On the Bott periodicity, J-homomorphisms, and $H_*Q_0S^{-k}$

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Abstract

The Curtis conjecture predicts that the only spherical classes in $H_*(Q_0S^0; \mathbb{Z}/2)$ are the Hopf invariant one and the Kervaire invariant one elements. We consider Sullivan's decomposition

$$Q_0S^0 = J \times \mathrm{coker}J$$

where J is the fibre of $\psi^q - 1$ ($q = 3$ at the prime 2) and observe that the Curtis conjecture holds when we restrict to J. We then use the Bott periodicity and the J-homomorphism $SO \to Q_0S^0$ to define some generators in $H_*(Q_0S^0; \mathbb{Z}/p)$, when p is any prime, and determine the type of subalgebras that they generate. For $p = 2$ we determine spherical classes in $H_*(\Omega_0^k J; \mathbb{Z}/2)$. We determine truncated subalgebras inside $H_*(Q_0S^{-k}; \mathbb{Z}/2)$. Applying the machinery of the Eilenberg-Moore spectral sequence we define classes that are not in the image of by the J-homomorphism. We shall make some partial observations on the algebraic structure of $H_*(\Omega_0^k \mathrm{coker}J; \mathbb{Z}/2)$. Finally, we shall make some comments on the problem in the case equivariant J-homomorphisms.

Contents

1 Introduction and statement of results

Let Q_0S^0 be the base point component of $QS^0 = \text{colim} \, \Omega^i S^i$. We have the following.

Theorem 1. *The image of the composite*

$$_2\pi_* J \rightarrowtail \, _2\pi_* Q_0 S^0 \xrightarrow{h} H_*(Q_0 S^0; \mathbb{Z}/2)$$

consists only of the image of the Hopf invariant one elements in positive dimensions where h denotes the Hurewicz homomorphism, i.e. only η, ν, σ survive under the above composite.

We write $_p\pi_*$ for p-primary homotopy where p is a prime number. We shall use $f_\#$ to denote the mapping induced in homotopy, and f_* for the mapping induced in homology, where f is a mapping of spaces. Given a class $\xi_i \in H_*(X; \mathbb{Z}/p)$, the subindex i will refer to the dimension of the class. We shall use $E_{\mathbb{Z}/p}$ for the exterior algebra functor over $\mathbb{Z}/p$, R for the Dyer-Lashof algebra (the algebra of Kudo-Araki operations Q^i and their iterations), and A for the Steenrod algebra; all mod p.

Let $Q_0 S^{-k}$ be the base-point component of $\Omega^k Q_0 S^0$. The real J-homomorphism $SO \to Q_1 S^0$ is defined using the reflection through a hyperplane. After composition with a translation map $*[-1] : Q_1 S^0 \to Q_0 S^0$, obtained by 'loop sum' with a map of degree -1, we obtain

$$J_{\mathbb{R}} : SO \longrightarrow Q_1 S^0 \xrightarrow{*[-1]} Q_0 S^0.$$

On the other hand, the original real Bott periodicity provides a homotopy equivalence $\mathbb{Z} \times BO \to \Omega^8 BO$. After looping once we have the homotopy equivalence $SO \to \Omega_0^8 SO$ which we refer to as real Bott periodicity where $\Omega_0^8 SO$ denotes the base-point component of $\Omega^8 SO$. Consider the iterated loops of $J_{\mathbb{R}}$ as mappings

$$\Omega^k J_{\mathbb{R}} : \Omega^k SO \longrightarrow QS^{-k} = \Omega^k Q_0 S^0.$$

The $\mathbb{Z}/p$-homology of the spaces $\Omega^k SO$ are completely known. Let $x_i^{-k} \in H_i(Q_0 S^{-k}; \mathbb{Z}/2)$ be the image of the algebra generator of $H_*(\Omega_0^k SO; \mathbb{Z}/2)$ in dimension i under $(\Omega^k J_{\mathbb{R}})_*$ if there is such a generator, e.g. $H_*(\Omega_0^{8j} SO; \mathbb{Z}/2) \simeq E_{\mathbb{Z}/2}(s_i : i > 0)$ and $x^{-8j} = (\Omega^{8j} J_{\mathbb{R}})_* s_i$. We then have the following.

Theorem 2. *The classes $x_i^{-k} \in H_i(Q_0 S^{-k}; \mathbb{Z}/2)$ are nontrivial if and only if: (1) If $k \not\equiv 0 \mod 8$; or (2) If $k = 8j$ and $i \neq 2^{\nu_j} k - 1$ where 2^{ν_j} is the largest power of 2 which divides $3^{4j} - 1$.*
The action of the mod 2 Steenrod operations on the classes $Q^I x_i^{-k}$ is determined by the Nishida relations together with their action on x_i^{-k}. If $k \equiv 0, 1, 2, 4$ then the subalgebra inside $H_* Q_0 S^{-k}$ spanned by the classes $Q^I x_i^{-k}$ is an exterior algebra. Moreover, if $k > 7$ then x_i^{-k} belongs to the kernel of the iterated homology suspension $\sigma_*^t : H_* Q_0 S^{-k} \to H_{*+t} Q_0 S^{-k+t}$ for some $t \leqslant 7$.

Next, note that any generator $\alpha \in \, _2\pi_1 Q_0 S^{-n+1}$ maps nontrivially under the Hurewicz homomorphism $_2\pi_1 Q_0 S^{-n+1} \to H_1(Q_0 S^{-n+1}; \mathbb{Z}/2)$, see Lemma 5.6. We determine how far any such class will survive under the homology suspension $H_1(Q_0 S^{-n+1}; \mathbb{Z}/2) \to H_{i+1} Q_0 S^{-n+i+1}$ when α is a generator in the image of the J-homomorphism, i.e. we determine largest such i.

Theorem 3. *The class x_d^{-n} is spherical if and only if $d \leqslant 8$ and there is an element $f \in \, _2\pi_d Q_0 S^{-n}$ which is in the image of $(\Omega^n J_{\mathbb{R}})_\#$ with $hf = x_d^{-n} \in H_d Q_0 S^{-n}$. Moreover, suppose that $n = 8t + k$ with $0 \leqslant k \leqslant 7$ and $f = (\Omega^n J_{\mathbb{R}}) f'$ for some $f' \in \, _2\pi_d \Omega_0^n SO$. Then under the composition*

$$_2\pi_d \Omega_0^n SO = \, _2\pi_d \Omega_0^{8t+k} SO \xrightarrow{\simeq} \, _2\pi_d \Omega_0^k SO \xrightarrow{\simeq} \, _2\pi_{d+k} SO \xrightarrow{J_{\mathbb{R}}} \, _2\pi_{d+k}^S$$

the class f' maps to a Hopf invariant one element.

The precise value for d depends on k, and can be determined as explained in Lemma 6.1. We shall also determine the Hurewicz image of the elements μ_{8i+1} and $\eta\mu_{8i+1}$ (see Lemma 6.2), which are not in the image of the J-homomorphism, and observe that 'they are detected on very low dimensional spheres'.

To state our next result, we fix some notation. Given a generator $\gamma \in \, _2\pi_k^S \simeq \, _2\pi_0 Q_0 S^{-k}$ let $[\gamma] = h\gamma$ where $h : \, _2\pi_0 QS^{-k} \to H_0(QS^{-k}; \mathbb{Z}/2)$ is the Hurewicz homomorphism.

Proposition 4. *Assume that $_2\pi_k^S \simeq \, _2\pi_0 QS^{-k}$ has a summand which is the cyclic group of order 2^d, and let γ^k be the generator of this component. Then the homology algebra $H_*(Q_0 S^{-k}; \mathbb{Z}/2)$ contains a*

truncated polynomial subalgebra whose truncation height is at most 2^d and the generators of this subalgebra are of the form $Q^I[\gamma_J^k]$ where

$$[\gamma_J^k] = Q^J[\gamma^k] * [-2^{l(J)}\gamma^k] \in H_{\dim(J)}Q_0S^{-k} \quad \text{if } l(J) < d,$$
$$[\gamma_J^k] = Q^J[\gamma^k] \in H_{\dim(J)}Q_0S^{-k} \quad\quad\quad\quad\; \text{if } l(J) = d.$$

We assume both I and J are admissible, and we allow I to be the empty sequence but not J. Moreover, the action of the Steenrod operations Sq_*^r on these subalgebras is determined by the Nishida relations.

Note that π_*J is completely known [R86, Proposition 1.5.22]. Hence, we know the type of the truncated subalgebras that they give rise to.

Next, consider the complex J-homomorphism $J_{\mathbb{C}} : U \to Q_0S^0$ and its iterated loops. Recall that for primes $p > 2$ we have $\text{im}(J_{\mathbb{C}})_\# \simeq {}_p\pi_*J$ [R86, Theorem 1.5.19] where $(J_{\mathbb{C}})_\#$ is the mapping induces in homotopy and J is the fibre of $\psi^q - 1 : BU \to BU$ with q a prime such that $q^i - 1 \not\equiv 0 \bmod p$ for $0 \leqslant i \leqslant p-2$, and p divides $q^{p-1} - 1$ exactly once. Notice that there are many primes like this, and we may choose the least one for simplicity and denote with $q(p)$.

Let $p \geqslant 2$ be a prime number, and $k \geqslant 0$ any integer and let $\epsilon(k) = 1$ if k is odd, and $\epsilon(k) = 0$ otherwise. Let $w_{2i+1-\epsilon}^{-k}$ be the image of the generator of $H_{2i+1-\epsilon(k)}(\Omega_0^k U; \mathbb{Z}/p)$ under $(\Omega^k J_{\mathbb{C}})_*$. Notice that these are primitive if k is even.

Theorem 5. *The classes $w_{2i+1-\epsilon}^{-k} \in H_{2i+1-\epsilon(k)}(Q_0S^{-k}; \mathbb{Z}/p)$ are nontrivial if and only if (1) $p = 2$ and $k \equiv 0, 1, 7 \bmod 8$; or (2) $p = 2$ and $k \equiv 2, 3 \bmod 8$ and i is even; or (3) $p = 2$ and $k \equiv 4 \bmod 8$ and i is odd; or (4) $p > 2$, $k = 2n - 1$, and $p|q(p)^{n+i} - 1$; or (5) $p > 2$, $k = 2n$, and $p|q(p)^{n+i} - 1$.*
The action of the mod p Steenrod operations on the classes $Q^I w_{2i+1-\epsilon}^{-k}$ is determined by their action on $w_{2i+1-\epsilon}^{-k}$ together with the Nishida relations. If k is even, then the classes $Q^I w_{2i+1-\epsilon}^{-k}$ generate exterior subalgebra in $H_*(Q_0S^{-k}; \mathbb{Z}/2)$ whereas $w_{2i+1-\epsilon}^{-k}$ generate an exterior subalgebra in $H_*(Q_0S^{-k}; \mathbb{Z}/2)$ for $p > 2$. Moreover, for k even and $p > 2$ the classes $Q^I w_{2i+1-\epsilon}^{-k}$ generate a subalgebra whose truncation height is p if I is odd dimensional, and an exterior algebra otherwise.
Finally, if $k > 3$ then $\sigma_*^t w_{2i+1-\epsilon}^{-k} = 0$ for some $t \leqslant 3$.

The action of the mod p Steenrod operations on the classes $w_{2i+1-\epsilon}^{-k}$ is determined by their action on the classes $c_{2i} \in H_{2i}(BU; \mathbb{Z}/p), u_{2i+1} \in H_{2i+1}(U; \mathbb{Z}/p)$.

