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Glendinning, Paul

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Paul Glendinning

School of Mathematics and CICADA (Centre for Computational and Dynamic Analysis)
University of Manchester, Oxford Road, Manchester M13 9PL, U.K.
p.a.glendinning@manchester.ac.uk

Abstract

The border collision normal form is a two dimensional continuous, piecewise affine map which arises naturally in models where the dynamics is defined by different systems of equations in different regions of phase space, but which are continuous across the boundaries between regions. There are theorems which establish the existence of invariant measures for chaotic attractors of these systems, but the conditions are hard to establish analytically. By verifying these conditions numerically it is possible to describe regions of parameter space for which invariant measures do exist (up to numerical confidence) and compare this with what is known about the dynamics in these regions.

Keywords: border collision bifurcation, chaotic attractor, invariant measure

1. Introduction

Some control systems use different strategies under different conditions. This leads to switched systems: dynamical systems where the defining equations change as a variable crosses some switching (or critical) surface. Switching also occurs in models of friction, some electric circuits and contact mechanics, see [2, 3, 4, 6, 16]. In this paper we consider maps which are piecewise smooth and have discrete time and continuous variables. There is a switching surface (or critical line) Σ dividing the regions in which the dynamics is determined by smooth maps, and the equations are continuous across Σ. Thus the left and right sides of Σ could be labelled by L and R respectively, and a discrete variable defined to take values in $\{L, R\}$ according to which side of

Σ the continuous variables are at time n. This discrete variable then determines which dynamical system is applied at the next time step. In particular we consider the normal form of the bifurcation which occurs as a fixed point of one of the systems moves to intersect the critical surface as a parameter is varied. This normal form, called the border collision normal form [17] is a two dimensional map defined by a pair of affine maps, one holding in L and the other in R. There maps have a very rich array of possible dynamics and novel bifurcations (e.g. [13]). Here we concentrate on parameters for which there are chaotic attractors, and we will seek to determine when these have invariant measures.

The existence of a 'natural' invariant measure for a chaotic attractor means that the attractor has a number of nice properties: the evaluation of (almost all) functions on (almost all) orbits of the attractor converge to the integral of the function over the measure, and the support of the measure defines the attractor itself as a geometric object. Thus the existence of an invariant measure gives the chaotic set a certain regularity and makes predictions possible for averaged quantities. Hunt *et al* [10] provide an admirable introduction to invariant measures of dynamical systems, and some of the techniques and results which are known. Unfortunately it is relatively hard to prove existence theorems for any particular example, and even when classes of examples are considered the conditions which need to be verified for the theorems to apply are often difficult to check by hand. The aim of this paper is to use numerical simulations to verify the conditions of existence theorems for a class of two-dimensional maps of the plane, and to investigate how this meshes with results about the dynamics of these maps.

In the context of switched systems, the normal form for the bifurcation which occurs if a fixed point of one of the maps strikes the switching surface was developed by Nusse and Yorke [17]. By transforming the switching surface to the $y-$axis ($x = 0$) the local dynamics with $\mathbf{x} = (x, y)^T$ is

$$\mathbf{x}_{n+1} = \begin{cases} A_L\mathbf{x} + \mathbf{m} & \text{if } x \leq 0 \\ A_R\mathbf{x} + \mathbf{m} & \text{if } x \geq 0 \end{cases} \tag{1}$$

where the matrices A_L and A_R, and the vector $\mathbf{m}$ are defined as

$$A_k = \begin{pmatrix} T_k & 1 \\ -D_k & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{m} = \begin{pmatrix} \mu \\ 0 \end{pmatrix} \tag{2}$$

for $k = L, R$. The constants T_k and D_k are the trace and determinant of the Jacobian of the defining equations evaluated at the bifurcation point on the left and right of Σ, whilst μ is the bifurcation parameter. By scaling, only the sign of μ changes the behaviour of the map, and in this paper we will assume that $\mu \in \{-1, 1\}$ (ignoring the 'bifurcation' value $\mu = 0$) and the aim is to understand the structure of attractors of these systems, not to make a comparison between the bifurcation states $\mu = -1$ and $\mu = 1$, so in any particular discussion μ will be fixed and the trace and determinant of the two defining maps A_R and A_L will act as parameters.

