-2} & (1-\gamma) a c D^{-1} & -D^{-1}(b c \gamma-a d)
\end{array}\right)
$$

and
$h g=\left(\begin{array}{ccc}\alpha \gamma+\bar{\beta} a D \delta & \beta \gamma-a \alpha D \delta & -b D \delta \\ \alpha d \bar{\delta} D^{-2}+\bar{\beta} D^{-1}(b c-a d \gamma) & \beta d \bar{\delta} D^{-2}-\alpha(b c-a d \gamma) D^{-1} & -b d D^{-1}(1-\gamma) \\ -\alpha c \bar{\delta} D^{-2}+\bar{\beta}(\gamma-1) a c D^{-1} & -c \beta \bar{\delta} D^{-2}-a c D^{-1} \alpha(\gamma-1) & -D^{-1}(b c \gamma-a d)\end{array}\right)$.
Now if $[g, h]=1$ then we have the following relations from the $(1,1),(1,2),(1,3)$ and $(3,1)$ entries respectively:

$$
\begin{aligned}
\alpha \gamma+d \beta \bar{\delta} D^{-2} & =\alpha \gamma+a \bar{\beta} \delta D ; \\
a \alpha \delta D+\beta D^{-1}(b c-a d \gamma) & =\beta \gamma-a \alpha \delta D ; \\
b \alpha \delta D+b d \beta D^{-1}(1-\gamma) & =-b \delta D ; \quad \text { and } \\
-c \alpha \bar{\delta} D^{-2}+a c \bar{\beta} D^{-1}(\gamma-1) & =c \bar{\delta} D^{-2} .
\end{aligned}
$$

The relations from the other entries are all equivalent to the four shown above.
Looking at the relation from the $(1,1)$ entry, we determine that $d=a \bar{\beta} \beta^{-1} \overline{\delta^{-1}} \delta D^{3}$, giving (i). By considering the relation from the $(1,2)$ entry we get

$$
2 a \alpha \delta D=\beta \gamma-\beta D^{-1}(b c-a d \gamma)
$$

and substituting $a d=b c+D$, we get

$$
2 a \alpha \delta D=2 \beta \gamma-\beta D^{-1} b c(1-\gamma)
$$

Rearranging again, we get $b c=2 D \beta^{-1}(1-\gamma)^{-1}(\beta \gamma-a \alpha \delta D)$. From the relation deduced from the $(1,3)$ entry we get

$$
b d \beta D^{-1}(1-\gamma)=-b \delta D(1+\alpha)
$$

Using the relation for $d$ determined in (i), we get

$$
b\left(a \bar{\beta} \beta^{-1} \overline{\delta^{-1}} \delta D^{3}\right) \beta D^{-1}(1-\gamma)=-b \delta D(1+\alpha)
$$

and so

$$
a b=-b(1+\alpha)(1-\gamma)^{-1} \overline{\beta^{-1} \delta} D^{-1} .
$$

Hence either $b=0$ or $a=-(1+\alpha)(1-\gamma)^{-1} \overline{\beta^{-1}} \bar{\delta} D^{-1}$. Similarly from the relation deduced from the $(3,1)$ entry, either $a$ is as shown above or $c=0$. This gives both (ii) and (iii).

Lemma 5.7. Let $y_{\alpha} \in \Delta_{2}^{\alpha}(t)$ for some $\alpha \in G F(q) \backslash\{ \pm 1\}$. Then $\left|\Delta_{1}\left(y_{\alpha}\right) \cap \Delta_{2}^{-\alpha}(t)\right|=1$.

Proof. Without loss of generality, choose $y_{\alpha}$ such that $\left[y_{\alpha}, x\right]=1$, so $\left(y_{\alpha}\right)_{11}=\alpha$ and set $\left(y_{\alpha}\right)_{12}=\beta$. Let $y_{-\alpha} \in \Delta_{2}^{-\alpha}(t)$ be as in (5.3) for suitable $a, b, c, d \in G F\left(q^{2}\right)$. We remark that if $\alpha=0$, we denote this element $y_{0}^{\prime}$ to distinguish it from $y_{0}$. Assuming $\left[y_{-\alpha}, y_{\alpha}\right]=1$, we apply Lemma 5.6 by setting $\alpha=-\gamma$, and note that $\bar{\beta} \beta=\bar{\delta} \delta$.

Suppose that $b, c \neq 0$, then $a$ and $b$ are as in Lemma 5.6(ii). Since $\alpha=-\gamma$, we have $a=-D^{-1} \overline{\beta^{-1} \delta}$, giving $b=2 D \beta^{-1}(1-\gamma)^{-1}\left(\beta \gamma-\overline{\beta^{-1} \delta} \delta \gamma\right) c^{-1}$. However, $\beta \gamma-\overline{\beta^{-1}} \delta \delta \gamma=\beta\left(\gamma-\overline{\beta^{-1}} \beta^{-1} \bar{\delta} \delta \gamma\right)=0$ since $\overline{\beta^{-1}} \beta^{-1} \bar{\delta} \delta=1$. This yields $b=0$,
contradicting our original assumption. Hence $b=c=0$, giving $a$ as in Lemma 5.6(iii) and thus $a \delta \alpha D=-\beta \alpha$. Hence either $\alpha=0$ or $a=-\beta \delta^{-1} D^{-1}$. If $\alpha \neq 0$, then $a D=-\beta \delta^{-1}$ and $d D^{-2}=-\overline{\beta \delta^{-1}}$ showing that

$$
y_{-\alpha}=\left(\begin{array}{cc|c}
-\alpha & -\beta^{2} \delta^{-1} & \\
-\overline{\beta^{2} \delta^{-1}} & \alpha & \\
\hline & & -1
\end{array}\right) .
$$

If $\alpha=\gamma=0$, then both $y_{0}$ and $y_{0}^{\prime}$ commute with $x$, where $\left(y_{0}\right)_{12}=\beta$ and $\left(y_{0}^{\prime}\right)_{12}=\delta$. If $y_{0}$ and $y_{0}^{\prime}$ commute, then an easy calculation shows that $\delta= \pm \beta$. Since $y_{0} \neq y_{0}^{\prime}$, we must have $\delta=-\beta$.

Hence in both cases, $y_{\alpha}$ commutes with a single element of $\Delta_{2}^{-\alpha}(t)$.
Lemma 5.8. Let $y_{\alpha} \in \Delta_{2}^{\alpha}(t)$. Then $\left|\Delta_{1}\left(y_{\alpha}\right) \cap \Delta_{2}^{\gamma}(t)\right|=q+1$ for $\alpha \neq-\gamma$.

Proof. As in Lemma 5.7, choose $y_{\alpha}$ such that $\left[y_{\alpha}, x\right]=1$ with $\left(y_{\alpha}\right)_{11}=\alpha$ and set $\left(y_{\alpha}\right)_{12}=\beta$. Let $y_{\gamma} \in \Delta_{2}^{\gamma}(t)$ be as in (5.3) for suitable $a, b, c, d \in G F\left(q^{2}\right)$. For brevity we remark that if $\alpha=\gamma$, then $y_{\alpha}$ and $y_{\gamma}$ will denote different elements. Assume $\left[y_{\alpha}, y_{\gamma}\right]=1$, so the relevant relations from Lemma 5.6 hold for fixed $\alpha, \beta, \gamma, \delta$ satisfying $\alpha, \gamma \in G F(q) \backslash\{ \pm 1\}, \bar{\beta} \beta=1-\alpha^{2}$ and $\bar{\delta} \delta=1-\gamma^{2}$.

Suppose $b=c=0$, so Lemma 5.6(iii) holds. Since $\beta \neq 0$ and if $\alpha=0$, then $\gamma=0$, contradicting the assumption that $\alpha \neq-\gamma$. Hence $a=\beta \gamma \alpha^{-1} \delta^{-1} D^{-1}$. Using Lemma 5.6(i), we get $d=\overline{\beta \delta^{-1}} D^{2} \gamma \alpha^{-1}$ and so $a d=\bar{\beta} \beta \overline{\delta^{-1}} \delta^{-1} \gamma^{2} \alpha^{-2} D$. Combining the expressions for $\bar{\beta} \beta, \bar{\delta} \delta$ and $D$, we get

$$
\left(\gamma^{2}-\alpha^{2} \gamma^{2}\right)\left(\alpha^{2}-\alpha^{2} \gamma^{2}\right)^{-1}=1
$$

giving $\gamma^{2}=\alpha^{2}$ resulting in $\gamma= \pm \alpha$. Since $\alpha \neq-\gamma$, we must have $\alpha=\gamma$. But then $a D \delta=\beta$ and so $y_{\gamma}=y_{\alpha}$. Therefore, we may assume $b, c \neq 0$.

By a long but routine check, substitutions of $\bar{\beta} \beta, \bar{\gamma} \gamma$ and the relations in Lemma 5.6 show that $a d-b c=D$ holds. These relations also clearly show that $a, b, c$ and $d$ are all non-zero. Hence by (5.1), we have $\bar{a} b=-\bar{c} d$ and so $\bar{c} c=-\bar{a} b c d^{-1}$, and there are $q+1$ values of $c$ that satisfy this equation.

It now suffices to check that the remaining conditions of (5.1) hold. Since $\alpha, \gamma \in G F(q)$, we have $\overline{(1-\alpha)}(1-\alpha)^{-1}=\overline{(1-\gamma)}(1-\gamma)^{-1}=1$. Together with the relations already determined, we have $\bar{a} a+\bar{c} c=\bar{a} a-\bar{a} d^{-1} b c=\overline{D^{-1}} D^{-1}$. However $\bar{D} D=1$, so the conditions of (5.1) hold. By considering $\overline{\bar{a} a+\bar{c} c}$, we get a similar result for $\bar{b} b+\bar{d} d$. Hence there is only one possible value of each of $a$ and $d$, there are $(q+1)$ different values of $c$ with $b$ depending on $c$, proving the lemma.

We may summarise Lemmas $5.7-5.8$ by the following.
Proposition 5.9. Let $y_{\alpha} \in \Delta_{2}^{\alpha}(t)$, and $f_{\alpha, \gamma}$ be the number of elements in $\Delta_{2}^{\gamma}(t)$ that commute with $y_{\alpha}$. Then

$$
f_{\alpha, \gamma}= \begin{cases}1 & \text { if } \gamma=-\alpha \\ q+1 & \text { if } \gamma \neq-\alpha\end{cases}
$$

As a consequence, we have the following.

Corollary 5.10. Let $y \in \Delta_{2}(t)$. Then $\left|\Delta_{1}(y) \cap \Delta_{3}(t)\right|=q+1$.

Proof. Since the valency of the graph is $q(q-1)$, Proposition 5.9 gives Corollary 5.10 .

For the remainder of this chapter, denote

$$
y=\left(\begin{array}{rrr}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{array}\right) \in \Delta_{2}^{0}(t)
$$

and define $z_{\gamma}=\left(\begin{array}{rrr}1 & -2 & \bar{\gamma} \\ -2 & 1 & -\bar{\gamma} \\ \gamma & -\gamma & -3\end{array}\right)$, for $\bar{\gamma} \gamma=-4$. An easy check shows that
$\left[z_{\gamma}, y\right]=1, \bar{z}_{\gamma}^{T}=z_{\gamma}$ and $z_{\gamma}$ is an involution, hence $z_{\gamma} \in Z_{0}$ and $d\left(t, z_{\gamma}\right) \leq 3$.
However, since $t$ is the sole element with top-left entry 1 that is at most distance 2 from $t$, we have $d\left(t, z_{\gamma}\right) \geq 3$ and thus equality.

Lemma 5.11. $\Delta_{1}(y) \cap \Delta_{3}(t)=\left\{z_{\gamma} \mid \gamma \in G F\left(q^{2}\right), \bar{\gamma} \gamma+4=0\right\}$.

Proof. By Lemma 5.1, there are $q+1$ values of $\gamma$ and $z_{\gamma}$ centralises $y$ for all such $\gamma$. By Corollary 5.10, $\left|\Delta_{1}(y) \cap \Delta_{3}(t)\right|=q+1$, and so the lemma follows.

Fix $\gamma$ and let $g \in C_{H}(t)$ be of the form as described in (5.2) for suitable $a, b, c, d \in G F\left(q^{2}\right)$. Then

$$
z_{\gamma} g=\left(\begin{array}{ccc}
D^{-1} & -2 a+c \bar{\gamma} & -2 b+d \bar{\gamma} \\
-2 D^{-1} & a-\bar{\gamma} c & b-d \bar{\gamma} \\
\gamma D^{-1} & -\gamma a-3 c & -b \gamma-3 d
\end{array}\right)
$$

and

$$
g z_{\gamma}=\left(\begin{array}{ccc}
D^{-1} & -2 D^{-1} & D^{-1} \bar{\gamma} \\
-2 a+b \gamma & a-b \gamma & -a \bar{\gamma}-3 b \\
-2 c+d \gamma & c-d \gamma & -c \bar{\gamma}-3 d
\end{array}\right) .
$$

If $\left[z_{\gamma}, g\right]=1$, then we equate the entries to get conditional relations. From the (2,2) entries, we see that $b=c \bar{\gamma} \gamma^{-1}$. This, combined with the $(2,3)$ entry, gives $d=a+4 c \gamma^{-1}$. The $(3,1)$ entry shows that $c=-2^{-1}\left(D^{-1}-d\right) \gamma$, and so $d=2 D^{-1}-a$. Hence

$$
\begin{aligned}
& b=-2^{-1}\left(a-D^{-1}\right) \bar{\gamma} \\
& c=-2^{-1}\left(a-D^{-1}\right) \gamma ; \quad \text { and } \\
& d=2 D^{-1}-a
\end{aligned}
$$

for $a \in G F\left(q^{2}\right)$. A routine check shows these relations are sufficient for $\left[z_{\gamma}, g\right]=1$. These relations, together with the conditions of (5.1) and $\bar{D} D=1$, give

$$
\begin{equation*}
a \overline{D^{-1}}+\bar{a} D^{-1}=2 . \tag{5.4}
\end{equation*}
$$

Clearly, the number of possible such $a$ is $\left|C_{H}\left(\left\langle t, z_{\gamma}\right\rangle\right)\right|$. Since $D=a d-b c$, we get $D^{3}=1$. Therefore $\bar{D} D=D^{3}=1$ which has a non-trivial solution if and only if $q \equiv 5(\bmod 6)$.

Lemma 5.12. If $q \not \equiv 5(\bmod 6)$, then $\left|C_{H}\left(\left\langle t, z_{\gamma}\right\rangle\right)\right|=q$. Moreover, $\mathcal{C}\left(H, Z_{0}\right)$ is connected of diameter 3 and $\left|\Delta_{3}(t)\right|=(q+1)\left(q^{2}-1\right)$.


Figure 5.1: The collapsed adjacency graph for $\mathcal{C}\left(H, Z_{0}\right)$ when $q \not \equiv 5(\bmod 6)$.

Proof. Since $q \not \equiv 5(\bmod 6)$, from (5.4) we have $D=1$ and $\bar{a}+a-2=0$. There are $q$ distinct values of $a$ satisfying this, so $\left|C_{H}\left(\left\langle t, z_{\gamma}\right\rangle\right)\right|=q$. Denote the $C_{H}(t)$-orbit containing $z_{\gamma}$ by $\Delta_{3}^{\gamma}(t)$. Hence,

$$
\left|\Delta_{3}^{\gamma}(t)\right|=\frac{\left|C_{H}(t)\right|}{\left|C_{H}\left(\left\langle t, z_{\gamma}\right\rangle\right)\right|}=(q+1)\left(q^{2}-1\right) .
$$

Combining Lemmas 5.2(ii)-(iii) and 5.5, we have

$$
\left|Z_{0} \backslash\left(\{t\} \cup \Delta_{1}(t) \cup \Delta_{2}(t)\right)\right|=\left|\Delta_{3}^{\gamma}(t)\right| .
$$

Hence $\mathcal{C}\left(H, Z_{0}\right)$ is connected of diameter 3, and $\Delta_{3}^{\gamma}(t)=\Delta_{3}(t)$ as required.

Remark: Since $\Delta_{3}(t)$ is a single $C_{H}(t)$-orbit and the valency of the graph is $q(q-1)$, for $w \in \Delta_{3}(t)$ we have $\left|\Delta_{1}(w) \cap \Delta_{3}(t)\right|=q$. This proves Theorem 1.5 when $q \not \equiv 5(\bmod 6)$ and moreover, the collapsed adjacency diagram for $\mathcal{C}\left(H, Z_{0}\right)$ is as in Figure 5.1.

We now turn our attention to the remaining case, when $q \equiv 5(\bmod 6)$.

Lemma 5.13. Suppose $q \equiv 5(\bmod 6)$.
(i) $\left|C_{H}\left(\left\langle t, z_{\gamma}\right\rangle\right)\right|=3 q$.
(ii) There are exactly three $C_{H}(t)$-orbits in $\Delta_{3}(t)$, each of length $\frac{1}{3}(q+1)\left(q^{2}-1\right)$.
(iii) $\mathcal{C}\left(H, Z_{0}\right)$ is connected of diameter 3 and $\left|\Delta_{3}(t)\right|=(q+1)\left(q^{2}-1\right)$.

Proof. From (5.4), we have $\bar{D} D=D^{3}=1$ and since $q \equiv 5(\bmod 6)$, there are three possible values for $D$. Since $a \overline{D^{-1}}+\bar{a} D^{-1}-2=\overline{\left(a \overline{D^{-1}}\right)}+a \overline{D^{-1}}-2=0$ then for each value of $D$, there are $q$ such values of $a \overline{D^{-1}}$. Hence there are $3 q$ values of $a \overline{D^{-1}}$ in total, proving (i).