Next, we use the machinery of the Eilenberg-Moore spectral sequence. The mapping $\Omega^{2k+1}J_{\mathbb{C}} : BU \to Q_0S^{-2k-1}$ extends to an infinite loop map $j_{2k+1}^{\mathbb{C}} : QBU \to Q_0S^{-2k-1}$. We consider

$$\Omega j_{2k+1}^{\mathbb{C}} : Q\Sigma^{-1}BU \to Q_0S^{-2k-2}.$$

The inclusion $\iota_{BU} : BU \to QBU$ gives the mapping $\Omega\iota_{BU} : U \to Q\Sigma^{-1}BU$, providing us with a factorisation of $\Omega^{2k+2}J_{\mathbb{C}}$ as

$$\Omega^{2k+2}J_{\mathbb{C}} : U \to Q\Sigma^{-1}BU \to Q_0S^{-2k-2}$$

where $Q\Sigma^{-1}X = \Omega QX$. Similarly, for an arbitrary positive integer k the mapping $\Omega^k J_{\mathbb{R}} : \Omega_0^k SO \to Q_0S^{-k}$ give rise to an infinite loop map $j_k^{\mathbb{R}} : Q\Omega_0^k SO \to Q_0S^{-k}$. Looping once, we obtain

$$\Omega j_k^{\mathbb{R}} : \Omega_0 Q\Omega_0^k SO \to Q_0S^{-k-1}.$$

Calculating $H_*(Q_0\Sigma^{-1}\Omega_0^k SO; \mathbb{Z}/2)$, for $k \equiv 0, 1, 3, 7 \bmod 8$, and $H_*(Q\Sigma^{-1}BU; \mathbb{Z}/2)$ is quite straightforward (see Section 8). We may define classes in $H_*(Q_0S^{-k-1}; \mathbb{Z}/2)$ by considering the image of the algebra generators of $H_*(Q_0\Sigma^{-1}\Omega_0^k SO; \mathbb{Z}/2)$ and $H_*(Q\Sigma^{-1}BU; \mathbb{Z}/2)$.

Theorem 6. *If $k \leqslant 7$ then $(\Omega j_k^{\mathbb{R}})_*$ define infinitely many nontrivial classes in $H_*(Q_0S^{-k-1}; \mathbb{Z}/2)$ in each dimension when $k \equiv 0, 1, 3, 7 \bmod 8$. Similarly, $(\Omega j_k^{\mathbb{C}})_*$ defines infinitely many nontrivial classes in $H_*(Q_0S^{-k-1}; \mathbb{Z}/2)$ for $k \leqslant 3$ odd.*

Notice that 'infinitely many nontrivial classes' is for the whole homology ring, in each dimension there will be only finitely many of such classes are nontrivial. The details and description of these classes are left to the proof. We suspect that similar work can be done for odd primes. At the last section we will comment on the equivariant versions of these results.

2 The fibre of the Adams operations

The homological degree of a stable map $S^0 \to S^0$ yields $\pi_0 Q S^0 \simeq \pi_0^S \simeq \mathbb{Z}$. Let $Q_d S^0$ be the component of stable maps $S^0 \to S^0$ of degree d. The loop sum with a stable map of degree $-d$ provides the translation map $*[-d] : Q_d S^0 \to Q_0 S^0$ resulting in homotopy equivalence between different components of $Q S^0$. By definition the space $Q S^0$ is an infinite loop space, as well as its base-point component $Q_0 S^0$, where the addition is just the addition of loops. The space $Q_1 S^0$ admits another loop space structure, once we consider the operation of composition of maps of degree 1. We write SG for $Q_1 S^0$ when regarded as an infinite loop space with the composition product.

At each prime p, we have Sullivan's splitting of the space SG [M75, Theorem 5.5] as a product

$$SG \simeq J \times \mathrm{coker} J$$

whereas unlike the odd primes, at the prime 2 this splitting is not an splitting of infinite loop spaces [M75], [S76, Theorem 4]. Nevertheless, by homotopy equivalence (of spaces and not of infinite loop spaces) $*[-1] : SG = Q_1 S^0 \to Q_0 S^0$ we have the splitting

$$Q_0 S^0 \to J \times \mathrm{coker} J.$$

The spaces J and $\mathrm{coker} J$ are relate to the real and complex J-homomorphisms throughout the affirmative solution of the Adams conjecture [Q71, Theorem 1.1], [A78]. For $p = 2$ the space J is defined to be the the 2-local fibre of the Adams operation $\psi^3 - 1 : BSO \to BSO$, whereas the space $\mathrm{coker} J$ is defined to be the fibre of a certain map $G/O \to BSO$.

The real J-homomorphism is an infinite loop map $SO \to SG$ [M70]. The Adams conjecture predicts that the row in the following diagram is null-homotopic

$$
\begin{array}{ccc}
 & G/O & \\
\overset{\beta}{\nearrow} & \downarrow & \\
BSO \xrightarrow{\psi^3-1} & BSO \xrightarrow{BJ} & BSG
\end{array}
$$

where β is shown to exist by Quillen [Q71, Theorem 1.1] and is referred to as 'a solution to the Adams conjecture'. Any solution to the Adams conjecture provides a factorisation of the J-homomorphism, up to homotopy, as [S79]

$$SO \xrightarrow{\partial} J \xrightarrow{\alpha} SG$$

where ∂ is the boundary map in the fibration sequence used to define J. At odd primes, the space J is defined to be the p-local fibre of the Adams operation $\psi^q - 1 : BU \to BU$ where q is a prime number chosen as explained at the previous section. There, again we have a similar commutative diagram and a factorisation of the J-homomorphism. Notice that using the translation map, at each prime p we have homotopy factorisations

$$J_{\mathbb{R}} : \quad SO \xrightarrow{\partial} J \xrightarrow{\alpha} Q_0 S^0, \qquad J_{\mathbb{C}} : \quad U \xrightarrow{\partial} J \xrightarrow{\alpha} Q_0 S^0$$

Since $\mathrm{im}(J_{\mathbb{C}})_{\#} \simeq {}_p \pi_* J$ for $p > 2$ [R86, Theorem 1.5.19] then at odd primes it is enough to use the complex J-homomorphism, replacing SO by U, and $J_{\mathbb{C}}$ by $J_{\mathbb{R}}$. At odd primes, the space $\mathrm{coker} J$ can be defined in a similar way as fibre of a certain map. For our purpose, the important is that at any prime p the mapping α is a split monomorphism in the p-local category, which is one of the mappings used to obtain Sullivan's splitting of $Q_0 S^0$, and hence it induces an (split) injection in $\mathbb{Z}/p$-homology.

Applying the iterated loop functor Ω^k to Sullivan's splitting, we obtain the splitting of k-fold loop spaces

$$\Omega^k Q_0 S^0 = Q S^{-k} \to \Omega^k (J \times \mathrm{coker} J) \simeq \Omega^k J \times \Omega^k \mathrm{coker} J \qquad (1)$$

as well as the following homotopy factorisations

$$\Omega^k J_{\mathbb{R}} : \quad \Omega_0^k SO \xrightarrow{\Omega^k \partial} \Omega^k J \xrightarrow{\Omega^k \alpha} \Omega_0^k Q_0 S^0, \qquad \Omega^k J_{\mathbb{C}} : \quad \Omega_0^k U \xrightarrow{\Omega^k \partial} \Omega_0^k J \xrightarrow{\Omega^k \alpha} \Omega_0^k Q_0 S^0$$

for $p = 2$, and $p > 2$ respectively. In this case, all of our mappings are loop maps, i.e. in homology they induce multiplicative maps. Moreover, for a similar reason as above the mapping $\Omega^k \alpha$ induces a split injection in $\mathbb{Z}/p$-homology.

3   Proof of Theorem 1

According to [R86, Proposition 1.5.22] at the prime 2 we have an injection $\mathrm{im}(J_\mathbb{R})_\# \rightarrowtail {}_2\pi_* J$ which is an isomorphism, apart from dimensions $8i+1$ and $8i+2$ where in these dimensions there are element $\mu_{8i+1}, \eta\mu_{8i+1}$ which are not given by the J-homomorphism. We break the proof of Theorem 1 into two lemmata.

Lemma 3.1. *Let $\mu_{8i+1} \in {}_2\pi_{8i+1} J$ be the element of order 2 which is not in $\mathrm{im}(J_\mathbb{R})_\#$. Then both μ_{8i+1} and $\eta\mu_{8i+1}$ map trivially under the composite ${}_2\pi_* J \rightarrowtail {}_2\pi_* Q_0 S^0 \xrightarrow{h} H_*(Q_0 S^0; \mathbb{Z}/2)$.*

Proof. Let σ_{8i-1} be the generator of ${}_2\pi_{8i-1} J \simeq \mathbb{Z}/2^j$ where $8i = 2^j(2s+1)$. Then $\alpha_{8i-1} := 2^{j-1}\sigma_{8i-1}$ has order 2 in this group.
According to Adams [A66, Theorem 12.13] we may construct μ_{8i+1} as a triple Toda bracket $\{\eta, 2, \alpha_{8i-1}\}$ where $\eta \in \pi_1^S$ is the Hopf invariant one element. We think of α_{8i-1} as a mapping $S^{8i-1} \to Q_0 S^0$. We have the composite

$$S^{8i} \xrightarrow{\eta} S^{8i-1} \xrightarrow{2} S^{8i-1} \xrightarrow{\alpha_{8i-1}} Q_0 S^0$$

where the successive compositions are trivial. Hence, we obtain

$$S^{8i+1} \xrightarrow{\eta^\flat} C_2 \xrightarrow{\alpha'_\natural} Q_0 S^0$$

representing the triple Toda bracket for μ_{8i+1} where $\alpha' = \alpha_{8i-1}$ and C_2 is the mapping one of 2. Note that C_2 has its top cell in dimension $8i$, hence $\eta^\flat$ is trivial in homology. Note that the mapping $\eta^\flat$ is in the stable range, hence can be taken as a genuine mapping. Therefore, the above composite is trivial in homology. Since the triviality in homology occurs only for dimensional reasons, then the indeterminacy in choosing the mappings $\eta^\flat$ and $\alpha_\natural$ will not make any change to this fact. Hence, μ_{8i+1} is trivial in homology, i.e. it maps trivially under the Hurewicz homomorphism ${}_2\pi_* Q_0 S^0 \to H_*(Q_0 S^0; \mathbb{Z}/2)$. Consequently, $\eta\mu_{8i+1}$ is trivial in homology (although this was trivial from the given decomposition as well). This completes the proof. $\square$

Next we consider the $\mathrm{im}(J_\mathbb{R})_\#$.

Lemma 3.2. *The image of ${}_2\pi_* SO \xrightarrow{(J_\mathbb{R})_\#} {}_2\pi_* Q_0 S^0 \xrightarrow{h} H_*(Q_0 S^0; \mathbb{Z}/2)$ consists of only the image of the Hopf invariant one elements.*

First, we fix some notation. Recall that for a given space X, if $x \in H_* X$ is spherical then it is primitive, and x is A-annihilated, i.e. $Sq_*^r x = 0$ for all $r > 0$ where Sq_*^r is dual to Sq^r.
It is quite standard that $H_*(SO; \mathbb{Z}/2) \simeq E_{\mathbb{Z}/2}(s_i : i \geqslant 1)$. The primitive classes in this ring are unique classes $p_{2n+1}^{SO} = s_{2n+1} + \sum s_i s_{2n+1-i}$. Define primitive classes $p_{2n+1}^{S^0} \in H_{2n+1}(Q_0 S^0; \mathbb{Z}/2)$ by $p_{2n+1}^{S^0} = (J_\mathbb{R})_* p_{2n+1}^{SO}$. The action of Sq_*^r on these classes [M70], see also [E80, Lemma 3.6], is given by

$$Sq_*^r p_{2n+1}^{SO} = \binom{2n+1-r}{r} p_{2n+1-r}^{SO}, \quad Sq_*^r p_{2n+1}^{S^0} = \binom{2n+1-r}{r} p_{2n+1-r}^{S^0}.$$

Therefore, the only A-annihilated primitive classes in $H_*(SO; \mathbb{Z}/2)$ are $p_{2^t-1}^{SO}$. Note that $(J_\mathbb{R})_*$ is a monomorphism because of the Sullivan decomposition, and hence $p_{2^t-1}^{SO}$ maps isomorphically onto $p_{2^t-1}^{S^0}$.

of Lemma 3.2. It is enough to show that the image of ${}_2\pi_* SO \xrightarrow{h} H_*(SO; \mathbb{Z}/2) \xrightarrow{(J_\mathbb{R})_*} H_*(Q_0 S^0; \mathbb{Z}/2)$ only consists of the Hopf invariant one elements.
If $p_{2^t-1}^{SO}$ is spherical so $p_{2^t-1}^{S^0}$ is. On the other hand $\sigma_*^{2^t-1} p_{2^t-1}^{S^0} = g_{2^t-1}^2 \in H_*(QS^{2^t-1}; \mathbb{Z}/2)$ with $\sigma_*^{2^t-1}$ being the iterated homology suspension $H_*(Q_0 S^0; \mathbb{Z}/2) \to H_{*+2^t-1}(QS^{2^t-1}; \mathbb{Z}/2)$. Therefore $g_{2^t-1}^2 \in H_*(QS^{2^t-1}; \mathbb{Z}/2)$ should be spherical. However, according to Boardman and Steer [BS67, Corollary 5.15] (see also [AE00, Proposition 5.8]), the class $g_{2^t-1}^2$ is spherical if and only if there exists a Hopf invariant one element in ${}_2\pi_{2^t-1}^S$. This then implies that $t = 1, 2, 3$. This the completes the proof. $\square$

We then have completed the proof of Theorem 1.