The border collision normal form (1,2) is a very natural set of continuous systems to write down, and the equations were discussed by mathematicians before the application to the border collision bifurcation was discovered. Hence special cases arise in the literature, and in the early 1980s they provided one of the first test-beds for the development of techniques to prove the existence of strange attractors with ergodic, absolutely continuous invariant measures; a program which eventually led to the breakthrough of Mora and Viana [15] proving the existence for some parameters of the Hénon map. Early work by Misiurewicz [14] centred on the Lozi map, which can be though of as (1) with

$$1 < T_L \leq 2, \quad T_R = -T_L, \quad D_R = D_L. \tag{3}$$

Young [19] extended these results to more general maps of the plane (cf. [8]), and it is her theorem that is used here. It is worth stressing that nothing genuinely new is added to her result in this paper, but in order to apply the result to the border collision normal form it is necessary to rescale the system and make small modifications to the statement. Although this is not theoretically deep, we believe it is valuable to understand explicitly how and where the theorems apply, rather than leaving them as implicit statements. The discussion of Young's Theorem and its adaptation to the border collision normal form is given in Section 2. In Section 3 the results of Misiurewicz [14] for the Lozi map are revisited and extended with numerical aid. Section 4 considers the case of robust chaos described in [4], and the final sections consider more abstruse examples.

We end this introduction by providing a more formal definition of invariant measures for a map $F : X \to X$.

A measure μ on X is an $F-$invariant measure (or just invariant measure if it is clear which map is being considered) if for every measurable set $U \subseteq X$, $\mu(U) = \mu(F^{-1}(U))$ where $F^{-1}(U) = \{x \in X \mid F(x) \in U\}$. It is a probability measure if $\mu(X) = 1$. The invariant measures proved to exist by Young [19] have nice properties on invariant manifolds (the measures have absolutely continuous conditional measures on unstable manifolds), and this implies that they are *Bowen-Ruelle* measures with the natural ergodic property that time averages over orbits equal spatial averages with respect to the measure. More precisely an invariant probability measure is a Bowen-Ruelle measure if there is a set $U \subseteq X$ of positive Lebesgue measure such that for every continuous function $\phi : X \to X$ and almost all $x \in U$

$$\frac{1}{n}\sum_{k=0}^{n-1}\phi(F^k(x)) \to \int_U \phi d\mu$$

as $n \to \infty$ (see [10] for more details). Intuitively it may be helpful to think of U as a basin of attraction for some complicated invariant set (the support of the invariant measure) which has nice statistical properties, i.e. it is a classic strange attractor.

2. The existence of invariant measures

Very little has been written about the existence of invariant measures for the dynamics of the normal form within the border collision community. However, Lozi maps and their generalizations have been considered in this light [8, 19, 10] and this provides a theoretical framework within which the existence of measures can be established. The key to this is Young's Theorem [19]. This theorem, and the minor adaptations needed to apply it to the border collision normal form are described in this section.

2.1. Young's Theorem

Consider a general piecewise affine continuous map of the form (1) but with linear parts A_R and A_L defined by

$$A_k = \begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix} \tag{4}$$

$k = R, L$. To apply Young's Theorem these coefficients must satisfy some simple inequalities:

$$\begin{array}{rl} (|\alpha_k| - |\beta_k|) - (|\gamma_k| - |\delta_k|) \geq 0, & k = R, L \\ |\alpha_k| - |\beta_k| > 1, & k = R, L \\ (|\beta_k| + |\delta_k|)/(|\alpha_k| - |\beta_k|)^2 < 1 & k = R, L. \end{array} \tag{5}$$