Fix $\gamma$, and let $\Delta_{3}^{\gamma}(t)$ be the $C_{H}(t)$-orbit containing $z_{\gamma}$. We have

$$
\begin{equation*}
\left|\Delta_{3}^{\gamma}(t)\right|=\frac{\left|C_{H}(t)\right|}{\left|C_{H}\left(\left\langle t, z_{\gamma}\right\rangle\right)\right|}=\frac{1}{3}(q+1)\left(q^{2}-1\right) . \tag{5.5}
\end{equation*}
$$

Let $h=\left(\begin{array}{c|cc}E & & \\ & \lambda & \mu \\ \sigma & \tau\end{array}\right) \in C_{H}(t)$ where $E=\lambda \tau-\mu \sigma$. Then
$h^{-1} z_{\gamma} h=\left(\begin{array}{ccc}1 & E(\bar{\gamma} \sigma-2 \lambda) & E(-2 \mu+\tau \bar{\gamma}) \\ -E^{-2}(2 \tau+\mu \gamma) & (\lambda \mu \gamma-\sigma \bar{\gamma} \tau+4 \mu \sigma) E^{-1}+1 & \left(-\bar{\gamma} \tau^{2}+\mu^{2} \gamma+4 \mu \tau\right) E^{-1} \\ E^{-2}(2 \sigma+\lambda \gamma) & \left(-\lambda^{2} \gamma+\sigma^{2} \bar{\gamma}-4 \lambda \sigma\right) E^{-1} & (\lambda \mu \gamma-\sigma \bar{\gamma} \tau+4 \mu \sigma) E^{-1}-3\end{array}\right)$.
Suppose $h^{-1} z_{\gamma} h=z_{\delta} \in \Delta_{3}(t) \cap \Delta_{1}(y)$ for some $\delta \neq \gamma$. Hence $\left(h^{-1} z_{\gamma} h\right)_{21}=-2=\left(h^{-1} z_{\gamma} h\right)_{12}$ gives $\tau=E^{2}-2^{-1} \mu \gamma$ and $\lambda=2^{-1} \bar{\gamma} \sigma+E^{-1}$. Since $E=\lambda \tau-\mu \sigma$, we have $2^{-1} \bar{\gamma} \sigma E^{2}-2^{-1} \mu \gamma E^{-1}=0$ and so $\mu=\bar{\gamma} \gamma^{-1} \sigma E^{3}$. Rewriting $\tau$, we get $\tau=E^{2}-2^{-1} \bar{\gamma} \sigma E^{3}$. To summarise,

$$
\begin{aligned}
& \lambda=2^{-1} \bar{\gamma} \sigma+E^{-1} ; \\
& \mu=\bar{\gamma} \gamma^{-1} \sigma E^{3} ; \quad \text { and } \\
& \tau=E^{2}-2^{-1} \bar{\gamma} \sigma E^{3} .
\end{aligned}
$$

Using these relations and $\bar{\gamma} \gamma=-4$, a simple check shows that $\left(h^{-1} z_{\gamma} h\right)_{22}=1$ and $\left(h^{-1} z_{\gamma} h\right)_{33}=-3$ hold, and $\left(h^{-1} z_{\gamma} h\right)_{31}=E^{-3} \gamma=\delta$. Easy substitutions and checks show that $\left(h^{-1} z_{\gamma} h\right)_{32}=-\left(h^{-1} z_{\gamma} h\right)_{31}$ and $\overline{\left(h^{-1} z_{\gamma} h\right)_{13}}=\left(h^{-1} z_{\gamma} h\right)_{31}$. Since $\bar{\delta} \delta=-4$, we have $\overline{E^{3}} E^{3}=1$. In particular, $E^{3}$ is a $(q+1)^{\text {th }}$ root of unity. There are $q+1$ such roots and only a third of them are cubes in $G F\left(q^{2}\right)^{*}$. Hence there are only $\frac{1}{3}(q+1)$ such values of $\delta=E^{-3} \gamma$. Therefore, we can pick $\gamma_{1}, \gamma_{2}$ and $\gamma_{3}$ such that $\overline{\gamma_{i}} \gamma_{i}=-4$ where the $z_{\gamma_{i}}$ are not pairwise $C_{H}(t)$-conjugate. Hence there are at least 3 orbits in $\Delta_{3}(t)$, and by (5.5) they all have length $\frac{1}{3}(q+1)\left(q^{2}-1\right)$. But (as in the
proof of Lemma 5.12), $\left|Z_{0} \backslash\left(\{t\} \cup \Delta_{1}(t) \cup \Delta_{2}(t)\right)\right|=(q+1)\left(q^{2}-1\right)$ and so this proves (ii), and (iii) follows immediately.

This now completes the proof of Theorem 1.5 for all $q$. We conclude this chapter by determining the collapsed adjacency diagram for $\mathcal{C}\left(H, Z_{0}\right)$ when $q \equiv 5(\bmod 6)$.

Lemma 5.14. Suppose $q \equiv 5(\bmod 6)$ and let $y_{\alpha} \in \Delta_{2}^{\alpha}(t)$. Then for any $w \in \Delta_{3}(t)$, $\left|\Delta_{1}\left(y_{\alpha}\right) \cap w^{C_{H}(t)}\right|=\frac{1}{3}(q+1),$.

Proof. If $y_{\alpha}$ commutes with $\mu_{1}$ elements of $\Delta_{3}^{\gamma}(t)$ for some $\gamma \in G F\left(q^{2}\right)$, then $z \in \Delta_{3}^{\gamma}(t)$ commutes with $\mu_{2}$ elements of $\Delta_{2}^{\alpha}(t)$, where

$$
\left|\Delta_{2}^{\alpha}(t)\right| \mu_{1}=\mu_{2}\left|\Delta_{3}^{\gamma}(t)\right| .
$$

Hence by Lemmas 5.5 and 5.13, we have $q \mu_{1}=\frac{1}{3}(q+1) \mu_{2}$. Since $q$ and $\frac{1}{3}(q+1)$ are coprime, $q$ divides $\mu_{2}$ and so $\mu_{2}=n q$ for some positive integer $n$. Hence $\mu_{1}=\frac{1}{3} n(q+1)$ and since $\mu_{1} \leq q+1$ by Lemma 5.10, $n \in\{1,2,3\}$. It suffices to prove that there exists an element from each $C_{H}(t)$-orbit in $\Delta_{3}(t)$ that commutes with $y_{\alpha}$, since this then forces $n=1$.
Recall $z_{\gamma} \in \Delta_{3}^{\gamma}(t)$ for some $\bar{\gamma} \gamma=-4$, and let $x_{\alpha, \beta}=\left(\begin{array}{cc|c}\alpha & \beta & \\ \bar{\beta} & -\alpha & \\ \hline & & -1\end{array}\right) \in \Delta_{2}(t) \cap \Delta_{1}(x)$ where $\bar{\beta} \beta=1-\alpha^{2}$ and $\alpha \neq 0$ (the case when $\alpha=0$ has been dealt with in Lemma 5.11). Consider

$$
g=\left(\begin{array}{c|cc}
1 & \\
& \beta^{-1} & 2^{-1} \alpha \bar{\gamma} \beta^{-1} \\
-2^{-1} \alpha \gamma \overline{\beta^{-1}} & \overline{\beta^{-1}}
\end{array}\right) \in C_{H}(t)
$$

and set $y_{\alpha}=x_{\alpha, \beta}^{g} \in \Delta_{2}^{\alpha}(t)$. A direct calculation shows that $\left[y_{\alpha}, z_{\gamma}\right]=1$ for all $\gamma$ such that $\bar{\gamma} \gamma=-4$ and hence $y_{\alpha}$ commutes with at least one element in each $C_{H}(t)$-orbit of $\Delta_{3}(t)$, proving the lemma.

Lemma 5.15. Suppose $q \equiv 5(\bmod 6)$ and let $z \in \Delta_{3}(t)$. Then for all $w \in \Delta_{3}(t) \cap \Delta_{1}(z), z$ and $w$ are $C_{H}(t)$-conjugate and $\left|\Delta_{3}(t) \cap \Delta_{1}(z)\right|=q$.

Proof. Any $z \in \Delta_{3}(t)$ commutes with $q$ elements in each of the $(q-2) C_{H}(t)$-orbits of $\Delta_{2}(t)$. Since the valency of the graph is $q(q-1)$, we have $\left|\Delta_{3}(t) \cap \Delta_{1}(z)\right|=q(q-1)-q(q-2)=q$. Recall $z_{\gamma} \in \Delta_{3}^{\gamma}(t)$ and without loss of generality, set $z=z_{\gamma}$. Let

$$
y_{-3}=\left(\begin{array}{ccc}
-3 & 1 & -3\left(2^{-1}\right) \bar{\gamma} \\
1 & -3\left(2^{-1}\right) & 3 \gamma^{-1} \\
-3\left(2^{-1}\right) \gamma & 3 \overline{\gamma^{-1}} & 7\left(2^{-1}\right)
\end{array}\right)
$$

which, by Lemma 5.14, is an element of $\Delta_{2}^{-3}(t)$ commuting with $z$. Set

$$
w=y_{-3} z=\left(\begin{array}{ccc}
1 & 1 & 2^{-1} \bar{\gamma} \\
1 & -2^{-1} & \gamma^{-1} \\
2^{-1} \gamma & \overline{\gamma^{-1}} & -3\left(2^{-1}\right)
\end{array}\right)
$$

which is an involution in $\Delta_{3}(t)$, since $(w)_{11}=1$ and $\left[z, y_{-3}\right]=1$.
First observe that $z$ and $w$ are $C_{H}(t)$-conjugate, via the element

$$
g=\left(\begin{array}{c|cc}
-1 & \\
& 2^{-1}(1+c \bar{\gamma}) & \gamma^{-1}(3-c \bar{\gamma}) \\
& c & -2+2^{-1} c \bar{\gamma}
\end{array}\right) \in C_{H}(t)
$$

where $c=3 \gamma\left(\gamma^{2}-4\right)^{-1}$. After some manipulation, one can also show $\bar{c}=c$. Let

$$
h=\left(\begin{array}{c|cc}
D^{-1} & \\
\hline & a & -2^{-1}\left(a-D^{-1}\right) \bar{\gamma} \\
& -2^{-1}\left(a-D^{-1}\right) \gamma & 2 D^{-1}-a
\end{array}\right)
$$

where $a \overline{D^{-1}}+\bar{a} D^{-1}=2$ and $D^{3}=1$. From the discussion prior to Lemma 5.12, $h \in C_{H}\left(\left\langle g, z_{\gamma}\right\rangle\right)$. By a direct calculation, if such an element were to commute with $w$, this will force $a=D^{-1}$ and so $C_{H}(\langle t, z, w\rangle)=Z(H)$, which has order 3. Hence, $\left|w^{C_{H}(\langle t, z\rangle)}\right|=q$ accounting for all involutions in $\Delta_{3}(t) \cap \Delta_{1}(z)$. Thus, all elements in $\Delta_{3}(t) \cap \Delta_{1}(z)$ are $C_{H}(\langle t, z\rangle)$-conjugate, and hence $C_{H}(t)$-conjugate to $z$.

With the addition of Lemmas 5.14 and 5.15, we can now determine the collapsed adjacency diagram for $\mathcal{C}\left(H, Z_{0}\right)$ where $q \equiv 5(\bmod 6)$ to be as in Figure 5.2. One


Figure 5.2: The collapsed adjacency graph for $\mathcal{C}\left(H, Z_{0}\right)$ when $q \equiv 5(\bmod 6)$. may note that the permutation action of $H$ on $Z_{0}$ is the same as the action of $G$ on the non-isotropic points $\{\langle v\rangle \mid(v, v)=1\}$. This is because for any $z \in Z_{0}, C_{V}(z)$ is a non-isotropic 1-space and by Lemma 10.14 of [38], $G$ is transitive on the set of non-isotropic 1-spaces.

## Chapter 6

## 4-Dimensional Unitary Groups <br> over Even Characteristic Fields

In a manner similar to Chapter 3, we now focus on the 4-dimensional unitary groups and, in particular, Theorems 1.6 and 1.7. Let $q$ be an even prime power and let $H=S U_{4}(q) \cong U_{4}(q)=G$. Let $V$ be the unitary $G F\left(q^{2}\right) G$-module and $(\cdot, \cdot)$ be the corresponding unitary form on $V$, defined by the Gram matrix

$$
J=\left(\begin{array}{llll}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{array}\right)=\binom{J_{0}}{J_{0}}
$$

with respect to some basis of $V$. As in Chapter 5, for any $a \in G F\left(q^{2}\right)$ we write $\bar{a}=a^{q}$ and $\overline{\left(a_{i j}\right)}=\left(\overline{a_{i j}}\right)$. Let

$$
t_{1}=\left(\begin{array}{llll}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) \quad \text { and } \quad t_{2}=\left(\begin{array}{cccc}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) .
$$

Since ${\overline{t_{i}}}^{T}=t_{i}^{T}$ and $G$ has two conjugacy classes of involutions (see, for example, Lemma 6.1 of [10]), a straightforward calculation shows that $t_{i} \in G$, and

$$
\begin{aligned}
& Z_{1}=\left\{x \in G \mid x^{2}=1, \operatorname{dim} C_{V}(x)=3\right\}, \\
& Z_{2}=\left\{x \in G \mid x^{2}=1, \operatorname{dim} C_{V}(x)=2\right\}
\end{aligned}
$$

being the two conjugacy classes of involutions in $G$, with $t_{i} \in Z_{i}$. We set

$$
S=\left\{\left.\left(\begin{array}{cccc}
1 & a & b & c  \tag{6.1}\\
0 & 1 & d & \bar{a} d+\bar{b} \\
0 & 0 & 1 & \bar{a} \\
0 & 0 & 0 & 1
\end{array}\right) \right\rvert\, \begin{array}{c} 
\\
a, b, c \in G F\left(q^{2}\right) \\
d \in G F(q) \\
c+\bar{c}=a \bar{b}+\bar{a} b
\end{array}\right\}
$$

and a routine check shows $S \in \operatorname{Syl}_{2}(G)$.

### 6.1 The Structure of $\mathcal{C}\left(G, Z_{1}\right)$

We begin by proving Theorem 1.6. Clearly, $\operatorname{dim} C_{V}\left(t_{1}\right)=3, \operatorname{dim}\left[V, t_{1}\right]=1$ and [ $V, t_{1}$ ] is an isotropic 1-subspace of $V$ stabilized by $t_{1}$. We set $Q=O_{2}\left(C_{G}\left(t_{1}\right)\right)$, and let $S$ be as in (6.1).

Lemma 6.1. (i) Let $R \in \operatorname{Syl}_{2}\left(\operatorname{Stab}_{G}\left[V, t_{1}\right]\right)$. Then $R \in \operatorname{Syl}_{2}\left(C_{G}\left(t_{1}\right)\right)$ and, in particular, if $x \in \operatorname{Stab}_{G}\left[V, t_{1}\right]$ is an involution, then $x \in C_{G}\left(t_{1}\right)$.
(ii) $\mathcal{C}\left(G, Z_{1}\right)$ is connected of diameter 2.

Proof. Showing $C_{G}\left(t_{1}\right) \leq \operatorname{Stab}_{G}\left[V, t_{1}\right]$ is a routine check (see Proposition 2.15). By Lemma 6.2 of [10], $C_{G}\left(t_{1}\right)$ contains a subgroup $M \cong S L_{2}(q)$ such that $Q \cap M=1$. Moreover, Proposition 4.1.18 of [32] gives the structure of $\operatorname{Stab}_{G}\left[V, t_{1}\right]$ to be $\operatorname{Stab}_{G}\left[V, t_{1}\right] \cong\left[q^{5}\right]:[a] . S L_{2}(q) .[b]$ where $[a]$ and $[b]$ are odd order subgroups that normalise the group isomorphic to $S L_{2}(q)$. Hence $|Q|=q^{5}$, and the subgroups with shape $[a]$ and $[b]$ normalise $Q M$. Therefore, $\left|\operatorname{Syl}_{2}\left(\operatorname{Stab}_{G}\left[V, t_{1}\right]\right)\right|=\left|\operatorname{Syl}_{2}(Q M)\right|$. In particular, any involution $x \in \operatorname{Stab}_{G}\left[V, t_{1}\right]$ must then centralise $t_{1}$, so proving (i). Let $U$ be a 2-subspace of $V$. If the unitary form on restriction to $U$ is degenerate, then it contains an isotropic vector. Suppose then the unitary form is
non-degenerate on restriction to $U$. Up to conjugacy, the unitary form is unique and so $U$ contains an isotropic vector (see Satz 8.8 of [28]). Let $x \in X$ such that $x \notin C_{G}\left(t_{1}\right)$, and $W$ be an isotropic 1-subspace of $C_{V}\left(\left\langle t_{1}, x\right\rangle\right)$. By Witt's Lemma, $G$ is transitive on the set of isotropic 1-subspaces of $V$. Since $\left[V, y_{0}\right.$ ] is an isotropic 1 -space for all $y_{0} \in X$, there exists $y \in X$ such that $W=[V, y]$. By an identical argument as that in Lemma 3.3(ii), $y$ stabilises any subspace of $V$ containing $W$. In particular, $y$ stabilises both $C_{V}(t)$ and $C_{V}(x)$. Moreover, $y$ is an involution and so by (i), $y \in C_{G}\left(t_{1}\right) \cap C_{G}(x)$. Since $t_{1} \neq y \neq x$ we have $d\left(t_{1}, x\right)=2$. Moreover, $x$ is arbitrary and so $\mathcal{C}\left(G, Z_{1}\right)$ is connected of diameter 2 , so proving (ii).

We now describe explicitly the structure of $Q=O_{2}\left(C_{G}\left(t_{1}\right)\right)$. Clearly $C_{S}\left(\left[V, t_{1}\right]\right)=S$, and Lemma 4.1.12 of [32] reveals that $Q$ centralizes the spaces $\left[V, t_{1}\right],\left[V, t_{1}\right]^{\perp} /\left[V, t_{1}\right]$ and $V /\left[V, t_{1}\right]^{\perp}$, so with respect to a suitable basis we have

$$
Q=\left\{\left.\left(\begin{array}{cccc}
1 & a & b & c  \tag{6.2}\\
0 & 1 & 0 & \bar{b} \\
0 & 0 & 1 & \bar{a} \\
0 & 0 & 0 & 1
\end{array}\right) \right\rvert\, \begin{array}{l} 
\\
a, b, c \in G F\left(q^{2}\right) \\
c+\bar{c}=a \bar{b}+b \bar{a}
\end{array}\right\} .
$$

A simple calculation shows

$$
Q \cap\left(Z_{1} \cup Z_{2}\right)=\left\{\left.\left(\begin{array}{cccc}
1 & a & b & c \\
0 & 1 & 0 & \bar{b} \\
0 & 0 & 1 & \bar{a} \\
0 & 0 & 0 & 1
\end{array}\right) \right\rvert\, \begin{array}{l} 
\\
a, b \in G F\left(q^{2}\right) \\
c, a \bar{b} \in G F(q)
\end{array}\right\}
$$

We define the following subsets of $S$ :

$$
Q_{0}=\left\{\left.\left(\begin{array}{cccc}
1 & 0 & 0 & c \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) \right\rvert\, c \in G F(q)^{*}\right\}, \quad Q_{1}=\left\{\left.\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & d & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) \right\rvert\, d \in G F(q)^{*}\right\}
$$

and

$$
Q_{2}=\left\{\left.\left(\begin{array}{cccc}
1 & 0 & b & c \\
0 & 1 & d & \bar{b} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) \right\rvert\, \begin{array}{c}
b \bar{b}-c d=0 \\
b \in G F\left(q^{2}\right)^{*}, c, d \in G F(q)^{*}
\end{array}\right\} .
$$

Lemma 6.2. (i) $Q_{0}=Q \cap Z_{1}$.
(ii) $Q_{0} \dot{\cup} Q_{1} \dot{\cup} Q_{2}=S \cap Z_{1}$.
(iii) $\left|\Delta_{1}\left(t_{1}\right)\right|=q^{4}-q^{2}+q-2$.
(iv) $\left|\Delta_{2}\left(t_{1}\right)\right|=q^{5}(q-1)$.