4 The Hopf invariant one result

The following should be well known, but we provide a proof.

Lemma 4.1. *The Hopf invariant one elements in $_2\pi_*^S$ belong to the image of the real J-homomorphism. Consequently, the following statements are equivalent: (1) There exists a Hopf invariant one element; (2) $p_{2^t-1}^{SO}$ is spherical; (3) $p_{2^t-1}^{S^0}$ is spherical.*

Proof. The J-homomorphism is the stablised version of homomorphisms $\pi_r SO(q) \to \pi_{r+q}S^q$. The mapping $f : S^{2n-1} \to S^n$ has Hopf invariant one, if and only if S^{n-1} is parallelisable, i.e. there exists a trivilisation of its tangent bundle. If S^{n-1} is parallelisable we define $\phi : S^{n-1} \to SO(n)$ by $x \mapsto [x, v_2, \ldots, v_n]$ where $\{v_2, \ldots, v_n\}$ is an $(n-1)$-frame orthogonal to x. The mapping $f : S^{n-1} \times S^{n-1} \to S^{n-1}$ defined by $f(x, y) = \phi(x) \cdot y$ has bidegree $(1, 1)$. The Hopf construction then yields a mapping $S^{2n-1} \to S^n$ of Hopf invariant one. Hence, the Hopf invariant one elements belong to the image of $J_{\mathbb{R}}$. According to the proof of Lemma 3.2, (3) implies (1) and hence (2). Recall from proof of Lemma 3.2 that $S^{2n-1} \to S^n$ has Hopf invariant one if and only if the adjoint map $S^{2n-2} \to \Omega S^n$ is nontrival in homology, hence $S^{n-1} \to \Omega^n S^n$ is nontrivial in homology. But it then the pull back $f' : S^{n-1} \to SO(n)$ has to be nontrivial in homology, i.e. $hf' = p_{2^t-1}^{SO}$ for some t. This later maps to $p_{2^t-1}^{S^0}$. This completes the proof. $\qquad\square$

The following is immediate from the Bott periodicity, and description of the homology suspension $\sigma_* : H_*\Omega_0^{k+1}SO \to H_{*+1}\Omega_0^k SO$ [C60].

Lemma 4.2. *Suppose $x \in H_*\Omega_0^k SO$ for some $k \geqslant 0$. Then $\sigma_*^t x = 0$ for some $t \leqslant 7$.*

We then have the following short proof of the Hopf invariant one result.

Corollary 4.3. *There are no Hopf invariant one elements in $_2\pi_{2^t-1}^S$ for $t > 3$.*

Proof. Suppose there exist a Hopf invariant one element in $_2\pi_{2^t-1}^S$ with $t > 3$. Then, according to Lemma 4.1, $p_{2^t-1}^{SO} \in H_{2^t-1}(SO; \mathbb{Z}/2)$ is spherical. The spherical class $p_{2^t-1}^{SO}$ desuspends to spherical classes $\xi_k \in H_{2^t-1-k}(\Omega_0^k SO; \mathbb{Z}/2)$ for any $k \leqslant 2^t - 1$ with the property that $\sigma_*^k \xi_k = p_{2^t-1}^{SO}$. As $t > 3$ we then may choose $k > 7$, but this contradicts the previous lemma. This completes the proof. $\qquad\square$

5 The homology of iterated loops of J-homomorphisms

The fact that $\Omega^k \alpha$ is an split injection in $\mathbb{Z}/p$-homology defines a set of generators in $H_*(Q_0 S^{-k}; \mathbb{Z}/p)$ in a geometric way, if we have a geometric description of α which is easy to understand. The next option is to show that $\Omega^k J_{\mathbb{R}}$ and $\Omega^k J_{\mathbb{C}}$ induce monomorphisms in homology, at least on certain elements. This is done by analysing well-known facts about $H_*(\Omega^k J; \mathbb{Z}/p)$. We deal with $p = 2$ and $p > 2$ separately.

5.1 The case $p = 2$: Proof of Theorem 2

Let us write H_* for $H_*(-; \mathbb{Z}/2)$. At the prime 2 we have the fibration sequence

$$\Omega_0^{k+1}BSO = \Omega_0^k SO \xrightarrow{\Omega^k \partial} \Omega_0^k J \longrightarrow \Omega_0^k BSO \xrightarrow{\psi^3 - 1} \Omega_0^k BSO.$$

The homology of $\Omega^k J$ is obtained by applying the Serre spectral sequence to the fibration $\Omega_0^k SO \xrightarrow{\Omega^k \partial} \Omega_0^k J \longrightarrow \Omega_0^k BSO$ or passing to connected covers and then applying the Serre spectral sequence if necessary. Recall from [C60], [CP89] that

$$\begin{aligned}
H^* SO &\simeq \mathbb{Z}/2[f_{2i-1} : i \geqslant 1], \\
H^* BO &\simeq \mathbb{Z}/2[w_i : i \geqslant 1], \\
H_* BO &\simeq \mathbb{Z}/2[a_i : i \geqslant 1], \\
H_* SU/SO &\simeq \mathbb{Z}/2[u_2, u_3, u_5, \ldots, u_{2n+1}, \ldots].
\end{aligned}$$

Moreover, given a positive integer j we define ν_j to be the largest positive integer that 2^{ν_j} divides $3^{4j} - 1$. The result of Cohen and Peterson [CLM, Theorem 1.6] on $H_*\Omega^k J$ reads as following.

6

Theorem 5.1. (1) *If $k \equiv 2, 4, 5, 6$ mod 8 then as vector spaces we have*

$$H^*\Omega_0^k J \simeq H^*\Omega^k BSO \otimes H^*\Omega^{k+1} BSO.$$

(2) *If $k \equiv 1$ mod 8 then as an algebra*

$$H^*\Omega_0^k J \simeq H^*Spin \otimes H^*SO/U.$$

(3) *If $k \equiv 3$ mod 8 then as an algebra*

$$H^*\Omega_0^k J \simeq H^*SU/Sp \otimes H^*BSp.$$

(4) *If $k = 8j - 1$ mod 8 then as an algebra*

$$H_*\Omega_0^k J \simeq \mathbb{Z}/2[u_{2n+1}^2 : n \equiv 0 \text{ mod } 2^{\nu_j}] \otimes \mathbb{Z}/2[u_2, u_{2n+1} : 2n \not\equiv 0 \text{ mod } 2^{\nu_j}] \otimes (\otimes_{n \geqslant 1}\mathbb{Z}/2[a_n]/a_n^{2^{\nu_j}}).$$

(5) *If $k = 8j$ then as an algebra*

$$H^*\Omega_0^k J \simeq (\otimes_{n \geqslant 1}\mathbb{Z}/2[w_n]/w_n^{2^{\nu_j}}) \otimes \mathbb{Z}/2[f_{k2^{\nu_j}-1}^2] \otimes \mathbb{Z}/2[f_{2n-1} : 2n \not\equiv 0 \text{ mod } 2^{\nu_j}].$$

of Theorem 2. First let $k \not\equiv 0, 7$ mod 8. Recall from [McC01, Theorem 5.9] that if $F \xrightarrow{i} E \to B$ is a fibration with B simply connected and path connected, the mapping $i^* : H^q E \to H^q F$ may be calculated as the composite

$$H^q E \to E_\infty^{0,q} = E_{q+1}^{0,q} \subseteq E_q^{0,q} \subseteq \cdots \subseteq E_2^{0,q} = H^q F.$$

There is a similar statement for homology spectral sequence. For $k \not\equiv 0, 7$ mod 8, Cohen and Peterson [CLM, Theorem 1.6] have shown that the Serre spectral sequence collapses, and there are no nontrivial extension problems. Hence, the mapping $(\Omega^k \partial)_*$ is an injection, or $(\Omega^k \partial)^*$ is a surjection which is the same as saying $(\Omega^k \partial)_*$ is an injection, as we work with field coefficients. Combining this with the fact that $(\Omega^k \alpha)_*$ is an injection in homology, then implies that $\Omega^k J_\mathbb{R} : \Omega_0^k SO \to Q_0 S^{-k}$ is an injection in $\mathbb{Z}/2$-homology. Hence, in this case, the classes x_i^{-k} defined in Theorem 2 are nontrivial.

Next, let $k = 8j - 1$ for some j. We have the fibration $BO \simeq \Omega^{8j} BSO \to \Omega_0^{8j-1} J \to \Omega^{8j-1} BSO$. Similar to the previous case, together with the calculations in Theorem 5.1 confirms that $(\Omega^k \partial)_*$ maps the elements $a_i \in H_i BO$ monomorphically into $H_*\Omega_0^k J$. Again the injectivity of $(\Omega^k \alpha)_*$ implies that $(\Omega^k J_\mathbb{R})_*$ is a monomorphism on the elements $a_i \in H_i BO$. Moreover, the fact that the mapping $(\Omega^k \partial)_*$ is a multiplicative map together with the above description of $H_*\Omega_0^k J$ tells is about $\ker(\Omega^k \partial)_*$ and hence about $\ker(\Omega^k J_\mathbb{R})_*$. In particular, $(\Omega^{8j-1} \partial)_* a_n^{2^{\nu_j}} = 0$ for each $n \geqslant 1$.

Finally, let $k = 8j$ where we have the fibration

$$SO \simeq \Omega_0^{8j+1} BO \to \Omega_0^{8j} J \to BO.$$

We consider the following commutative diagram

$$
\begin{array}{ccc}
H_{*+1}BO & \longrightarrow & H_{*+1}\Omega_0^{8j-1}J \\
\uparrow{\scriptstyle \sigma_*} & & \uparrow{\scriptstyle \sigma_*} \\
H_*SO & \longrightarrow & H_*\Omega_0^{8j}J.
\end{array}
$$

The elements $s_{i-1} \in H_{i-1}SO$ are determined uniquely by the behavior of the homology suspension, by

$$\sigma_* s_{i-1} = p_i^{BO} = \begin{cases} a_{2n+1} + \text{decomposable terms} & \text{if } i = 2n+1, \\ (p_{2l+1}^{BO})^{2^{\rho(i)}} & \text{if } i = 2^{\rho(i)}(2l+1) \end{cases}$$

where $p_{2i+1}^{BO} = a_{2i+1}$ modulo decomposable terms. Notice that in $(\Omega^{8j-1} \partial)_*$ maps the terms a_n monomorphically, and kills terms of the form $a_n^{2^{\nu_j}}$. This implies that $\ker(\Omega^{8j} \partial)_*$ as an ideal of H_*SO is determined by

$$\langle s_i : i = 2^{\nu_j} k - 1, k \geqslant 0 \rangle.$$

Hence, for $i \neq 2^{\nu_j} k - 1$ we have

$$x_i^{-8j} \neq 0.$$

Recall from [C60] that for $k \equiv 0, 1, 2, 4 \mod 8$, $H_*\Omega^k SO$ is exterior. This together with the fact that $(\Omega^k J_{\mathbb{R}})_*$ is multiplicative implies that $(x_i^{-k})^2 = 0$ in these cases. Moreover, the Cartan formula $(Q^i \xi)^2 = Q^{2i}\xi^2$ implies that $(Q^I x_i^{-k})^2 = 0$ for any nonempty sequence I. Hence, we see that the subalgebra of $H_* Q_0 S^{-k}$ spanned by classes of the form $Q^I x_i^{-k}$ is an exterior algebra if $k \equiv 0, 1, 2, 4 \mod 8$. The action of the Steenrod operations Sq_*^r on the classes x_i^{-k} is determined by their actions on the generators of $H_*\Omega_0^k SO$. This together with the Nishida relations determines the action of these operations on the classes $Q^I x_i^{-k}$. Finally, x_i^{-k} is the image of a generator in $H_i \Omega_0^k SO$. Applying Lemma 4.2, then implies that $\sigma_*^t x_i^{-k} = 0$ for some $t \leqslant 7$. $\qquad \square$

