Geometric structure in the tempered dual of the p-adic group SL(4)

Chao, Kuok Fai and Plymen, Roger

2010

MIMS EPrint: **2010.106**

ISSN 1749-9097

Submitted exclusively to the London Mathematical Society
doi:10.1112/0000/000000

Geometric structure in the tempered dual of SL(4)

Kuok Fai Chao and Roger Plymen

Abstract

We exhibit a definite geometric structure in the tempered dual of $\mathrm{SL}(4, \mathbb{Q}_p)$. Especially interesting is the case of $\mathrm{SL}(4, \mathbb{Q}_2)$, when we reveal a tetrahedron of reducibility in the tempered dual. This conforms to a recent geometric conjecture.

1. *Introduction*

Let G be a reductive p-adic group. The tempered dual $\mathrm{Irr}^{\mathrm{temp}}(G)$ of G admits a natural topology in which it is locally compact. A connected component of $\mathrm{Irr}^{\mathrm{temp}}(G)$ will be denoted by $\mathrm{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}}$. The label $\mathfrak{s}$ encodes a complex torus $D^{\mathfrak{s}}$, a compact torus $E^{\mathfrak{s}}$, and a finite group $W^{\mathfrak{s}}$. The group $W^{\mathfrak{s}}$ acts on $E^{\mathfrak{s}}$ and $D^{\mathfrak{s}}$.

This data allows us to construct the *extended quotient* $E^{\mathfrak{s}}//W^{\mathfrak{s}}$. The extended quotient $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ has more than one connected component, unless the action of $W^{\mathfrak{s}}$ is free (which is rare).

It follows that the compact spaces $\mathrm{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}}$ and $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ cannot, in general, be homeomorphic. The cuspidal support map **Sc** assigns to each point in $\mathrm{Irr}^{\mathrm{temp}}(G)$ its cuspidal support in the algebraic variety $D^{\mathfrak{s}}/W^{\mathfrak{s}}$, see [**11**, VI.7.1]. The geometric conjecture developed in [**1**],[**2**], [**3**],[**4**] asserts that there exists a continuous bijection

$$\mu^{\mathfrak{s}} : E^{\mathfrak{s}}//W^{\mathfrak{s}} \to \mathrm{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}}$$

which behaves well with respect to the map **Sc**, in the sense that

$$\mathbf{Sc} \circ \mu^{\mathfrak{s}} = \pi^{\mathfrak{s}}_{\sqrt{q}} \tag{1.1}$$

where $\pi^{\mathfrak{s}}_{\sqrt{q}}$ is a deformation of the standard projection $\pi^{\mathfrak{s}} : E^{\mathfrak{s}}//W^{\mathfrak{s}} \to E^{\mathfrak{s}}/W^{\mathfrak{s}}$ to the so-called q-projection $\pi^{\mathfrak{s}}_{\sqrt{q}} : E^{\mathfrak{s}}//W^{\mathfrak{s}} \to D^{\mathfrak{s}}/W^{\mathfrak{s}}$. The natural number q is the cardinality of the residue field of the underlying local field F; if $F = \mathbb{Q}_p$ then $q = p$. We should point out that the geometric conjecture predicts a geometric structure in the dual of a p-adic group which was previously unknown (apart from an early intimation in [**5**]).

We will prove Eqn.(1.1) for the special linear group $\mathrm{SL}_4(\mathbb{Q}_p)$ when $\mathfrak{s} = [T, \sigma]_G$, with T a maximal torus in G. The case $p = 2$ is especially interesting. In this case, there is a *tetrahedron of reducibility* in the tempered dual of SL_4 which does not occur when $p > 2$. The extended quotient performs a deconstruction: it creates the ordinary quotient and six unit intervals. The six intervals are then assembled into the six edges of a tetrahedron, and create a perfect model of reducibility, see §4.

By a *cocharacter* we shall mean a morphism $\mathbb{C}^\times \to T^\vee$ of algebraic groups, where $T^\vee$ is the dual torus in the Langlands dual $G^\vee$. The q-projection $\pi^{\mathfrak{s}}_{\sqrt{q}}$ is constructed from a finite set of cocharacters (depending on $\mathfrak{s}$), see [**3**, §1]. The cocharacters which enter the definition of the q-projection $\pi^{\mathfrak{s}}_{\sqrt{q}}$ depend only on two-sided cells $\mathbf{c}$, see §3 and §4.

2000 *Mathematics Subject Classification* 20G05, 22E50.