Proof. Let $v=(\alpha, \beta, \gamma, \delta) \in V$ and let

$$
x=\left(\begin{array}{cccc}
1 & a & b & c \\
0 & 1 & 0 & \bar{b} \\
0 & 0 & 1 & \bar{a} \\
0 & 0 & 0 & 1
\end{array}\right) \in Q \cap\left(Z_{1} \cup Z_{2}\right)
$$

with $a, b \in G F\left(q^{2}\right)$ such that $a \bar{b} \in G F(q)$, and $c \in G F(q)$. It is easily seen that $v^{x}=(\alpha, a \alpha+\beta, b \alpha+\gamma, c \alpha+\bar{b} \beta+\bar{a} \gamma+\delta)$. If $v^{x} \in C_{V}(x)$, then $\alpha=0$ and $\bar{b} \beta+\bar{a} \gamma=0$. Routine calculations show that $\operatorname{dim} C_{V}(x)=3$ if and only if $a=b=0$ and $c \neq 0$. This proves (i) and $\left|Q_{0}\right|=q-1$.

Let $y \in(S \backslash Q) \cap\left(Z_{1} \cup Z_{2}\right)$ and so $y$ is of the form

$$
y=\left(\begin{array}{llll}
1 & 0 & b & c \\
0 & 1 & d & \bar{b} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

where $b \in G F\left(q^{2}\right)$ and $c, d \in G F(q)$ with $d \neq 0$. Moreover, $v^{y}=(\alpha, \beta, b \alpha+d \beta+\gamma, c \alpha+\bar{b} \beta+\delta)$. If $v^{y} \in C_{V}(y)$ then $b \alpha+d \beta=0$ and $c \alpha+\bar{b} \beta=0$. Routine analysis of the cases reveal that $\operatorname{dim} C_{V}(y)=3$ if and only if $b=c=0$ and $d \neq 0$; or $b, c, d \neq 0$ and $b \bar{b}=c d$. The former case gives rise to $Q_{1}$,
while the latter case yields $Q_{2}$. This accounts for all involutions in $S$. Since the union is clearly disjoint, (ii) follows.

Clearly $\left|Q_{1}\right|=q-1$ so it remains to determine $\left|Q_{2}\right|$. By Lemma 5.1, $x^{q+1}-\lambda$ splits over $G F\left(q^{2}\right)$ for any $\lambda \in G F(q)$ and so there are $q+1$ such $b \in G F\left(q^{2}\right)$ satisfying $\bar{b} b-c d=0$ for any given $c, d \in G F(q)^{*}$. Hence, $\left|Q_{2}\right|=(q-1)^{2}(q+1)$ and so

$$
\left|(S \backslash Q) \cap Z_{1}\right|=\left|Q_{1} \cup Q_{2}\right|=\left(q^{2}-1\right)(q-1)+(q-1)=q^{2}(q-1)
$$

For any $R_{1}, R_{2} \in \operatorname{Syl}_{2}(Q M)$, we have $R_{1} \cap R_{2}=Q$ and $\left|\operatorname{Syl}_{2} M\right|=q+1$. Hence

$$
\begin{aligned}
\left|\Delta_{1}\left(t_{1}\right)\right| & =\left|\operatorname{Syl}_{2} M\right|\left|(S \backslash Q) \cap Z_{1}\right|+\left|Q \cap Z_{1}\right|-1 \\
& =\left|\operatorname{Syl}_{2} M\right|\left|Q_{1} \cup Q_{2}\right|+\left|Q_{0}\right|-1 \\
& =q^{2}(q-1)(q+1)+(q-1)-1 \\
& =q^{4}-q^{2}+q-2,
\end{aligned}
$$

so proving (iii).
As determined in Theorem 4 of $[24],\left|Z_{1}\right|=\left(q^{2}-q+1\right)\left(q^{4}-1\right)$ and since the diameter of $\mathcal{C}\left(G, Z_{1}\right)$ is 2 by Lemma 6.1(ii), we have

$$
\begin{aligned}
\left|\Delta_{2}\left(t_{1}\right)\right| & =\left|Z_{1}\right|-\left|\Delta_{1}\left(t_{1}\right)\right|-1 \\
& =\left(q^{2}-q+1\right)\left(q^{4}-1\right)-\left(q^{4}-q^{2}+q-2\right)-1 \\
& =q^{5}(q-1),
\end{aligned}
$$

and (iv) follows.

Lemmas 6.1 and 6.2 combined prove Theorem 1.6.

### 6.2 The Structure of $\mathcal{C}\left(G, Z_{2}\right)$

We now concentrate on proving Theorem 1.7. Set $t=t_{2}$ and it is easily seen that $[V, t]=C_{V}(t)$ is a 2-dimensional totally isotropic subspace of $V$. Proposition 4.1.18 of [32] reveals that $\operatorname{Stab}_{G} C_{V}(t) \cong\left[q^{4}\right]:[a] . L_{2}\left(q^{2}\right) .[b]$ for odd order subgroups [a] and [b]. Moreover, Lemma 6.2 of [10] shows that $P=O_{2}\left(C_{G}(t)\right)$ has order $q^{4}$ and $C_{G}(t)$
contains a subgroup $L \cong L_{2}(q)$ such that $P \cap L=1$. Theorem 4 of [24] determines that $\left|Z_{2}\right|=q\left(q^{3}+1\right)\left(q^{4}-1\right)$ and so by comparing orders, $C_{G}(t) \cong\left[q^{4}\right]: L_{2}(q)$. We write $C_{G}(t)=P L$ and let $S \in \operatorname{Syl}_{2}(G)$ be as in (6.1). Let

$$
s=\left(\begin{array}{cccc}
1 & a & b & c \\
0 & 1 & d & \bar{a} d+\bar{b} \\
0 & 0 & 1 & \bar{a} \\
0 & 0 & 0 & 1
\end{array}\right) \in S
$$

where $a, b, c \in G F\left(q^{2}\right), d \in G F(q)$ and $c+\bar{c}=a \bar{b}+b \bar{a}$. By Lemma 4.1.12 of [32], $P$ centralises the spaces $C_{V}(t)$ and $V / C_{V}(t)$ (since $C_{V}(t)=[V, t]$ is totally isotropic).

Hence, if $s \in P$ then $\bar{a}=0$ and thus $a=0$. Therefore,

$$
P=\left\{\left.\left(\begin{array}{cccc}
1 & 0 & b & c \\
0 & 1 & d & \bar{b} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) \right\rvert\, c, d \in G F(q), b \in G F\left(q^{2}\right)\right\}
$$

and is elementary abelian of order $q^{4}$. Define

$$
S_{t}=\left\{\left.\left(\begin{array}{cccc}
1 & a & b & c \\
0 & 1 & d & a d+\bar{b} \\
0 & 0 & 1 & a \\
0 & 0 & 0 & 1
\end{array}\right) \right\rvert\, \begin{array}{c} 
\\
a, d \in G F(q) \\
b, c \in G F\left(q^{2}\right) \\
c+\bar{c}=a(b+\bar{b})
\end{array}\right\} \subseteq S
$$

A direct calculation shows that $S_{t}=C_{S}(t)$ and clearly $P \unlhd S_{t}$. Moreover, let

$$
P_{1}=\left\{\left.\left(\begin{array}{cccc}
1 & a & b & c \\
0 & 1 & 0 & b \\
0 & 0 & 1 & a \\
0 & 0 & 0 & 1
\end{array}\right) \right\rvert\, a, b, c \in G F(q)\right\}
$$

which is elementary abelian of order $q^{3}$ and routine calculations reveal that $S_{t}=P P_{1}$ and $Z\left(S_{t}\right)=P \cap P_{1}$.

Lemma 6.3. (i) $P=C_{G}\left(C_{V}(t)\right)$.
(ii) $L=\left\{\left.\left(\begin{array}{l|l}A & \\ \hline & A\end{array}\right) \right\rvert\, A \in S L_{2}(q)\right\} \cong S L_{2}(q)$.
(iii) $\operatorname{Stab}_{G} C_{V}(t)=N_{G}(P)$.

Proof. By Lemma 4.1.12 of [32], $P \leq C_{G}\left(C_{V}(t)\right)$. Let $v=(0,0, \alpha, \beta) \in C_{V}(t)$ and let

$$
g=\left(\begin{array}{l|l}
A & B \\
\hline C & D
\end{array}\right) \in G
$$

where each block is a $2 \times 2$ matrix. Thus $v^{g}=((\alpha, \beta) C,(\alpha, \beta) D)$ and so if $g \in C_{G}\left(C_{V}(t)\right)$ then $C=0$ and $D=I_{2}$. Moreover, $\operatorname{det} g=1=\operatorname{det} A$ and $\bar{g}^{T} J g=J$ reveals $J_{0} A=J_{0}$. Since $\operatorname{det} A$ and $\operatorname{det} J_{0}$ are both non-zero, this forces $A=I_{2}$. This now forces $g \in P$, so proving the reverse inclusion and thus (i).

It is clear that the description of $L$ given in the statement of the lemma is isomorphic to $S L_{2}(q)$ and a quick check shows that $L \leq C_{G}(t)$ and $L \cap P=1$, so proving (ii).

By Theorem 1 of [31], $\operatorname{Stab}_{G} C_{V}(t)$ is maximal in $G$. Since $P \unlhd \operatorname{Stab}_{G} C_{V}(t) \leq N_{G}(P)$ and $G$ is simple, (iii) follows.

Lemma 6.4. (i) $\left|P \cap Z_{2}\right|=q(q-1)\left(q^{2}+1\right)$.
(ii) $\left|\Delta_{1}(t)\right|=q(q-1)\left(2 q^{2}+q+1\right)-1$.

Proof. Recall $Q$ from (6.2). Comparing $(S \backslash Q) \cap\left(Z_{1} \cup Z_{2}\right)$ with $P^{\#}$, it is easy to see we have equality. As shown in Lemma 6.2,
$\left|(S \backslash Q) \cap Z_{1}\right|=\left|P \cap Z_{1}\right|=(q-1)\left(q^{2}+1\right)$. Since $P$ is elementary abelian of order $q^{4}$, there are $q^{4}-1$ possible involutions in $P$ and so $\left|P \cap Z_{2}\right|=\left(q^{4}-1\right)-2(q-1)-\left(q^{2}-1\right)(q-1)=q(q-1)\left(q^{2}+1\right)$, proving (i).

Let

$$
y=\left(\begin{array}{cccc}
1 & a & b & c \\
0 & 1 & d & a d+\bar{b} \\
0 & 0 & 1 & a \\
0 & 0 & 0 & 1
\end{array}\right) \in S_{t} \backslash P
$$

and so $a \neq 0, a, d \in G F(q), b, c \in G F\left(q^{2}\right)$ and $c+\bar{c}=a(b+\bar{b})$. Let $v=(\alpha, \beta, \gamma, \delta) \in V$. Then $v^{y}=(\alpha, a \alpha+\beta, b \alpha+d \beta+\gamma, c \alpha+(a d+\bar{b}) \beta+a \gamma+\delta)$
and so $v \in C_{V}(y)$ if and only if $\alpha=0($ since $a \neq 0)$ and $d \beta=(a d+\bar{b}) \beta+a \gamma=0$. Therefore $C_{V}(y)$ is 3-dimensional if and only if $d=0$, hence (as can be seen in Lemma 6.2(i)) $a, b=0$ and $c \neq 0$. However, since $a \neq 0$, any involution in $S_{t} \backslash P$ lies in $Z_{2}$. By direct calculation $y^{2}=I_{4}$ if and only if $d=0$ and $a \bar{b}=a b$. Hence, $b \in G F(q)$ and so

$$
\left(S_{t} \backslash P\right) \cap Z_{2}=\left\{\left.\left(\begin{array}{cccc}
1 & a & b & c \\
0 & 1 & 0 & b \\
0 & 0 & 1 & a \\
0 & 0 & 0 & 1
\end{array}\right) \right\rvert\, a, b, c \in G F(q), a \neq 0\right\}
$$

of order $q^{2}(q-1)$. Let $S_{0} \in \operatorname{Syl}_{2} L$ such that $P S_{0}=S_{t}$. All $q+1$ conjugates of $S_{0}$ pairwise-intersect trivially, $\left|Z_{2} \cap P\right|=q(q-1)\left(q^{2}+1\right)$ and $\left|Z_{2} \cap\left(S_{t} \backslash P\right)\right|=q^{2}(q-1)$, so combining these facts we get

$$
\begin{aligned}
\left|\Delta_{1}(t)\right| & =q(q-1)\left(q^{2}+1\right)+(q+1)(q-1) q^{2}-1 \\
& =q(q-1)\left(2 q^{2}+q+1\right)-1,
\end{aligned}
$$

proving the lemma.

We introduce the following notation. For any $x \in Z_{2}$, define $P_{x}=C_{G}\left(C_{V}(x)\right)$, $C_{x}=C_{G}(x)$ and $N_{x}=\operatorname{Stab}_{G}\left(C_{V}(x)\right)$ and observe that $P=P_{t}$.

Lemma 6.5. Let $x, y \in Z_{2}$.
(i) If $C_{V}(x)=C_{V}(y)$ then $[x, y]=1$.
(ii) $P_{x}=P_{y}$ if and only if $P_{x} \cap P_{y} \cap Z_{2} \neq \varnothing$.
(iii) If $y \in N_{t}$ then $d(t, y) \leq 2$.

Proof. Part (i) follows immediately by definition of $P_{x}$ and $P_{y}$ and that they are both abelian. Suppose $P_{x} \cap P_{y} \cap Z_{2} \neq \varnothing$ and let $z$ be an element in this intersection. Then $z$ centralises both $C_{V}(x)$ and $C_{V}(y)$ and so must centralise their sum. However, since $z \in Z_{2}, \operatorname{dim} C_{V}(z)=2$ and contains $C_{V}(x)+C_{V}(y)$. This can only happen if $C_{V}(x)=C_{V}(y)$ and so $P_{x}=P_{y}$. The reverse implication is trivial, so proving (ii).

Recall $S \in \operatorname{Syl}_{2} N_{t}$ as in (6.1). Let

$$
\left(\begin{array}{cccc}
1 & a & b & c \\
0 & 1 & d & \bar{a} d+\bar{b} \\
0 & 0 & 1 & \bar{a} \\
0 & 0 & 0 & 1
\end{array}\right) \in S \cap Z_{2} \quad \text { and } \quad\left(\begin{array}{cccc}
1 & 0 & a & e \\
0 & 1 & f & \bar{a} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) \in P_{t} \cap Z_{2}
$$

for $a, b, c \in G F\left(q^{2}\right), d, e, f \in G F(q)$ and $c+\bar{c}=a \bar{b}=b \bar{a}$. An easy check shows that these matrices commute, and so for any $x \in S \cap Z_{2}$, there exists $z \in P_{t} \cap Z_{2}$ such that $[x, z]=1$. Since $y \in N_{t} \cap Z_{2}, y$ must lie in some Sylow 2-subgroup of $N_{t}$, say $T$. By Sylow's Theorems, there exists $g \in N_{t}$ such that $T^{g}=S$. Hence, there exists $x \in P_{t} \cap Z_{2}$ such that $\left[y^{g}, x\right]=1$ and so $\left[y, x^{g^{-1}}\right]=1$. However, since $g \in N_{t}=N_{G}\left(P_{t}\right)$ we have $x^{g^{-1}} \in P_{t}$. Since $P_{t}$ is abelian, (iii) follows.

Let $\mathcal{U}$ be the totality of 2 -subspaces of $V$. For any matrix $A$, define $\operatorname{RREF}(A)$ to be the row reduced echelon form of $A$ and let

$$
\mathcal{M}=\left\{\begin{array}{l|l}
U \in M_{2,4}\left(q^{2}\right) & \begin{array}{c}
U=\operatorname{RREF}(U) \\
U \text { has no zero rows }
\end{array}
\end{array}\right\}
$$

Define the following map $\rho: \mathcal{U} \rightarrow \mathcal{M}$ by $\rho\left(\left\langle v_{1}, v_{2}\right\rangle\right)=\operatorname{RREF}\left(\binom{v_{1}}{v_{2}}\right)$.
Lemma 6.6. $\rho$ is a well-defined bijective map.

Proof. Let $\left\langle v_{1}, v_{2}\right\rangle \in \mathcal{U}$. Since $v_{1}$ and $v_{2}$ are linearly independent, elementary linear algebra shows that RREF $\left(\binom{v_{1}}{v_{2}}\right) \in \mathcal{M}$, hence $\rho$ is well-defined. The map
RREF : $M_{2,4}\left(q^{2}\right) \rightarrow \mathcal{M}$ is clearly surjective and every element of $M_{2,4}\left(q^{2}\right)$ can be constructed by using linearly independent vectors of $V$ as rows, proving surjectivity. Injectivity is immediate by the definition of the elementary row operations, and so Lemma 6.6 follows.