5.2 Proof of Theorem 5: The case of odd primes

Let us write H_* for $H_*(-; \mathbb{Z}/p)$. At the prime $p > 2$ we have the fibration sequence

$$\Omega_0^{k+1} BU = \Omega_0^k U \xrightarrow{\Omega^k \partial} \Omega_0^k J \longrightarrow \Omega_0^k BU \xrightarrow{\psi^q - 1} \Omega_0^k BU.$$

In this case, the homology of $\Omega_0^k J$ is obtained by applying the Serre spectral sequence to the fibration

$$\Omega_0^k U \xrightarrow{\Omega^k \partial} \Omega_0^k J \longrightarrow \Omega_0^k BU.$$

Notice that for the spaces in the complex Bott periodicity, i.e. U and BU, we have [C60, Assertions 1,3]

$$\begin{aligned} H_*(U; \mathbb{Z}) &\simeq E_{\mathbb{Z}}(u_{2i-1} : i \geqslant 1, u_{2i-1} \text{ primitive}), \\ H_*(BU; \mathbb{Z}) &\simeq \mathbb{Z}[c_{2i} : i \geqslant 1]. \end{aligned}$$

Hence, at each prime p the homology will have the 'same' structure over $\mathbb{Z}/p$ with the same generators at the same dimensions. The homology of $\Omega_0^k J$ at odd primes is calculated, due to Cohen and Peterson [CLM, Theorem 1.3, Theorem 1.4]. We recall the result.

Theorem 5.2. (i) *There is an isomorphism of algebras*

$$H_*\Omega_0^{2n} J \to \bigotimes_{p | q^{n+i}-1} E_{\mathbb{Z}/p}((\Omega^{2n}\partial)_* u_{2i-1}) \otimes \mathbb{Z}/p[c_{2i}]$$

where c_{2i} is the unique element mapping to $c_{2i} \in H_{2i}\Omega^{2n} BU$.
(ii) *There is an isomorphism of algebras*

$$H_*\Omega_0^{2n-1} J \to \bigotimes_{p | q^{n+i}-1} \mathbb{Z}/p[(\Omega^{2n-1}\partial)_* c_{2i}] \otimes E_{\mathbb{Z}/p}(u_{2i+1})$$

where u_{2i+1} is the unique element mapping to $u_{2i+1} \in H_{2i+1}\Omega_0^{2n-1} U$.
In both cases the uniqueness of the generators comes from their definitions and from the Serre spectral sequence arguments respectively.

The proof of Theorem 5, at the case of odd primes, is similar to the proof of Theorem 2. Applying calculation of $H_*\Omega_0^k J$ yields to information $\Omega^k J_{\mathbb{C}}$ and that in particular $(\Omega^k J_{\mathbb{C}})_*$ is injective on the generators. Note that the injectivity of $(\Omega^k J_{\mathbb{C}})_*$ is a consequence of the injectivity of $(\Omega^k \partial)$ in $\mathbb{Z}/p$-homology, which is clear in the above presentation. Hence, the iterated complex J-homomorphism $\Omega^k J_{\mathbb{C}}$ defines nontrivial classes $w_{2i+1-\epsilon}^{-k} \in H_* Q_0 S^{-k}$ by

$$w_{2i+1-\epsilon(k)}^{-k} = \begin{cases} (\Omega^k J_{\mathbb{C}})_* c_{2i} & \text{if } k \text{ is odd}, \\ (\Omega^k J_{\mathbb{C}})_* u_{2i+1} & \text{if } k \text{ is even}, \end{cases}$$

Moreover, when k is even the homology ring $H_*\Omega_0^k U \simeq H_* U$ is exterior. This then means that

$$(w_{2i+1}^{-k})^2 = 0.$$

The ring $H_* Q_0 S^{-k}$ is a bicommutative and biassociative Hopf algebra which are completely known [W78, Chpater III (8.7)]. The classes w_{2i+1}^{-k}, and consequently the classes $Q^I w_{2i+1}^{-k}$ are primitive. At $p > 2$, this implies $(Q^I w_{2i+1}^{-k})^2 = 0$ if it is an odd dimensional class. Hence, the classes $Q^I w_{2i+1}^{-k}$ generate an exterior subalgebra in $H_*(Q_0 S^{-k}; \mathbb{Z}/p)$.

On the other hand, if $Q^I w_{2i+1}^{-k}$ is even dimensional then the classes $Q^I w_{2i+1}^{-k}$ either have to generate a polynomial algebra or a truncated algebra of truncation height p^n. However, we have the Cartan formula $(Q^i \xi)^p = Q^{pi} \xi^p$. This implies that $(Q^I w_{2i+1}^{-k})^p = 0$. Hence, the classes $Q^I w_{2i+1}^{-k}$, with I odd dimensional, generate a truncated polynomial subalgebra of truncated height p inside $H_*(Q_0 S^{-k}; \mathbb{Z}/p)$.

The action of the mod p Steenrod operations on the classes $w_{2i+1-\epsilon}^{-k}$ coupled with the Nishida relations tells us the action of the operation Sq_*^r on the classes $Q^I w_{2i+1-\epsilon}^{-k}$ and hence on the subalgebra of $H_* Q_0 S^{-k}$ spanned by these classes. Finally, note that the behavior of w_{2i+1}^{-k} under the homology suspension is determined by the behavior of the homology suspension $\sigma_* : H_* \Omega_0^{k+1} U \to H_{*+1} \Omega_0^k U$. The following is immediate from [C60] (see also [D60]).

Lemma 5.3. *Let $p \geqslant 2$ be a prime, and $x \in H_*(\Omega_0^k U; \mathbb{Z}/p)$ for some $k \geqslant 0$. Then $\sigma_*^t x = 0$ for some $t \leqslant 3$.*

This lemma then shows that $\sigma_*^t w_{2i+1}^{-k} = 0$ for some $t \leqslant 3$, if $k > 3$. This completes the proof of Theorem 5 at the case of odd primes.

5.3 The $\mathbb{Z}/2$-homology of $\Omega^k J_{\mathbb{C}}$: Proof of Theorem 5 when $p = 2$.

We write H_* for $H_*(-; \mathbb{Z}/2)$. The mapping $J_{\mathbb{C}}$ factors through $J_{\mathbb{R}}$ as $J_{\mathbb{C}} : U \to SO \to Q_0 S^0$. For $0 \leqslant t \leqslant 7$, applying the functor Ω^{8k+t} to this factorisation and restricting to the base point components, we obtain

$$
\begin{array}{ccccccc}
U & \xrightarrow{\iota_0} & \Omega_0^{8k} SO & \simeq SO & \longrightarrow & \Omega_0^{8k} J \\
BU & \xrightarrow{\iota_1} & \Omega_0^{8k+1} SO & \simeq SO/U & \longrightarrow & \Omega_0^{8k+1} J \\
U & \xrightarrow{\iota_2} & \Omega_0^{8k+2} SO & \simeq U/Sp & \longrightarrow & \Omega_0^{8k+2} J \\
BU & \xrightarrow{\iota_3} & \Omega_0^{8k+3} SO & \simeq BSp & \longrightarrow & \Omega_0^{8k+3} J \\
U & \xrightarrow{\iota_4} & \Omega_0^{8k+4} SO & \simeq Sp & \longrightarrow & \Omega_0^{8k+4} J \\
BU & \xrightarrow{\iota_5} & \Omega_0^{8k+5} SO & \simeq Sp/U & \longrightarrow & \Omega_0^{8k+5} J \\
U & \xrightarrow{\iota_6} & \Omega_0^{8k+6} SO & \simeq U/O & \longrightarrow & \Omega_0^{8k+6} J \\
BU & \xrightarrow{\iota_7} & \Omega_0^{8k+7} SO & \simeq BO & \longrightarrow & \Omega_0^{8k+7} J.
\end{array}
$$

The calculation of $(\Omega^k J_{\mathbb{C}})_*$ then reduces to understanding the homology of ι_t where $0 \leqslant t \leqslant 7$. We have the following observation.

Lemma 5.4. *For $t = 5, 6$ we have $(\iota_t)_* = 0$. For $t \neq 5, 6$ the nontrivial action of $(\iota_t)_*$ is determined as following (modulo decomposable terms)*

$$
\begin{array}{lllll}
(\iota_0)_* u_{2i+1} & = & s_{2i+1}, & (\iota_3)_* c_{4i} & = & p_{4i}, \\
(\iota_1)_* c_{2i} & = & c_{2i}, & (\iota_4)_* u_{4i-1} & = & z_{4i-1}, \\
(\iota_2)_* u_{4i+1} & = & u_{4i+1}, & (\iota_7)_* c_{2i} & = & a_i^2,
\end{array}
$$

where $c_{2i} \in H_{2i} SO/U$, $z_{4i-1} \in H_i SP$, $p_{4i} \in H_{4i} BSp$, and $a_i \in H_i BO$ are generators.

Notice that the generators $u_{2i+1} \in H_{2i+1} U$ are primitive, hence their image under $(\Omega^k J_{\mathbb{C}})_*$ for k even is also a primitive element which tells one how to calculate the decomposable terms.

Proof. We only show that $(\iota_t)_* = 0$ for $t = 5, 6$ and the other cases are straightforward. Notice that ι_6 is a loop map, and in homology its domain is an exterior algebra whereas its target is a polynomial algebra, hence $(\iota_6)_* = 0$. Next, consider $\iota_5 : BU \to Sp/U$. Recall that $H^* Sp/U$ is an exterior algebra with generators in dimensions $4i + 2$ [C60, Assertaion 27] whereas $H^* BU$ is a polynomial algebra. This implies that $(\iota_5)^* = 0$, and hence $(\iota_5)_* = 0$. This completes the proof. $\square$

The above lemma shows that $(\Omega^k J_{\mathbb{C}})_* = 0$ if $k \equiv 5, 6 \mod 8$. Recall that $w_{2i+1-\epsilon}^{-k} \in H_* Q_0 S^{-k}$ in Theorem 2 are defined by

$$
w_{2i+1-\epsilon}^{-k} = \begin{cases} (\Omega^k J_{\mathbb{C}})_* c_{2i} & \text{if } k \text{ is odd,} \\ (\Omega^k J_{\mathbb{C}})_* u_{2i+1} & \text{if } k \text{ is even,} \end{cases}
$$

where for k odd we have $\Omega^k J_{\mathbb{C}} : BU \to Q_0 S^{-k}$ and for k even $\Omega^k J_{\mathbb{C}} : U \to Q_0 S^{-k}$. We then have the following observation.

Corollary 5.5. *The image of $(\Omega^k J_{\mathbb{C}})_*$ in $\mathbb{Z}/2$-homology is determined as following: The classes $w_{2i+1-\epsilon}^{-k}$ are nontrivial only in one of the following cases: (1) If $k \equiv 0, 1, 7 \mod 8$; (2) If $k \equiv 2, 3 \mod 8$ and i is even; (3) If $k \equiv 4 \mod 8$ and i is odd.*

Lemma 5.3 shows that $\sigma_*^t w_{2i+1}^{-k} = 0$ for some $t \leqslant 3$, if $k > 3$. Since $H_* U$ is an exterior algebra then for $k \equiv 0, 2, 4$ the subalgebra of $H_* Q_0 S^{-k}$ spanned by the classes of the for $Q^I w_{2i+1-\epsilon}^{-k}$ is an exterior algebra. The action of the Steenrod operation is also similar. These observations complete the proof of Theorem 5 when $p = 2$.

5.4 An application to the complex transfer

The following will be used here, and later on.