There is an abundance of L-packets in the tempered dual of SL_4. There are, for example, L-packets in the tempered dual of $\mathrm{SL}_4(\mathbb{Q}_2)$ which are parametrized by the 1-skeleton of a tetrahedron. The L-packets which occur in this article all conform to the *L-packet conjecture* in [**4**, §10].

2. *Some background material*

The extended quotient. We recall the definition of the extended quotient. Let X be a Hausdorff topological space. Let Γ be a finite group acting on X as homeomorphisms. Let

$$I(X) = \{(x,\gamma) \in X \times \Gamma : \gamma x = x\}$$

with group action on $I(X)$ given by

$$\alpha \cdot (x,\gamma) = (\alpha x, \alpha\gamma\alpha^{-1})$$

for $\alpha \in \Gamma$. Then the extended quotient is given by

$$X//\Gamma := I(X)/\Gamma = \bigsqcup_{\gamma\in\Gamma} X^\gamma/\Gamma \tag{2.1}$$

with one γ in each conjugacy class of Γ. The map $(x,\gamma) \mapsto x$ induces the *standard projection*

$$\pi : X//\Gamma \to X/\Gamma.$$

Extended affine Weyl groups. Let $G = \mathrm{SL}_4(F)$. Let $\mathfrak{s} = [T,\sigma]_G$ be a Bernstein component with respect to a character σ of T. In this section, we will write $W_\mathfrak{s} = W^\mathfrak{s}$ the isotropy group of $\mathfrak{s}$. We let $\pi_\sigma|_T = \sigma$ where π_σ is a (unitary) character of the standard maximal torus $\tilde{T} = F^\times \times F^\times \times F^\times \times F^\times$ of $\tilde{G} = \mathrm{GL}_4(F)$.

We denote by $W_\mathfrak{s}^0$ the isotropy of $\tilde{\mathfrak{s}} = [\tilde{T},\pi_\sigma]_{\tilde{G}}$. The group $W_\mathfrak{s}^0$ is a finite Weyl group. Let $\Phi_\mathfrak{s}$ denote a root system for $W_\mathfrak{s}^0$, and let $\Phi_\mathfrak{s}^+ = \Phi_\mathfrak{s} \cap \Phi^+$, where Φ^+ is a positive root system for the Weyl group of G. Then $\Phi_\mathfrak{s}^+$ is a positive system in $\Phi_\mathfrak{s}$. The group $W_\mathfrak{s}$ is not a Weyl group in general. However, we have the following relation (see for instance [**8**, Prop. 2.3]):

$$W_\mathfrak{s} = W_\mathfrak{s}^0 \rtimes C_\mathfrak{s}, \tag{2.2}$$

where

$$C_\mathfrak{s} = \left\{ w \in W_\mathfrak{s}^0 \ : \ w \cdot \Phi_\mathfrak{s}^+ = \Phi_\mathfrak{s}^+ \right\}.$$

Let $T^\vee$ denote the dual torus of T in the Langlands dual $G^\vee = \mathrm{PGL}_4(\mathbb{C})$ of G. Let $X(T^\vee)$ be the group of characters of $T^\vee$. We have

$$X(T^\vee) \simeq \left\{ (l_1,l_2,l_3,l_4) \in \mathbb{Z}^4 \ : \ l_1+l_2+l_3+l_4 = 0 \right\}.$$

We set

$$W_\mathfrak{s}^\mathrm{e} = X(T^\vee) \rtimes W_\mathfrak{s}^0.$$

Then $W_\mathfrak{s}^\mathrm{e}$ is the extended affine Weyl group of the p-adic group $H_\mathfrak{s}^0$. The group $H_\mathfrak{s}^0$ arises from [**12**, § 8]. Let $\Phi_\mathfrak{s}^\vee$ denote the set of coroots of the root system $\Phi_\mathfrak{s}$. The quadruple $(X(T), \Phi_\mathfrak{s}, X(T^\vee), \Phi_\mathfrak{s}^\vee)$ is the root datum of $H_\mathfrak{s}^0$. There is a canonical bijection (due to Lusztig [**9**]) between two-sided cells in $W_\mathfrak{s}^\mathrm{e}$ and unipotent classes in the Langlands dual of $H_\mathfrak{s}^0$.

Here are two relevant cases (the character τ is defined in §3):
- $\mathfrak{s} = [T,1]_G$, $\quad W_\mathfrak{s}^0 = \mathfrak{S}_4$, $\quad C_\mathfrak{s} = 1$, $\quad H_\mathfrak{s}^0 = \mathrm{SL}(4)$
- $\mathfrak{s} = [T,\tau]_G$, $\quad W_\mathfrak{s}^0 = 1$, $\quad C_\mathfrak{s} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\quad H_\mathfrak{s}^0 = T$.