Let $\mathcal{U}_{\text {iso }}$ be the subset of $\mathcal{U}$ consisting of all totally isotropic 2 -subspaces of $V$. We
also define the following elements of $\mathcal{M}$ :

$$
\begin{aligned}
& U=\left(\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) ; \quad U_{\beta}=\left(\begin{array}{cccc}
0 & 1 & \beta & 0 \\
0 & 0 & 0 & 1
\end{array}\right) ; \\
& U_{\alpha, \beta}=\left(\begin{array}{cccc}
1 & \alpha & 0 & \beta \\
0 & 0 & 1 & \bar{\alpha}
\end{array}\right) ; \quad U_{\alpha, \beta, \gamma}=\left(\begin{array}{cccc}
1 & 0 & \alpha & \beta \\
0 & 1 & \gamma & \bar{\alpha}
\end{array}\right) ;
\end{aligned}
$$

for $\alpha, \beta, \gamma \in G F\left(q^{2}\right)$. Moreover, we set

$$
\begin{aligned}
& \mathcal{M}_{1}=\{U\} \\
& \mathcal{M}_{2}=\left\{U_{\beta}, U_{\alpha, \beta} \mid \alpha, \beta \in G F(q)\right\} ; \\
& \mathcal{M}_{3}=\left\{U_{\alpha, \beta} \mid \alpha \in G F\left(q^{2}\right) \backslash G F(q), \beta \in G F(q)\right\} ; \text { and } \\
& \mathcal{M}_{4}=\left\{U_{\alpha, \beta, \gamma} \mid \alpha \in G F\left(q^{2}\right), \beta, \gamma \in G F(q)\right\} .
\end{aligned}
$$

and define $\mathcal{M}_{0}=\bigcup_{i=1}^{4} \mathcal{M}_{i}$.
Lemma 6.7. $\rho\left(\mathcal{U}_{\text {iso }}\right)=\mathcal{M}_{0}$.

Proof. Using elementary calculations, one can show that every element $W$ in each of the $\mathcal{M}_{i}$ satisfies $\bar{W}^{T} J W=0$, and any element $W^{\prime}$ in $\mathcal{M} \backslash \mathcal{M}_{0}$ satisfies ${\overline{W^{\prime}}}^{T} J W^{\prime} \neq 0$.

Remark: The action of $G$ on $\mathcal{M}$ is given by $W^{g}=\operatorname{RREF}(W g)$ for all $W \in \mathcal{M}$ and $g \in G$. If $W=\rho\left(\left\langle v_{1}, v_{2}\right\rangle\right)$ and $g \in \operatorname{Stab}_{G}\left\langle v_{1}, v_{2}\right\rangle$ then $W^{g}=A W$ for some $A \in G L_{2}\left(q^{2}\right)$. If $A=I_{2}$, then $g \in C_{G}\left(\left\langle v_{1}, v_{2}\right\rangle\right)$.

Since $G$ is transitive on $\mathcal{U}_{\text {iso }}$ (by Witt's Lemma), $\mathcal{U}_{\text {iso }}=\left\{C_{V}(x) \mid x \in Z_{2}\right\}$. In light of Lemma 6.6, for all $x \in Z_{2}$ we have $\rho\left(C_{V}(x)\right) \in \mathcal{M}_{0}$ and in particular, $\rho\left(C_{V}(t)\right)=U$. Moreover, $G$ acts transitively on $\mathcal{M}_{0}$.

Lemma 6.8. (i) $\rho\left(C_{V}(x)\right)=U \in \mathcal{M}_{1}$ if and only if $x \in P \cap Z_{2}$.
(ii) Let $U_{\beta} \in \mathcal{M}_{2}$ for some $\beta \in G F(q)$ and let $y \in Z_{2}$ be such that $\rho\left(C_{V}(y)\right)=U_{\beta}$. Then $d(t, y) \leq 2$.

Proof. Part (i) follows as an immediate consequence of Lemma 6.3(i). Let

$$
x=\left(\begin{array}{cccc}
1 & 1 & \beta & 0 \\
0 & 1 & 0 & \beta \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{array}\right) \in S_{t} \cap Z_{2}
$$

Clearly $U_{\beta}^{x}=U_{\beta}$ and so $U_{\beta}=\rho\left(C_{V}(x)\right)$. Hence $x, y \in P_{x}=P_{y}$ and so $[x, y]=1$ by Lemma 6.5(i). Since $S_{t} \leq C_{t}$, we have $[x, t]=1$ and hence $d(t, y) \leq 2$.

Using Lemma 6.3(ii), we see that if $g \in C_{t}$ then

$$
g=\left(\begin{array}{c|c}
A & B A  \tag{6.3}\\
\hline & A
\end{array}\right), \quad A \in S L_{2}(q), B=\left(\begin{array}{ll}
\bar{b} & c \\
d & b
\end{array}\right)
$$

for $b \in G F\left(q^{2}\right)$ and $c, d \in G F(q)$. Moreover,

$$
g^{2}=\left(\begin{array}{c|c}
A^{2} & A B A+B  \tag{6.4}\\
\hline & A^{2}
\end{array}\right)=I_{4} \text { if and only if } A^{2}=I_{2} \text { and } B^{A}=B
$$

Lemma 6.9. Let $U_{\alpha, \beta} \in \mathcal{M}_{2} \cup \mathcal{M}_{3}$, for some $\alpha \in G F\left(q^{2}\right)$ and $\beta \in G F(q)$. Then there exists $x \in C_{G}(t) \cap Z_{2}$ such that $\rho\left(C_{V}(x)\right)=U_{\alpha, \beta}$ if and only if $U_{\alpha, \beta} \in \mathcal{M}_{2}$.

Proof. Let $g \in C_{G}(t) \cap Z_{2}$, and so

$$
g=\left(\begin{array}{c|c}
A & B A \\
\hline & A
\end{array}\right)=\left(\begin{array}{c|c}
A & A B \\
\hline & A
\end{array}\right)
$$

where $A$ and $J_{1}$ are as in (6.3). It is easy to see that

$$
\left(\begin{array}{cccc}
1 & \alpha & 0 & \beta \\
0 & 0 & 1 & \bar{\alpha}
\end{array}\right)\left(\begin{array}{c|c}
A & A B \\
& A
\end{array}\right)=\left(\left(\begin{array}{ll}
1 & \alpha \\
0 & 0
\end{array}\right) A \left\lvert\,\left(\begin{array}{ll}
1 & \alpha \\
0 & 0
\end{array}\right) A B+\left(\begin{array}{ll}
0 & \beta \\
1 & \bar{\alpha}
\end{array}\right) A\right.\right) .
$$

Suppose

$$
\left(\begin{array}{ll}
1 & \alpha \\
0 & 0
\end{array}\right)\left(\begin{array}{ll}
a_{1} & a_{2} \\
a_{3} & a_{4}
\end{array}\right)=\left(\begin{array}{cc}
a_{1}+\alpha a_{3} & a_{2}+\alpha a_{4} \\
0 & 0
\end{array}\right)=\left(\begin{array}{ll}
1 & \alpha \\
0 & 0
\end{array}\right),
$$

then $a_{1}=1+\alpha a_{3}$ and $a_{2}=\left(a_{4}+1\right) \alpha$. Since $\operatorname{det} A=1$, we also have $a_{1} a_{4}+a_{2} a_{3}=a_{4}+a_{3} \alpha=1$. Hence $a_{4}=1+a_{3} \alpha$ and thus $a_{2}=a_{3} \alpha^{2}$. Therefore,
$A=\left(\begin{array}{cc}1+a \alpha & a \alpha^{2} \\ a & 1+a \alpha\end{array}\right)$ for some $a \in G F\left(q^{2}\right)$, and an easy check shows that $A^{2}=I_{2}$. Moreover,

$$
\left(\begin{array}{cc}
1 & \alpha \\
0 & 0
\end{array}\right)\left(\begin{array}{cc}
b & c \\
d & \bar{b}
\end{array}\right)=\left(\begin{array}{cc}
\bar{b}+\alpha d & c+\alpha b \\
0 & 0
\end{array}\right)
$$

and

$$
\left(\begin{array}{cc}
0 & \beta \\
1 & \bar{\alpha}
\end{array}\right)\left(\begin{array}{cc}
1+a \alpha & a \alpha^{2} \\
a & 1+a \alpha
\end{array}\right)=\left(\begin{array}{cc}
a \beta & \beta(1+a \alpha) \\
1+a \alpha+a \bar{\alpha} & a \alpha^{2}+\bar{\alpha}+a \alpha \bar{\alpha}
\end{array}\right)
$$

so summing the two yields

$$
\left(\begin{array}{cc}
a \beta+\bar{b}+d \alpha & \beta+a \beta \alpha+c+b \alpha \\
1+a \alpha+a \bar{\alpha} & a \alpha^{2}+\bar{\alpha}+a \alpha \bar{\alpha}
\end{array}\right)=\left(\begin{array}{cc}
0 & \beta \\
1 & \bar{\alpha}
\end{array}\right) .
$$

Equating the $(2,1)$ entries, we get $a(\alpha+\bar{\alpha})=0$. If $a=0$ then $g \in P$ and thus $\rho\left(C_{V}(g)\right) \neq U_{\alpha, \beta}$, by Lemma 6.8(i). Hence $a \neq 0$ and so $\alpha \in G F(q)$.

Let

$$
x=\left(\begin{array}{cccc}
1+a \alpha & a \alpha^{2} & a \beta+\alpha d & \alpha^{2} d \\
a & 1+a \alpha & d & a \beta+\alpha d \\
0 & 0 & 1+a \alpha & a \alpha^{2} \\
0 & 0 & a & 1+a \alpha
\end{array}\right)
$$

where $a, d, \alpha, \beta \in G F(q)$. A routine check reveals $x \in C_{G}(t) \cap Z_{2}$ and $U_{\alpha, \beta}^{x}=U_{\alpha, \beta}$ and hence $U_{\alpha, \beta}=\rho\left(C_{V}(x)\right)$ as required.

Corollary 6.10. Let $x \in Z_{2}$ be such that $\rho\left(C_{V}(x)\right)=U_{\alpha, \beta} \in \mathcal{M}_{2}$. Then $d(t, x) \leq 2$.

Proof. By Lemma 6.8, there exists $y \in C_{t} \cap Z_{2}$ such that $\rho\left(C_{V}(y)\right)=U_{\alpha, \beta} \in \mathcal{M}_{1}$ and $[t, y]=1$. Hence $[x, y]=1$ by Lemma 6.3 and so $d(t, x) \leq 2$.

Lemma 6.11. Let $x \in Z_{2} \cap C_{t}$. Then $\rho\left(C_{V}(x)\right) \in \mathcal{M}_{1} \cup \mathcal{M}_{2}$.

Proof. By Lemmas 6.8 and 6.9, there exists $y \in C_{t} \cap Z_{2}$ such that $\rho\left(C_{V}(y)\right) \in \mathcal{M}_{1} \cup \mathcal{M}_{2}$. Moreover, Lemma 6.9 reveals that for any $x \in C_{t} \cap Z_{2}$,
$\rho\left(C_{V}(x)\right) \notin \mathcal{M}_{3}$. It suffices to show that $\rho\left(C_{V}(x)\right) \notin \mathcal{M}_{4}$ for any $x \in C_{t} \cap Z_{2}$. Let $U_{\alpha, \beta, \gamma} \in \mathcal{M}_{4}$ and $z \in C_{t} \cap Z_{2}$, so

$$
z=\left(\begin{array}{c|c}
A & B A \\
& A
\end{array}\right)
$$

where $A$ and $B$ are as in (6.3) subject to the conditions (6.4). Suppose $\rho\left(C_{V}(z)\right)=U_{\alpha, \beta, \gamma}$, then $U_{\alpha, \beta, \gamma}^{z}=U_{\alpha, \beta, \gamma}$. However, this forces $z \in P$ and so $\rho\left(C_{V}(z)\right) \in \mathcal{M}_{1}$ by Lemma 6.8(i). This provides our contradiction and so Lemma 6.11 holds.

Lemma 6.12. Let $U_{\alpha, \beta} \in \mathcal{M}_{3}$.
(i) There exists $y \in N_{t} \cap Z_{2}$ such that $\rho\left(C_{V}(y)\right)=U_{\alpha, \beta}$.
(ii) If $x \in Z_{2}$ is such that $\rho\left(C_{V}(x)\right)=U_{\alpha, \beta}$, then $d(t, x) \leq 3$.

Proof. First observe that if $y=\left(\begin{array}{l|l}A & \\ \hline & D\end{array}\right)$ is such that $\bar{A}^{T} J_{0} D=J_{0}, A^{2}=D^{2}=I_{2}$, then $y \in N_{t} \cap Z_{2}$, since $U^{y}=D U$. Let

$$
z=\left(\begin{array}{cc|cc}
\alpha & \alpha+\alpha^{2} & & \\
\alpha^{-1}+1 & \alpha & & \\
\hline & & \bar{\alpha} & \bar{\alpha}+\bar{\alpha}^{2} \\
& & \bar{\alpha}^{-1} & \bar{\alpha}
\end{array}\right)
$$

for $\alpha \in G F\left(q^{2}\right) \backslash G F(q)$. An easy calculation shows that $z \in N_{t} \cap Z_{2}$ and $\rho\left(C_{V}(z)\right)=U_{\alpha, 0}$. Let $g \in P_{t}$ and since $P_{t} \leq N_{t}, N_{t}^{g}=N_{t}$, hence $z^{g} \in N_{t}$. Moreover, $d(t, z)=d\left(t^{g}, z^{g}\right)=d\left(t, z^{g}\right)$ since $P_{t}$ is abelian, and $C_{V}\left(z^{g}\right)=C_{V}(z)^{g}$. Let

$$
g=\left(\begin{array}{llll}
1 & 0 & b & c \\
0 & 1 & d & \bar{b} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

for $b \in G F\left(q^{2}\right)$ and $c, d \in G F(q)$. Observe that $\rho\left(C_{V}\left(z^{g}\right)\right)=\rho\left(C_{V}(z)^{g}\right)=U_{\alpha, 0}^{g}$. Hence

$$
U_{\alpha, 0} g=\left(\begin{array}{cccc}
1 & \alpha & b+d \alpha & c+\bar{b} \alpha \\
0 & 0 & 1 & \bar{\alpha}
\end{array}\right)
$$

and so

$$
U_{\alpha, 0}^{g}=\left(\begin{array}{cccc}
1 & \alpha & 0 & c+\bar{b} \alpha+(b+d \alpha) \bar{\alpha} \\
0 & 0 & 1 & \bar{\alpha}
\end{array}\right)
$$

By choosing suitably different $g$ by varying $c$, we obtain all possible $U_{\alpha, \beta} \in \mathcal{M}_{3}$, proving (i).
For any $x \in Z_{2}$ such that $\rho\left(C_{V}(x)\right)=U_{\alpha, \beta} \in \mathcal{M}_{3},\left[x, z^{g}\right]=1$ by Lemma 6.8(i). Since $d\left(t, z^{g}\right) \leq 2$ by Lemma 6.5, (ii) follows immediately.

Lemma 6.13. Let $z \in Z_{2}$ be such that $\rho\left(C_{V}(z)\right) \in \mathcal{M}_{4}$. Then $z \notin N_{t}$.
Proof. Let $\rho\left(C_{V}(z)\right)=U_{\alpha, \beta, \gamma}$ for some $\alpha \in G F\left(q^{2}\right)$ and $\beta, \gamma \in G F(q)$, and let

$$
g=\left(\begin{array}{l|l}
A & B \\
\hline C & D
\end{array}\right)
$$

where each block is a $2 \times 2$ matrix. Recall $N_{t}=\operatorname{Stab}_{G} C_{V}(t)$ and hence $g \in N_{t}$ if and only if $U^{g}=E U$ for some $E \in G L_{2}\left(q^{2}\right)$. This occurs precisely when $C=0$ and $D=E$. In addition, $U_{\alpha, \beta, \gamma}^{g}=U_{\alpha, \beta, \gamma}$ and $\bar{g}^{T} J g=J$ forces $A=D=I_{2}$. However, this results in $z \in P$ and so $\rho\left(C_{V}(z)\right)=U \neq U_{\alpha, \beta, \gamma}$ by Lemma 6.8(i), contradicting our assumption. Therefore, Lemma 6.13 holds.

Lemma 6.14. Let

$$
x_{b, c}=\left(\begin{array}{cccc}
1 & 1 & b & c \\
0 & 1 & 0 & b \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{array}\right) \in P_{1} \leq C_{t}
$$

(i) For any $U_{\alpha, \beta, \gamma} \in \mathcal{M}_{4}$, there exists $y \in C_{x_{b, c}} \cap Z_{2}$ such that $\rho\left(C_{V}(y)\right)=U_{\alpha, \beta, \gamma}$.
(ii) Let $x \in Z_{2}$ be such that $\rho\left(C_{V}(x)\right) \in \mathcal{M}_{4}$. Then $d(t, x) \leq 3$.

Proof. Let

$$
h_{b, f}=\left(\begin{array}{cccc}
1 & 0 & f+1 & 0 \\
1 & 0 & f & 0 \\
0 & 1 & b & c+f+1 \\
0 & 1 & b & c+f
\end{array}\right)
$$

for $b, c \in G F(q), c=\bar{f}+f$ and $f \in G F\left(q^{2}\right)$. Consider the group homomorphism $N_{2}$ defined in Lemma 5.1. Since $q$ is even, $\operatorname{ker} N_{2}=G F(q)$ and so $N_{2}$ is surjective with $\left|\operatorname{ker} N_{2}\right|=q$. Hence, there are $q$ possible $h_{b, f}$ for a fixed $b$ and $c$. An easy check shows that ${\overline{h_{b, f}}}^{T} J h_{b, f}=J$ and det $h_{b, f}=1$ and so $h_{b, f} \in G$. Moreover, $t^{h_{b, f}}=x_{b, c}$. If $x \in C_{t}$, then $x^{g} \in C_{t}=C_{x_{b, c} .}$ Let $U_{\beta} \in \mathcal{M}_{2}$ and so by Lemma 6.8(ii), $U_{\beta}=\rho\left(C_{V}(y)\right)$ for some $y \in C_{t}$. We have

$$
U_{\beta} h_{b, f}=\left(\begin{array}{cccc}
1 & \beta & f+b \beta & (c+f+1) \beta \\
0 & 1 & b & c+f
\end{array}\right)
$$

and

$$
U_{\beta}^{h_{b, f}}=\left(\begin{array}{cccc}
1 & 0 & c+\bar{f} & \alpha \\
0 & 1 & b & c+f
\end{array}\right)
$$

as $c=\bar{f}+f$. Each of the $q$ distinct elements $h_{b, f}$ give rise to distinct $U_{\beta}^{h_{b, f}}=\rho\left(C_{V}\left(x^{h_{b, f}}\right)\right)$. Moreover, since $b, \beta, c \in G F(q)$, for all $U_{\alpha, \beta, \gamma} \in \mathcal{M}_{4}$ there exists $b \in G F(q), f \in G F\left(q^{2}\right)$ such that $U_{\beta}^{h_{b, f}}=U_{\alpha, \beta, \gamma}$, proving (i). Part (ii) is immediate by repeated applications of Lemma 6.8(i), yielding $\left[t, x_{b, c}\right]=1$,
$\left[x_{b, c}, y\right]=1$ and $[x, y]=1$.

