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1 Introduction

In the representation theory of reductive $p-$adic groups, the issue of reducibility of induced representations is an issue of great intricacy. It is the contention of Aubert-Baum-Plymen, expressed as a conjecture [1, 2, 3, 4], that there exists a simple geometric structure underlying this intricate theory.

Let G be a reductive p-adic group. Let $\mathfrak{s}$ be the point in the Bernstein spectrum of G which contains the cuspidal pair (M, σ). We will suppose that the irreducible cuspidal representation σ has unitary central character. Let $\Psi^t(M)$ denote the set of unramified *unitary* characters of M. Then $\Psi^t(M)$ has the structure of a compact torus. Attached to the point $\mathfrak{s}$ there is a compact torus $E^{\mathfrak{s}}$:

$$E^{\mathfrak{s}} := \{\psi \otimes \sigma : \psi \in \Psi^t(M)\}$$

The standard projection will be denoted

$$\pi^{\mathfrak{s}} : E^{\mathfrak{s}}//W^{\mathfrak{s}} \to E^{\mathfrak{s}}/W^{\mathfrak{s}}$$

and the q-projection will be denoted

$$\pi^{\mathfrak{s}}_{\sqrt{q}} : E^{\mathfrak{s}}//W^{\mathfrak{s}} \to E^{\mathfrak{s}}/W^{\mathfrak{s}}.$$

The q-projection is constructed from cocharacters, one for each connected component of $E^{\mathfrak{s}}//W^{\mathfrak{s}}$, see Eqn.(2) in §2. The extended quotient $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ is a compact Hausdorff space.

The reduced C^*-algebra of G is liminal, and its primitive ideal space is in canonical bijection with the tempered dual $\mathrm{Irr}^t(G)$ of G. Transporting the Jacobson topology on the primitive ideal space, we obtain a locally compact topology on the tempered dual, see [5, 3.1.1, 4.4.1, 18.3.2]. The set of tempered representations of G determined by $\mathfrak{s}$ will be denoted $\mathrm{Irr}^t(G)^{\mathfrak{s}}$, and the infinitesimal character will be denoted $inf.ch.$ We have

$$\mathrm{Irr}^t(G)^{\mathfrak{s}} \subset \mathrm{Irr}^t(G)$$

and, in the induced topology, $\mathrm{Irr}^t(G)^{\mathfrak{s}}$ is compact. It is not necessarily Hausdorff.

The evidence for the ABP conjecture begins to accumulate. The conjecture is true for $\mathrm{GL}_n(F)$, see [1]; the proof uses the local Langlands correspondence for $\mathrm{GL}(n)$. In [3], the authors prove that it is also true for the principal series of the exceptional group G_2. In [8], Jawdat and Plymen have proved the conjecture for the elliptic representations of the special linear group $\mathrm{SL}_n(F)$. We must also mention the comprehensive article by Solleveld [13], in which the emphasis is placed on affine Hecke algebras.

We note in passing that the R-groups for SL_N are abelian [6], and so representations induced from discrete series of Levi subgroups are multiplicity-free. This will apply in particular to the induced representations $\mathrm{Ind}(\tau \otimes 1)$ which we will consider.

In this paper, we will prove part (3) of the ABP conjecture for $\mathrm{SL}_4(\mathbb{Q}_p)$ when $\mathfrak{s} = [T, \sigma]_G$. The case $p = 2$ is especially interesting. In this case, there is a *tetrahedron of reducibility* in the tempered dual of SL_4 which does not occur when $p > 2$. The extended quotient performs a deconstruction: it creates the ordinary quotient and six unit intervals. The six intervals are then assembled into the six edges of a tetrahedron, and create a perfect model of reducibility.

Theorem 1.1. *Let* $G = \mathrm{SL}_4(\mathbb{Q}_p)$. *Let* $\mathfrak{s} = [T, \sigma]_G$. *There is a continuous bijection* $\mu^{\mathfrak{s}} : E^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \to \mathrm{Irr}^t(G)^{\mathfrak{s}}$ *for which the following diagram is commutative:*

$$\begin{array}{ccc} E^{\mathfrak{s}}/\!/W^{\mathfrak{s}} & \xrightarrow{\mu^{\mathfrak{s}}} & \mathrm{Irr}^t(G)^{\mathfrak{s}} \\ {\scriptstyle \pi^{\mathfrak{s}}_{\sqrt{q}}}\downarrow & & \downarrow{\scriptstyle inf.ch.} \\ E^{\mathfrak{s}}/W^{\mathfrak{s}} & = & E^{\mathfrak{s}}/W^{\mathfrak{s}} \end{array}$$

The number of points in the fibre of $\pi^{\mathfrak{s}}_{\sqrt{q}}$ *above* $\tau \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$ *equals the number of irreducible constituents in the induced representation* $\mathrm{Ind}(\tau \otimes 1)$. *This confirms, in a special case, part (3) of the conjecture in [2]. The cocharacters which enter the definition of the q-projection* $\pi^{\mathfrak{s}}_{\sqrt{q}}$ *depend only on two-sided cells* $\mathbf{c}$.

There is an abundance of L-packets in the tempered dual of SL_4. There are, for example, L-packets in the tempered dual of $\mathrm{SL}_4(\mathbb{Q}_2)$ which are parametrized by the 1-skeleton of a tetrahedron. The L-packets which occur in this article all conform to the *L-packet conjecture* in [4, §10].

Acknowledgements. We thank Paul Baum for helping us to resolve a crucial issue in §3, and Anne-Marie Aubert for her contributions to §4.