The R-groups. We have 5 conjugacy classes of Levi subgroups of SL_4, one for each partition of 4. Let $P = MU$ be a standard parabolic subgroup of $G = \mathrm{SL}_4(F)$. Let $\tilde{M}$ be the corresponding Levi subgroup of $\tilde{G} = \mathrm{GL}_4(F)$ so that $M = \tilde{M} \cap \mathrm{SL}_4(F)$. We will use the framework, notation and results in [**7**]. Let $\sigma \in E_2(M)$ and $\pi_\sigma \in E_2(\tilde{M})$ with $\pi_\sigma \supset \sigma$. Let $W(M)$ be the Weyl group of M. Let

$$\overline{L}(\pi_\sigma) := \{\eta \in \widehat{F^\times} | \pi_\sigma \otimes \eta \simeq w\pi_\sigma \text{ for some } w \in W\}$$
$$X(\pi_\sigma) := \{\eta \in \widehat{F^\times} | \pi_\sigma \otimes \eta \simeq \pi_\sigma\}$$

By [**7**, Theorem 2.4], the R-group of σ is given by

$$R(\sigma) \cong \overline{L}(\pi_\sigma)/X(\pi_\sigma).$$

From now on, we will restrict ourselves to the case $M = T$ the standard maximal torus. For the Bernstein component $\mathfrak{s} = [T, \sigma]_G$, we let $\pi_\sigma|_T = \sigma$ where π_σ is a unitary character of $\tilde{M}$. From now on, we denote $(\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \cdots \times \mathrm{GL}_{n_r}) \cap \mathrm{SL}_n$ by $n_1 + n_2 + \cdots + n_r$ where $\Sigma n_i = n$. For example, $1 + 1 + 1 + 1$ means $(\mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1) \cap \mathrm{SL}_4$.

Recalling the definition of $E^\mathfrak{s}$, we see that $E^\mathfrak{s}$ may be identified with $\mathbb{T}^4/\mathbb{T}$, the maximal compact subgroup of the dual torus $T^\vee$ in the Langlands dual $G^\vee = \mathrm{PGL}_4(\mathbb{C})$.

For the principal series of SL_4, an explicit bijection between $E^\mathfrak{s}//W^\mathfrak{s}$ and $\mathrm{Irr}^{\mathrm{temp}}(G)^\mathfrak{s}$ is constructed in [**6**]. The method comprises a case-by-case analysis. In order to avoid the repetitive arguments [**6**], we have selected two cases which sufficiently illustrate the method. All remaining details are set out in [**6**]. As for the dependence of cocharacters on two-sided cells, these cases neatly illustrate the two extremes: in one case, we have $W_\mathfrak{s} = W^0_\mathfrak{s} = \mathfrak{S}_4$; in the other case, we have $W_\mathfrak{s} = C_\mathfrak{s} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

3. *The arithmetically unramified case*

We will focus on the case when $\mathfrak{s} = [T, \sigma]_G$ with $\sigma = 1$, the trivial character of T.

Theorem 3.1. *Let $\mathfrak{s} = [T, 1]_G$. There exists a continuous bijection*

$$\mu^\mathfrak{s} : E^\mathfrak{s}//W^\mathfrak{s} \to \mathrm{Irr}^{\mathrm{temp}}(G)^\mathfrak{s}$$

such that

$$\mathbf{Sc} \circ \mu^\mathfrak{s} = \pi^\mathfrak{s}_{\sqrt{q}} \tag{3.1}$$

where $\pi^\mathfrak{s}_{\sqrt{q}}$ is a deformation of the standard projection $\pi^\mathfrak{s} : E^\mathfrak{s}//W^\mathfrak{s} \to E^\mathfrak{s}/W^\mathfrak{s}$ to the q-projection $\pi^\mathfrak{s}_{\sqrt{q}} : E^\mathfrak{s}//W^\mathfrak{s} \to D^\mathfrak{s}/W^\mathfrak{s}$.

Proof. The group $W^\mathfrak{s}$ is the symmetric group $\mathfrak{S}_4$. This group has five conjugacy classes, one for each cycle type. We now compute the extended quotient. We view $\mathfrak{S}_4$ as the permutation group of 4 letters $(abcd)$.