Lemma 6.15. $\mathcal{C}\left(G, Z_{2}\right)$ is connected of diameter 3.

Proof. For every $x \in Z_{2}$, there exists $W \in \mathcal{M}_{0}$ such that $\rho\left(C_{V}(x)\right)=W$. Together, Lemmas 6.8-6.14 show that for all $x \in Z_{2}, d(t, x) \leq 3$.

We now turn to calculating the disc sizes.

Lemma 6.16. Let $x \in P_{t^{T}}$. Then $x \in Z_{1}$ if and only if

$$
x=\left(\begin{array}{l|l}
I_{2} & \\
\hline B & I_{2}
\end{array}\right)
$$

where $B \in\left\{\left(\begin{array}{ll}0 & a \\ 0 & 0\end{array}\right),\left(\begin{array}{ll}0 & 0 \\ a & 0\end{array}\right),\left(\begin{array}{ll}b & c \\ d & \bar{b}\end{array}\right) \left\lvert\, \begin{array}{c}a, c, d \in G F(q), b \in G F\left(q^{2}\right) \\ b \bar{b}-c d=0\end{array}\right.\right\}$.
Proof. This is shown via analogous case analysis as performed in Lemma 6.2.

Lemma 6.17. Each $\mathcal{M}_{i}$ is a $C_{t}$-orbit, for $i=1, \ldots, 4$.
Proof. That $\mathcal{M}_{1}$ is a $C_{t}$-orbit is obvious since $C_{t} \leq N_{t}=\operatorname{Stab}_{G} C_{V}(t)$. For any $y \in C_{t}$, there exists $W \in \mathcal{M}_{1} \cup \mathcal{M}_{2}$ such that $\rho\left(C_{V}(y)\right)=W$. Since $\mathcal{M}_{1}$ is $C_{t}$-invariant, so must $\mathcal{M}_{2}$ be. Let

$$
g_{1}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & d & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) \quad \text { and } \quad g_{2}=\left(\begin{array}{cc|cc}
1+a b & a b^{2} & 0 & 0 \\
a & 1+a b & c(1+a b) & c a b^{2} \\
\hline & & 1+a b & a b^{2} \\
& & a & 1+a b
\end{array}\right)
$$

be elements of $C_{t}$, for $a, b, c, d \in G F(q), a, d \neq 0$. Then a simple calculation shows that $U_{0}^{g_{1}}=U_{d}$ and

$$
U_{0} g_{2}=\left(\begin{array}{cccc}
a & 1+a b & c(1+a b) & c a b^{2} \\
0 & 0 & a & 1+a b
\end{array}\right)
$$

so $U_{0}^{g_{2}}=U_{a^{-1}+b, c a^{-2}}$. Since $a, b, c, d \in G F(q)$, clearly $C_{t}$ is transitive on $\mathcal{M}_{2}$.
As $C_{t} \leq N_{t}$ and for all $W \in \mathcal{M}_{3}$ there exists $y \in N_{t}$ such that $\rho\left(C_{V}(y)\right)=W$ by Lemma 6.12, $\mathcal{M}_{3}$ is $C_{t}$-invariant. Let

$$
g_{3}=\left(\begin{array}{cc|c}
0 & b & \\
b^{-1} & d & \\
\hline & 0 & b \\
& b^{-1} & d
\end{array}\right) \text { and } g_{4}=\left(\begin{array}{cccc}
1 & 0 & 0 & c \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

be elements of $C_{t}$ for $b, c, d \in G F(q)$ and $b, c \neq 0$. Then for $\alpha \in G F\left(q^{2}\right) \backslash G F(q)$,

$$
U_{\alpha, 0} g_{3}=\left(\begin{array}{cccc}
b^{-1} \alpha & b+d \alpha & 0 & 0 \\
0 & 0 & b^{-1} \alpha^{q} & b+d \alpha^{q}
\end{array}\right)
$$

and $U_{\alpha, 0}^{g_{3}}=U_{b^{2} \alpha^{-1}+d b, 0}$. Every element in $G F\left(q^{2}\right)$ can be written as $\lambda \alpha^{-1}+\mu$ for some $\lambda, \mu \in G F(q)$. Indeed, an easy check shows the map $G F(q) \times G F(q) \rightarrow G F\left(q^{2}\right)$ such that $(\lambda, \mu) \mapsto \lambda \alpha^{-1}+\mu$ is injective. Since $q$ is even, every element in $G F(q)$ is a square and hence $b^{2}$ can take all possible values in $G F(q)$. Thus $b^{2} \alpha^{-1}+d b$ takes all possible values in $G F\left(q^{2}\right) \backslash G F(q)$. Moreover, $U_{\alpha, 0}^{g_{4}}=U_{\alpha, c}$ and $c \in G F(q)$. Hence $C_{t}$ is transitive on $\mathcal{M}_{3}$.

Finally, let

$$
g_{5}=\left(\begin{array}{cccc}
1 & 0 & b & c \\
0 & 1 & d & \bar{b} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) \in C_{t}
$$

for $b \in G F\left(q^{2}\right)$ and $c, d \in G F(q)$, observing that $U_{0,0,0}^{g_{5}}=U_{b, c, d}$. Hence $C_{t}$ is transitive on $\mathcal{M}_{4}$, so proving the lemma.

Let $w_{1}, w_{2} \in Z_{2}$ and $W_{i}=\rho\left(C_{V}\left(w_{i}\right)\right)$. Let $x \in P_{w_{1}}$ and suppose $d(t, x)=i$. If $C_{V}\left(w_{1}\right)$ and $C_{V}\left(w_{2}\right)$ are $C_{t}$-conjugate by some element $g$, say, then $d(t, x)=d\left(t, x^{g}\right)=i$. Hence, $\left|P_{w_{1}} \cap \Delta_{i}(t)\right|=\left|P_{w_{2}} \cap \Delta_{i}(t)\right|$.

Lemma 6.18. Let $x \in Z_{2}$ be such that $\rho\left(C_{V}(x)\right)=U_{0} \in \mathcal{M}_{2}$. Then $\left|P_{x} \cap \Delta_{1}(t)\right|=q(q-1)$ and $\left|P_{x} \cap \Delta_{2}(t)\right|=q^{3}(q-1)$.

Proof. By (6.3) and (6.4), if $y \in C_{t} \cap Z_{2}$ then

$$
y=\left(\begin{array}{c|c}
A & B A \\
\hline & A
\end{array}\right)
$$

where $A^{2}=[A, B]=I_{2}$ and $B=\left(\begin{array}{ll}b & c \\ d & \bar{b}\end{array}\right)$ for $b \in G F\left(q^{2}\right)$, and $c, d \in G F(q)$.
Suppose $y \in P_{x} \cap C_{t} \cap Z_{2}$, then $U_{0}^{y}=U_{0} y=U_{0}$. Clearly

$$
U_{0}^{y}=\left(\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right) A \left\lvert\,\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right) A B+\left(\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right) A\right.\right)
$$

Letting $A=\left(\begin{array}{ll}a_{1} & a_{2} \\ a_{3} & a_{4}\end{array}\right)$, we must have $a_{3}=0$ and $a_{4}=a_{1}=1$. So $A=\left(\begin{array}{ll}1 & a \\ 0 & 1\end{array}\right)$ for some $a \in G F(q)$. Also, calculation of the second component gives

$$
\left(\begin{array}{ll}
d & \bar{b} \\
0 & 1
\end{array}\right)=\left(\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right)
$$

and thus $d, b=0$. Moreover, $y \in Z_{2}$ and by the case analysis performed in Lemma 6.4(ii), we must have $a \neq 0$ and hence

$$
P_{x} \cap C_{t} \cap Z_{2}=\left\{\left.\left(\begin{array}{cccc}
1 & a & 0 & c \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & a \\
0 & 0 & 0 & 1
\end{array}\right) \right\rvert\, a, c \in G F(q), a \neq 0\right\} .
$$

Therefore $\left|P_{x} \cap C_{t} \cap Z_{2}\right|=\left|P_{x} \cap \Delta_{1}(t)\right|=q(q-1)$. By conjugacy and Lemma 6.4(i), $\left|P_{x} \cap Z_{2}\right|=q(q-1)\left(q^{2}+1\right)$ and by Lemma 6.8(ii), if $z \in P_{x} \cap Z_{2}$ then $d(t, z) \leq 2$. Hence

$$
\begin{aligned}
\left|P_{x} \cap \Delta_{2}(t)\right| & =\left|P_{x} \cap Z_{2}\right|-\left|P_{x} \cap \Delta_{1}(t)\right| \\
& =q^{3}(q-1),
\end{aligned}
$$

proving the lemma.

From Lemma 6.16, recall that

$$
P_{t^{T}}=\left\{\binom{I_{2}}{\hline B} \left\lvert\, B=\left(\begin{array}{ll}
b & c  \tag{6.5}\\
d & \bar{b}
\end{array}\right)\right., b \in G F\left(q^{2}\right), c, d \in G F(q)\right\}
$$

and $\rho\left(C_{V}\left(t^{T}\right)\right)=U_{0,0,0}$.
Lemma 6.19. $\left|P_{t^{T}} \cap \Delta_{2}(t)\right|=q^{2}(q-1)$.
Proof. By Lemma 6.13, if $y \in P_{t^{T}}$ then $2 \leq d(t, y) \leq 3$. If $d(t, y)=2$ then there exists $x \in C_{t} \cap Z_{2}$ such that $[x, y]=1$. Let

$$
x=\left(\begin{array}{c|c}
A & C A \\
\hline & A
\end{array}\right) \in C_{t} \cap Z_{2} \quad \text { and } \quad y=\left(\begin{array}{c|c}
I_{2} & \\
\hline B & I_{2}
\end{array}\right) \in P_{t^{T}} \cap Z_{2},
$$

where $x$ satisfies the conditions of (6.3) and (6.4), and $y$ satisfies the conditions of (6.5). Now $[x, y]=1$ if and only if $C A B=B C A=0$ and $B A+A B=B C A B=0$, which occurs if and only if $A B=B A$ and $B C=C B=0$. From Lemmas 6.9 and 6.18 we must have

$$
A=\left(\begin{array}{cc}
1+a b & a b^{2} \\
a & 1+a b
\end{array}\right) \quad \text { or } \quad A=\left(\begin{array}{ll}
1 & a \\
0 & 1
\end{array}\right) .
$$

Observe that $a \neq 0$, otherwise $x \in P_{t}=P_{x}$ and by Lemma 6.12, $P_{t^{T}} \cap N_{x} \cap Z_{2}=\varnothing$, contradicting the assumption that $[x, y]=1$. If $B=\left(\begin{array}{cc}c & d \\ e & \bar{c}\end{array}\right)$ for $c \in G F\left(q^{2}\right)$, $d, e \in G F(q)$ then for the latter choice of $A$ we have

$$
B A=\left(\begin{array}{ll}
c & a c+d \\
e & a e+\bar{c}
\end{array}\right)=\left(\begin{array}{cc}
c+a e & d+a \bar{c} \\
e & \bar{c}
\end{array}\right)=A B
$$

if and only if $e=0$ and $c \in G F(q)$. For the former choice of $A$ we get

$$
\begin{aligned}
B A & =\left(\begin{array}{ll}
c(1+a b)+a d & a b^{2} c+d(1+a b) \\
e(1+a b)+a \bar{c} & a b^{2} e+\bar{c}(1+a b)
\end{array}\right) \\
& =\left(\begin{array}{cc}
c(1+a b)+a b^{2} e & d(1+a b)+\bar{c} a b^{2} \\
a c+e(1+a b) & a d+\bar{c}(1+a b)
\end{array}\right)=A B
\end{aligned}
$$

if and only if $d=b^{2} e$ and $c \in G F(q)$. In both cases, for any such $y$ satisfying the respective conditions, $[x, y]=1$ where $C=0$. Therefore, if

$$
\mathcal{F}=\left\{\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
c & d & 1 & 0 \\
0 & c & 0 & 1
\end{array}\right), \left.\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
c & b^{2} e & 1 & 0 \\
e & c & 0 & 1
\end{array}\right) \right\rvert\, b, c, d, e \in G F(q), e \neq 0\right\}
$$

then $|\mathcal{F}|=q^{2}+(q-1) q^{2}=q^{3}$ and $P_{t^{T}} \cap \Delta_{2}(t) \subseteq \mathcal{F}$. Using Lemma 6.16, we can determine how many elements of $\mathcal{F}$ lie in $Z_{1}$ by a similar tack as in the proof of Lemma 6.2. Therefore, $\left|\mathcal{F} \cap Z_{1}\right|=(q-1)^{2}+(q-1)+q=q^{2}$ and so

$$
\left|P_{t}^{T} \cap \Delta_{2}(t)\right|=|\mathcal{F}|-\left|\mathcal{F} \cap Z_{1}\right|=q^{2}(q-1)
$$

proving Lemma 6.19.
Lemma 6.20. Let $y \in Z_{2}$ be such that $\rho\left(C_{V}(y)\right)=U_{\alpha, 0} \in \mathcal{M}_{3}$.
(i) $P_{y} \cap \Delta_{2}(t) \subseteq N_{t}$.
(ii) $\left|P_{y} \cap \Delta_{2}(t)\right|=q\left(q^{2}-1\right)$.

Proof. Let

$$
g=\left(\begin{array}{llll}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & \alpha & 0 & 0 \\
0 & 0 & 1 & \bar{\alpha}
\end{array}\right)
$$

for some $\alpha \in G F\left(q^{2}\right) \backslash G F(q)$, while an easy calculation shows $g \in G$. Since $U^{g}=U_{\alpha, 0}$ we have $P_{y}=P_{t}^{g}=C_{G}\left(C_{V}\left(t^{g}\right)\right)$. By Lemma 6.5(iii), if $x \in P_{t}$ and $z \in C_{x}$ then $\rho\left(C_{V}(z)\right) \in \mathcal{M}_{1} \cup \mathcal{M}_{2} \cup \mathcal{M}_{3}$. Hence, if $x^{g} \in P_{y}$ and $z^{g} \in C_{x^{g}}$ then $\rho\left(C_{V}\left(z^{g}\right)\right) \in \mathcal{M}_{1}^{g} \cup \mathcal{M}_{2}^{g} \cup \mathcal{M}_{3}^{g}$. Note that if $d\left(t, x^{g}\right)=2$ then there exists $w \in \Delta_{1}(t)$ such that $\left[x^{g}, w\right]=1$ and so $x^{g} \in N_{w}$, and $\rho\left(C_{V}(w)\right) \in \mathcal{M}_{1} \cup \mathcal{M}_{2}$ by Lemma 6.11. However, by easy calculations, $U^{g}=U_{\alpha, 0} \notin \mathcal{M}_{1} \cup \mathcal{M}_{2}$ and $U_{\beta}^{g}=U_{\alpha, \beta} \notin \mathcal{M}_{1} \cup \mathcal{M}_{2}$ since $\alpha \in G F\left(q^{2}\right) \backslash G F(q)$. Moreover,

$$
U_{\beta, \gamma}^{g}=\left(\begin{array}{cccc}
1 & 0 & \bar{\beta}+\alpha \gamma & \bar{\alpha} \bar{\beta}+\alpha \beta+\bar{\alpha} \alpha \gamma \\
0 & 1 & \gamma & \beta+\bar{\alpha} \gamma
\end{array}\right) \notin \mathcal{M}_{1} \cup \mathcal{M}_{2}
$$

for $\beta \in G F\left(q^{2}\right), \gamma \in G F(q)$. Finally, $U_{0}^{g}=U \in \mathcal{M}_{1}$ and so for any $x^{g} \in P_{y}$ and $z^{g} \in C_{x^{g}} \cap C_{t} \cap Z_{2}$, we must have $C_{V}\left(z^{g}\right)=C_{V}(t)$. Hence $x^{g} \in N_{t}$, so proving (i). Recall

$$
P_{t}=P=\left\{\left.\left(\begin{array}{cccc}
1 & 0 & b & c \\
0 & 1 & d & \bar{b} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) \right\rvert\, b \in G F\left(q^{2}\right), c, d \in G F(q)\right\} .
$$

Clearly $P_{y}=P_{t}^{g}$ and so direct calculation yields

$$
P_{y}=\left\{\left.\left(\begin{array}{cccc}
\alpha b+1 & \alpha^{2} b & \alpha c & \bar{\alpha} \alpha c  \tag{6.6}\\
b & \alpha b+1 & c & \bar{\alpha} c \\
\bar{\alpha} d & \bar{\alpha} \alpha d & \bar{\alpha} \bar{b}+1 & \bar{\alpha}^{2} \bar{b} \\
d & \alpha d & \bar{b} & \bar{\alpha} \bar{b}+1
\end{array}\right) \right\rvert\, \begin{array}{c}
\alpha \in G F\left(q^{2}\right) \backslash G F(q) \\
b \in G F\left(q^{2}\right) \\
c, d \in G F(q)
\end{array}\right\}
$$

and by (i), any $z \in P_{y}$ such that $d(t, z)=2$ must lie in $N_{t}$. By inspection, any such $z$ must have $d=0$ in (6.6). Clearly, $z \in Z_{2}$ if and only if $z^{g^{-1}} \in Z_{2}$ and by Lemma 6.2 , this occurs if and only if $b \neq 0$. Hence

$$
\left|P_{y} \cap N_{t} \cap Z_{2}\right|=\left|P_{y} \cap \Delta_{2}(t)\right|=q\left(q^{2}-1\right),
$$

and (ii) holds.

Lemma 6.21. (i) $\left|\Delta_{2}(t)\right|=q^{3}(q-1)\left(q^{3}+2 q^{2}+q-1\right)$.
(ii) $\left|\Delta_{3}(t)\right|=q^{4}(q-1)\left(q^{3}-q+1\right)$.