Theorem 1. *(Young's Theorem [19]) Let* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *be a map of the form (1) defined by matrices given by (4) with coefficients which satisfy (5). Suppose that the map takes some rectangle* $D = [a, b] \times [c, d]$ *with* $a < 0 < b$ *into itself and let* $S = \{0\} \times [c, d]$ *(the segment of the* $y-$*axis in* D*). Let* $u = \min_{k=R,L}(|\alpha_k| - |\beta_k|)$*. If there exists an integer* $N > 0$ *such that* $u^N > 2$ *and* $F^p(S) \cap S = \emptyset$ *for* $1 \leq p < N$ *then* F *has an attractor with a Bowen-Ruelle measure.*

N describes how many iterates of the map are applied before returning to the critical set, and hence provides a bound on the horizontal expansion of segments. If $N = 1$ then we interpret the final condition to be automatically true (as a property of the empty set) and the condition becomes simply $|T_k| > 2$. Note that in [19] the Theorem is stated with a small misprint: the inequalities for p is given as $1 \leq p \leq N$. The remark that the final condition is automatically true if $N = 1$ makes no sense in this case, and the replacement of F by F^N is natural provided equality is excluded, so we are confident this is the correct statement of the Theorem.

Young's result applies to a much broader class of maps than those defined here, but this is the application of her result to the piecewise affine continuous maps defined here. For the border collision normal form where the matrices A_R and A_L are defined in (2) the conditions (5) become

$$\begin{aligned} |T_k| - 1 - |D_k| \geq 0, &\quad k = R, L \\ |T_k| > 2, &\quad k = R, L \\ 1/(|T_k| - 1)^2 < 1 &\quad k = R, L. \end{aligned} \tag{6}$$

The third if these equations is implied by the second, so only the the first two of these equations need to be satisfied. Unfortunately the second condition is restrictive in a way which means that (for example) the Lozi map, with parameters (3), cannot be considered. But, as Young herself points out, this problem can be avoided by a simple scaling, and this is part of what is done in the next subsection. Two other problems need to be addressed before the result can be applied: the rectangular region R must be identified and then the exponent N computed. The first of these problems needs some thought, and it is easier to use a non-rectangular region with a view to maximizing N. The second is where computer simulations come into their own. These three factors: scaling, existence of an invariant region (i.e. a connected set $U \subset \mathbb{R}^2$ which is the closure of its interior – a *region* – such that if $x \in U$ then $F(x) \in U$ – *invariance*) and the calculation of the exponent N are the subject of the next subsection.

2.2. Scaling for the Border Collision Normal Form

Given $\varepsilon > 0$ let $y = \varepsilon z$, then in terms of the new coordinates (x, z) the normal form becomes

$$\begin{pmatrix} x_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} T_k & \varepsilon \\ -D_k/\varepsilon & 0 \end{pmatrix} \begin{pmatrix} x_n \\ z_n \end{pmatrix} + \begin{pmatrix} \mu \\ 0 \end{pmatrix} \tag{7}$$

with $k = R$ if $x_n \geq 0$ and $k = L$ if $x_n \leq 0$. Young's conditions on the derivatives, (5), therefore become

$$\begin{aligned} \varepsilon(|T_k| - \varepsilon) - |D_k| \geq 0, &\quad k = R, L \\ |T_k| > 1 + \varepsilon, &\quad k = R, L \\ \varepsilon/(|T_k| - \varepsilon)^2 < 1 &\quad k = R, L. \end{aligned} \tag{8}$$

These inequalities can be satisfied for all pairs (T_k, D_k) with $|T_k| > 1 + \varepsilon$ and $|D_k| < \varepsilon$ for some $\varepsilon \in (0, 1)$ – a great improvement on the previous case. Note that the third of equations (8) is satisfied if the second equation is satisfied and $\varepsilon < 1$.

Suppose A is an invariant region for the border collision normal form in the standard coordinates (1), and A' is the corresponding region in the new coordinates (x, z). The clearly A' is an invariant region, and the intersection, C, of A with the critical line $x = 0$ is mapped by the coordinate transformation to the intersection, C', of the transformed critical line (still $x = 0$) with A'. Since the map $(x, y) \to (x, z)$ is a differentiable conjugacy for the dynamics if $\varepsilon \neq 0$, the geometric condition of Young's Theorem can either be written in the old coordinates or the new coordinates, and we will choose to continue to work in the new coordinates.