- $\gamma = (abcd)$, $E^\gamma/Z(\gamma) = E^\mathfrak{s}/W^\mathfrak{s}$
- $\gamma = (acbd)$, $E^\gamma/Z(\gamma) \cong \mathbb{T}^2$
- $\gamma = (bcad)$, $E^\gamma/Z(\gamma) \cong \mathbb{T}$
- $\gamma = (badc)$, $E^\gamma/Z(\gamma) \cong \mathbb{T} \sqcup \mathbb{T}$
- $\gamma = (bcda)$,

$$\begin{aligned} E^\gamma/Z(\gamma) &= \{(1,1,1,1), (1,-1,1,-1), (1,i,-1,-i), (1,-i,-1,i)\} \\ &= pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4 \end{aligned}$$

Hence, we have

$$E^\mathfrak{s}//W^\mathfrak{s} = E^\mathfrak{s}/W^\mathfrak{s} \sqcup \mathbb{T}^2 \sqcup \mathbb{T} \sqcup \mathbb{T} \sqcup \mathbb{T} \sqcup pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4 \tag{3.2}$$

Now we identify each element in the compact subspace $\mathrm{Irr}^{\mathrm{t}}(G)^{\mathfrak{s}}$. In the following, St_k will denote the Steinberg representation of GL_k. The induced representations

$$\{\mathrm{Ind}^4_{(2+1+1)}(z_1^{\mathrm{val}}\mathrm{St}_2 \otimes z_2^{\mathrm{val}} \otimes 1) : z_1, z_2 \in \mathbb{T}\}$$

are irreducible, tempered, their infinitesimal characters lie in $\mathfrak{s}$, and they are parametrized by $\mathbb{T}^2$.

The induced representations

$$\{\mathrm{Ind}^4_{(3+1)}(z \cdot \mathrm{St}_3 \otimes 1) : z \in \mathbb{T}\}$$

are irreducible, tempered, have central characters in $\mathfrak{s}$ and are parametrized by $\mathbb{T}$.

The induced representations

$$\{\mathrm{Ind}^4_{(2+2)}(z \cdot \mathrm{St}_2 \otimes \mathrm{St}_2) : z \in \mathbb{T}\}$$

have central characters in $\mathfrak{s}$, and are parameterized by $\mathbb{T}$. They are irreducible except when $z = -1$. The R-group is as follows:

$$R((-1)^{\mathrm{val}} \cdot \mathrm{St}_2 \otimes \mathrm{St}_2) =< (-1)^{\mathrm{val}} > .$$

There are two irreducible components, denoted by ρ^+ and ρ^-. We will locate ρ^- in the second copy of $\mathbb{T}$ and identify ρ^+ by pt_1.

The Steinberg representation $\mathrm{St}(\mathrm{SL}_4)$ has central character in $\mathfrak{s}$. We identify this representation by pt_2.

The unramified unitary principal series of SL_4 contains points of reducibility. In fact, there is a *circle of reducibility*, as we now proceed to explain. Let $t = (z, -z, 1, -1)$ except $z = i$ and let χ_t be the corresponding unramified unitary character. Then the representation χ_t is given by

$$z^{\mathrm{val}} \otimes (-z)^{\mathrm{val}} \otimes 1 \otimes (-1)^{\mathrm{val}}.$$

Then $(-1)^{\mathrm{val}}$ is the generator of $\bar{L}(\chi_t)$ and $X(\chi_t) = 1$. Hence

$$R(\chi_t) = \mathbb{Z}/2\mathbb{Z}$$

and the induced representation

$$\lambda(t) := \mathrm{Ind}_B^G(\chi_t)$$

is reducible and admits two irreducible subrepresentations:

$$\lambda(t) = \lambda(t)^+ \oplus \lambda(t)^-.$$

We assign $\lambda(t)^+$ to $[z, -z, 1, -1] \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$ and $\lambda(t)^-$ to $z \in \mathbb{T}$.

Now, we turn to the point $t = (i, -i, 1 - 1)$. Then

$$\bar{L}(\chi_t) =< i^{\mathrm{val}} >$$

and $X(\chi_t) = 1$. We infer that $R(\chi_t) = \mathbb{Z}/4\mathbb{Z}$. The induced representation $\tau = \mathrm{Ind}_B^G(\chi_t)$ is reducible with 4 irreducible constituents τ_1, τ_2, τ_3, τ_4. We assign τ_1 to $[i, -i, 1, -1] \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$ and τ_2 to the point i in the third copy of $\mathbb{T}$ and assign τ_3 and τ_4 to pt_3 and pt_4 respectively.