Proof. Let $x, y, z \in Z_{2}$ be such that $\rho\left(C_{V}(x)\right)=U_{0} \in \mathcal{M}_{2}, \rho\left(C_{V}(y)\right)=U_{\alpha, 0} \in \mathcal{M}_{3}$ and $\rho\left(C_{V}(z)\right)=U_{0,0,0} \in \mathcal{M}_{4}$. Then Lemmas 6.18, 6.19 and 6.20(ii) yield

$$
\begin{aligned}
\left|\Delta_{2}(t)\right| & =\left|\mathcal{M}_{2}\right|\left|P_{x} \cap \Delta_{2}(t)\right|+\left|\mathcal{M}_{3}\right|\left|P_{y} \cap \Delta_{2}(t)\right|+\left|\mathcal{M}_{4}\right|\left|P_{z} \cap \Delta_{2}(t)\right| \\
& =\left(q+q^{2}\right) q^{3}(q-1)+\left(q^{2}-q\right) q\left(q^{2}-1\right) q+q^{4} q^{2}(q-1) \\
& =q^{3}(q-1)\left(q^{3}+2 q^{2}+q-1\right),
\end{aligned}
$$

proving (i). Since $\mathcal{C}\left(G, Z_{2}\right)$ has diameter 3 by Lemma 6.15 , we have

$$
\begin{aligned}
\left|\Delta_{3}(t)\right| & =\left|Z_{2}\right|-\left|\Delta_{0}(t)\right|-\left|\Delta_{1}(t)\right|-\left|\Delta_{2}(t)\right| \\
& =q^{4}(q-1)\left(q^{3}-q+1\right),
\end{aligned}
$$

proving (ii).

Combined, Lemmas 6.4(ii), 6.15 and 6.21 complete the proof of Theorem 1.7.

## Chapter 7

## Classical Group Extensions and Affine Linear Groups

Breaking away from the scene of finite simple groups, this chapter is devoted to the study of commuting involution graphs of some non-simple groups.

### 7.1 2-dimensional Projective General Linear Groups

We open this study by investigating the structure of the commuting involution graphs of $G=P G L_{2}(q)$ for $q$ odd, in particular proving Theorem 1.8. Let $H=G L_{2}(q)$ and $G=H / Z(H) \cong P G L_{2}(q)$. For any element $g \in H$ we denote its image in $G$ by $\bar{g}$. There are two classes of involutions in $G$, one of which is contained in $G^{\prime} \cong L_{2}(q)$. Let $I$ be the set of elements in $H \backslash Z(H)$ that square to a non-trivial element of $Z(H)$. Clearly, for any $g \in I, \bar{g}$ is an involution in $G$ and so denote by $\bar{I}$ the set of involutions in $G$. Suppose $\operatorname{det}(g)$ is a square in $G F(q)$. Then there exists an element $z \in Z(H)$ such that $\operatorname{det}(z g)=1$, and hence $\bar{g} \in G^{\prime}$. Let $X$ be the image in $G$ of the subset of $I$ whose elements have a non-square determinant. Clearly, $X$ is the conjugacy class of involutions of $G$ such that $X \cap G^{\prime}=\varnothing$. The proof of Theorem 1.8, which we present for clarity, is analogous to the proof of

Theorem 2.10, which can be found in [15]. For the rest of this section, all matrices are to be read modulo $Z(H)$. Set $\delta= \pm 1$ where $q \equiv \delta(\bmod 4)$.

Lemma 7.1. We have

$$
X=\left\{\left.\left(\begin{array}{cc}
a & b \\
c & -a
\end{array}\right) \right\rvert\,-a^{2}-b c \text { a non-square in } G F(q)\right\} .
$$

and $|X|=\frac{1}{2} q(q-\delta)$.
Proof. Let $g=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ and a routine calculation shows that $g^{2} \in Z(H)$ if and only if $a^{2}=d^{2}$ and $b(a+d)=c(a+d)=0$. If $a=d \neq 0$ then $b=c=0$ and so $g \in Z(H)$. Therefore we must have $a=-d$. Hence,

$$
\bar{I}=\left\{\left.\left(\begin{array}{cc}
a & b \\
c & -a
\end{array}\right) \right\rvert\, a^{2}+b c \neq 0\right\},
$$

and thus $X$ can be described as in the statement of the lemma.
Using Lemma 2.1 of [15], we see that $|\bar{I}|-|X|=\frac{1}{2} q(q+\delta)$. It suffices to show $|I|$, which is precisely the number of triples $(a, b, c)$, where $a, b, c \in G F(q)$, that satisfy $a^{2}+b c \neq 0$. Suppose $a^{2}+b c=0$. When $a=0$, either $b$ or $c$ must be zero (or both), giving rise to $2 q-1$ solutions. When $a \neq 0$ we have $a^{2}=-b c$ and $b, c \neq 0$. This gives rise to $(q-1)^{2}$ solutions. Therefore the number of triples $(a, b, c)$ that satisfy $a^{2}+b c=0$ is $2 q-1+(q-1)^{2}=q^{2}$. Hence, the number of triples $(a, b, c)$ that satisfy $a^{2}+b c \neq 0$ is $q^{3}-q^{2}=q^{2}(q-1)$, and so $|\bar{I}|=\frac{q^{2}(q-1)}{(q-1)}=q^{2}$. Thus, $|\bar{I}|-\frac{1}{2} q(q+\delta)=\frac{1}{2} q(q-\delta)=|X|$ and the result follows.

We first deal with the case when $\delta=1$. Without loss of generality, let

$$
t=\left(\begin{array}{ll}
0 & \mu \\
1 & 0
\end{array}\right) \in X
$$

for $\mu$ a non-square in $G F(q)$.

Proposition 7.2. Suppose $\delta=1$. Then $\mathcal{C}(G, X)$ has diameter 3 with disc sizes

$$
\begin{aligned}
\left|\Delta_{1}(t)\right| & =\frac{1}{2}(q+1) \\
\left|\Delta_{2}(t)\right| & =\frac{1}{4}\left(q^{2}-1\right) ; \text { and } \\
\left|\Delta_{3}(t)\right| & =\frac{1}{4}(q-5)(q+1) .
\end{aligned}
$$

Proof. It is well-known (see, for example, Lemma 3.1 of [41]) that the centraliser of any involution in $G$ is dihedral. By Lemma 7.1, $C_{G}(t) \cong \operatorname{Dih}(2(q+1))$. Let

$$
r=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) \quad \text { and } \quad s=\left(\begin{array}{ll}
0 & \nu \\
1 & 0
\end{array}\right)
$$

where $\nu$ is a non-square in $G F(q) \backslash\{\mu\}$. It is an easy check to see that $r \in G^{\prime}, s \in X$ and both commute with $t$. Since $q+1$ is even, there exist 2 non-trivial conjugacy classes of involutions in $C_{G}(t)$. Hence exactly one of these classes lie in $X$ and so

$$
\begin{equation*}
\left|\Delta_{1}(t)\right|=\frac{1}{2}(q+1) . \tag{7.1}
\end{equation*}
$$

Since any dihedral subgroup of $G$ lies in a unique maximal dihedral subgroup of $G$, for any $x, y \in X$ such that $[x, y] \neq 1$, we have $\left|C_{G}(x) \cap C_{G}(y) \cap X\right|=1$. Moreover, any fours-group in a dihedral group is self-centralising, and for commuting involutions $y, z \in X$, we have $y z \notin X$. Hence, $\Delta_{1}(y) \cap \Delta_{1}(z)=\varnothing$. Therefore an easy count gives

$$
\begin{equation*}
\left|\Delta_{2}(t)\right|=\frac{1}{4}\left(q^{2}-1\right) . \tag{7.2}
\end{equation*}
$$

Suppose there exists $z \in X$ such that $d(t, z) \geq 4$, and let $x \in \Delta_{1}(t) \cup\{t\}$. If $z x$ has order dividing $q+1$ then there exists $y \in X$ such that $\langle z, x\rangle \leq C_{G}(y)$, contradicting our choice of $z$. Hence $z x$ must have order dividing $q-1$, and so $\langle z, x\rangle$ is contained in a unique dihedral group, $D_{x}$, of order $2(q-1)$. Any involution in $D_{x}$ not equal to $x$ is at least distance 3 from $x$ and thus does not lie in $\Delta_{1}(t) \cup\{t\}$. Hence, distinct elements $x$ must lie in distinct such $D_{x}$, and each $D_{x}$ contains $\frac{1}{2}(q-3)$ involutions other than $z$, all of which are at least distance 3 from $z$. This accounts for
$\frac{1}{4}(q-3)(q+3)$ involutions. However,

$$
\begin{aligned}
|X|-\left|\Delta_{2}(t)\right|-\left|\Delta_{1}(t)\right|-1 & =\frac{1}{4}(q+1)(q-5) \\
& <\frac{1}{4}(q+3)(q-3)
\end{aligned}
$$

by Lemma 7.1, (7.1) and (7.2). This provides our contradiction and so there is no involution $z \in X$ such that $d(t, z) \geq 4$. Proposition 7.2 then follows.

We now deal with the case when $\delta=-1$. Let $t=\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)$ and since -1 is non-square in $G F(q), t \in X$. Set $t_{\alpha}=\left(\begin{array}{cc}1 & 0 \\ \alpha & -1\end{array}\right)$ and observe that $t_{\alpha} \in X$ and $t_{0}=t$.

Lemma 7.3. Let $x \in X$. Then there exists a unique $t_{\alpha}$ such that $\left[x, t_{\alpha}\right]=1$.
Moreover,

$$
\Delta_{1}(t)=\left\{\left.\left(\begin{array}{cc}
0 & \lambda^{2} \\
1 & 0
\end{array}\right) \right\rvert\, \lambda \in G F(q)^{*}\right\} .
$$

Proof. Let $x=\left(\begin{array}{cc}a & b \\ c & -a\end{array}\right)$ and suppose $\left[x, t_{\alpha}\right]=1$. Then $t_{\alpha} x=\lambda x t_{\alpha}$ in $H$ for some $\lambda \in G F(q)^{*}$, that is

$$
\left(\begin{array}{cc}
a & b \\
\alpha a-c & \alpha b+a
\end{array}\right)=\left(\begin{array}{cc}
\lambda(a+b \alpha) & -\lambda b \\
\lambda(c-a \alpha) & \lambda a
\end{array}\right) .
$$

If $\lambda=1$ then $b=0$ and $c=a \alpha$, resulting in $x=t_{\alpha}$. So assume $\lambda \neq 1$, thus $a=\lambda(a+b \alpha)$ and hence $(1-\lambda) a=\lambda b \alpha$. Clearly, if $b=0$ then $a=0$ which is impossible so $b \neq 0$. But then $b=-\lambda b$ which implies $\lambda=-1$ and so $\alpha=-2 a b^{-1}$. Therefore, $\alpha$ is determined by the entries of $x$ and thus there is a unique $t_{\alpha}$ that commutes with $x$ in $G$. If $\alpha=0$ then $a=0$ and so $x=\left(\begin{array}{ll}0 & b \\ c & 0\end{array}\right) \equiv\left(\begin{array}{cc}0 & b c^{-1} \\ 1 & 0\end{array}\right)$ modulo $Z(H)$. Since $x \in X,-b c^{-1}$ must be non-square and therefore $b c^{-1}$ must be square in $G F(q)$, since $q \equiv 3(\bmod 4)$. This proves Lemma 7.3.

Lemma 7.4. (i) $\left|\Delta_{1}(t)\right|=\frac{1}{2}(q-1)$.
(ii) $\Delta_{2}(t)=\left\{\left(\begin{array}{cc}1 & -\lambda^{2} \tau \\ \tau & -1\end{array}\right) \left\lvert\, \begin{array}{cc}-1+\lambda^{2} \tau^{2} & \text { a } \text { non-square in } G F(q) \\ \lambda \neq 0\end{array}\right.\right\}$ and $\left|\Delta_{2}(t)\right|=\frac{1}{4}(q-1)(q-3)$.

Proof. The first part is an immediate consequence of Lemma 7.3. Let $y=\left(\begin{array}{cc}a & \sigma \\ \mu & -a\end{array}\right) \in X$. If $a=0$ then $y \in \Delta_{1}(t)$ so we may assume $a \neq 0$ and thus, modulo $Z(H), y=\left(\begin{array}{cc}1 & b \\ c & -1\end{array}\right)$. If $y \in \Delta_{2}(t)$ then there exists $x=\left(\begin{array}{cc}0 & \lambda^{2} \\ 1 & 0\end{array}\right) \in \Delta_{1}(t)$ such that $[x, y]=1$. That is, $x y=\mu y x$ in $H$, for some $\mu \in G F(q)^{*}$. A routine check reveals we must have $b=-\lambda^{2} c$.

Just as in Proposition 7.2, for any non-commuting involutions $x$ and $y$ we have $\left|C_{G}(x) \cap C_{G}(y) \cap X\right|=1$ and if $x \in C_{G}(t)$ then $\Delta_{1}(x) \cap \Delta_{1}(t)=\varnothing$. Hence, an easy calculation gives $\left|\Delta_{2}(t)\right|=\frac{1}{4}(q-1)(q-3)$.

Lemma 7.5. Let $x, y \in X$ such that $x \neq y$. Then $d(x, y) \leq 2$ if and only if the order of $x y$ divides $q-1$. Moreover, $d(x, y)=2$ if and only if $x y$ fixes two points of the projective line.

Proof. The order of $x y$ divides $q-1$ if and only if $\langle x, y\rangle$ lies in a dihedral group of order $2(q-1)$, which contains a central involution $z \in X$. Hence, $d(x, y) \leq 2$. Since $G$ acts 3-transitively on the projective line, a point-stabiliser has order $q(q-1)$ and the pointwise stabiliser of 2 points has order $q-1$. Hence, if $x y$ fixes two points of the projective line, then the order of $x y$ divides $q-1$. If $d(x, y)=2$, then the order of $x y$ divides $\frac{1}{2}(q-1)$ and so $x y$ fixes two points of the projective line by Satz 8.3 of [28]. If the order of $x y$ is 2 , then $\operatorname{det}(x y)=\operatorname{det}(x) \operatorname{det}(y)$ is a square in $G F(q)$ and hence $x y \notin X$. So $x y$ is conjugate to $\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$ which fixes no points of the projective line. Therefore, if $d(x, y)=1, x y$ fixes no points of the projective line.

Proposition 7.6. Suppose $\delta=-1$ and $q \geq 19$. Then $\mathcal{C}(G, X)$ has diameter 3 with
disc sizes

$$
\begin{aligned}
\left|\Delta_{1}(t)\right| & =\frac{1}{2}(q-1) \\
\left|\Delta_{2}(t)\right| & =\frac{1}{4}(q-1)(q-3) ; \text { and } \\
\left|\Delta_{3}(t)\right| & =\frac{1}{4}(q-1)(q+5)
\end{aligned}
$$

Proof. The first two disc sizes are given in Lemma 7.4(i) and (ii). Using Lemmas 7.3 and $7.4(\mathrm{ii})$, it is easy to see that
$X \backslash\left(\Delta_{1}(t) \cup \Delta_{2}(t) \cup\{t\}\right)=\left\{\left(\begin{array}{cc}1 & -\mu \tau \\ \tau & -1\end{array}\right) \left\lvert\, \begin{array}{c}\mu \text { either } 0 \text { or a non-square in } G F(q) \\ -1+\mu \tau^{2} \text { a non-square in } G F(q)^{*} \\ \tau \neq 0\end{array}\right.\right\}$.
Let $x=\left(\begin{array}{cc}1 & -\mu \tau \\ \tau & -1\end{array}\right) \in X \backslash\left(\Delta_{1}(t) \cup \Delta_{2}(t) \cup\{t\}\right)$ and $x_{\lambda}=\left(\begin{array}{cc}0 & \lambda^{2} \\ 1 & 0\end{array}\right) \in \Delta_{1}(t)$. Then

$$
x_{\lambda} x=\left(\begin{array}{cc}
\lambda^{2} \tau & -\lambda^{2} \\
1 & -\mu \tau
\end{array}\right)
$$

Using Lemma 7.5, $x$ is at most distance 3 from $t$ if there exists $\lambda \in G F(q)^{*}$ for which $x_{\lambda} x$ fixes two points on the projective line. That is, if there exists two linearly independent vectors $(\alpha, \beta)$ such that $(\alpha, \beta) x_{\lambda} x=(k \alpha, k \beta)$. We then have

$$
\begin{align*}
& k \alpha=\lambda^{2} \tau \alpha+\beta  \tag{7.3}\\
& k \beta=-\lambda^{2} \alpha-\mu \tau \beta . \tag{7.4}
\end{align*}
$$

Since $\lambda$ and $\tau$ are both non-zero, (7.3) shows that if $\alpha=0$ then $\beta=0$, and similarly (7.4) show the converse, both of which are invalid. Upon rearrangement of (7.3) and (7.4), we get

$$
\alpha \beta^{-1}=\left(k-\lambda^{2} \tau\right)^{-1}=-(k-\mu \tau) \lambda^{-2}
$$

observing that we must have $k \neq \lambda^{2} \tau$ and $k \neq \mu \tau$. We then have

$$
k^{2}-\left(\mu \tau+\lambda^{2} \tau\right) k+\left(\mu \tau^{2}+1\right) \lambda^{2}=0
$$

Provided the discriminant of this quadratic equation, say $\Phi(\lambda)$, is non-zero, there are two distinct values of $k$ and hence of $\alpha \beta^{-1}$. There are $q-1$ values of $\lambda$ and $\Phi(\lambda)$


Figure 7.1: The Diameter 4 Commuting Involution Graph of $P G L_{2}(7)$.


Figure 7.2: The Diameter 4 Commuting Involution Graph of $P G L_{2}(11)$.
is a quartic in $\lambda$, hence at worst $\frac{1}{4}(q-1)$ values of $\Phi(\lambda)$. However, we must insist $\Phi(\lambda) \neq 0$ and the earlier restrictions that $k \neq \lambda^{2} \tau$ and $k \neq \mu \tau$. This removes at most three possible values of $\Phi(\lambda)$, and so there are at least $\frac{1}{4}(q-1)-3=\frac{1}{4}(q-13)$ suitable values of $\Phi(\lambda)$. Hence, for $q>13$, the diameter of $\mathcal{C}(G, X)$ is 3 . The size of the third disc follows immediately. Since $q=19$ is the least such $q>13$ when $\delta=-1$, Proposition 7.6 holds.

Propositions 7.2-7.6 combined complete the proof of Theorem 1.8. One may also note that this is just the action on $\binom{q+1}{2}$ unordered pairs of points of the projective line.