2 Geometric structure

We recall the definition of the extended quotient. Let X be a Hausdorff topological space. Let Γ be a finite group acting on X as homeomorphisms. Let

$$\widetilde{X} = \{(x, \gamma) \in X \times \Gamma : \gamma x = x\}$$

with group action on $\widetilde{X}$ given by

$$\alpha \cdot (x, \gamma) = (\alpha x, \alpha\gamma\alpha^{-1})$$

for $\alpha \in \Gamma$. Then the extended quotient is given by

$$X//\Gamma := \widetilde{X}/\Gamma = \bigsqcup_{\gamma\in\Gamma} X^{\gamma}/\Gamma \tag{1}$$

with one γ in each conjugacy class of Γ.

For each connected component $c \subset E^{\mathfrak{s}}//W^{\mathfrak{s}}$ we will construct a (bounded) cocharacter $h_c : \mathbb{T} \to E^{\mathfrak{s}}$ and then define

$$\pi_t(x) = h_c(t) \cdot \pi(x) \tag{2}$$

with $x \in c$, $t \in \mathbb{T}$. Evaluation at $\sqrt{q}$ creates the q-projection $\pi_{\sqrt{q}}$.

We fix the local field F to be $\mathbb{Q}_p$. We have 5 conjugacy classes of Levi subgroups of SL_4, one for each partition of 4. Let $P = MU$ be a standard parabolic subgroup of $G = \mathrm{SL}_4(F)$. Let $\tilde{M}$ be the corresponding Levi subgroup of $\tilde{G} = \mathrm{GL}_4(F)$ so that $M = \tilde{M} \cap \mathrm{SL}_4(F)$. We will use the framework, notation and results in [6]. Let $\sigma \in E_2(M)$ and $\pi_\sigma \in E_2(\tilde{M})$ with $\pi_\sigma \supset \sigma$. Let $W(M)$ be the Weyl group of M. Let

$$\begin{aligned} \tilde{L}(\pi_\sigma) &:= \{\eta \in \widehat{F^\times} | \pi_\sigma \otimes \eta \simeq w\pi_\sigma \text{ for some } w \in W\} \\ X(\pi_\sigma) &:= \{\eta \in \widehat{F^\times} | \pi_\sigma \otimes \eta \simeq pi_\sigma\} \end{aligned}$$

By [6, Theorem 2.4], the R-group of σ is given by

$$R(\sigma) \cong \tilde{L}(\pi_\sigma)/X(\pi_\sigma).$$

From now on, we will restrict ourselves to the case $M = T$ the standard maximal torus. For the Bernstein component $\mathfrak{s} = [T, \sigma]_G$, we let $\pi_\sigma|_T = \sigma$ where π_σ is a unitary character of $\tilde{M}$. Then we write

$$\pi_\sigma = \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$$

where π_i is not an unramified twist of π_j with $i \neq j$. In this section, we will discuss the extended quotient $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ with respect to each Bernstein component $\mathfrak{s} = [T, \sigma]_G$ and prove the geometric conjecture for the principal series of $\mathrm{SL}_4(F)$. Hence, we will construct the explicit bijection between $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ and $\mathrm{Irr}^{\mathfrak{t}}(G)^{\mathfrak{s}}$. From now on, we denote $(\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \cdots \times \mathrm{GL}_{n_r}) \cap \mathrm{SL}_n$ by $n_1 + n_2 + \cdots + n_r$ where $\Sigma n_i = n$. For example, $1+1+1+1$ means $(\mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1) \cap \mathrm{SL}_4$. Recalling the definition of $E^{\mathfrak{s}}$, we see that $E^{\mathfrak{s}}$ may be identified with $\mathbb{T}^4/\mathbb{T}$, the maximal compact subgroup of the dual torus $T^\vee$ in the Langlands dual $G^\vee = \mathrm{PGL}_4(\mathbb{C})$.

2.1 Case 1: $\pi_\sigma = \pi \otimes \pi \otimes \pi \otimes \pi$

The group $W^{\mathfrak{s}}$ is the symmetric group $\mathfrak{S}_4$. This group has five conjugacy classes, one for each cycle type. The following is the structure of each component in the extended quotient.

- $\gamma = (abcd) = 1$,

  $E^\gamma/Z(\gamma) = E^{\mathfrak{s}}/W^{\mathfrak{s}}$

- $\gamma = (acbd)$,

  $E^\gamma = \{(a,b,b,d) : a,c,d \in \mathbb{T}\}/\mathbb{T} \cong \{(z_1,z_2,z_2,1) : z_1,z_2 \in \mathbb{T}\} \cong \mathbb{T}^2$

  $$E^\gamma/Z(\gamma) \cong \mathbb{T}^2$$

- $\gamma = (bcad)$,

  $E^\gamma = \{(a,a,a,b) : a,b \in \mathbb{T}\}/\mathbb{T} \cong \{(z,z,z,1) : z \in \mathbb{T}\} \cong \mathbb{T}$

  $$E^\gamma/Z(\gamma) \cong \mathbb{T}$$

- $\gamma = (badc)$,

  $E^\gamma = \{(a,a,b,b), (a,-a,b,-b) : a,b \in \mathbb{T}\}/\mathbb{T} \cong \{(1,1,z,z), (1,-1,z,-z) : z \in \mathbb{T}\} \cong \mathbb{T} \sqcup \mathbb{T}$

  $$E^\gamma/Z(\gamma) \cong \mathbb{T} \sqcup \mathbb{T}$$

- $\gamma = (bcda)$,

  $E^\gamma = \{(1,1,1,1), (1,-1,1,-1), (1,i,-1,-i), (1,-i,-1,i)\} \cong pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4$

  $$E^\gamma/Z(\gamma) \cong pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4$$

Hence, we have

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T}^2 \sqcup \mathbb{T} \sqcup \mathbb{T} \sqcup \mathbb{T} \sqcup pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4 \tag{3}$$

Now we identify each element in the compact subspace $\mathrm{Irr}^{\mathrm{t}}(G)^{\mathfrak{s}}$. The induced representations

$$\{\mathrm{Ind}^4_{(2+1+1)}(z_1^{\mathrm{val}}\mathrm{St}_2 \otimes z_2^{\mathrm{val}} \otimes 1) : z_1, z_2 \in \mathbb{T}\}$$

are irreducible, tempered, their infinitesimal characters lie in $\mathfrak{s}$, and they are parametrized by $\mathbb{T}^2$.