For $t = (z_1, z_2, z_3, 1) \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$ except $t = (z, -z, 1, -1)$, the induced representation $\mathrm{Ind}_M^G(\chi_t)$ is irreducible.

We build a map

$$\mu : E^{\mathfrak{s}}//W^{\mathfrak{s}} \longrightarrow \mathrm{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}}$$

and here are the details:

Point in $E^{\mathfrak{s}}/W^{\mathfrak{s}}$	Irreducible representation	Cocharacter $h(t)$
$(z_1, z_2) \in \mathbb{T}^2$	$\mathrm{Ind}^4_{(2+1+1)}(z_1 \cdot \mathrm{St}_2 \otimes z_2^{\mathrm{valodet}}\pi \otimes \pi)$	$(t, t^{-1}, 1, 1)$
$z \in \mathbb{T}$	$\mathrm{Ind}^4_{(3+1)}(z \cdot \mathrm{St}_3 \otimes \pi)$	$(t^2, 1, t^{-2}, 1)$
$z \in \mathbb{T}$	$\mathrm{Ind}^4_{(2+2)}(z \cdot \mathrm{St}_2 \otimes \mathrm{St}_2)$	(t, t^{-1}, t, t^{-1})
$z \in \mathbb{T}$	$\lambda(t)^+$	1
pt_1	ρ^+	(t, t^{-1}, t, t^{-1})
pt_2	$\mathrm{St}(\mathrm{SL}_4)$	(t^3, t, t^{-1}, t^{-3})
pt_3	τ_3	1
pt_4	τ_4	1
$t \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$	$\mathrm{Ind}^G_B(\chi_t)$	1

It is clear that Eqn.(1.1) is satisfied. We note that the compact space $\mathrm{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}}$ is non-Hausdorff. One connected component contains a double-point, and another connected component contains a double-circle (and a quadruple point), see [**10**].

The Langlands dual group of $H^0_{\mathfrak{s}} = \mathrm{SL}_4(F)$ is the complex Lie group $\mathrm{PGL}_4(\mathbb{C})$. There are five unipotent classes in $\mathrm{PGL}_4(\mathbb{C})$:

$$\mathbf{u}_0 \le \mathbf{u}_3 \le \mathbf{u}_2 \le \mathbf{u}_1 \le \mathbf{u}_e,$$

which are respectively parametrized by the following partitions of 4:

$$(1^4) \le (2, 1^2) \le (2^2) \le (3, 1) \le (4).$$

They correspond (see for instance [**13**]) to the two-sided cells

$$\mathbf{c}_0 \le \mathbf{c}_3 \le \mathbf{c}_2 \le \mathbf{c}_1 \le \mathbf{c}_e.$$

We write

$$\begin{array}{cc} \mathrm{pt}_1 = (1, 1, 1, 1) & \mathrm{pt}_2 = (1, -1, 1, -1) \\ \mathrm{pt}_3 = (1, i, -1, -i) & \mathrm{pt}_4 = (1, -i, -1, i) \end{array} \tag{3.3}$$

and define

$$(E^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_0 := E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \{(z, -z, 1, -1) : z \in \mathbb{T}\} \sqcup \mathrm{pt}_3 \sqcup \mathrm{pt}_4 \simeq E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T} \sqcup \mathrm{pt}_3 \sqcup \mathrm{pt}_4$$

$$(E^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_3 := \{(z_1, z_1, z_2, 1) \,:\, z_1, z_2 \in \mathbb{T}\} \simeq \mathbb{T}^2$$

$$(E^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2 := \{(z, z, 1, 1) \,:\, z \in \mathbb{T}\} \sqcup \mathrm{pt}_1 \simeq \mathbb{T} \sqcup \mathrm{pt}_1$$

$$(E^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_1 := \{(z, z, z, 1) \,:\, z \in \mathbb{T}\} \simeq \mathbb{T}$$

$$(E^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_e := \mathrm{pt}_2.$$

From Eqn.(3.2), we get the following cell-decomposition of $E^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$:

$$E^{\mathfrak{s}}/\!/W^{\mathfrak{s}} = (E^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_0 \sqcup (E^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_3 \sqcup (E^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2 \sqcup (E^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_1 \sqcup (E^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_e,$$

with cocharacters

$$h_0 = 1, \quad h_3(t) = (t, t^{-1}, 1, 1), \quad h_2(t) = (t, t^{-1}, t, t^{-1})$$

$$h_1(t) = (t^2, 1, t^{-2}, 1), \quad h_e(t) = (t^3, t, t^{-1}, t^{-3}).$$

We have included pt_1 in the subset $(E^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$ in order to attach the two elements ρ^+ and ρ^- to the same unipotent class. It should be a general fact that all the elements in a given L-packet are attached to the same unipotent class.