In the statement of Young's Theorem the invariant region is a rectangle, but the proof relies only on the expansion properties of near-horizontal segments, and so works for any invariant region which intersects the critical line and its images nicely. In particular we may take any convex invariant region instead of R. These comments lead to the following reformulation of Young's Theorem for the border collision normal form.

Theorem 2. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a map of the form (1) and suppose F has a convex invariant region A which intersects the critical line $\{x = 0\}$ in a closed, non-empty line segment C. If there exists $\varepsilon > 0$ and $N \in \mathbb{Z}^+$ such that inequalities (8) hold, and $(|T_k| - \varepsilon)^N > 2$ with $F^p(C) \cap C = \emptyset$ for $1 \leq p < N$, then F has an attractor with a Bowen-Ruelle measure.*

As before, note that if $N = 1$ then we interpret the final condition to be automatically true (as a property of the empty set) and the condition becomes simply that inequalities (8) hold and $|T_k| - \varepsilon > 2$. We shall use the conditions from Theorem 2 to verify the existence of invariant measures for the border collision normal form.

3. The Lozi Map revisited

Throughout the late 1970s and the 1980s, a major question of research was to describe the existence and prevalence of strange attractors in Hénon maps, which are two-dimensional maps of the plane

$$\begin{aligned} x_{n+1} &= f(x_n) + y_n \\ y_{n+1} &= bx_n \end{aligned} \tag{9}$$

where f is a smooth map of the real line, for example the logistic map. A general framework within which these questions could be answered was developed by Benedicks and Carleson [5] and then Mora and Viana [15], but early successful attempts to describe strange attractors used piecewise linear models for f, and in particular the Lozi maps [12], which has

$$f(x) = 1 - a|x|, \quad a \in (1, 2] \tag{10}$$

i.e. the Lozi map is

$$\begin{aligned} x_{n+1} &= -a|x_n| + y_n + 1 \\ y_{n+1} &= bx_n \end{aligned} \tag{11}$$

with $1 < a \leq 2$ (cf. the border collision normal form (1,2)).

Misiurewicz [14] proved the existence of strange attractors as the closure of the unstable manifold of a saddle fixed point. The Lozi map is equivalent to the border collision normal form with

$$T_R = -a, \quad T_L = a, \quad D_L = D_R = -b \tag{12}$$

and we will begin by stating Misiurewicz's result (which does not include the existence of an invariant probability measure, but pre-dates the work of Young [19] and Collet and Levy [8]).

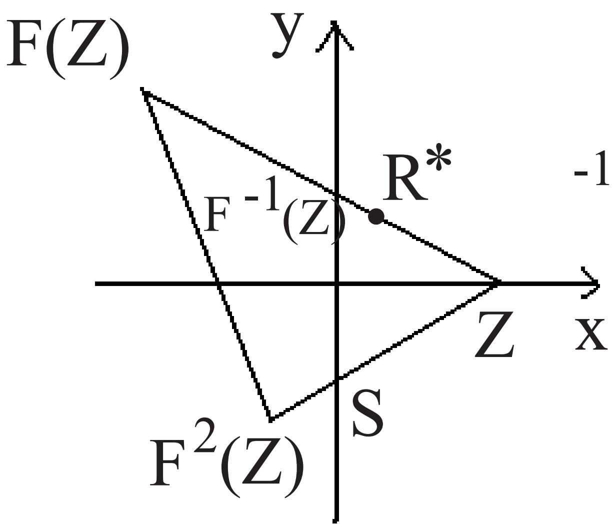

Figure 1: Geometry of the invariant region for parameters in $\mathcal{P}$. Note that $F(Z)$ can also lie to the right of the $y-$axis.

3.1. Misiurewicz's parameters

Misiurewicz [14] considers the parameter region $\mathcal{P}$ defined by the three inequalities

$$0 < b < \frac{a^2 - 1}{2a + 1}, \quad a\sqrt{2} > b + 2, \quad 2a + b < 4 \tag{13}$$

Note that for these parameters F has a fixed point X in $x > 0$ which is a saddle.