The induced representations

$$\{\mathrm{Ind}^4_{(3+1)}(z \cdot \mathrm{St}_3 \otimes 1) : z \in \mathbb{T}\}$$

are irreducible, tempered, have central characters in $\mathfrak{s}$ and are parametrized by $\mathbb{T}$.

The induced representations

$$\{\mathrm{Ind}^4_{2+2}(z \cdot \mathrm{St}_2 \otimes \mathrm{St}_2) : z \in \mathbb{T}\}$$

have central characters in $\mathfrak{s}$, and are parameterized by $\mathbb{T}$. They are irreducible except when $z = -1$. The R-group is as follows:

$$R((-1)^{\mathrm{val}} \cdot \mathrm{St}_2 \otimes \mathrm{St}_2) =< (-1)^{\mathrm{val}} > .$$

There are two irreducible components, denoted by ρ^+ and ρ^-. We will locate ρ^- in the second copy of $\mathbb{T}$ and identify ρ^+ by pt_1.

The Steinberg representation $\mathrm{St}(\mathrm{SL}_4)$ has central character in $\mathfrak{s}$. We identify this representation by pt_2.

The unramified unitary principal series of SL_4 contains points of reducibility. In fact, there is a *circle of reducibility*, as we now proceed to explain. Let $t = (z, -z, 1, -1)$ except $z = i$ and let χ_t be the corresponding unramified unitary character. Then the representation $\chi_t \otimes \pi_\sigma$ is given by

$$z^{\mathrm{val}}\pi \otimes (-z)^{\mathrm{val}}\pi \otimes \pi \otimes (-1)^{\mathrm{val}}\pi.$$

Then $(-1)^{\mathrm{val}}\pi$ is an element in $\bar{L}(\chi_t \otimes \pi_\sigma)$ and $X(\chi_t \otimes \pi_\sigma) = 1$. Hence

$$R(\chi_t \otimes \sigma) \cong \mathbb{Z}/2\mathbb{Z}$$

and the induced representation

$$\lambda(t) := \mathrm{Ind}^G_T(\chi_t \otimes \pi_\sigma)$$

is reducible and admits two irreducible subrepresentations:

$$\lambda(t) = \lambda(t)^+ \oplus \lambda(t)^-.$$

We assign $\lambda(t)^+$ to $[z, -z, 1, -1] \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$ and $\lambda(t)^-$ to $z \in \mathbb{T}$.

Now, we turn to the point $t = (i, -i, 1-1)$. Then

$$\bar{L}(\chi_t \otimes \pi_\sigma) =< i^{\mathrm{val}} >$$

and $X(\pi_\sigma) = 1$. Hence, we know $R(\chi_t \otimes \sigma) = \mathbb{Z}/4\mathbb{Z}$. The induced representation $\tau = \mathrm{Ind}_T^G(\chi_t \otimes \sigma)$ is reducible with 4 irreducible constituents τ_1, τ_2, τ_3, τ_4. We locate τ_1 to $[i, -i, 1, -1] \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$ and τ_2 to the point i in the third copy of $\mathbb{T}$ and identify τ_3 and τ_4 by pt_3 and pt_4 respectively.

For $t = (z_1, z_2, z_3, 1) \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$ except $t = (z, -z, 1, -1)$, the induced representation $\mathrm{Ind}_M^G(\chi_t \otimes \sigma)$ is irreducible.

We build a map

$$\mu : E^{\mathfrak{s}}//W^{\mathfrak{s}} \longrightarrow \mathrm{Irr}^{\mathrm{t}}(G)^{\mathfrak{s}}$$

and here are the details:

Point in $E^{\mathfrak{s}}/W^{\mathfrak{s}}$	Irreducible representation	Cocharacter $h(t)$
$(z_1, z_2) \in \mathbb{T}^2$	$\mathrm{Ind}^4_{(2+1+1)}(z_1 \cdot \mathrm{St}_2 \otimes z_2^{\mathrm{valodet}} \pi \otimes \pi)$	$(t, t^{-1}, 1, 1)$
$z \in \mathbb{T}$	$\mathrm{Ind}^4_{(3+1)}(z \cdot \mathrm{St}_3 \otimes \pi)$	$(t^2, 1, t^{-2}, 1)$
$z \in \mathbb{T}$	$\mathrm{Ind}^4_{(2+2)}(z \cdot \mathrm{St}_2 \otimes \mathrm{St}_2)$	(t, t^{-1}, t, t^{-1})
$z \in \mathbb{T}$	$\lambda(t)^+$	1
pt_1	ρ^+	(t, t^{-1}, t, t^{-1})
pt_2	$\mathrm{St}(\mathrm{SL}_4)$	(t^3, t, t^{-1}, t^{-3})
pt_3	τ_3	1
pt_4	τ_4	1
$t \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$	$\mathrm{Ind}_T^G(\chi_t \otimes \sigma)$	1

It is clear that $\pi_{\sqrt{q}} = inf.ch \circ \mu$. We note that the compact space $\mathrm{Irr}^t(G)^{\mathfrak{s}}$ is non-Hausdorff. One connected component contains a double-point, and another connected component contains a double-circle (and a quadruple point), see [9].