□

4. *A tetrahedron of reducibility*

We exhibit a *tetrahedron of reducibility* in the tempered dual of $\mathrm{SL}_4(\mathbb{Q}_2)$ which does not occur in the tempered dual of $\mathrm{SL}_4(\mathbb{Q}_p)$ when $p > 2$. Let F denote the p-adic field $\mathbb{Q}_p$, and let $U_F^n := 1 + p^n\mathbb{Z}_p$, $n \geq 1$ denote the standard congruence unit groups. Let $U_F = \mathfrak{o}_F^\times$. For $p = 2$ define homomorphisms $\eta, \chi : U/U^3 \to \mathbb{Z}/2\mathbb{Z}$ as follows:

$$\eta(x) = 0, x \equiv 1 \mod 4$$

$$\eta(x) = 1, x \equiv -1 \mod 4$$

$$\chi(x) = 0, x \equiv \pm 1 \mod 8$$

$$\chi(x) = 1, x \equiv \pm 5 \mod 8$$

The map η defines an isomorphism of U/U^2 onto $\mathbb{Z}/2\mathbb{Z}$ and the map χ defines an isomorphism of U^2/U^3 onto $\mathbb{Z}/2\mathbb{Z}$. The *level* of a character ψ of $F^\times$ is the least integer $n \geq 0$ such that ψ is trivial on U_F^{n+1}. Then we have η is level 1 and χ is level 2. The product $\eta \cdot \chi$ is also level 2.

The three ramified quadratic characters of $\mathbb{Q}_2^\times$ create a unitary character of the standard Borel subgroup in $\mathrm{SL}_4(\mathbb{Q}_2)$:

$$\tau : \begin{bmatrix} x_1 & * & * & * \\ 0 & x_2 & * & * \\ 0 & 0 & x_3 & * \\ 0 & 0 & 0 & x_4 \end{bmatrix} \mapsto \eta(x_2)\chi(x_3)(\eta \cdot \chi)(x_4)$$

THEOREM 4.1. *Let $G = \mathrm{SL}_4(\mathbb{Q}_2)$ and $\mathfrak{s} = [T, \tau]_G$. There exists a continuous bijection*

$$\mu^{\mathfrak{s}} : E^{\mathfrak{s}}//W^{\mathfrak{s}} \to \mathrm{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}}$$

such that

$$\mathbf{Sc} \circ \mu^{\mathfrak{s}} = \pi^{\mathfrak{s}} \tag{4.1}$$

where $\pi^{\mathfrak{s}}$ is the standard projection $\pi^{\mathfrak{s}} : E^{\mathfrak{s}}//W^{\mathfrak{s}} \to E^{\mathfrak{s}}/W^{\mathfrak{s}}$. The orbifold $E^{\mathfrak{s}}/W^{\mathfrak{s}}$ contains a tetrahedron of reducibility.

Proof. We twist the character τ by an unramified unitary character ψ and form the induced representation $\mathrm{Ind}_B^G(\psi\tau)$. Let $E^{\mathfrak{s}}$ be the corresponding compact torus. The subgroup of the Weyl group which fixes $E^{\mathfrak{s}}$ is the finite group $W^{\mathfrak{s}} := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We have the standard projection

$$\pi : E^{\mathfrak{s}}//W^{\mathfrak{s}} \to E^{\mathfrak{s}}/W^{\mathfrak{s}}$$

of the extended quotient onto the ordinary quotient. The extended quotient $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ is the disjoint union of 6 unit intervals a, b, c, d, e, f and the ordinary quotient $E^{\mathfrak{s}}/W^{\mathfrak{s}}$. In the projection $\pi^{\mathfrak{s}}$, these 6 intervals assemble themselves into the 6 edges of a tetrahedron in $E^{\mathfrak{s}}/W^{\mathfrak{s}}$. The cardinality of each fibre of $\pi^{\mathfrak{s}}$ creates a perfect model of reducibility. The locus of reducibility is the 1-skeleton $\mathfrak{R}$ of a tetrahedron, and we have

$$|\pi^{-1}(\psi\tau)| = |Ind_B^G(\psi\tau)|$$

for all unramified unitary characters ψ of T. On the interior of each edge $\pi(a), \ldots, \pi(f)$ of $\mathfrak{R}$, each induced representation admits 2 distinct irreducible constituents; on each vertex of $\mathfrak{R}$, each induced representation admits 4 distinct irreducible components.