We finish this section by giving the collapsed adjacency diagrams of the diameter 4 commuting involution graphs of $P G L(2,7)$ and $P G L(2,11)$ as calculated in Magma [19] (Figures 7.1 and 7.2).

### 7.2 Affine Orthogonal Groups

Our final detailed study concerns the affine orthogonal groups, in particular Theorem 1.9.

Theorem 4.18 deals with the commuting involution graph of the affine orthogonal group $\mathrm{AO}_{3}(q)$, and is instrumental in the proof of Theorem 1.4. In the next chapter, the commuting involution graph of $A O_{4}^{-}(q)$ is utilised in a similar way. This section also highlights the fact that a quotient of a group does not necessarily preserve distance in group itself. Indeed, Theorem 4.18 showed that for $A O_{3}(q) \cong Q \rtimes L$ where $L \cong O_{3}(q)$ and $t, x$ conjugate involutions in $L,[t, x]=1$ did not necessarily imply that $[t, u x]=1$ for some $u \in Q,(u x)^{2}=1$. Thus, careful examination of these groups is needed. We begin with a preliminary result about groups with more than one class of involutions.

Lemma 7.7. Let $H$ be a finite group with more than one class of involutions. Let $x, y \in H$ be non-conjugate involutions in $H$. Then there exists an involution $z \in H$ such that $[x, z]=[y, z]=1$.

Proof. Any two involutions generate a dihedral group, $D$, of order $2 n$ for some $n$. If $n$ is odd, then all involutions in such a dihedral group are $D$-conjugate. Since $\langle x, y\rangle \leq H$ and the generators are not $H$-conjugate, $\langle x, y\rangle$ must be a dihedral group of order $2 n$, for $n$ even. Hence $\langle x, y\rangle$ contains a central involution, and the lemma follows.

Let $G$ be an affine orthogonal group. So $G=V \rtimes L$ where $L$ is an $n$-dimensional simple orthogonal group, and $V$ is the orthogonal $n$-dimensional $L$-module. In a spirit similar to that of Theorem 4.18, we identify $G$ with $V L$ and any element $g \in G$ can be expressed as a product $g=g_{V} g_{L}$ where $g_{V} \in V$ and $g_{L} \in L$. For the remainder of this section, any such subscripts will describe such an expression. If $X$ is a conjugacy class of involutions in $G$, we denote $X_{L}=X V / V$. Clearly, $X_{L}$ is a conjugacy class of involutions in $L$. We set $d^{L}$ to be the standard distance metric on $\mathcal{C}\left(L, X_{L}\right)$ and $\Delta_{i}^{L}(t)=\left\{x_{L} \in X_{L} \mid d^{L}\left(t, x_{L}\right)=i\right\}$. As usual, we let $d$ be the distance
metric on $\mathcal{C}(G, X)$ and $\Delta_{i}(t)$ to be the $i^{\text {th }}$ disc of $t$. The following lemma is a direct consequence of the action of $L$ on $V$.

Lemma 7.8. (i) $x^{2}=1$ if and only if $x_{V} \in\left[V, x_{L}\right]$.
Let $x, y \in X$ be such that $\left[x_{L}, y_{L}\right]=1$.
(ii) $\left[x_{L}, y\right]=1$ if and only if $y_{V} \in C_{V}\left(x_{L}\right) \cap\left[V, y_{L}\right]$.
(iii) $[x, y]=1$ if and only if $x_{V}^{y_{L}}-x_{V}=y_{V}^{x_{L}}-y_{V}$.

Let $(\cdot, \cdot)$ be the orthogonal form on $V$.
Lemma 7.9. Let $x_{L}, y_{L} \in X_{L}$.
(i) $C_{V}\left(x_{L}\right)=\left[V, x_{L}\right]^{\perp}$.
(ii) $C_{V}\left(x_{L}\right)=C_{V}\left(y_{L}\right)$ if and only if $x_{L}=y_{L}$.
(iii) For any subspace $U \leq V, V=U \oplus U^{\perp}$.

Proof. See Lemmas 4.5(ii)-(iii) and 4.6(i) for analogous proofs.

We introduce the following map: for any $x \in X_{L}$, define $\varphi_{x}: V \rightarrow[V, x]$ by $v \mapsto v^{x}-v$. This is a well-defined, surjective map and $\operatorname{ker} \varphi_{x}=C_{V}(x)$. For $U \leq V$, we denote the restriction of $\varphi_{x}$ to $U$ by $\left.\varphi_{x}\right|_{U}$.

Lemma 7.10. Let $x_{L}, y_{L} \in X_{L}$ be such that $\left[x_{L}, y_{L}\right]=1$.
(i) $\left.\operatorname{Im} \varphi_{y_{L}}\right|_{\left[V, x_{L}\right]}=\left[V, x_{L}\right] \cap\left[V, y_{L}\right]$.
(ii) For any $x_{V} \in\left[V, x_{L}\right]$, there exists $y_{V} \in\left[V, y_{L}\right]$ such that $[x, y]=1$.

Proof. Since $\left[x_{L}, y_{L}\right]=1$, we certainly have $\left.\operatorname{Im} \varphi_{y_{L}}\right|_{\left[V, x_{L}\right]} \subseteq\left[V, x_{L}\right] \cap\left[V, y_{L}\right]$. By Lemma 7.9,

$$
\left[V, x_{L}\right]=\left(\left[V, x_{L}\right] \cap C_{V}\left(y_{L}\right)\right) \oplus\left(\left[V, x_{L}\right] \cap\left[V, y_{L}\right]\right)
$$

and since $\left.\operatorname{ker} \varphi_{y_{L}}\right|_{\left[V, x_{L}\right]}=\left[V, x_{L}\right] \cap C_{V}\left(y_{L}\right)$, the appropriate isomorphism theorem shows equality, proving (i). For any $x_{V} \in\left[V, x_{L}\right]$, we have $x_{V}^{y_{L}}-x_{V} \in\left[V, x_{L}\right] \cap\left[V, y_{L}\right]$. By (i) using $\left.\varphi_{x_{L}}\right|_{\left[V, y_{L}\right]}$, there exists $y_{V} \in\left[V, y_{L}\right]$ such that $y_{V}^{x_{L}}-y_{V}=x_{V}^{y_{L}}-x_{V}$. Hence $[x, y]=1$ by Lemma 7.8(iii).

## 3-Dimensional Affine Orthogonal Groups

Let $G$ be the 3-dimensional affine orthogonal group, $A O_{3}(q) \cong q^{3}: L_{2}(q)$. The structure of the commuting involution graph was determined earlier in Theorem 4.18 via a direct method. To illustrate the generality of the theory developed in this chapter, we provide an alternative proof for the diameter. Let $X$ be the conjugacy class of involutions of $G$ and let $t=t_{L} \in X$. For any $x \in X, \operatorname{dim}\left(C_{V}(x)\right)=1$ and $\operatorname{dim}([V, x])=2$.

Lemma 7.11. $\operatorname{dim}\left(C_{V}\left(t_{L}\right) \cap\left[V, x_{L}\right]\right)=1$ for all $x_{L} \in \Delta_{1}^{L}\left(t_{L}\right)$.
Proof. Suppose $C_{V}\left(t_{L}\right) \cap\left[V, x_{L}\right]=\varnothing$. Since $C_{V}\left(t_{L}\right)^{x_{L}}=C_{V}\left(t_{L}^{x_{L}}\right)=C_{V}\left(t_{L}\right)$, for any $v \in C_{V}\left(t_{L}\right)$ either $v^{x_{L}}=v$ or $v^{x_{L}}=v^{-1}$. However, $C_{V}\left(t_{L}\right) \cap\left[V, x_{L}\right]=\varnothing$, so for all $v \in C_{V}\left(t_{L}\right), v^{x_{L}}=x_{L}$. Thus $C_{V}\left(t_{L}\right)=C_{V}\left(x_{L}\right)$ and so $t_{L}=x_{L}$, a contradiction. As $\operatorname{dim}\left(C_{V}\left(t_{L}\right)\right)=1$, the result follows.

Lemma 7.12. Let $x_{L} \in X_{L}$ be such that $\left[t, x_{L}\right]=1$. Then $d(t, x) \leq 2$.
Proof. If $x_{V} \in\left[V, x_{L}\right] \cap C_{V}(t)$ then $[t, x]=1$ by Lemma 7.8(ii). Assume $x_{V} \in\left[V, x_{L}\right] \backslash C_{V}(t)$, and let $y_{L}=t x_{L}$. Now $\left.\varphi_{x_{L}}\right|_{C_{V}(t) \cap\left[V, y_{L}\right]} \subseteq\left[V, x_{L}\right] \cap\left[V, y_{L}\right]$ by Lemma 7.10(i). If $C_{V}(t) \cap\left[V, y_{L}\right] \subseteq \operatorname{ker} \varphi_{x_{L}}$ then $\operatorname{dim}\left(C_{V}(t) \cap\left[V, y_{L}\right] \cap C_{V}\left(x_{L}\right)\right)=1$. Hence, $C_{V}(t)=C_{V}\left(x_{L}\right)$ by dimensions and by Lemma 7.9(ii), $t=x_{L}$, providing us with a contradiction. Therefore,

$$
\begin{equation*}
\left.\operatorname{Im} \varphi_{x}\right|_{C_{V}(t) \cap\left[V, y_{L}\right]}=\left[V, x_{L}\right] \cap\left[V, y_{L}\right] . \tag{7.5}
\end{equation*}
$$

Now $x_{V} \in\left[V, x_{L}\right] \backslash C_{V}(t)$ and $x_{V}^{y_{L}}-x_{V} \in\left[V, x_{L}\right] \cap\left[V, y_{L}\right]$. However by (7.5), there exists $y_{V} \in C_{V}(t) \cap\left[V, y_{L}\right]$ such that $y_{V}^{x_{L}}-y_{V}=x_{V}^{y_{L}}-x_{V}$. Hence by Lemma 7.8(iii), $[y, x]=1$ and Lemma 7.8(ii) gives $[t, y]=1$, so $d(t, x) \leq 2$.

Lemma 7.13. Let $y_{L} \in \Delta_{2}^{L}(t)$. Then for any $y_{V} \in\left[V, y_{L}\right], d(t, y)=2$.
Proof. Since $\operatorname{dim}\left(C_{V}(t)\right)=1$ and by Lemma 7.9(ii), $C_{V}(t) \cap C_{V}\left(y_{L}\right)=\varnothing$. Hence $C_{V}(t) \cap\left[V, x_{L}\right] \cap C_{V}\left(y_{L}\right)=\varnothing$, and

$$
\begin{equation*}
\left.\operatorname{Im} \varphi_{y_{L}}\right|_{C_{V}(t) \cap\left[V, x_{L}\right]}=\left[V, x_{L}\right] \cap\left[V, y_{L}\right] . \tag{7.6}
\end{equation*}
$$

Let $y_{V} \in\left[V, y_{L}\right]$. Then $y_{V}^{x_{L}}-y_{V} \in\left[V, y_{L}\right] \cap\left[V, x_{L}\right]$ and hence there exists $x_{V} \in C_{V}(t) \cap\left[V, x_{L}\right]$ such that $x_{V}^{y_{L}}-x_{V}=y_{V}^{x_{L}}-y_{V}$ by (7.6). Therefore $d(t, y)=2$.

Proposition 7.14. $\mathcal{C}(G, X)$ is connected of diameter 3.

Proof. Since $\mathcal{C}\left(L, X_{L}\right)$ has diameter 3 (see Theorem 2.10), the result follows immediately from Lemmas 7.10(ii) and Lemma 7.13.

## 4-Dimensional Affine Orthogonal Groups

Let $G=V L$ be one of the following 4-dimensional affine orthogonal groups of $\varepsilon$-type, for $\varepsilon= \pm 1$ :

$$
G \cong\left\{\begin{array}{l}
A O_{4}^{+}(q) \cong q^{4}:\left(S L_{2}(q) \circ S L_{2}(q)\right) \\
A O_{4}^{-}(q) \cong q^{4}: L_{2}\left(q^{2}\right) .
\end{array}\right.
$$

We first deal with a trivial case.

Proposition 7.15. Let $G \cong A O_{4}^{+}(q)$ and $Y_{L}$ be the conjugacy class of $L$ consisting of one involution. Then $\mathcal{C}(G, Y)$ is totally disconnected.

Proof. Let $Y_{L}=\{s\}$ and observe that $[V, s]=V$. All elements of $x \in Y$ are of the form $x=x_{V} s$. An easy calculation shows that $[u s, v s]=1$ if and only if $u=v$ and the result follows.

For the rest of the section we assume the following. If $G$ is of + -type, let $X$ be the non-trivial conjugacy class of involutions in $G$. If $G$ is of --type, let $X$ be the sole conjugacy class of involutions in $G$. Let $t=t_{L} \in X$, and observe that for any $x \in X, \operatorname{dim}\left(C_{V}(x)\right)=\operatorname{dim}([V, x])=2$.

Lemma 7.16. Let $x_{L} \in \Delta_{1}^{L}(t)$.
(i) If $G$ is of --type, or $G$ is of +-type and $x_{L} \neq-t$, then $\operatorname{dim}\left(C_{V}(t) \cap\left[V, x_{L}\right]\right)=1$.
(ii) If $G$ is of +-type and $x_{L}=-t$, then $C_{V}(t)=\left[V, x_{L}\right]$.

Proof. We subdivide (i) into two cases. The possibility where $C_{V}(t) \cap\left[V, x_{L}\right]=\varnothing$ is proved analogously as in Lemma 7.11. It remains to disprove the case when $C_{V}(t)=\left[V, x_{L}\right]$. For all $v \in C_{V}(t), v^{t}=v$ and $v^{t x_{L}}=v^{x_{L}}=v^{-1}$, so $\left[V, x_{L}\right] \subseteq\left[V, t x_{L}\right]$. If $G$ is of --type, or $G$ is of +-type and $x_{L} \neq-t$, then $t x_{L} \in X$ and we have equality and thus $x=t x_{L}$ by Lemma 7.9(ii), providing a contradiction, so (i) holds. Assume now $G$ is of + -type and $x_{L}=-t$. For $u \in C_{V}(t)$, we have $u=u^{t}=u^{-x}=(-u)^{x}$ and so $u^{x}=-u$, proving (ii).

Lemma 7.17. Let $x_{L} \in X \cap L$ be such that $\left[t, x_{L}\right]=1$. Then $d(t, x) \leq 2$.
Proof. Suppose Lemma 7.16(i) holds. If $x_{V} \in\left[V, x_{L}\right] \cap C_{V}(t)$ then $[t, x]=1$ by Lemma 7.8(ii). Assume $x_{V} \in\left[V, x_{L}\right] \backslash C_{V}(t)$, and let $y_{L}=t x_{L} \in X$. Now $\left.\varphi_{x_{L}}\right|_{C_{V}(t) \cap\left[V, y_{L}\right]} \subseteq\left[V, x_{L}\right] \cap\left[V, y_{L}\right]$ by Lemma 7.10(i). If $C_{V}(t) \cap\left[V, y_{L}\right] \subset \operatorname{ker} \varphi_{x_{L}}$ then $\operatorname{dim}\left(C_{V}(t) \cap\left[V, y_{L}\right] \cap C_{V}\left(x_{L}\right)\right)=1$. Since $y_{L}=t x_{L}$, and for any $u \in C_{V}(t) \cap C_{V}\left(x_{L}\right)$, we have $u=u^{t}=u^{x}=u^{t} x$, this is impossible. Therefore,

$$
\begin{equation*}
\left.\operatorname{Im} \varphi_{x}\right|_{C_{V}(t) \cap\left[V, y_{L}\right]}=\left[V, x_{L}\right] \cap\left[V, y_{L}\right] . \tag{7.7}
\end{equation*}
$$

Now $x_{V} \in\left[V, x_{L}\right] \backslash C_{V}(t)$ and $x_{V}^{y_{L}}-x_{V} \in\left[V, x_{L}\right] \cap\left[V, y_{L}\right]$. However by (7.7), there exists $y_{V} \in C_{V}(t) \cap\left[V, y_{L}\right]$ such that $y_{V}^{x_{L}}-y_{V}=x_{V}^{y_{L}}-x_{V}$. Hence by Lemma 7.8(iii), $[y, x]=1$ and Lemma 7.8(ii) gives $[t, y]=1$, so $d(t, x) \leq 2$. Suppose instead Lemma 7.16(ii) holds. Then $C_{V}(t)=\left[V, x_{L}\right]$ and so by Lemma 7.8(ii), $[t, x]=1$ for all $x_{V} \in\left[V, x_{L}\right]$, so proving the lemma.

Lemma 7.18. Let $y_{L} \in \Delta_{2}^{L}(t)$ and $x_{L} \in \Delta_{1}^{L}(t)$ be such that $\left[x_{L}, y_{L}\right]=1$. Then $C_{V}(t) \cap\left[V, x_{L}\right] \cap C_{V}\left(y_{L}\right)=\varnothing$ if and only if $d\left(t, y_{V} y_{L}\right)=2$ for all $y_{V} \in\left[V, y_{L}\right]$. Proof. Assume $C_{V}(t) \cap\left[V, x_{L}\right] \cap C_{V}\left(y_{L}\right)=\varnothing$. Then

$$
\begin{equation*}
\left.\operatorname{Im} \varphi_{y_{L}}\right|_{C_{V}(t) \cap\left[V, x_{L}\right]}=\left[V, x_{L}\right] \cap\left[V, y_{L}\right] . \tag{7.8}
\end{equation*}
$$

Let $y_{V} \in\left[V, y_{L}\right]$. Then $y_{V}^{x_{L}}-y_{V} \in\left[V, y_{L}\right] \cap\left[V, x_{L}\right]$ and hence there exists $x_{V} \in C_{V}(t) \cap\left[V, x_{L}\right]$ such that $x_{V}^{y_{L}}-x_{V}=y_{V}^{x_{L}}-y_{V}$ by (7.8), so $d(t, y)=2$. To show the converse, suppose $\operatorname{dim}\left(C_{V}(t) \cap\left[V, x_{L}\right] \cap C_{V}\left(y_{L}\right)\right)=1$. Let $u \in\left[V, y_{L}\right] \backslash C_{V}\left(x_{L}\right)$. Then any $v \in C_{V}(t) \cap\left[V, x_{L}\right]$ also lies in $C_{V}\left(y_{L}\right)$ by dimensions. So $v^{y_{L}}-v=0$. However, $u^{x}-u \neq 0$ and so $d(t, y)>2$.