Lemma 5.6. *Suppose $\alpha \in {}_2\pi_k^S$ is a generator. Then the stable adjoint of α as an element of ${}_2\pi_1 Q_0 S^{-k+1}$ maps nontrivially under the Hurewicz homomorphism*

$$h^1 : {}_2\pi_1 Q_0 S^{-k+1} \to H_1 Q_0 S^{-k+1}.$$

Proof. Consider the following commutative diagram

$$
\begin{array}{ccc}
{}_2\pi_1 Q_0 S^{-k+1} & \longrightarrow & H_1 Q_0 S^{-k+1} \\
{\scriptstyle \simeq} \uparrow & & \uparrow {\scriptstyle \sigma_*} \\
{}_2\pi_0 Q S^{-k} & \longrightarrow & H_0 Q S^{-k}.
\end{array}
$$

The fact that $\alpha \in {}_2\pi_k^S$ is a generator implies that it maps nontrivially under the Hurewicz map $h^0 : {}_2\pi_0 Q S^{-k} \to H_0 Q S^{-k}$, representing a class with odd coefficient in $H_0(Q_0 S^{-k}; \mathbb{Z})$. Hence, if $\sigma_* h^0 \alpha \neq 0$ then it will have odd coefficient in $H_*(Q_0 S^{-k+1}; \mathbb{Z})$. On the other hand, notice that $Q_0 S^{-k+1}$ is a loop space which means its fundamental group is Abelian, so $\pi_1 Q_0 S^{-k+1} \simeq H_1(Q_0 S^{-k+1}; \mathbb{Z})$. Hence, $\sigma_* h^0 \alpha$ has odd coefficient in $H_1(Q_0 S^{-k+1}; \mathbb{Z})$, therefore its $\mathbb{Z}/2$-reduction is nontrivial, i.e. α maps nontrivially under the Hurewicz homomorphism

$$ {}_2\pi_1 Q_0 S^{-k+1} \to H_1(Q_0 S^{-k+1}; \mathbb{Z}/2). $$

$\square$

Next, we apply the $\mathbb{Z}/2$-homology of $\Omega^k J_{\mathbb{C}}$ to give a proof of the following well known fact in terms of homology.

Lemma 5.7. *The complex transfer*

$$ \lambda_{\mathbb{C}} : Q\Sigma\mathbb{C}P_+ \longrightarrow Q_0 S^0 $$

does not induce an epimorphism on the 2-primary homotopy groups.

Proof. The complex transfer is the infinite loop extension of the composite

$$ \Sigma\mathbb{C}P_+ \longrightarrow U \xrightarrow{J_{\mathbb{C}}} Q_0 S^0. $$

Consider the generator $\sigma_{8i-1} \in {}_2\pi_{8i-1} J$. If σ_{8i-1} is in the image of the complex transfer then it admits a factorisation $S^{8i-1} \xrightarrow{\sigma'_{8i-1}} Q\Sigma\mathbb{C}P_+ \xrightarrow{\lambda_{\mathbb{C}}} Q_0 S^0$ which adjoints down to

$$ S^1 \longrightarrow \Omega^{8i-2} Q\Sigma\mathbb{C}P_+ \xrightarrow{\Omega^{8i-2}\lambda_{\mathbb{C}}} Q_0 S^{-8i+2} \ . $$

According to the previous lemma this composition should be nontrivial in $\mathbb{Z}/2$-homology, whereas Lemma 5.4 implies that $(\Omega^{8i-2}\lambda_{\mathbb{C}})_* = 0$. This is a contradiction. Hence,

$$ (\lambda_{\mathbb{C}})_\# : {}_2\pi_*^S \Sigma\mathbb{C}P_+ \simeq {}_2\pi_* Q\Sigma\mathbb{C}P_+ \longrightarrow {}_2\pi_* Q_0 S^0 \simeq {}_2\pi_*^S $$

is not an epimorphism. $\square$

6   Spherical classes in $H_*(\Omega_0^k J; \mathbb{Z}/2)$: Proof of Theorem 3

Throughout this section we write H_* for $\mathbb{Z}/2$-homology. We determine the spherical classes in $H_*\Omega_0^k J$. This in particular determines which of the classes x_d^{-k} defined in Theorem 2 are spherical. First, we record the following observation on spherical classes in $H_*\Omega_0^k SO$, which should be well-known.

Lemma 6.1. *Let $x \in H_d\Omega_0^{8t+k} SO$ be spherical with $hf = x$ for some $f \in {}_2\pi_d\Omega_0^{8t+k} SO$. Then one of the following cases holds:*
(1) $k = 0$ and $d = 1, 3, 7$; (2) $k = 1$ and $d = 2, 6$; (3) $k = 2$ and $d = 1, 5$; (4) $k = 3, 4, 5$ and $d = 7 - k$;
(6) $k = 6$ and $d = 1, 2$; (7) $k = 7$ and $d = 1, 2, 4, 8$.
In any of these cases, the mapping f maps to one of the Hopf invariant one elements under the composite

$$ {}_2\pi_d\Omega_0^{8t+k} SO \xrightarrow{\simeq} {}_2\pi_d\Omega_0^k SO \xrightarrow{\simeq} {}_2\pi_{d+k} SO \xrightarrow{J_{\mathbb{R}}} {}_2\pi_{d+k}^S. $$

Proof. By Bott periodicity, it is enough to prove the lemma when $t = 0$. The lemma is an outcome of the Adams's Hopf invariant one result.

If $k = 0$ this is just Lemma 4.1. For $k = 7$, $a_1 \in H_1 BO$ is spherical by inclusion of the bottom cell whereas a spherical class in $H_d\Omega_0^7 SO \simeq BO$ with $d > 1$ desuspends to a spherical class in $H_{d-1}SO$. Therefore $d = 2, 4, 8$.

Other cases, also follow similarly by using the homology suspension or desuspension arguments. For instance, if $k = 1$, then a spherical class $x \in H_d SO/U$ lives in an exterior algebra which means that it cannot be a square, i.e x determines a nontrivial indecomposable element in H_*SO/U which suspends to a spherical class in $H_{d+1}SO$. Bearing in mind that SO/U has its bottom cell in dimension 2 proves our claim.

For $k = 2$, notice that H^*SO/U is polynomial. The Eilenberg-Moore spectral sequence argument then shows that the homology suspension maps indecomposable elements of H_*U/Sp isomorphically onto primitive elements in $H_{*+1}SO/U$. Moreover, notice that H_*U/Sp is exterior. Hence, a spherical class in $H_d U/Sp$ must correspond to an indecomposable element which means that it maps isomorphically to a spherical class in $H_{d+1}SO/U$. This implies that $d = 1, 5$.

For $k = 3$, a spherical class in $x \in H_d BSp$, with $d > 4$, desuspends to a spherical class in $H_{d-4}BO$. This implies that $d - 4 \leqslant 8$, i.e. $d \leqslant 12$. On the hand the A-annihilated primitive classes in H_*BSp live in dimensions $4(2^i - 1)$. This implies that only 4 and 12 dimensional classes in H_*BSp have chance to be spherical. If there is a spherical class in $H_{12}BSp$ it then must desuspend to a primitive class in $H_8 BO$. However, the only primitive class in $H_8 BO$ is given by a_1^8 which dies under the homology suspension. Hence, we are only left with a spherical class in dimension 4 for H_*BSp.

For $k = 4$, again we observe that H^*BSp is polynomial, hence a spherical class in $H_d Sp$ must survives to a spherical class in $H_{d+1}BSp$, i.e. we must have $n = 3$.

If $k = 5$, a spherical class in $H_d Sp/U$ with $d > 2$ must desuspend to a spherical class in $H_{d-2}BO$ which mean $d - 2 \leqslant 8$. A reasons simlar to previous cases leads to a contradiction. Hence, $d = 2$ which is given by the inclusion of the bottom cell.

Finally, in the case of $k = 6$ the inclusion of the bottom cell shows that u_1 is spherical, where as in dimension 2 the spherical class is u_1^2 with $\sigma_* a_1 = u_1^2$. We note that this is related to η, as delooping once more we will obtain a mapping $S^3 \to Sp/U$ which is detected by Sq^2 on the bottom cell. The other candidates can be shown not to be spherical similar to the previous cases. $\square$

We apply this observation to the following problem. From the splitting (1.2) it is clear that if a spherical class $x \in H_*\Omega_0^k J$ maps nontrivially to a spherical class in $H_*Q_0 S^{-k}$, and projects back onto a spherical class in $H_*\Omega_0^k J$. Hence, in order to classify such spherical classes in $H_*Q_0 S^{-k}$, it is enough to determine spherical classes in $H_*\Omega_0^k J$, i.e. it is enough to calculate the image of the Hurewicz homomorphism

$$ h^d : {}_2\pi_d\Omega_0^k J \longrightarrow H_d\Omega_0^k J. $$

We first, deal with μ_{8i+1} and $\eta\mu_{8i+1}$ which are not in the image of the J-homomorphism. Notice that according to Lemma 5.6 they will map nontrivially under the Hurewicz homomorphism h^1.

Lemma 6.2. *For the classes μ_{8i+1} and $\eta\mu_{8i+1}$ we have the following equations:*

$$
\begin{array}{llll}
(1) & h^4(\mu_{8i+1}) & = & 0, \\
(2) & h^2(\mu_{8i+1}) & \neq & 0, \\
(3) & h^3(\eta\mu_{8i+1}) & = & 0, \\
(4) & h^2(\eta\mu_{8i+1}) & \neq & 0.
\end{array}
$$

Proof. For the first equation, consider the composition

$$S^{8i} \xrightarrow{\eta} S^{8i-1} \xrightarrow{2} S^{8i-1} \xrightarrow{\alpha_{8i-1}} Q_0 S^0.$$

This adjoints down to

$$S^3 \xrightarrow{\eta} S^2 \xrightarrow{2} S^2 \xrightarrow{\alpha''} Q_0 S^{-8i+3}$$

where α'' is the adjoint of α_{8i-1}. The successive compositions 2η and $\alpha''2$ are trivial. Hence, we obtain

$$S^4 \xrightarrow{\eta^\flat} C'_2 \xrightarrow{\alpha''_\sharp} Q_0 S^{-8i+3}$$

representing the adjoint of μ_{8i+1} as an element in $_2\pi_4 Q_0 S^{-8i+3}$. Notice that $(\eta^\flat)_* = 0$ for dimensional reasons, hence $h^4(\mu_{8i+1}) = 0$.

On the other hand, according to Lemma 5.6 $\mu_{8i+1} : S^1 \to Q_0 S^{-8i}$ is detected by homology. Since μ_{8i+1} does not belong to the image of the J-homomorphism, hence it maps nontrivially under the mapping $\partial_\# : _2\pi_1 \Omega_0^{8i} J \to _2\pi_1 \Omega_0^{8i} BSO \simeq _2\pi_1 BO$ where ∂ was introduced in Section 5.1. Applying Lemma 5.6 we observe that $h^1(\partial_\# \mu_{8i+1}) = a_1$. Notice that in the composition

$$S^1 \longrightarrow \Omega_0^{8i} J \longrightarrow \Omega_0^{8i} BSO \longrightarrow BO$$

which realises $\partial_\# \mu_{8i+1}$, all maps are loop maps. Hence, we may adjoint up once. In homology we then obtain

$$\sigma_* h^1(\partial_\# \mu_{8i+1}) = \sigma_* a_1 = u_1^2 \in H_* U/O$$

hence showing that $h^2(\mu_{8i+1}) = \sigma_* h^1(\mu_{8i+1}) \neq 0$, where $H_* U/O \simeq \mathbb{Z}/2[u_{2i+1} : i \geqslant 0]$.

For the third equality, notice that $\mu_{8i+1} : S^{8i+1} \to S^0$ has stable adjoint $S^2 \to Q_0 S^{-8i+1}$. Hence, $\eta\mu_{8i+1}$ can be represented as

$$S^3 \xrightarrow{\eta} S^2 \xrightarrow{\mu_{8i+1}} Q_0 S^{-8i+1}$$

where $\eta_* = 0$. Hence, $h^3(\eta\mu_{8i+1}) = 0$.