Hence, we have

Lemma 2.1. *Part (3) of the geometric conjecture is true for*

$$\mathfrak{s} = [T, \pi \otimes \pi \otimes \pi \otimes \pi]_G.$$

2.2 Case 2: $\pi_\sigma = \pi_1 \otimes \pi_1 \otimes \pi_1 \otimes \pi_2$

For this case, the isotropy group is the symmetric group $\mathfrak{S}_3$. There are three conjugacy classes. They are $\{\gamma_1\}$, $\{\gamma_2, \gamma_3, \gamma_6\}$, $\{\gamma_4, \gamma_5\}$. Now we choose γ_1, γ_2 and γ_4 as the representatives for their own conjugacy classes. Now we analysis case by case.

- $\gamma = (abcd) = 1$, $E^\gamma/Z(\gamma) = E^\mathfrak{s}/W^\mathfrak{s}$
- $\gamma = (bacd)$, $E^\gamma = \{(a,a,c,d) : a,c,d \in \mathbb{T}\}/\mathbb{T} \cong \{(z_1,z_1,z_2,1) : z_1, z_2 \in \mathbb{T}\} \cong \mathbb{T}^2$
- $\gamma = (bcad)$, $E^\gamma = \{(a,a,a,c) : a,c \in \mathbb{T}\}/\mathbb{T} \cong \{(z,z,z,1) : z \in \mathbb{T}\} \cong \mathbb{T}$

Hence, we have

$$E^\mathfrak{s}//W^\mathfrak{s} = E^\mathfrak{s}/W^\mathfrak{s} \sqcup \mathbb{T}^2 \sqcup \mathbb{T} \tag{4}$$

The representations

$$\{\mathrm{Ind}^4_{(2+1+1)}(z_1 \cdot St_2(\pi_1) \otimes z_2^{val \circ det}\pi_1 \otimes \pi_2) : z_1, z_2 \in \mathbb{T}\}$$

are irreducible, tempered, and have infinitesimal character in $\mathfrak{s}$. These representations are parametrized by $\mathbb{T}^2$ and we set the cocharacter by

$$h_c(t) = (t, t^{-1}, 1, 1).$$

The representations

$$\{\mathrm{Ind}^4_{(3+1)}(z \cdot St_3(\pi_1) \otimes \pi_2) : z \in \mathbb{T}\}$$

are irreducible, tempered and have infinitesimal character in $\mathfrak{s}$. These representations are parametrized by $\mathbb{T}$. For this component, we set the cocharacter:

$$h_c(t) = (t^2, 1, t^{-2}, 1).$$

Now we consider the point $t = (z_1, z_2, z_3, 1) \in E^\mathfrak{s}/W^\mathfrak{s}$ and χ_t is the unramified unitary character determined by t. The induced representation $\mathrm{Ind}_M^G(\chi_t \otimes \sigma)$ is irreducible. Each unramified unitary character determines a tempered representation of G with respect to $\mathfrak{s}$.

We turn to build up the map μ satisfying the geometric conjecture, i.e.

$$\mu : E^\mathfrak{s}/W^\mathfrak{s} \sqcup \mathbb{T}^2 \sqcup \mathbb{T} \longrightarrow \mathrm{Irr}^\mathrm{t}(G)^\mathfrak{s}$$

The details of this map are as follows:

Point in $E^{\mathfrak{s}}/W^{\mathfrak{s}}$	Irreducible representation	Cocharacter $h(t)$
$(z_1,z_2)\in\mathbb{T}^2$	$\mathrm{Ind}^4_{(2+1+1)}(z_1\cdot \mathrm{St}_2(\pi_1)\otimes z_2^{\mathrm{valodet}}\pi_1\otimes\pi_2)$	$(t,t^{-1},1,1)$
$z\in\mathbb{T}$	$\mathrm{Ind}^4_{(3+1)}(z\cdot \mathrm{St}_3(\pi_1)\otimes\pi_2)$	$(t^2,1,t^{-2},1)$
$t\in E^{\mathfrak{s}}/W^{\mathfrak{s}}$	$\mathrm{Ind}^G_T(\chi_t\otimes\sigma)$	1

It is clear that $\pi_{\sqrt{q}} = inf.ch\circ\mu$. Hence, we have

Lemma 2.2. *Part (3) of the geometric conjecture is true for*

$$\mathfrak{s}=[M,\pi_1\otimes\pi_1\otimes\pi_1\otimes\pi_2]_G.$$

2.3 Case 3: $\pi_\sigma=\pi_1\otimes\pi_1\otimes\pi_2\otimes\pi_2$

There are two distinct cases to be considered. In case 3.1, the corresponding π_σ is $\pi_\sigma=\pi_1\otimes\pi_1\otimes\pi_2\otimes\pi_2$. The R-group $R(\sigma)$ is trivial and the isotropy subgroup $W^{\mathfrak{s}}$ is given by $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Now we investigate the extended quotient $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ respect to $\mathfrak{s}$. Since $W^{\mathfrak{s}}$ is abelian in this case, then we know that each element contribute one conjugacy class and the centralizer $Z(\gamma_i)$ of γ_i is W itself. Then, we have

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\gamma_1}/W^{\mathfrak{s}}\sqcup E^{\gamma_2}/W^{\mathfrak{s}}\sqcup E^{\gamma_3}/W^{\mathfrak{s}}\sqcup E^{\gamma_4}/W^{\mathfrak{s}}$$

We will analysis case by case. Now we compute each component in extended quotient $E^{\mathfrak{s}}//W^{\mathfrak{s}}$.