Theorem 3. *[14] Consider the Lozi map (11) with parameters in $\mathcal{P}$. Then F has a hyperbolic, transitive attractor which is the closure of the unstable manifold of the fixed point R^*.*

Our first result in this section applies Theorem 2 to this region of parameters.

Theorem 4. *(Numerical) The attractor of the Lozi map described in Theorem 3 has a Bowen-Ruelle measure.*

Proof: Let Z be the intersection of the unstable manifold of the fixed point R^* with the $x-$axis. Then for parameters in $\mathcal{P}$ $F(Z)$ is in $x < 0$ and the triangle $ZF(Z)F^2(Z)$ is an invariant region [14]. The point $F^2(Z)$ may be in $x < 0$ or $x > 0$; the geometry for cases with $F(Z)$ in $x < 0$ is sketched in Figure 1.

The intersection of this invariant triangle with the critical line (the $y-$axis) is a vertical line segment $SF^{-1}(Z)$, where S is the intersection of $F^2(Z)Z$ with the $y-$axis if $F^2(Z)$ is in $x < 0$, or the intersection of $F(Z)F^2(Z)$ with the $y-$axis if $F^2(Z)$ is in $x > 0$ (In the case of intersection at $x = 0$ then $S = F^2(Z)$).

By checking numerically we are convinced that $F(S)$ lies on the $x-$axis in $x > 0$ and $F^2(S)$ lies in $x < 0$ for parameters in $\mathcal{P}$.

To apply Young's Theorem in the form of Theorem 2 let $\varepsilon = b$ in (8). Then the three inequalities are all satisfied provided

$$a > 1 + b \tag{14}$$

since $0 < b < 1$. It is easy to prove (analytically this time) that (14) holds for all parameters in $\mathcal{P}$ and that in fact $a - b > v$ where

$$v = 2 - \frac{6(\sqrt{2} - 1)}{2 + \sqrt{2}} \tag{15}$$

(this is derived by considering $a - b$ at the intersection the lines $b = 4 - 2a$ and $b = \sqrt{2}a - 2$, which lies just outside $\mathcal{P}$ and gives a lower value for $a - b$ than any other point. Note (*numerically*) that $v^3 > 0$.

Thus if $F^{-1}(Z)S = C$, we can apply Theorem 2 if $F(C)$ and $F^2(C)$ are both disjoint from C. Now, $F(C) = ZF(S)$ which is clearly in $x > 0$, and $F^2(C) = F(Z)F^2(S)$ which lies in $x < 0$ provided $F^2(S)$ lies in $x < 0$. This can easily (but painfully) be calculated and yields a polynomial inequality, but *we have checked numerically that $F^2(S)$ is in $x < 0$ for all parameters in $\mathcal{P}$* and hence Theorem 2 can be applied to show the existence of a Bowen-Ruelle measure.

□

The numerical verifications referred to were neither sophisticated or exhaustive: a 100×100 grid was set up in parameter space containing $\mathcal{P}$ and the properties were verified on this grid. Much more sophisticated approaches could clearly be used, but the importance of the result does not seem to merit that degree of effort!