Finally, note that adjointing down $S^3 \to S^2 \to Q_0 S^{-8i+1}$ only once, we may represented $\eta\mu_{8i+1}$ as

$$S^2 \xrightarrow{\eta} \Omega S^2 \xrightarrow{\Omega\mu_{8i+1}} Q_0 S^{-8i}$$

which shows that $h^2(\eta\mu_{8i+1}) = (\Omega\mu_{8i+1})_* \eta_* g_2 = (\Omega\mu_{8i+1})_* g_1^2 = (h^1 \mu_{8i+1})^2 = a_1^2 \in H_* BO$ whereas $\eta_* g_2 = g_1^2$ as η has Hopf invariant one. This equality then show that $h^2(\eta\mu_{8i+1}) \neq 0$. $\qquad\square$

We note that the above lemma does not tell us about $h^3(\mu_{8i+1})$, although we guess that $h^3(\mu_{8i+1}) \neq 0$. Now we can complete proof of Theorem 3.

of Theorem 3. Assume that x_d^{-k} is spherical. This project onto a spherical class in $H_* \Omega_0^k J$. Notice that the composite $\Omega_0^k SO \to \Omega_0^k J \to \Omega^k BSO$ is null homotopic, i.e. the class x_d^{-k} maps trivially into $H_* \Omega^k BSO$. Hence, it can not arise from the classes μ_{8i+1} and $\eta\mu_{8i+1}$ (detailed calculated for this also can be carried out). On the other hand, x_d^{-k} cannot arise from $_2\pi_* \mathrm{coker} J$. To see this note that Sullivans decomposition yields a (split) fibration

$$\Omega^k \mathrm{coker} J \longrightarrow \Omega^k Q_0 S^0 \longrightarrow \Omega^k J.$$

If x_d^{-k} does arise from $_2\pi_* \mathrm{coker} J$, it then must be trivial in $H_* \Omega_0^k J$ which is a contradiction. Hence, x_d^{-k} is spherical only if it arises from the image of the J-homomorphism. Lemma 6.1 then finishes the proof. $\qquad\square$

7 Truncated subalgebras in $H_*(Q_0 S^{-k}; \mathbb{Z}/2)$:Proof of Proposition 4

We keep H_* for $H_*(-; \mathbb{Z}/2)$ unless otherwise stated. Our use of the iterated loops of the real and complex J-homomorphism comes with a cost. In both of the real and complex case, we used to focus on the base point components of $\Omega^k SO$ and $\Omega^k U$. But there are some elements in $H_* Q_0 S^{-k}$ which are not seen in

this way, neither detecting them with the Eilenberg-Moore spectral sequence is the easiest way to do this.

As a motivating example, recall that [D60, Part I, Lemma 4.10]

$$H_*Q_0S^0 \simeq \mathbb{Z}/2[Q^Ix_i : \mathrm{excess}(Q^Ix_i) > 0, (I,i) \text{ admissible}],$$

where $x_i = Q^i[1] * [-2]$ and, denotes the loop sum in H_*QS^0, and $I = (i_1,\ldots,i_r)$ is admissible if $i_j \leqslant 2i_{j+1}$. More important for us us that $[n]$ is the image of $n \in \pi_0QS^0$ under the Hurewicz homomorphism $\pi_0QS^0 \to H_0(QS^0;\mathbb{Z})$. Note that in dimension 0 both π_0QS^0 and H_0QS^0 are groups under the loop sum and the Hurewicz map respects this operation, i.e. $[n] * [m] = [n+m]$. The map $n : S^0 \to QS^0$ extends to an infinite loop map $n : QS^0 \to QS^0$, providing necessary maps for homotopy equivalence between different path components of QS^0.

7.1 Some remarks on the geometry of QS^{-k}

In general, we know that $\pi_0QS^{-k} \simeq \pi_k^S$ is a direct sum of cyclic groups of the form $\mathbb{Z}/2^d(2s+1)$, although we are far from knowing these summands in general. If we fix an automorphism of π_k^S, then we may label different components of QS^{-k} by writing $Q_\alpha S^{-k}$ for $\alpha \in \pi_0QS^{-k} \simeq \pi_k^S$.

For $k = 0$, this automorphism is given by the degree mapping. In fact, we have a mapping $\deg : QS^0 \to \mathbb{Z} \simeq \Omega_0^{\mathrm{fr}}$ which induces the isomorphism $\pi_0QS^0 \to \mathbb{Z}$. For $k > 0$, a choice is provided by the framing homomorphism (up to homotopy)

$$\mathrm{fr} : QS^{-k} \to \Omega_k^{\mathrm{fr}}.$$

If we take γ^k to be generator of a summand in π_0QS^{-k} we then may represent it as an element in QS^{-k}. We then define a mapping $[-\gamma^k] : Q_{\gamma^k}S^{-k} \to Q_0S^{-k}$ by $\alpha \mapsto \alpha - \gamma^k$. This induces a homotopy equivalence between different components of QS^{-k}.

If we consider the 2-primary homotopy groups, we are then only concerned with $\mathbb{Z}/2^d$ part of each summand of the form $\mathbb{Z}/2^d(2s+1)$. If we write $_2QS^{-k}$ for the 2-local version of QS^{-k}, we then may consider the projection map $QS^{-k} \to {}_2QS^{-k}$ as the identification map induced by $\mathbb{Z}/2^d(2s+1) \to \mathbb{Z}/2^d$. That is to say, in the level of spaces, for α, β belonging to the summand $\mathbb{Z}/2(2s+1)$ if $\alpha \equiv \beta \mod {}_2\pi_k^S$ then $Q_\alpha S^{-k}$ and $Q_\beta S^{-k}$ are identified under the projection map $QS^{-k} \to {}_2QS^{-k}$. Of course, $_2QS^{-k}$ is more complicated than this, but we allow ourselves to think of different components of $_2QS^{-k}$ as equivalence classes of 2-local versions of components of QS^{-k}, and we drop the sub-preindex 2 from the notation.

Next, the Hurewicz homomorphism $h : {}_2\pi_0QS^{-k} \to H_0QS^{-k}$. For any $\alpha \in {}_2\pi_k^S$, we define $[\alpha] = h\alpha \in H_0QS^{-k}$. Then it is clear that $[\alpha] * [\beta] = [\alpha + \beta]$ for $\alpha, \beta \in \pi_0QS^{-n}$ where $*$ denotes the Pontrjagin product in H_*QS^{-k} induces by the loop sum. We note that $[0]$ acts as the identity element for the Pontrjagin product in H_*QS^{-k}. Applying the Cartan formula to the relation $[0] * [0] = [0 + 0] = [0]$ together with induction shows that $Q^i[0] = 0$ for all $i > 0$; whereas $Q^0[0] = [0] * [0] = [0]$.

For a given class $\alpha \in {}_2\pi_k^S$ we then may consider the homology classes $Q^I[\alpha]$. If α and β are two generators for two summands of $_2\pi_k^S$ then the fact that there is no additive relation between these generators implies that there is no multiplicative relation between the classes of the form $Q^I[\alpha]$ and $Q^J[\beta]$. Finally, notice that the value of $Q^I[\alpha + \beta] = Q^I([\alpha] * [\beta])$ is calculated by the Cartan formula.

Proposition 4, then determines the type of subalgebras in $H_*Q_0S^{-k}$ that one can obtain from elements of the form $\gamma^k \in {}_2\pi_*^S$ of order 2^d. Before continuing with the proof we recall the Cartan formula, for $t > 0$,

$$(Q^I\xi)^{2^t} = Q^{2^tI}\xi^{2^t}$$

where $2^tI = (2^ti_1,\ldots,2^ti_l)$ for $I = (i_1,\ldots,i_l)$.

of Proposition 4. Observe that $[\gamma^k]$ is the generator of $H_0Q_{\gamma^k}S^{-k}$ where $Q_{\gamma^k}S^{-k}$ denotes the component of QS^{-k} corresponding to γ^k. By construction of the Kudo-Araki operations, Q^i, applying an operation Q^I with $l(I) = l$ to $[\gamma^k]$ we land in the component 2^l of QS^{-k} say $Q_{2^l\gamma^k}S^{-k}$, i.e. $Q^I[\gamma^k] \in H_{\dim(I)}Q_{2^l\gamma^k}S^{-k}$. Hence, we use the translation isomorphism $*[-2^l\gamma^k] : Q_{2^l\gamma^k}S^{-k} \to Q_0S^{-k}$ to land in the base point component of QS^{-k}. Finally, notice that once we are in the base point component then applying operations will not change the component, as $20 = 0$!

Since, we are concerned with a cyclic group it suffices to consider sequences of length at most d, that is

we consider classes of the form

$$
\begin{aligned}
&[\gamma_J^k] = Q^J[\gamma^k] * [-2^{l(J)}\gamma^k] \in H_{\dim(J)}Q_0 S^{-k} \quad \text{if } l(J) < d,\\
&[\gamma_J^k] = Q^J[\gamma^k] \in H_{\dim(J)}Q_0 S^{-k} \qquad\qquad\quad \text{if } l(J) = d,
\end{aligned}
$$

Notice that for different choices of admissible sequences J, the classes $[\gamma_J^k]$ are not decomposable elements as they are only characterised by the sequence J. We then like to show that the classes $Q^I[\gamma_J^k]$ form a truncated polynomial algebra of truncation height not more than 2^d. This comes from the Cartan formula mentioned above, and the fact $Q^I[0] = 0$ for any nonempty sequence with a nonzero entry. More precisely, if $l(I) < d$ we have

$$
\begin{aligned}
(Q^I[\gamma_J^k])^{2^d} &= Q^{2^d I}[\gamma_J^k]^{2^d}\\
&= Q^{2^d I}(Q^J[\gamma^k] * [-2^{l(J)}\gamma^k])^{2^d}\\
&= Q^{2^d I}(Q^{2^d J}[2^d\gamma^k] * [-2^{l(J)+d}\gamma^k]\\
&= Q^{2^d I}(Q^{2^d J}[0] * [-2^{l(J)}\gamma^k])\\
&= 0.
\end{aligned}
$$

Similarly, one can show that if $l(J) = d$, then $(Q^I[\gamma_J^k])^{2^d} = 0$.
The observation that $[\gamma^k]$ is a 0-dimensional class then implies that the Nishida relations are enough to describe the A-module structure of these subalgebras. This then completes the proof. $\qquad\square$

Since, for different choices of the generators γ^k there are no multiplicative, neither additive, relations among the classes of the form $Q^I[\gamma_J^k]$ we then can deduce that in general the homology of $H_*Q_0 S^{-k}$ may contain tensor product of truncated polynomial algebras of different truncation heights.
Note that in the special case of the $_2\pi_* J$ we know the order of the generators, and hence we are able to determined the subalgebras more precisely. Finally, notice that $\sigma_*[\gamma^k] \neq 0$.