- $\gamma=(abcd)=1$, $E^\gamma/Z(\gamma)=E^{\mathfrak{s}}/W^{\mathfrak{s}}$.
- $\gamma=(bacd)$, $E^\gamma=\{(a,a,c,d): a,c,d\in\mathbb{T}\}/\mathbb{T}\cong\{(1,1,z_1,z_2): z_1,z_2\in\mathbb{T}\}\cong\mathbb{T}^2$. Then we have
$$E^\gamma/Z(\gamma)\cong\mathbb{T}^2.$$
- $\gamma=(abdc)$, $E^\gamma=\{(a,b,c,c): a,b,c\in\mathbb{T}\}/\mathbb{T}\cong\{(z_1,z_2,1,1): z_1,z_2\in\mathbb{T}\}\cong\mathbb{T}^2$.
$$E^\gamma/Z(\gamma)\cong\mathbb{T}^2.$$
- $\gamma=(badc)$, $E^\gamma=\{(a,a,c,c),(a,-a,c,-c): a,c\in\mathbb{T}\}/\mathbb{T}\cong\{(1,1,z,z): z\in\mathbb{T}\}\sqcup\{(1,-1,z,-z): z\in\mathbb{T}\}\cong\mathbb{T}\sqcup\mathbb{T}$.
$$E^\gamma/Z(\gamma)\cong\mathbb{T}\sqcup\mathbb{T}.$$

This leads to the equation

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T}^2 \sqcup \mathbb{T}^2 \sqcup \mathbb{T} \sqcup \mathbb{T} \tag{5}$$

and we now construct the bijection μ between $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ and the set $\mathrm{Irr}^{\mathrm{t}}(G)^{\mathfrak{s}}$.

The representations

$$\{\mathrm{Ind}^4_{(2+1+1)}(z_1 \cdot St_2(\pi_1) \otimes z_2^{\mathrm{val} \circ det}\pi_2 \otimes \pi_2) : z_1, z_2 \in \mathbb{T}\}$$

are tempered, irreducible and have inf.character in $\mathfrak{s}$. Parameter space is $\mathbb{T}^2$.

The representations

$$\{\mathrm{Ind}^4_{(1+1+2)}(\pi_1 \otimes z_1^{\mathrm{val} \circ det}\pi_1 \otimes z_2 \cdot St_2(\pi_2)) : z_1, z_2 \in \mathbb{T}\}$$

are irreducible, tempered, have inf.ch. in $\mathfrak{s}$, parameter space $\mathbb{T}^2$.

The representations

$$\mathrm{Ind}^4_{(2+2)}(z \cdot St_2(\pi_1) \otimes St_2(\pi_2))$$

are irreducible, tempered, have inf.ch. in $\mathfrak{s}$, parameter space $\mathbb{T}$.

Let $t = (1, -1, z, -z)$ and χ_t be the corresponding character. We have $\chi_t \otimes \pi_\sigma = \pi_1 \otimes (-1)^{val}\pi_1 \otimes z^{\mathrm{val}}\pi_2 \otimes (-z)^{\mathrm{val}}\pi_2$. It is not hard to get $\bar{L}(\chi_t \otimes \pi_\sigma) =< (-1)^{\mathrm{val}} >$ and $X(\chi_t \otimes \pi_\sigma) = 1$. Thus, $R(\chi_t \otimes \sigma) = \mathbb{Z}/2\mathbb{Z}$. It leads the induced representation $\mathrm{Ind}_M^G(\chi_t \otimes \sigma)$ is reducible and there are two irreducible subrepresentations $\pi(t)^+$, $\pi(t)^-$ of G, i.e.

$$\mathrm{Ind}_M^G(\chi_t \otimes \sigma) = \pi(t)^+ \oplus \pi(t)^-$$

Hence, $\pi(t)^+$ and $\pi(t)^-$ are tempered. They will contribute the elements in $\mathrm{Irr}^{\mathrm{t}}(G)^{\mathfrak{s}}$. Such induced representations will be identified by $\mathbb{T}$ and we set the character

$$h_c(t) = 1.$$

Indeed, each unitary unramified twist of σ except the type $t = (1, -1, z, -z)$ will be an irreducible tempered representation. In other word, every point t in $E^{\mathfrak{s}}/W^{\mathfrak{s}}$, which does not belong to $(1, -1, z, -z)$, will generate an unitary unramified character of $\mathbb{T}$, i.e. $\chi_t = (z_1^{val}, z_2^{val}, z_3^{val}, z_4^{val})$ and the representation $\mathrm{Ind}_M^G(\chi_t \otimes \sigma)$ induced by $\chi_t \otimes \sigma$ is irreducible. Thus such induced representation contributes an element in $\mathrm{Irr}^{\mathrm{t}}(G)^{\mathfrak{s}}$. The detail of this map is as follow:

Point in $E^{\mathfrak{s}}/W^{\mathfrak{s}}$	Irreducible representation	Cocharacter $h(t)$
$(z_1, z_2) \in \mathbb{T}^2$	$\mathrm{Ind}^4_{(2+1+1)}(z_1 \cdot \mathrm{St}_2(\pi_1) \otimes z_2 \cdot \pi_2 \otimes \pi_2)$	$(t, t^{-1}, 1, 1)$
$(z_1, z_2) \in \mathbb{T}^2$	$\mathrm{Ind}^4_{(1+1+2)}(\pi_1 \otimes z_1 \cdot \pi_1 \otimes z_2 \cdot \mathrm{St}_2(\pi_2)$	$(1, 1, t, t^{-1})$
$z \in \mathbb{T}$	$\mathrm{Ind}^4_{(2+2)}(z \cdot \mathrm{St}_2(\pi_1) \otimes \mathrm{St}_2(\pi_2))$	(t, t^{-1}, t, t^{-1})
$z \in \mathbb{T}$	$\pi(t)^+$	1
$t \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$	$\mathrm{Ind}_T^G(\chi_t \otimes \sigma)$	1