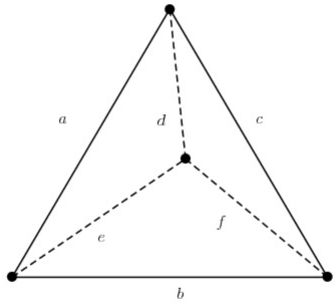

FIGURE 1. *The tetrahedron of reducibility.*

We have $R(\tau) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $W^{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We compute the extended quotient:

– $\gamma = (abcd)$, $\quad E^\gamma/W^{\mathfrak{s}} = E/W^{\mathfrak{s}}$.

– $\gamma = (badc)$, $\quad E^\gamma = \{(1,1,z,z),(1,-1,z,-z) : z \in \mathbb{T}\}$.
$$E^\gamma/W^\gamma \cong \mathbb{I} \sqcup \mathbb{I}$$
– $\gamma = (cdab)$, $\quad E^\gamma = \{(1,z,1,z),(1,z,-1,-z) : z \in \mathbb{T}\}$.
$$E^\gamma/W^\gamma \cong \mathbb{I} \sqcup \mathbb{I}$$
– $\gamma = (dcab)$, $\quad E^\gamma = \{(1,z,z,1),(1,z,-z,-1) : z \in \mathbb{T}\}$.
$$E^\gamma/W^\gamma \cong \mathbb{I} \sqcup \mathbb{I}$$
This leads to the following formula for the extended quotient:
$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup \mathbb{I}$$
where $\mathbb{I}$ is the interval $[-1, 1]$.

Let $T^\vee$ be the dual torus in $\mathrm{PGL}(4,\mathbb{C})$. Then $(1,1,z,z) \in T^\vee$ and it is appropriate to use homogeneous coordinates, in which case we write $(1 : 1 : z : z)$. Suppose that $z \pm 1$. Note that
$$\chi(1 : \chi : \eta z^{\mathrm{val}} : \chi\eta z^{\mathrm{val}}) = (\chi : 1 : \eta z^{\mathrm{val}} : \chi\eta z^{\mathrm{val}}) = (1 : \chi : \eta z^{\mathrm{val}} : \chi\eta z^{\mathrm{val}}).$$
We infer that the $\overline{L}$-group of the corresponding character is
$$\overline{L}(1 \otimes \chi \otimes \eta z^{\mathrm{val}} \otimes \chi\eta z^{\mathrm{val}}) =< \chi >= \mathbb{Z}/2\mathbb{Z}.$$
From the theory of the R-group, outlined in §2, we deduce that the representation induced from the point $(1 : 1 : z : z) \in T^\vee$ is reducible with two irreducible constituents.

The remaining $\overline{L}$-groups are as follows:
$$\overline{L}(1 \otimes (-1)^{\mathrm{val}}\chi \otimes \eta z^{\mathrm{val}} \otimes \chi\eta(-z)^{\mathrm{val}}) =< (-1)^{\mathrm{val}}\chi >= \mathbb{Z}/2\mathbb{Z}$$
$$\overline{L}(1 \otimes z^{\mathrm{val}} \otimes 1 \otimes z^{\mathrm{val}}) =< \eta >= \mathbb{Z}/2\mathbb{Z}$$
$$\overline{L}(1 \otimes z^{\mathrm{val}} \otimes (-1)^{\mathrm{val}} \otimes (-z)^{\mathrm{val}}) =< (-1)^{\mathrm{val}}\eta >= \mathbb{Z}/2\mathbb{Z}$$
$$\overline{L}(1 \otimes z^{\mathrm{val}} \otimes z^{\mathrm{val}} \otimes 1) =< \chi\eta >= \mathbb{Z}/2\mathbb{Z}$$
$$\overline{L}(1, z^{\mathrm{val}}, (-z)^{\mathrm{val}}, (-1)^{\mathrm{val}}) =< (-1)^{\mathrm{val}}\eta\chi >= \mathbb{Z}/2\mathbb{Z}.$$

In each case, the corresponding induced representation has two irreducible constituents.