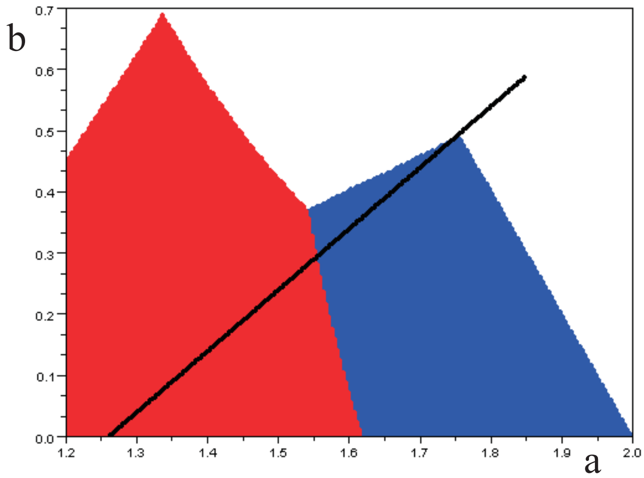

Figure 2: Parameter space for the Lozi map: the is a Bowen-Ruelle measure for the shaded region which lies below the line, which is $b = a - 2^{\frac{1}{3}}$. This line leaves the shaded region just to the left of the local maximum in b. The right hand shaded region has an invariant region with $F(Z)$ to the right of the $y-$axis, the left hand region has an invariant region with $F(Z)$ on the left of the $y-$axis.

3.2. Beyond Misiurewicz

The triangle $ZF(Z)F^2(Z)$ described above is an invariant region over a much larger range of parameter values than the set $\mathcal{P}$ of the previous subsection. Indeed, the results of Young [19] show the existence of a Bowen-Ruelle measure for all parameters in the shaded region of Fig. 2 which lie below the line $(a-b)^3 = 2$. This, and the significance of the different shading in Fig. 2, are explained below.

Theorem 5. *Suppose that a Lozi map has a convex invariant region A which intersects the critical line $x = 0$ on a line segment C. If there exists $N > 0$ such that*

$$(a-b)^N > 2 \tag{16}$$

and $F^k(C) \cap C = \emptyset$, $1 \leq k < N$, the F has an attractor in A with a Bowen-Ruelle measure.

Proof: If $(a-b)^N > 2$ then $a-b > 1$ and so the inequalities (8) are satisfied with $\varepsilon = b$. This, together with (16) and the self-intersection condition for C

imply that the conditions of Theorem 2 are satisfied and the result follows.

□

Corollary 1. *Define S and Z as in the proof of Theorem 4. If $ZF(Z)F^2(Z)$ is an invariant region, $(a-b)^3 > 2$, $F(S)$ lies in $x > 0$ and $F^2(S)$ lies in $x < 0$, then F has a Bowen-Ruelle measure.*

Proof: Since C, the intersection of the critical line with the invariant region, is the line segment $F^{-1}(Z)S$, $f(C) = ZF(S)$ and since Z is in $x > 0$, $f(C)$ is in $x > 0$ provided $F(S)$ is in $x > 0$. Similarly, as $F^2(C) = F(Z)F^2(S)$ and $F(Z)$ is in $x < 0$, the interval $F^2(C)$ lies in $x < 0$ if $F^2(S)$ is also in $x < 0$.

□

Note that $(a-b)^3 > 2$ is the same as $b < a - 2^{\frac{1}{3}}$; the straight line in Fig. 2 is the boundary of this region, in the left hand shaded region $ZF(Z)F^2(Z)$ is an invariant region with $F^2(Z)$ in $x > 0$ with the conditions on $F(S)$ and $F^2(S)$ satisfied, and the right hand shaded region is the same but with $F^2(Z)$ in $x < 0$.

3.3. Lozi-like Border Collisions

In the language of the border collision normal form, $T_L = |T_R| > 0$ and $-1 < D_L = D_R < 0$. The more general problem with simpler geometry relaxes these constraints whilst keeping $-1 < D_L, D_R < 0$ and retaining the signs of the traces. This case is treated briefly in [4], and to maintain comparability with existing results we will not go into this case here, but simply note that the same geometric ideas can be applied.

4. Robust Chaos

Suppose

$$0 \leq D_R, D_L < 1, \quad T_L > 1 + D_L, \quad \text{and} \quad T_R < -(1 + D_R) \qquad (17)$$

Then if $\mu > 0$ then the border collision normal form has two fixed points, L^* in $x < 0$ and R^* in $x > 0$. Both are saddles. The linear part of the map has two positive eigenvalues, $s_\pm$ with $0 < s_- < 1 < s_+$ whilst the fixed point in $x > 0$ has negative eigenvalues. This case is considered in some detail by Banerjee, Yorke and Grebogi [4], who christen the chaotic attractors which