Therefore, the condition $\pi_{\sqrt{q}} = inf.ch \circ \mu$ is satisfied. We conclude

Lemma 2.3. *Part (3) of the geometric conjecture is true for*

$$\mathfrak{s} = [T, \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_2]_G.$$

The next we will discuss the case 3.2. In this case, the representation π_σ is of the form $\pi \otimes \pi \otimes \eta\pi \otimes \eta\pi$. The R-group is $\mathbb{Z}/2\mathbb{Z}$ and the isotropy group is $\mathbb{Z}/\mathbb{Z}2 \wr \mathbb{Z}/2\mathbb{Z}$ generated by $< \gamma_2, \gamma_3, \gamma_4 >$ where $\gamma_2, \gamma_3, \gamma_4$ are explicitly given by the table below:

$W^{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$			
γ	$\gamma(abcd)$	γ	$\gamma(abcd)$
γ_1	$abcd$	γ_5	$badc$
γ_2	$bacd$	γ_6	$dcab$
γ_3	$abdc$	γ_7	$cdba$
γ_4	$cdab$	γ_8	$dcba$

There are five conjugacy classes:

$$\{\gamma_1\}, \{\gamma_2, \gamma_3\}, \{\gamma_4, \gamma_8\}, \{\gamma_5\}, \{\gamma_6, \gamma_7\}$$

From above, there are five conjugacy classes. The extended quotient $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ is given by

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = \bigsqcup_{\gamma} E^{\gamma}/Z(\gamma)$$

and we choose γ_1, γ_2, γ_4, γ_5 and γ_6 to be the representatives in their own conjugacy classes and their centralizers are as follows:

$$Z(\gamma_1) = W^{\mathfrak{s}}, Z(\gamma_2) = \{\gamma_1, \gamma_2, \gamma_3\}, Z(\gamma_4) = \{\gamma_1, \gamma_4, \gamma_5, \gamma_8\} \tag{6}$$

$$Z(\gamma_5) = W^{\mathfrak{s}}, Z(\gamma_6) = \{\gamma_1, \gamma_6, \gamma_7\} \tag{7}$$

Now we analysis case by case.

- $\gamma = \gamma_1$, $E^{\gamma}/W^{\mathfrak{s}} = E/W^{\mathfrak{s}}$.
- $\gamma = \gamma_2$, $E^{\gamma} = \{(a, a, c, d) : a, c, d \in \mathbb{T}\}/\mathbb{T} \cong \{(z_1, z_1, z_2, 1) : z_1, z_2 \in \mathbb{T}\} \cong \mathbb{T}^2$.
- $\gamma = \gamma_4$, $E^{\gamma} = \{(a, b, a, b), (a, b, -a, -b) : a, b \in \mathbb{T}\}/\mathbb{T} \cong \{(z, 1, z, 1), (z, 1, -z, -1) : z \in \mathbb{T}\} \cong \mathbb{T} \sqcup \mathbb{T}$.
- $\gamma = \gamma_5$, $E^{\gamma} = \{(a, a, b, b), (a, -a, b, -b) : a, b \in \mathbb{T}\}/\mathbb{T} \cong \{(1, 1, z, z), (1, -1, z, -z) : z \in \mathbb{T}\} \cong \mathbb{T} \sqcup \mathbb{T}$.

- $\gamma = \gamma_6,\ E^\gamma = \{(1,1,1,1),(1,-1,1,-1),(1,i,-1,-i),(1,-i,-1,i)\} \cong pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4$.

This leads to the equation

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T}^2 \sqcup \mathbb{T} \sqcup \mathbb{T} \sqcup \mathbb{T} \sqcup \mathbb{T} \sqcup pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4 \tag{8}$$

The representations

$$\{\mathrm{Ind}^4_{(2+1+1)}(z_1 \cdot St_2(\pi) \otimes z_2^{\mathrm{val}}\eta\pi \otimes \eta\pi) : z_1, z_2 \in \mathbb{T}\}$$

are irreducible, tempered, have inf.ch. in $\mathfrak{s}$, parameter space $\mathbb{T}^2$.

The representations

$$\mathrm{Ind}^4_{(2+2)}(z \cdot St_2(\pi) \otimes St_2(\eta\pi))$$

are irreducible, tempered, have inf.ch. in $\mathfrak{s}$, parameter space $\mathbb{T}$.

In the following, we consider the point $t = (z, 1, -z, -1)$. Thus, $\chi_t \otimes \pi_\sigma = z^{\mathrm{val}}\pi \otimes \pi \otimes (-z)^{\mathrm{val}}\eta\pi \otimes (-1)^{\mathrm{val}}\eta\pi$. We have $\bar{L}(\chi_t \otimes \pi_\sigma) =< (-1)^{\mathrm{val}}\eta >$ and $X(\pi_\sigma) = 1$. Thus $R(\chi_t \otimes \sigma) = \mathbb{Z}/2\mathbb{Z}$. It leads the induced representation $\rho(z)$ is reducible and there are two irreducible constituents $\rho(z)^+$, $\rho(z)^-$, i.e.

$$\rho(z) = \mathrm{Ind}^G_T(z^{val\circ det}\pi \otimes \pi \otimes (-z)^{\mathrm{val}}\eta\pi \otimes (-1)^{\mathrm{val}}\eta\pi = \rho(z)^+ \oplus \rho(z)^-$$

We identify these representations by $\mathbb{T}$.