We will assign the following labels:

$$(1,1,z,z) - (a)$$
$$(1,-1,z,-z) - (b)$$
$$(1,z,1,z) - (c)$$
$$(1,z,-1,-z) - (d)$$
$$(1,z,z,1) - (e)$$
$$(1,z,-z,-1) - (f)$$

We will now allow $z = \pm 1$ and investigate the points $(1,1,1,1), (1,-1,1,-1), (1,1,-1,-1)$ and $(1,-1,-1,1)$. It is easy to check that

$$(1,1,1,1) \in (a),(c),(e)$$
$$(1,-1,1,-1) \in (b),(c),(f)$$
$$(1,1,-1,-1) \in (a),(d),(f)$$
$$(1,-1,-1,1) \in (b),(d),(e)$$

For such points, the R-group $R(\psi\tau)$ is given by $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. This implies that, for each $\mathrm{Ind}_B^G(\psi\tau)$, there are 4 irreducible constituents. The extended quotient is the disjoint union of the ordinary quotient and six unit intervals. The six intervals are sent to the edges of a tetrahedron by the canonical projection

$$\pi : E^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \to E^{\mathfrak{s}}/W^{\mathfrak{s}}$$

The Langlands dual group of $H^0_{\mathfrak{s}} = T$ is the complex torus $T^\vee$ in $\mathrm{PGL}_4(\mathbb{C})$. There is only one unipotent class in $T^\vee$: the trivial class $\mathbf{u}_0$. Hence we set

$$(E^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_0 = E^{\mathfrak{s}}/\!/W^{\mathfrak{s}}.$$

There only one cocharacter, the trivial cocharacter.

□

The pre-image of the interior of one edge is the union of two open intervals (the one corresponding to the given edge and one in the ordinary quotient), replicating the fact that the R-group has order 2, while the pre-image of a vertex is the union of three endpoints of intervals and one point in the ordinary quotient, replicating the fact that the R-group has order 4 here. The 1-skeleton of the tetrahedron is a perfect model of reducibility and confirms the geometric conjecture in this case.

Acknowledgements. We thank Paul Baum for helping us to resolve a crucial issue in §4, Anne-Marie Aubert for her contributions to §2 and her careful reading of the manuscript. We thank Maarten Solleveld for alerting us to his preprint [**14**], in which the emphasis is placed on affine Hecke algebras.

References

1. A-M. Aubert, P. Baum and R.J. Plymen, The Hecke algebra of a reductive p-adic group: a geometric conjecture, Aspects of Mathematics 37, Vieweg Verlag (2006) 1-34.

2. A-M. Aubert, P. Baum and R.J. Plymen, Geometric structure in the representation theory of p-adic groups, C.R. Acad. Sci. Paris, Ser. I 345 (2007) 573-578.

3. A-M. Aubert, P. Baum and R.J. Plymen, Geometric structure in the principal series of the p-adic group G_2, Represent. Theory 15 (2011) 126 –169.

4. A-M. Aubert, P. Baum and R.J. Plymen, Geometric structure in the representation theory of p-adic groups II, Contemporary Math. 543 (2011), to appear.

5. J. Brodzki and R.J. Plymen, Geometry of the smooth dual of GL(n), C. R. Acad. Sci. Paris, Ser. I 331 (2000) 213 –218.
6. K.F. Chao, Geometric structure in the tempered dual of SL(N), Ph.D. Thesis, University of Manchester, 2010. http://eprints.ma.man.ac.uk/1525
7. D. Goldberg, R-groups and elliptic representations for SL_n, Pacific J. Math. 165 (1994) 77-92.
8. D. Goldberg and A. Roche, Hecke algebras and SL_n-types, Proc. London Math. Soc. 90 (2005) 87–131.
9. G. Lusztig, Cells in affine Weyl groups IV, J. Fac. Sci. Univ. Tokyo, Sect 1A, Math. 36 (1989) 297 – 328.
10. R.J. Plymen, Reduced C*-algebra for reductive p-adic groups, J. Functional Analysis 88 (1990) 251-266.
11. D. Renard, Représentations des groupes réductifs p-adiques, Cours spécialisés, Collection SMF 2010.
12. A. Roche, Types and Hecke algebras for principal series representations of split reductive p-adic groups, Ann. scient. Éc. Norm. Sup. 31 (1998) 361–413.
13. J.-Y. Shi, The partial order on two-sided cells of certain affine Weyl groups, J. Algebra **176** (1996) 607–621.
14. M. Solleveld, On the classification of irreducible representations of affine Hecke algebras with unequal parameters, arXiv:1008.0177v2[mathRT].

Kuok Fai Chao and Roger Plymen
School of Mathematics, Alan Turing building
University of Manchester
Manchester M13 9PL
England

kchao@ma.man.ac.uk
plymen@manchester.ac.uk