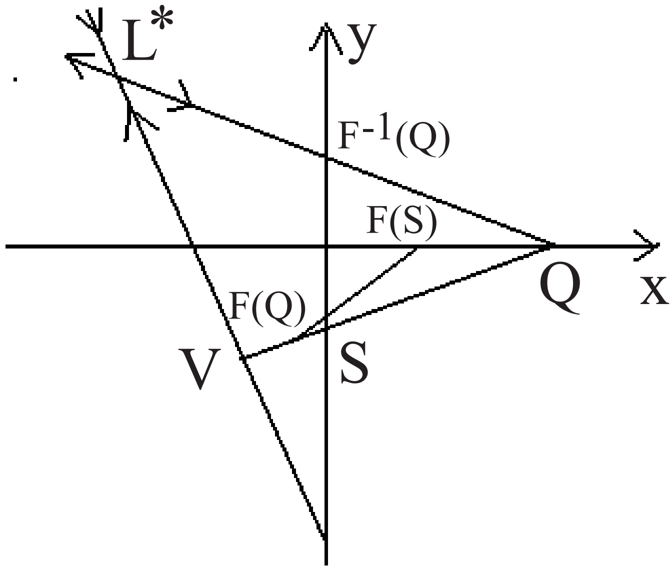

Figure 3: Geometry of the invariant region for robust chaos.

exist over open sets of parameter values as 'robust chaos'. They argue that if (17) holds and in addition

$$T_R > \frac{D_L(s_- - 1) + D_L(1 - T_L + D_L) + D_R s_-(s_+ - 1)}{D_L(s_+ - 1)} \tag{18}$$

then there is a convex invariant region. (Note that (18) is not exactly the expression in [4]: their expression contains an algebraic error, however the reasoning behind the argument is sound.)

It is interesting to consider how far Young's Theorem can be used to provide extra information about the attractor in this region. In contrast to the case of the Lozi map we find that a smaller region of parameter space can be shown to have a strange attractor with an invariant measure.

The geometry of this situation is shown in Fig. 3. The local unstable manifold of L^* has a fold on the $x-$axis at the point $Q = (x_Q, 0)$ where

$$x_Q = \frac{s_+ - 1}{T_L - 1 - D_L} \tag{19}$$

For the choice of parameters satisfying (17), $x_Q > 1$ and so (as $T_R < -1$), $F(Q)$ lies in $x < 0$ with $y < 0$. Let S be the intersection of the line $QF(Q)$

with the (negative) $y-$axis, and let V be the intersection of the extension of this line with the local stable manifold of L^*, so V lies to the left of the switching surface in $y < 0$.

Lemma 1. *If (17) and (18) hold then the closed triangle VL^*Q is an invariant region.*

Proof: Condition (18) ensures that $F(Q)$ lies to the right of V. Note that L^* is a saddle with positive eigenvalues if equalities (17) hold, and hence that the quadrants bounded by the stable and unstable manifolds of L^* are invariant under the left hand map. In particular both $F^2(Q)$ and $F(S)$ must lie in the infinite cone containing the angle $\angle VL^*Q$. Since the $x-$coordinate of $F(Q)$ is negative, the $y-$coordinate of $F^2(Q)$ is positive and so $F^2(Q)$ lies in VL^*Q. $F(S)$ is also in the invariant quadrant and has $y = 0$, hence it is also in VL^*Q. By definition, $F(V)$ lies closer to L^* on the line L^*V and so it too is in VL^*Q. These statements are enough to prove that the images of both $VL^*F^{-1}(Q)S$ under the left hand map, and $F^{-1}(Q)QS$ under the right hand map, lie in VL^*Q, and hence this is invariant.

□

Numerical experiments suggest that both $F^2(Q)$ and $F(S)$ lie in the region $QF(Q)L^*$ for our choice of parameters, and so this could equally well have been used as a smaller invariant region. This does not change the intersection of the set with the critical line, so no improvement to the results stated here is obtained using this smaller alternative.