Then we consider the point $t = (z, 1, z, 1)$. We get $\chi_t \otimes \pi_\sigma = z^{\mathrm{val}}\pi \otimes \pi \otimes z^{\mathrm{val}}\eta\pi \otimes \eta\pi$. We have $\bar{L}(\chi_t \otimes \pi_\sigma) =< \eta >$ and $X(\pi_\sigma) = 1$. Thus $R(\chi_t \otimes \sigma) = \mathbb{Z}/2\mathbb{Z}$. It leads the induced representation $\varrho(z)$ is reducible. There are two irreducible constituents $\varrho(z)^+$, $\varrho(z)^-$, i.e.

$$\varrho(z) = \mathrm{Ind}^G_T(z^{\mathrm{val}}\pi \otimes \pi \otimes z^{\mathrm{val}}\eta\pi \otimes \eta\pi = \varrho(z)^+ \oplus \varrho(z)^-$$

We identify these representations by $\mathbb{T}$.

Finally, we consider the point $t = (1, -1, z, -z)$ except $z = 1, i$. Thus, $\chi_t \otimes \pi_\sigma = \pi \otimes (-1)^{val}\pi \otimes (z)^{val}\eta\pi \otimes (-z)^{val}\eta\pi$. We have $\bar{L}(\chi_t \otimes \pi_\sigma) =< (-1)^{val}\eta >$ and $X(\chi_t \otimes \pi_\sigma) = 1$. Thus $R(\chi_t \otimes \sigma) = \mathbb{Z}/2\mathbb{Z}$. It leads the induced representation $\theta(z)$ is reducible. There are two irreducible constituents $\theta(z)^+$, $\theta(z)^-$, i.e.

$$\theta(z) = \mathrm{Ind}^G_M(\pi \otimes (-1)^{val}\pi \otimes (z)^{val}\eta\pi \otimes (-z)^{val}\eta\pi = \theta(z)^+ \oplus \theta(z)^-$$

We identify these representations by $\mathbb{T}$.

Now we still consider the point $t = (1, -1, z, -z)$ and fix $z = 1$. We have $t = (1, -1, 1, -1)$. Then $\bar{L}(\chi_t \otimes \pi_\sigma) =< (-1)^{val\circ det}, \eta >$ and $X(\chi_t \otimes \pi_\sigma) = 1$.

Hence, we know $R(\chi_t \otimes \sigma) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then the induced representation $\xi = \mathrm{Ind}_M^G(\chi_t \otimes \sigma)$ is reducible and there are 4 irreducible constituents ξ_1, ξ_2, ξ_3 and ξ_4. We locate ξ_1 to $E^{\mathfrak{s}}/W^{\mathfrak{s}}$ and ξ_2 to component c_5 and identify ξ_3 and ξ_4 by pt_1 and pt_2 respectively.

In the next, we fix $z = i$. We have $t = (1, -1, i, -i)$. Then $\bar{L}(\chi_t \otimes \pi_\sigma) =< i^{valodet}\eta >$ and $X(\chi_t \otimes \pi_\sigma) = 1$. Hence, we know $R(\chi_t \otimes \sigma) = \mathbb{Z}/4\mathbb{Z}$. The order of $R(\chi_t \otimes \sigma)$ is 4. Then the induced representation $\tau = \mathrm{Ind}_M^G(\chi_t \otimes \sigma)$ is reducible and there are 4 irreducible constituents τ_1, τ_2, τ_3 and τ_4. We locate τ_1 to $E^{\mathfrak{s}}/W^{\mathfrak{s}}$ and τ_2 to component c_5 and identify τ_3 and τ_4 by pt_3 and pt_4 respectively.

Point in $E^{\mathfrak{s}}/W^{\mathfrak{s}}$	Irreducible representation	Cocharacter $h(t)$
$(z_1, z_2) \in \mathbb{T}^2$	$\mathrm{Ind}^4_{(2+1+1)}(z_1 \cdot \mathrm{St}_2(\pi) \otimes z_2^{\mathrm{valodet}}\eta\pi \otimes \eta\pi)$	$(t, t^{-1}, 1, 1)$
$z \in \mathbb{T}$	$\mathrm{Ind}^4_{(2+2)}(z \cdot \mathrm{St}_2(\pi) \otimes \mathrm{St}_2(\eta\pi)$	(t, t^{-1}, t, t^{-1})
$z \in \mathbb{T}$	$\rho(z)^+$	1
$z \in \mathbb{T}$	$\varrho(t)^+$	1
$z \in \mathbb{T}$	$\theta(z)^+$	1
$t \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$	$\mathrm{Ind}_T^G(\chi_t \otimes \sigma)$	1
pt_1	ξ_3	1
pt_2	ξ_4	1
pt_3	τ_3	1
pt_4	τ_4	1

Indeed, the condition

$$\pi_{\sqrt{q}} = inf.ch \circ \mu$$

is satisfied. Hence, we have

Lemma 2.4. *Part (3) of the geometric conjecture is true for*

$$\mathfrak{s} = [T, \pi \otimes \pi \otimes \eta\pi \otimes \eta\pi]_G.$$

2.4 Case 4: $\pi_\sigma = \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_3$

In this section, we will consider the case $\pi_\sigma \cong \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_3$. Hence, we know the isotropy subgroup $W^{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z}$. Analysis case by case:

- $\gamma = (abcd) = 1$, $E^\gamma/Z(\gamma) = E^{\mathfrak{s}}/W^{\mathfrak{s}}$.
- $\gamma = (bacd)$, $E^\gamma = \{(a, a, c, d) : a, c, d \in \mathbb{T}\}/\mathbb{T} \cong \{(z_1, z_1, z_2, 1) : z_1, z_2 \in \mathbb{T}\} \cong \mathbb{T}^2$.

$$E^\gamma/Z(\gamma) \cong \mathbb{T}^2$$

Then, we have

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T}^2 \tag{9}$$

The representations

$$\{\mathrm{Ind}_{(2+1+1)}^{(4)}(z_1 \cdot St_2(\pi_1) \otimes z_2^{\mathrm{val}}\pi_2 \otimes \pi_3) : z_1, z_2 \in \mathbb{T}\}$$

are irreducible, tempered, have $inf.ch.$ in $\mathfrak{s}$, parameter space $\mathbb{T}^2$.

