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Differential formulations of Maxwell's equations in anisotropic materials

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The differential form of Maxwell equations on a 3-manifold have the invariant formulation

$$dE = -\partial_t B \quad (1)$$

$$dH = \partial_t D + J \quad (2)$$

Where E is 1-form, B a 2-form, H a twisted 1-form and D and J twisted 2-forms. The distinction between 1-forms for fields evaluated as line integrals, and 2 forms for fluxes over surfaces is self evident, the treatment of H , D and J as twisted is perhaps more subtle and can be seen from the orientation independence of the magnetic field around a loop. See Tonti [1] and Bossavit[2] for more details. A twisted differential form is a differential form taking its values in the orientation line bundle. Note that the forms have dimensions free of length as the formulation is metric independent, for example J is in Amperes rather than A/m^2 .

For isotropic materials the material properties are treated as scalar multiples of the Hodge star operator from 1 to 2 forms, with respect to the flat metric on \mathbb{R}^3 . However Maxwell's equations are independent of metric and this fails to account for anisotropic materials. While material properties can be considered as symmetric matrices in preferred bases, or as separate Riemannian metrics, it is more natural to consider them as operators analogous to Hodge stars in their own right.

Recall that for an orientable (pseudo)-Riemannian n -manifold the Hodge star from 1-forms to $n - 1$ -forms is defined by the property

$$*\alpha \wedge \beta = \langle \alpha, \beta \rangle_g V_g$$

for any 1-forms α and β , where $\langle \cdot, \cdot \rangle_g$ is the inner product on one-forms and V_g the volume form associated with the metric and the orientation.

This is clearly non-degenerate as a bilinear map and positive definite for a Riemannian metric, where positivity is defined by the orientation.

We can retain the essential properties defining an abstract Hodge star as a linear map $*$: $\Lambda^1 \rightarrow \Lambda^{n-1}$ which satisfies

$$*\alpha \wedge \beta = *\beta \wedge \alpha$$

which is non-degenerate in the sense that

$$*\alpha \wedge \alpha \neq 0$$

for $\alpha \neq 0$. The energy density stored by the electric or magnetic field has to be a twisted 3-form so it can be integrated over a volume and is independent of orientation. The electric energy density is $\frac{1}{2}E \wedge D$, the magnetic $-\frac{1}{2}H \wedge B$ and the Ohmic power dissipated is $E \wedge J$.

For media which are linear (eg Ohmic), reciprocal (which means they are not actively producing energy), stationary (response depend on field at this time not on its history), and not chiral (B depends only on H , D only on E) the material properties permeability μ , permittivity ϵ and conductivity σ are generalized Hodge stars and the constituent laws are

$$D = \epsilon E \tag{3}$$

$$B = \mu H \tag{4}$$

$$J = \sigma E. \tag{5}$$

(The J here is the conduction current, there may be current sources as well).

We now have Maxwell's equations as

$$dE = -\partial_t \mu H \tag{6}$$

$$dH = \partial_t \epsilon E + \sigma E \tag{7}$$

To see the relation between material properties and energy we consider the derivative of the Poynting form $E \wedge H$ which gives the power flux through a surface

$$\begin{aligned} d(E \wedge H) &= E \wedge \epsilon \partial_t E - H \wedge \mu \partial_t H + E \wedge \sigma E \\ &= \frac{1}{2} \partial_t (E \wedge \epsilon E - H \wedge \mu H) + E \wedge \sigma E \end{aligned}$$

hence by Stokes theorem

$$\int_{\partial M} i^*(E \wedge H) = \frac{1}{2} \partial_t \left(\int_M E \wedge \epsilon E - H \wedge \mu H \right) + E \wedge \sigma E$$

We notice that the material properties all take either twisted forms to straight, or straight to twisted, so that the energy densities are twisted forms as required. This means that the material properties define a metric ¹ but not an orientation. This is quite right as materials (except chiral materials) are invariant under a change of orientation.

Chiral, or more generally bi-anisotropic, materials have an additional material property which (for a fixed angular frequency) replaces 3 and 4 by

$$D = \epsilon E + \alpha HB = \beta E + \mu H \quad (8)$$

where now α and β map twisted to twisted and straight to straight forms.

Having produced a metric independent invariant formulation of Maxwell's equations in an inhomogenous anisotropic medium considered as a three-manifold, what of the space time formulation. The Faraday 2-form formulation on space time is traditionally done on a homogeneous medium in units where epsilon and mu are one. This is clearly undesirable as it disguises the geometric nature of the equations.

We consider frame in which the medium is at rest, and we have a chart (x^1, x^2, x^3, t) . Define a 2-form on the 4-manifold by

$$F = E \wedge dt + B$$

as usual so that

$$dF = 0$$

is simultaneously $dE = -\partial_t B$ when projected onto the subspace of 3 forms containing dt (time-like forms) and $dB = 0$ when projected on two forms not involving dt (space-like) (is time/space-like the right way to describe this decomposition?) . Similarly we define

$$\tilde{F} = H \wedge dt - D$$

so that

$$d\tilde{F} = J \wedge dt + \rho$$

is simultaneously $dH = \partial_t D$ and $dD = \rho$ where ρ is the charge density. We now now seek a linear material map γ , perhaps a Hodge star, so that

$$\tilde{F} = \gamma F. \quad (9)$$

¹but not on a 2-manifold, Hodge stars on the middle dimensional forms, such as 1 forms on a two manifold, always have determinant one. A constraint we don't impose on material properties, also any scalar multiple of a metric produces the same Hodge on a two manifold

Clearly γ maps straight 2-forms to twisted ones. Let $E = \sum E_i dx^i$ and $D = \sum E_i \epsilon dx^i$ and so on, we see by equating coefficients of E and D in Eq 9 that

$$\gamma(dx^i \wedge dt) = \epsilon dx^i$$

and

$$\gamma(\mu dx^i) = -dx^i \wedge dt$$

which is sufficient to define the action of γ on both time-like and space-like forms. Unlike the Hodge star associated with a metric our γ is not generally idempotent. As a matrix on the space-like and time-like components we have

$$\gamma = \begin{pmatrix} 0 & \epsilon \\ -\mu^{-1} & 0 \end{pmatrix}$$

and its inverse

$$\gamma^{-1} = \begin{pmatrix} 0 & -\mu \\ \epsilon^{-1} & 0 \end{pmatrix}$$

and Maxwell's equations are now

$$dF = 0 \tag{10}$$

$$d\gamma F = J \wedge dt + \rho \tag{11}$$

Note that for chiral and bianisotropic media the zero blocks in my expression for γ may be non-zero.

References

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- [2] Alain Bossavit, On "Generalized Finite Differences": Discretization of Electromagnetic Problems. Preprint 2002.