Example 7.1. *Consider $\eta \in {}_2\pi_0 QS^{-1} \simeq \pi_1^S$ which is of order 2, i.e. $Q^I[\eta] \in H_*Q_0 S^{-1}$ for all nonempty I. Hence, classes of the form $Q^I[\eta]$ span an exterior algebra inside $H_*Q_0 S^{-1}$. Applying the homotopy equivalence $*[\eta] : Q_0 S^{-1} \to Q_\eta S^{-1}$ allows us to consider the exterior subalgebra of $H_*Q_\eta S^{-1}$ generated by $Q^I[\eta] * [\eta]$. We then obtain a subalgebra of H_*QS^{-1} given by polynomials in $[\eta]$ and $[0]$ with coefficients of the form $Q^I[\eta]$.*
Note that in general, we don't know if all classes $Q^I[\gamma^k]$ are nontrivial. However, in this particular example, the classes $Q^I[\eta]$ with I admissible are nontrivial. To see this notice that the mapping $\eta : S^0 \to QS^{-1}$ extends to an infinite loop map $\eta : QS^0 \to QS^{-1}$. On the level of π_0, or equivalently on the level of H_0, this induces the projection $\eta_* : \mathbb{Z} \to \mathbb{Z}/2$, i.e.

$$
\eta_*[n] = \begin{cases} [\eta] & \text{if } n \text{ is odd,}\\ [0] & \text{if } n \text{ is even.} \end{cases}
$$

Hence one may work out the image of $\eta_* : H_*QS^0 \to H_*QS^{-1}$ as following

$$
\eta_* x_i = \eta_*(Q^i[1] * [-2]) = \eta_* Q^i[1] * \eta_*[-2] = Q^i[\eta] * [0] = Q^i[\eta].
$$

Applying the homology suspension we obtain

$$
\sigma_* Q^i[\eta] = \eta_* \sigma_* x_i = \eta_* Q^i g_1 = Q^i x_1 \neq 0
$$

which shows that $Q^i[\eta] \neq 0$. Notice that $Q^i x_1$ depends on the Adem relation for $Q^i Q^1$ [Z09, Lemma 5.10] and will be trivial or maybe nontrivial modulo decomposable terms. However, the decomposable part always has nontrivial terms, for example $Q^3 x_1 = x_1^4$. In fact, as x_1 is primitive, $Q^i x_1$ is primitive. If $Q^i Q^1 = 0$, then the decomposable part of $Q^i x_1$ is the square of a primitive and one may work out what this primitive is nontrivial.
We recall from [Z09, Theorem 5.34] that as an R-module

$$
H_*Q_0 S^{-1} \simeq E_{\mathbb{Z}/2}(Q^I w_{2i} : I \text{ admissible, excess}(I) > 2i, i \geqslant 0)
$$

with $I \neq \phi$ if $i = 0$, and $w_{2i} = (\Omega\lambda_{\mathbb{C}})_* c_{2i}$ where $\lambda : Q\Sigma\mathbb{C}P_+ \to Q_0 S^0$ is the complex transfer, and c_{2i} is the generator in $H_{2i}\mathbb{C}P_+$. Notice that $(I, 2i)$ is not necessarily admissible. The classes w_{2i} satisfy

$$
\sigma_* w_{2i} = p_{2i+1}^{S^0}.
$$

Moreover, two classes of the form $Q^I w_{2i}$ maybe identified if they map to the same element under the homology suspension $\sigma_* : H_{*-1} Q_0 S^{-1} \to H_* Q_0 S^0$.

Now observe that instead of $\lambda_{\mathbb{C}}$ we could have used the iterated J-homomorphisms to define

$$w_{2i} = \left\{ \begin{array}{l} (\Omega J_{\mathbb{R}})_* c_{2i} \\ (\Omega J_{\mathbb{C}})_* c_{2i} \end{array} \right.$$

where we have used the same notation c_{2i} to denote the corresponding generators in homology ring of BU and SO/U. But observe that BU as well as SO/U both have bottom cells in dimension 2, hence the cost of using these alternative definitions is that we do not see $w_0 \in H_0 Q_0 S^{-1}$. But we can recover this by using the class $[\eta]$ where we have

$$H_* Q_0 S^{-1} \simeq E_{\mathbb{Z}/2}(Q^I w_{2i}, Q^J[\eta] : I, J \text{ admissible}, \text{excess}(I) > 2i, \dim J > 0)$$

with J nonempty and $w_{2i} = (\Omega J_{\mathbb{R}})_* c_{2i}$.

We also have the following example from [Z09, Note 5.41].

Example 7.2. *Consider QS^{-9} with $\pi_0 QS^{-9} \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Let γ_i, $i = 1, 2, 3$, be the generators for the first, second and third copies of $\mathbb{Z}/2$ in $\pi_0 QS^{-9}$ respectively. The classes $Q^I[\gamma_i]$ live in the component $Q_{2\gamma_i} S^{-9} = Q_0 S^{-9}$, for $I \neq \phi$. We then have exterior subalgebras inside $H_* Q_0 S^{-9}$ generated by the symbols of the following forms,*

$$Q^{I_1}[\gamma_1], \quad Q^{I_2}[\gamma_2], \quad Q^{I_3}[\gamma_3], \quad I_1, I_2, I_3 \text{ admissible}.$$

As there are no relations among the generators γ_i we see that there are no multiplicative relations among the three set of generators provided above. This might be seen as an evidence for some stable splitting of the space $Q_0 S^{-9}$. We observe that within the homology algebra $H_* QS^{-9}$ terms of the form $Q^J[\gamma_i + \gamma_j] = Q^J([\gamma_i] * [\gamma_j])$ with $i \neq j$ can be calculated using the Cartan formula. We can apply similar techniques as we did before and use the translation maps $*[\gamma_i] : Q_0 S^{-9} \to Q_{\gamma_i} S^{-9}$ and $*[\gamma_i + \gamma_j] : Q_0 S^{-9} \to Q_{\gamma_i + \gamma_j} S^{-9}$ to obtain subalgebras of the other components, and hence we obtain a subalgebra of $H_* QS^{-9}$ which in this case will be an exterior algebra.

Notice that in the both of the above examples, we had cyclic groups of order 2 which was resulting in appearance of an exterior algebra in homology. If we consider the Hurewicz homomorphism

$$h : {}_2 \pi_0 QS^{-3} \to H_0 QS^{-3}$$

then the generator of $\mathbb{Z}/8$ maps to a homology class say $[\nu]$ with $[\nu]^{*8} = [0]$. This then will implies that in the $\mathbb{Z}/2$-homology of QS^{-3} there is a subalgebra generated by classes of the form $Q^I[\nu]$ each of which are of height $\leqslant 8$ in the homology algebra, i.e.

$$(Q^I[\nu])^8 = 0.$$

We conclude by saying that it seems possible to generalise the results here to odd primes with required changes and probably with more complications.

8   Proof of Theorem 6

We continue to work with $p = 2$ and write H_* for $H_*(-; \mathbb{Z}/2)$. The Eilenberg-Moore spectral sequence machinery [G04, Lemma 7.2, Proposition 7.3] implies that if X is a simply connected space and $H^* X$ is polynomial, then

$$H_* Q_0 \Sigma^{-1} X \simeq E_{\mathbb{Z}/2}(\sigma_* PH_* QX)$$

and the homology suspension $\sigma_* : QH_* Q_0 \Sigma^{-1} X \to PH_* QX$ is an isomorphism where P and Q are the primitive submodule and quotient indecomposable module functors.

According to [CLM, Part I] if $\{x_\alpha\}$ is an additive basis for $\overline{H}_* X$, the reduced homology of X with X path connected, then

$$H_* QX \simeq \mathbb{Z}/2[Q^I x_\alpha : I \text{ admissible}, \text{excess}(I) > \dim x_\alpha]$$

where a sequence of positive integers $I = (i_1, \ldots, i_r)$, called admissible, with $\text{excess}(I) = i_1 - (i_2 + \cdots + i_r)$. We allow I to be the empty sequence ϕ to be an admissible sequence, with $Q^\phi x = x$, and $\text{excess}(\phi) = +\infty$.

According to [MM65, Propositon 4.23], over $\mathbb{Z}/2$, a primitive class in H_*QX is either a square of a primitive, or determines an indecomposable class belonging to the kernel of the square root map $QH_*QX \to QH_*QX$. The square root map $r : H_*X \to H_*X$ is given by $rx = Sq_*^t x$ if $\dim x = 2t$ and is trivial on the odd dimensional classes. The square root map $r : H_*QX \to H_*QX$ is described by

$$rQ^{2i+1}\xi = 0, \qquad rQ^{2i}\xi = Q^i r\xi,$$

where $\xi \in H_*QX$.

Note that any class $Q^I x \in H_*QX$ with $\mathrm{excess}(Q^I x) > 0$, modulo decomposable terms, represents a unique indecomposable in QH_*QX. Since, r is a multiplicative map, then $r : QH_*QX \to QH_*QX$ is determined by its action on the terms $Q^I x$. We also note that in the cases of our consideration $r(Q^I x + Q^J y) = 0$ only if $rQ^I x = rQ^J y = 0$. Finally, for Q^I with i_j odd, we may write

$$Q^I = Q^J Q^{2k+1} + \text{ other terms },$$

where J is an admissible term, and not necessarily $(J, 2k+1)$, and other terms are determined by the 'Adem relations' for the operations Q^I. It simply says that if I has an odd entry at the middle then we can move it to the right end of the sequence.

Note that $X = BU, \Omega_0^k SO, k \equiv 0, 1, 3, 7$, have polynomial $\mathbb{Z}/2$-cohomology. Hence, we may apply the above machinery to calculate $H_*Q_0\Sigma^{-1}X$, where one passes to the universal cover of $Q_0\Sigma^{-1}X$ and then apply the machinery if necessary. We only need to describe the submodule of primitives in H_*QX. We sketch the calculations for the case of $X = BU$ and leave the rest to the reader.

Note that $H_*BU \simeq \mathbb{Z}/2[c_{2i} : i > 0]$ has an additive basis $\{c_J : J = (2j_1, \ldots, 2j_t)\}$ with $c_J = c_{2j_1}c_{2j_2}\cdots c_{2j_t}$ and $0 < 2j_1 \leqslant 2j_2 \leqslant \cdots \leqslant 2j_t$. Hence,

$$H_*QBU \simeq \mathbb{Z}/2[Q^I c_J : I \text{ admissible}, \mathrm{excess}(I) > \dim J].$$

Let us write $\odot$ for the Pontrjagin product in H_*QBU induced by the loop sum arising from applying Q to BU. Notice that c_J is not decomposable with respect to the $\odot$-product, e.g. $c_2 \odot c_4$ is a $\odot$-decomposable whereas $c_{(2,4)} = c_2 c_4$ is not a $\odot$-decomposable class.

For the square root map $r : QH_*QBU \to QH_*QBU$ we have $r(c_{2j_1}\cdots c_{2j_t}) = (rc_{2j_1})\cdots(rc_{2j_t})$. For $J = (j_1, \ldots, j_t)$ we write $4|J$ if each entry of J is a multiple of 4, otherwise $4 \nmid J$. The above formula then implies that that $c_J \in \ker r$ if and only if $4 \nmid J$. Hence, c_J with $4 \nmid J$ determines a unique primitive class in H_*QBU, say p_J^{BU}. For $4|J$ applying Q^{2i+1} gives a term belonging to $\ker r$, hence we have a unique primitive class which modulo decomposable terms are defined by

$$p_{i,J}^{BU} = Q^{2i+1}c_J.$$

In order to make a distinction from primitive classes coming from H_*BU we refer to primitive classes in H_*QBU as $\odot$-primitive classes. We then have the following.

Proposition 8.1. *Any $\odot$-primitive class in H_*QBU which is not a square can be written as a linear combination of classes of the form $Q^I p_{i,J}^{BU}$ and $Q^K p_L^{BU}$ with $4|J$ and $4 \nmid L$.*

Proof. Suppose $\xi \in H_*BU$ is a primitive class which is not a square. This implies that it has nontrivial image in QH_*QBU, and we may write

$$\xi = \sum Q^A c_J + \odot - \text{decomposable terms}.$$

The fact that ξ is $\odot$-primitive, implies that $\sum Q^A c_J$ has to belong to the kernel of the square root map $r : QH_*QBU \to QH_*QBU$. An examination shows that if we have two terms of the form $Q^I c_J$ and $Q^K c_L$ with $r(Q^I c_J) \neq 0 \neq r(Q^K c_L)$ then $r(Q^I c_J + Q^K c_L) \neq 0$. Hence, each term $Q^A c_J$ in the above expression for ξ must belong to $\ker r$. This then means that either A contains an odd entry, or $4 \nmid J$. It is clear that if $4 \nmid J$ then modulo $\odot$-decomposable terms

$$c_J = p_J^{BU}.$$

We focus on the cases when $4|J$ and $A = (a_1, \ldots, a_t)$ has an odd entry. If $a_t = 2i+1$ is odd, then modulo $\odot$-decomposable terms we have

$$Q^{a_t}c_J = p_{i,J}^{BU}.$$

Finally, if a_t is even and the odd entry of A is at the middle of the sequence we may use the Adem relations to move the odd entry to write and write

$$Q^A = Q^L Q^{2k+1} + \text{other terms}$$

where these "other terms" are of 'excess less than I'. The proof then is complete by an induction on the excess. This then implies that we may write Q^I as sum given by

$$Q^A = \sum Q^L Q^{2k+1}$$

which means that

$$Q^A c_J = \sum Q^L Q^{2k+1} c_J.$$

We then observe that modulo $\odot$-decomposable terms this later equality yields

$$Q^A c_J = \sum Q^L p_{k,J}.$$

Hence, we have shown that

$$\xi = \sum Q^I p_{i,J}^{BU} + \sum Q^K p_J^{BU} + D$$

where D is a sum of $\odot$-decomposable terms. But notice D is a decomposable primitive, so it must be square of a primitive. An induction on the length of the sequences A in the initial expression for ξ completes the proof. $\qquad\square$