Theorem 6. *Suppose (17) and (18) hold and let $\varepsilon = \max\{D_R, D_L\}$. Then the border collision normal form with $\mu > 1$ has an invariant region QVL^* as defined above. Furthermore, if there exists $N \geq 1$ such that*

$$(|T_k| - \varepsilon)^N > 2, \quad k = R, L \tag{20}$$

and both $F^{k+1}(S)$ and $F^k(Q)$ lie on the same side of the critical line $x = 0$ for $1 \leq k < N$, then F has an attractor with a Bowen-Ruelle measure.

Proof: Suppose that $D_L \geq D_R$. Let $\varepsilon = D_L$ and consider the change of variable in section 2.2 which leads to (7). Equation (20) implies that $|T_k| - \varepsilon > 1$ $k = R, L$ and so the second of inequalities (8) is satisfied. The third is satisfied as $\varepsilon < 1$ by (17), and the first of (8) is satisfied if $k = L$ by definition of ε and if $k = R$ since $D_L > D_R$. Hence the inequalities (8) hold

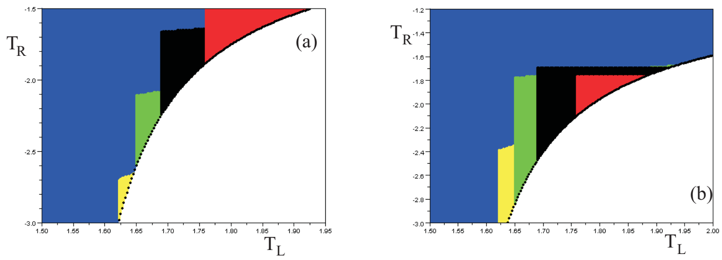

Figure 4: $(T_L, T_R)-$parameter space for robust: there is a convex invariant set for parameters in the shaded region, and the additional shading indicates those regions for which an invariant measure can be proved to exist with $N = 3, 4, 5, 6$ moving down the figures. In (a) $D_R = D_L = 0.5$, so the issue of choosing ε to be the larger of D_L and D_R creates no restrictions; in (b) $D_L = 0.5$ and $D_R = 0.2$ so we need to take $\varepsilon = D_L$.

and the remainder follows immediately from the statement of Theorem 2. The argument for the inequalities is entirely analogous if $D_R > D_L$.

□

Fig. 4 shows the numerically computed regions where the conditions of this Theorem apply for $N = 3, 4, 5, 6$ (the regions exist for decreasing T_R). Where more than one value of N can be chosen we shade it in keeping with the lower value of N. The dark boundary corresponds to equality in (18). Note that for larger $|T_R|$ our application of Young's Theorem does not even cover the whole of this boundary in the case $D_R = D_L = 0.5$ illustrated in Fig. 4(a). It would be interesting to know how far the results could be extended.

5. Invariant measure with two-dimensional support

Glendinning and Wong [9] describe a countable set of examples of border collision normal forms with attractors which are two dimensional convex polygons. The special feature of these examples which allows them to prove the existence of these regions (on which the dynamics is topologically transitive and periodic orbits are dense) is that there is a finite Markov partition. Thus these examples lead naturally to a semi-conjugacy with symbol sequences under the shift map, and the symbol spaces have invariant measures in the usual way [1, 7, 11, 18]. Lifting this measure back to the polygon via cylinder sets (whose diameter tends to zero) and noting that the semi-conjugacy is one-to-one except on a set of measure zero, this provides an invariant measure of the original map with support on the polygon. It is natural to take the measure of maximal entropy [7, 18].

6. Conclusion

Invariant measures provide a natural way of characterizing statistical properties of chaotic attractors. We have used a combination of rescaling and numerical verification of conditions to determine regions of parameter values for which the border collision normal form has a natural invariant measure. This has been used to re-examine attractors of the Lozi map and the example of robust chaos. In particular, we have shown that not all the region for which an invariant region can be shown to exist can be treated using the variant of Young's Theorem of section 2, but of course Young's result provides sufficient rather than necessary conditions for the existence of the invariant measure. It will be interesting to see whether the entire regions do indeed have natural invariant measures or whether other influences become important.

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