Lemma 7.19. For any $x_{L} \in X_{L}$, let $B_{x_{L}} \in\left\{C_{V}\left(x_{L}\right),\left[V, x_{L}\right]\right\}$. Let $x_{L}, y_{L}, z_{L} \in X_{L}$ be distinct such that $\left[x_{L}, y_{L}\right]=\left[y_{L}, z_{L}\right]=1$. Then $\operatorname{dim}\left(B_{x_{L}} \cap B_{y_{L}} \cap B_{z_{L}}\right)=1$ if and only if $\operatorname{dim}\left(B_{x_{L}}^{\perp} \cap B_{y_{L}} \cap B_{z_{L}}^{\perp}\right)=1$.

Proof. Suppose $B_{x_{L}} \cap B_{y_{L}} \cap B_{z_{L}} \neq \varnothing$. By Lemma 7.16, $\operatorname{dim}\left(B_{y_{L}} \cap B_{x_{L}}\right)=\operatorname{dim}\left(B_{y_{L}} \cap B_{z_{L}}\right)=1$, and so $B_{y_{L}} \cap B_{x_{L}}=B_{y_{L}} \cap B_{z_{L}}$. Since $\left[x_{L}, y_{L}\right]=\left[y_{L}, z_{L}\right]=1$, we must also have $\operatorname{dim}\left(B_{y_{L}} \cap B_{x_{L}}^{\perp}\right)=\operatorname{dim}\left(B_{y_{L}} \cap B_{z_{L}}^{\perp}\right)=1$. Let $u \in B_{y_{L}} \cap B_{z_{L}}$. For any $v \in B_{y_{L}} \cap B_{z_{L}}^{\perp}$, we have $(u, v)=0$. Since $B_{y_{L}} \cap B_{z_{L}}=B_{y_{L}} \cap B_{x_{L}}$, we have $v \in B_{x_{L}}^{\perp}$ and the result follows.

Proposition 7.20. $\mathcal{C}(G, X)$ is connected of diameter 3.

Proof. Let $z_{L} \in \Delta_{3}^{L}(t)$ and let $x \in \Delta_{1}^{L}(t)$ and $y \in \Delta_{2}^{L}(t)$ be such that $\left[x_{L}, y_{L}\right]=\left[y_{L}, z_{L}\right]=1$. If $C_{V}\left(x_{L}\right) \cap\left[V, y_{L}\right] \cap C_{V}\left(z_{L}\right)=\varnothing$ then for all $z_{V} \in\left[V, z_{L}\right]$, $d(x, z)=2$ and thus $d(t, z)=3$ by Lemma 7.10. We first suppose $G$ is of --type. If $C_{V}\left(x_{L}\right) \cap\left[V, y_{L}\right] \cap C_{V}\left(z_{L}\right) \neq \varnothing$, then by Lemma 7.19,

$$
\begin{equation*}
\left[V, x_{L}\right] \cap\left[V, y_{L}\right] \cap\left[V, z_{L}\right] \neq \varnothing . \tag{7.9}
\end{equation*}
$$

If $C_{V}(t) \cap\left[V, x_{L}\right] \cap C_{V}\left(y_{L}\right)=\varnothing$ then by Lemma 7.18, $d(t, y)=2$ and so by Lemma $7.10(\mathrm{ii}), d(t, z)=3$.

It remains to consider the case $C_{V}(t) \cap\left[V, x_{L}\right] \cap C_{V}\left(y_{L}\right) \neq \varnothing$. By Lemma 7.19,

$$
\begin{equation*}
[V, t] \cap\left[V, x_{L}\right] \cap\left[V, y_{L}\right] \neq \varnothing . \tag{7.10}
\end{equation*}
$$

Since $\operatorname{dim}\left(\left[V, x_{L}\right] \cap\left[V, y_{L}\right]\right)=1$, together with (7.9) and (7.10),

$$
\operatorname{dim}\left([V, t] \cap\left[V, x_{L}\right] \cap\left[V, y_{L}\right] \cap\left[V, z_{L}\right]\right)=1
$$

Set $U=[V, t] \cap\left[V, x_{L}\right] \cap\left[V, y_{L}\right] \cap\left[V, z_{L}\right]$.
Since $C_{G}(U) \unlhd \operatorname{Stab}_{G}(U)$ and $t x \in C_{G}(U)$, $t x$ and $z$ are not conjugate in $\operatorname{Stab}_{G}(U)$. Hence by Lemma 7.7 , there exists $r_{L} \in X_{L} \cap \operatorname{Stab}_{G}(U)$ such that $\left[t x_{L}, r_{L}\right]=\left[r_{L}, z_{L}\right]=1$. Since $G$ has only one class of involutions, $r_{L} \in X$. Suppose $U \leq C_{V}\left(r_{L}\right)$. Then $\operatorname{dim}\left(\left[V, z_{L}\right] \cap C_{V}\left(r_{L}\right) \cap C_{V}\left(t x_{L}\right)\right)=1$. Hence
$\operatorname{dim}\left(C_{V}\left(z_{L}\right) \cap C_{V}\left(r_{L}\right) \cap\left[V, t x_{L}\right]\right)=1$ by Lemma 7.19. If $C_{V}\left(z_{L}\right) \cap\left[V, r_{L}\right] \cap C_{V}\left(t x_{L}\right) \neq \varnothing$ then

$$
C_{V}\left(z_{L}\right)=\left(C_{V}\left(r_{L}\right) \cap\left[V, t x_{L}\right]\right) \oplus\left(\left[V, r_{L}\right] \cap C_{V}\left(t x_{L}\right)\right) .
$$

Hence, $t x_{L} \in \operatorname{Stab}_{G}\left(C_{V}\left(z_{L}\right)\right)$ and by Lemma 7.9(ii), $t x_{L} \in C_{G}\left(z_{L}\right)$, contradicting our choice of $z_{L}$. Therefore $U \leq\left[V, r_{L}\right]$. Then $\operatorname{dim}\left(\left[V, z_{L}\right] \cap\left[V, r_{L}\right] \cap C_{V}\left(t x_{L}\right)\right)=1$ and so $\operatorname{dim}\left(C_{V}\left(z_{L}\right) \cap\left[V, r_{L}\right] \cap\left[V, t x_{L}\right]\right)=1$ by Lemma 7.19. Since $\operatorname{dim}\left(\left[V, r_{L}\right] \cap C_{V}\left(t x_{L}\right)\right)=1$ and $C_{V}\left(z_{L}\right) \cap\left[V, z_{L}\right]=\varnothing, C_{V}\left(z_{L}\right) \cap\left[V, r_{L}\right] \cap C_{V}\left(t x_{L}\right)=\varnothing$. Therefore, for all $z_{V} \in\left[V, z_{L}\right], d\left(t x_{L}, z\right)=2$ and hence $d(t, z)=3$.

Suppose now $G$ is of +-type. We have $-x \in X$ and $[-x, y]=[y, z]=1$. Suppose $C_{V}\left(x_{L}\right) \cap\left[V, y_{L}\right] \cap C_{V}\left(z_{L}\right) \neq \varnothing$. Then by Lemma 7.16(ii), $C_{V}\left(-x_{L}\right)=\left[V, x_{L}\right]$ and so $C_{V}\left(-x_{L}\right) \cap C_{V}\left(x_{L}\right)=\varnothing$. By dimensions, we must then have $C_{V}\left(-x_{L}\right) \cap\left[V, y_{L}\right] \cap C_{V}\left(z_{L}\right)=\varnothing$, so Lemma 7.18 gives $d\left(x_{L}, z\right) \leq 2$ and so $d(t, z) \leq 3$.

This proves Lemma 7.20.

Together Propositions 7.14 and 7.20 complete the proof of Theorem 1.9.

## Chapter 8

## Prelude to Future Work

We close this thesis by laying the groundwork for three avenues of continued research in this area of study relating to commuting involution graphs.

### 8.1 Projective Symplectic Groups of Arbitrary Dimension

We dealt with 4-dimensional projective symplectic groups in Chapters 3 and 4. In the footsteps of Section 4 of [15], we look at $G=S p_{2 n}(q)$ where $n>2, q=p^{a}$ and $p=2$. Let $V$ denote the $G F(q) G$-symplectic module of dimension $2 n$ and let $t$ be an involution in $G$ for which $\operatorname{dim} C_{V}(t)=2 n-1$. Put $X=t^{G}$, the $G$-conjugacy class containing $t$.

Theorem 8.1. $\mathcal{C}(G, X)$ is connected of diameter 2 .

Proof. For $x \in X, C_{G}(x) \leq \operatorname{Stab}_{G}\left(C_{V}(x)\right)$ with

$$
\operatorname{Stab}_{G}\left(C_{V}(x)\right) \cong q^{2 n-1} S L_{2 n-2}(q)(q-1)
$$

Set $K_{x}=O^{2^{\prime}}\left(\operatorname{Stab}_{G}\left(C_{V}(x)\right)\right)$. Then $K_{x} \cong q^{2 n-1} S L_{2 n-2}(q)$ and $C_{G}(x)=K_{x}$. Let $x \in X \backslash\{t\}$. If $C_{V}(t)=C_{V}(x)$, then $x \in K_{t}$ and so $x \in \Delta_{1}(t)$. Now suppose that $C_{V}(t) \neq C_{V}(x)$. Then $\operatorname{dim}\left(C_{V}(t) \cap C_{V}(x)\right)=2 n-2$. Let $U$ be a 1-dimensional subspace of $C_{V}(t) \cap C_{V}(x)$. Since $[V, t]$ is a 1 -space and $G$ acts transitively on the

1 -subspaces of $V$, there exists $y \in X$ such that $[V, y]=U$. So $[V, y] \leq C_{V}(t) \cap C_{V}(x)$ and hence $y$ leaves both $C_{V}(t)$ and $C_{V}(x)$ invariant. Thus $y \in K_{t} \cap K_{x}=C_{G}(t) \cap C_{G}(x)$ and so $d(t, x) \leq 2$ and we see that $\mathcal{C}(G, X)$ is connected. Since $\mathcal{C}(G, X)$ cannot have diameter 1 (as then $\langle X\rangle$ would be abelian), the theorem follows.

For a symplectic group of dimension $2 n$ over a field of characteristic 2 , there are $\left[\frac{n}{2}\right]+n$ conjugacy classes of involutions (see Lemma 7.7 of [10]). Theorem 2.12 shows that the diameters of the commuting involution graphs of arbitrary dimensional special linear groups over a field of characteristic 2 have diameter at most 6 . Theorem 8.1 shows the simplest case for arbitrary dimensional symplectic groups over fields of characteristic 2 , when the involutions centralise a proper submodule of maximal dimension. It remains to show whether the remaining commuting involution graphs for arbitrary dimensional symplectic groups behave in a similar way to those of the arbitrary dimensional special linear groups.

### 8.2 4-Dimensional Projective Special Unitary Groups over Fields of Odd Characteristic

Chapters 5 and 6 dealt with $U_{3}(q)$ for all $q$, and $U_{4}(q)$ for $q$ even. Naturally, the next case to consider is $U_{4}(q)$ for $q$ odd. As with the symplectic groups previously, this case is a much tougher nut to crack. Moreover, there is a greater issue with congruence than that of Theorem 1.4. For example, $U_{4}(q)$ has a centre of order 2 when $q \equiv 1(\bmod 4)$ and a centre of order 4 when $q \equiv 3(\bmod 4)$ so involutions in $U_{4}(q)$ may actually be elements of order 8 in $S U_{4}(q)$. For Theorem 1.4, we utilised the exceptional isomorphism $\operatorname{PSp}_{4}(q) \cong O_{5}(q)$ and all involutions in $O_{5}(q)$ were involutions in $\mathrm{SO}_{5}(q)$. One then may consider the exceptional isomorphism $U_{4}(q) \cong O_{6}^{-}(q)$, but this endeavour is met with only limited success. For example, $\Omega_{6}^{-}(q)=O_{6}^{-}(q)$ has a trivial centre only when $q \equiv 1(\bmod 4)$. When $q \equiv 3(\bmod 4)$, $\Omega_{6}^{-}(q) \neq O_{6}^{-}(q)$ has a centre of order 2 , so we still end up dealing with elements of
order 4 that square to the non-trivial element in the centre. We at least begin the examination of this problem by tackling the case most similar to that of Theorem 1.4.

Theorem 8.2. Let $G \cong U_{4}(q)$ where $q \equiv 1(\bmod 4)$ and $t$ an involution in $G$ such that $\left|C_{G}(t)\right|=q^{2}(q-1)\left(q^{4}-1\right)$. Then $\mathcal{C}(G, X)$ is connected of diameter 3.

Proof. We use the well-known exceptional isomorphism $U_{4}(q) \cong O_{6}^{-}(q)$. Let $V$ be a 6 -dimensional vector space equipped with an orthogonal form defined by the Gram matrix

$$
J=\left(\begin{array}{l|l}
I_{5} & \\
\hline & \lambda
\end{array}\right)
$$

for $\lambda$ non-square in $G F(q)$. Set $H=S O_{6}^{-}(q)=\left\{A \mid A^{T} J A=J\right.$, $\left.\operatorname{det}(A)=1\right\}$. Since $q \equiv 1(\bmod 4),-I_{6} \notin H^{\prime}$ and so $Z\left(H^{\prime}\right)=1$. Thus we set $G=H^{\prime}=O_{6}^{-}(q) \cong U_{4}(q)$. Let

$$
t=\left(\begin{array}{l|l}
-I_{2} & \\
\hline & I_{4}
\end{array}\right)
$$

and so $\operatorname{dim}\left(C_{V}(t)\right)=4$. Proposition 4.1.6 of [32] shows that

$$
\operatorname{Stab}_{G} C_{V}(t) \cong\left(C_{\frac{q-1}{2}} \times L_{2}\left(q^{2}\right)\right) \cdot R
$$

where $R$ is a group of order 4 . Since $C_{G}(t)=\operatorname{Stab}_{G} C_{V}(t)$, it is easy to see $t$ satisfies the hypotheses of the theorem. We continue in a similar vein to that of Theorem 1.4. Let $X=t^{G}$ and for any $x \in X \backslash\{t\}, \operatorname{dim}\left(C_{V}(x) \cap C_{V}(x)\right) \geq 1$. Hence,

$$
X \subseteq \bigcup_{\substack{U \leq C_{V}(t) \\ \operatorname{dim}(U)=1}} C_{G}(U)
$$

and for each $U, t \in C_{G}(U) . C_{G}(t)$ acts orthogonally on $C_{V}(t)$ and so acts on the set of 1-subspaces of $C_{V}(t)$ in three orbits. The centralisers of a representative of each orbit can be determined using Propositions 4.1.6 and 4.1.20 of [32]. Hence, for any 1-subspace $U$ of $C_{V}(t)$, either $C_{G}(U) \cong O_{5}(q)$ or $C_{G}(U) \cong A O_{4}^{-}(q)$. Theorem 1.4 bounds the diameters of the commuting involution graphs of $O_{5}(q) \cong P S p_{4}(q)$ by 3 . Moreover, if $X$ covers the two conjugacy classes of $O_{5}(q)$, then for any $x \in X$ not

| $q$ | Class | $\left\|\Delta_{1}(t)\right\|$ | $\left\|\Delta_{2}(t)\right\|$ | $\left\|\Delta_{3}(t)\right\|$ |
| :---: | :--- | :---: | :---: | :---: |
| 3 | 2A | 66 | 1,232 | 1,536 |
| 5 | 2A | 560 | 90,864 | 79,200 |
|  | 2B | 585 | 114,816 | 120,848 |
| 7 | 2A | 2,107 | 100,792 |  |
|  | 2B | 2,227 | $1,514,400$ | $1,064,447$ |
| 9 | 2A | 6,192 | $12,171,680$ | $7,459,200$ |
|  | 2B | 6,273 | $12,380,032$ | $11,561,344$ |

Table 8.1: Disc sizes of the commuting involution graphs of $\operatorname{PSU}_{4}(q), q \leq 9$.
$C_{G}(U)$-conjugate to $t$, there exists $y \in C_{G}(U) \cap X$ such that $[t, y]=[y, x]=1$ by Lemma 7.7. Theorem 1.9 also shows that the diameter of the commuting involution graphs of $A O_{4}^{-}(q)$ is at most 3 . Hence for any $x \in X, d(t, x) \leq 3$.

To finish this section, we present the disc sizes of the commuting involution graphs of $U_{4}(q)$ for $q \leq 9$, as calculated in Magma [19] (Table 8.1). The notation for the conjugacy classes follows AtLas [22] convention.

### 8.3 Rank 2 Twisted Exceptional Groups of Lie Type

Projective unitary groups are twisted Chevalley groups of type ${ }^{2} A_{n}$. The 4-dimensional projective symplectic groups studied in Chapters 3 and 4 are Chevalley groups of type $C_{2}=B_{2}$. The simple groups of Suzuki arise from a twisted Dynkin diagram of type ${ }^{2} B_{2}={ }^{2} C_{2}$. These simple groups arise when the characteristic of the field is 2 , and has an automorphism $\sigma$ such that $x^{\sigma^{2}}=x^{2}$ for all $x \in G F\left(2^{a}\right)$. This only occurs when $a$ is odd, and the simple groups are denoted ${ }^{2} B_{2}\left(2^{2 m+1}\right)=\mathrm{Sz}\left(2^{2 m+1}\right)$. The Suzuki groups are one of the easiest of the twisted exceptional groups of Lie-type to understand, and so study in this area naturally starts here.

Theorem 8.3. Let $G \cong \mathrm{Sz}\left(2^{2 m+1}\right)$. Then $\mathcal{C}(G, X)$ is disconnected with each connected component a clique.

Proof. By a result of Bender [16], $G$ contains a strongly embedded subgroup, $K$. Then by definition, $K \cap K^{x}$ has odd order for any $x \in G \backslash K$, and for any involution $t$ in $G, C_{G}(t) \leq K$ (see, for example, Lemma 4.3 of [37]). The result follows immediately.

One may wish to study the commuting involution graphs of other twisted exceptional groups of Lie-type. The simple groups of at most Lie rank 2 not yet studied are the triality twisted exceptional groups ${ }^{3} D_{4}(q)$, the Ree groups $R\left(q^{2 m+1}\right)$ for $q=2,3$, and the untwisted Chevalley groups of type $G_{2}$. To describe the commuting involution graphs of such groups will complete the study for all the simple groups of Lie rank at most 2 .

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