Next, define unique elements $c_{i,J}^{-1} \in QH_{2i+\dim J-1}Q\Sigma^{-1}BU$ and $c_L^{-1} \in QH_{\dim L-1}Q\Sigma^{-1}BU$ by

$$\sigma_* c_{i,J}^{-1} = p_{i,J}^{BU}, \qquad \sigma_* c_L^{-1} = p_L^{BU}.$$

where $4|J$ and $4 \nmid L$. We then have

$$H_* Q\Sigma^{-1}BU \simeq E_{\mathbb{Z}/2}(Q^I c_{i,J}^{-1}, Q^K c_L^{-1} : \text{excess}(I) > 2i + \dim L, \text{excess}(K) > \dim L - 1),$$

as an R-module, where we assume I, K are admissible. Note that the classes $c_{i,J}^{-1}$ are not in the image of $(\Omega\iota_{BU})_* : H_*U \to H_*Q\Sigma^{-1}BU$, whereas c_L^{-1}s belong to the image of $(\Omega\iota_{BU})_*$ if and only if $L = (l_1,\ldots,l_t)$ is strictly increasing. Note that we require L to be strictly increasing as then we can see that $u_L = u_{l_1}\cdots u_{l_t} \in H_*U$ maps to c_L^{-1}. But if we allow equal entries in L we can not do this, as H_*U is an exterior algebra. Finally, we note that two classes of the form $c_{i,J}^{-1}$ may be identified if they map to the same primitive class in H_*QBU under the homology suspension $H_*Q\Sigma^{-1}BU \to H_{*+1}QBU$. Now we define define $w_{i,J}^{-2k-2} \in H_{2i+\dim J}Q_0S^{-2k-2}$ and $w_L^{-2k-2} \in H_{\dim L-1}Q_0S^{-2k-2}$ by

$$w_{i,J}^{-2k-2} = (\Omega j_{2k+1}^{\mathbb{C}})_* c_{i,J}^{-1}, \qquad w_L^{-2k-2} = (\Omega j_{2k+1}^{\mathbb{C}})_* c_L^{-1}.$$

The next observation is the clear from the above theorem and the fact that $(\Omega j_{2k+1}^{\mathbb{C}})_*$ is a multiplicative map.

Theorem 8.2. *The homology algebra $H_*Q_0S^{-2k-2}$ contains an exterior algebra of the form*

$$E_{\mathbb{Z}/2}(Q^I w_{i,J}^{-2k-2}, Q^K w_L^{-2k-2} : \text{excess}(I) > 2i + \dim J, \text{excess}(K) > \dim L - 1)$$

where we assume I, K are admissible. The classes $w_{i,J}^{-2k-2}$ are not in the image of $(\Omega^{2k+2}J_{\mathbb{C}})_*$. Moreover, the classes w_L^{-2k-2} are in the image of $(\Omega^{2k+2}J_{\mathbb{C}})_*$ if and only if L is strictly increasing. Notice that the classes belonging to $\text{im}(\Omega^{2k+2}J_{\mathbb{C}})_*$ are trivial if $2k \equiv 3, 4 \mod 8$.

(sketch). We only note that our claim on the image of $(\Omega^{2k+2}J_{\mathbb{C}})_*$ are easy to verify once we recall that $\Omega^{2k+2}J_{\mathbb{C}}$ factors through $\Omega\iota_{BU} : U \to Q\Sigma^{-1}BU$. The fact that certain classes in $H_*Q\Sigma^{-1}BU$ are not in the image of $\Omega\iota_{BU}$ shows that the claimed classes do not belong to the image of $(\Omega^{2k+2}J_{\mathbb{C}})_*$. We just observe that the numerical conditions at the end of the above theorem come from the fact that $\Omega^k J_{\mathbb{C}}$ is trivial in $\mathbb{Z}/2$-homology if $k \equiv 5, 6 \mod 8$. $\qquad\square$

Now, let $k = 0$ and choose $(I, 2i + 1, 4j + 1)$ to be an admissible sequence. We have $\sigma_* Q^I c_{i,(4j)}^{-1} = Q^I Q^{2i+1} u_{2i+1}$ which maps to $Q^I Q^{2i+1} p_{2i+1}^{S^0}$ under $(J_{\mathbb{C}})_*$, hence $Q^I w_{i,(4j)}^{-2} \in H_*Q_0S^{-2}$ is nontrivial. This then proves the last part of our claim in Proposition 6. Notice that the above theorem defines classes that are not in the image of J-homomorphism, neither it is in the R-subalgebra generated by the image of $(\Omega^{2k+2}J_{\mathbb{C}})_*$. This then proves Theorem 6, in the complex case. The case of $\Omega j_k^{\mathbb{R}}$ for $k \equiv 0, 1, 3, 7$ is similar.

9 Comments on $H_*(\Omega_0^k \mathrm{coker}J; \mathbb{Z}/2)$

At $p = 2$, the homology algebra $H_*Q_0S^0$ is a polynomial algebra over generators $Q^I x_i$. According to [KP72, Theorem 3.1] we may define $x_i = (J_{\mathbb{R}})_* s_i$ where $s_i \in H_i SO$ is the generator, and $(J_{\mathbb{R}})_*$ is an injection. This shows that in a loose sense, $\mathrm{im}(J_{\mathbb{R}})_*$ and the Dyer-Lashof algebra R are enough to generate $H_*Q_0S^0$; any $\xi \in H_*Q_0S^0$ can be represented as a polynomial in variables x_i and coefficients from the Dyer-Lashof algebra R (mod Adem relations). Notice that $H^*J \simeq H^*SO \otimes H^*BSO$, i.e. additively we know the homology of J, and all of this homology injects in $H_*Q_0S^0$. This then implies that as a space (at the prime 2) almost any class of the form $Q^I x_i$, expect those one mapping to H_*BSO, belong to $H_*\mathrm{coker}J$, i.e. we know about the additive homology of $\mathrm{coker}J$.

By 'counting arguments' it is clear that in $H_*Q_0S^{-k}$ we may use similar technique at least to define infinitely many classes in $H_*\Omega_0^k \mathrm{coker}J$ which are not necessarily nontrivial, although we suspect that there are infinitely many of them are nontrivial. For $k > 0$, the splitting (2.1) provides an equivalence of k-fold loop spaces which implies as algebras $H_*\mathrm{coker}J$ must be a subalgebra of $H_*Q_0S^{-k}$, unlike $k = 0$. For instance, when $k = 1$ we know that almost any class $Q^I w_{2i}$ belongs to $H_*\Omega_0 \mathrm{coker}J$ as well as we know that $H_*\Omega_0 \mathrm{coker}J$ is an exterior algebra. A similar reasoning provides us with the following observation.

Proposition 9.1. *Almost any class of the form $Q^I x_i^{-k}$ belongs to $H_*\Omega_0^k \mathrm{coker}J$.*

We note that in general we are unable to tell whether or not if $Q^I x_i^{-k} \neq 0$, however, if $k \leqslant 7$ then the definition of the classes x_i^{-k} together with Theorem 2 tells us that some of these classes do survive to nontrivial classes in $H_*Q_0S^0$. In these cases, we then obtain infinitely many classes in $H_*\Omega_0^k \mathrm{coker}J$. Notice that this does not tell us about the E_∞-algebra structure of $H_*\Omega_0^k \mathrm{coker}J$, but it tells us how operations coming from the k-fold loop structure should behave.

10 Equivariant J-homomorphisms

The effect of the $\mathbb{Z}/2$- and S^1-equivariant J-homomorphisms

$$J_{\mathbb{Z}/2} : SO \longrightarrow Q_0P_+, \qquad J_{S^1} : U \longrightarrow Q\Sigma\mathbb{C}P_+,$$

in $\mathbb{Z}/p$-homology is known due to Mann-Miller-Miller [MMM86, Corollary 5.3]. Analogous to the non-equivariant case, we may consider

$$\Omega^k J_{\mathbb{Z}/2} : \Omega_0^k SO \longrightarrow \Omega_0^k Q_0P_+, \qquad \Omega_0^k J_{S^1} : \Omega_0^k U \longrightarrow \Omega_0^k Q\Sigma\mathbb{C}P_+.$$

Similar to Theorem 2, using the Bott periodicity maps, we may define a set of classes in the $\mathbb{Z}/p$-homology of $\Omega_0^k Q_0P_+$ and $\Omega_0^k Q\Sigma\mathbb{C}P_+$ in a geometric way, which allows to determine their action of the Steenrod operations Sq_*^r on these classes.

The inclusion $\mathbb{Z}/p^n \to S^1$ yields equivariant transfer maps

$$t_2 : \Sigma\mathbb{C}P_+ \longrightarrow Q_0P_+ \qquad t_{p,n} : \Sigma\mathbb{C}P_+ \longrightarrow QB\mathbb{Z}/p_+^n$$

extending to infinite loop maps

$$t_2 : Q\Sigma\mathbb{C}P_+ \longrightarrow Q_0P_+ \qquad t_{p,n} : Q\Sigma\mathbb{C}P_+ \longrightarrow QB\mathbb{Z}/p_+^n.$$

We refer the reader to [MMM86] for homology of these maps. Similarly, the iterated loops of these mappings, composed with the iterated equivariant J-homomorphisms above, can be used to define classes in the $\mathbb{Z}/p$-homology algebras of $\Omega^k Q_0 B\mathbb{Z}/p_+^n$ with $p \geqslant 2$. For example, if $p > 2$ the mapping $t_{p,1} : \Sigma\mathbb{C}P_+ \to QB\mathbb{Z}/p_+^n$ is an injection in homology. The effect of $t_{p,1} : Q\Sigma\mathbb{C}P_+ \longrightarrow QB\mathbb{Z}/p_+$ in homology on the generators of $H_*(\Sigma\mathbb{C}P_+; \mathbb{Z}/p)$ is the same as the effect of $t_{p,1} : \Sigma\mathbb{C}P_+ \to QB\mathbb{Z}/p_+$ in homology. On the other hand observe that the generators of $H_*(\Sigma\mathbb{C}P_+; \mathbb{Z}/p)$ are in the image of $(J_{S^1})_*$. These together prove the following fact.

Lemma 10.1. *The mapping*

$$U \longrightarrow Q\Sigma\mathbb{C}P_+ \longrightarrow QB\mathbb{Z}/p_+$$

induces an injection in $\mathbb{Z}/p$-homology on the generators $u_{2i+1} \in H_{2i+1}(U; \mathbb{Z}/p)$, when $p > 2$.

We may also use homology suspension arguments to see that the $\mathbb{Z}/p$-homology of $\Omega^k J_{S^1}$ define infinitely many classes in homology of $\Omega_0^k QB\mathbb{Z}/p_+$ at least for $k \leqslant 3$.

We note that in the context of equivariant homotopy theory, there seem to be more complications. For instance, $\pi_0^{S^1} U \not\simeq 0$ where $\pi_*^{S^1}$ is the S^1-equivariant homotopy. This means that while restricting to the base point in the non-equivariant case, we had $U \to SG$, where in the equivariant case, the 'equivariant base point component' of U is not quite that straightforward. If one takes the component corresponding to the identity, then we get the ring of real representations. Similar complications arise, when we consider possibility of having S^1-equivariant versions of J, and Sullivan's splitting and so on. For example, J at $p = 2$ may be defined as the fibre of $\psi^3 - 1 : \Omega_0^\infty ko \to \Omega_0^\infty ko$ where ko denotes the connective KO-theory. However, the spaces representing this equivariant theory are more complicated, and are not like the spaces in the real or complex Bott periodicity, even in the case of ku [G04]. We leave further investigation on this to a future work.

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