We construct the bijection:

$$\mu : E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T}^2 \longrightarrow \mathrm{Irr}^{\mathrm{t}}(G)^{\mathfrak{s}}$$

in accordance with the following table:

Point in $E^{\mathfrak{s}}/W^{\mathfrak{s}}$	Irreducible representation	Cocharacter $h(t)$
$(z_1, z_2) \in \mathbb{T}^2$	$\mathrm{Ind}_{(2+1+1)}^4(z_1 \cdot \mathrm{St}_2(\pi_1) \otimes z_2 \cdot \pi_2 \otimes \pi_3)$	$(t, t^{-1}, 1, 1)$
$t \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$	$\mathrm{Ind}_T^G(\chi_t \otimes \sigma)$	1

Indeed, the condition

$$\pi_{\sqrt{q}} = inf.ch \circ \mu$$

is satisfied. Hence, we have

Lemma 2.5. *Part (3) of the geometric conjecture is true for*

$$\mathfrak{s} = [T, \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_3]_G.$$

2.5 Case 5: $\pi_\sigma = \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$

In this section, we will discuss the case when $\pi_\sigma \cong \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$. In fact, from the table of R-group above, we have known that there are four types in this case. First of all, we focus on the case 5.1. Indeed, π_σ is given by the form $\pi \otimes \eta\pi \otimes \eta^2\pi \otimes \eta^3\pi$ where η is ramified. We have proved that the R-group with resect to this is the cyclic group $\mathbb{Z}/4\mathbb{Z}$. Furthermore, the isotropy group $W^{\mathfrak{s}}$ is given by $\mathbb{Z}/4\mathbb{Z}$. In the following,we figure out the extended quotient with respect to $W^{\mathfrak{s}}$. The cyclic group is abelian, each element comprises a single conjugacy classes and the centralizer of each element is the cyclic group itself. Then we can immediately get the extended quotient

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\gamma_1}/W^{\mathfrak{s}} \sqcup E^{\gamma_2}/W^{\mathfrak{s}} \sqcup E^{\gamma_3}/W^{\mathfrak{s}} \sqcup E^{\gamma_4}/W^{\mathfrak{s}}$$

We analyze case by case.

- $\gamma = (abcd) = 1$, $E^\gamma/W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}}$.

- $\gamma = (bcda)$, $E^\gamma = \{(1,1,1,1),(1,-1,1,-1),(1,i,-1,-i),(1,i,-1,-i)\} \cong pt \sqcup pt \sqcup pt \sqcup pt$. Hence, we have

$$E^\gamma/W^\gamma \cong pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4$$

- $\gamma = (cdab)$, $E^\gamma = \{(a,b,-a,b),(a,b,a,b) : a,b \in \mathbb{T}\}/\mathbb{T} \cong \{(1,z,-1,-z),(1,z,1,z) : z \in \mathbb{T}\} \cong \mathbb{T} \sqcup \mathbb{T}$.

$$E^\gamma/W^\gamma \cong \mathbb{T} \sqcup \mathbb{T}$$

- $\gamma = (dabc)$, $E^\gamma = \{(1,1,1,1),(1,-1,1,-1),(1,i,-1,-i),(1,i,-1,-i)\} \cong pt \sqcup pt \sqcup pt \sqcup pt$.

$$E^\gamma/W^\gamma \cong pt_5 \sqcup pt_6 \sqcup pt_7 \sqcup pt_8$$

Then we have the decomposition

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T} \sqcup \mathbb{T} \sqcup \mathrm{pt}_1 \sqcup \mathrm{pt}_2 \sqcup \mathrm{pt}_3 \sqcup \mathrm{pt}_4 \sqcup \mathrm{pt}_5 \sqcup \mathrm{pt}_6 \sqcup \mathrm{pt}_7 \sqcup \mathrm{pt}_8 \tag{10}$$

It coincides the result in [8]. By [8, Theorem 5.3], we have

Lemma 2.6. *Part (3) of the conjecture is true for* $\mathfrak{s} = [M, \pi \otimes \eta\pi \otimes \eta^2\pi \otimes \eta^3\pi]$.

Then, we will concentrate on the case 5.2. Recall that the representation π_σ is given by $\pi \otimes \chi\pi \otimes \eta\pi \otimes \chi\eta\pi$ where χ and η are ramified quadratic characters. In fact, we consider the field $F = \mathbb{Q}_p$, $p \neq 3$. There are two ramified quadratic characters, the Legendre symbol $(\frac{u}{p})$ and its twist with $(-1)^{\mathrm{val}}$. For convenience, we denote the Legendre symbol by λ and its twist by $(-1)^{\mathrm{val}}\lambda$. Since η and χ have to be distinct, we set

$$\eta = \lambda,\ \chi = (-1)^{\mathrm{val}}\lambda.$$

Indeed, this case would belong to

$$\pi \otimes (-1)^{\mathrm{val}}\lambda\pi \otimes \lambda\pi \otimes (-1)^{\mathrm{val}}\pi$$

It means it should be in case 3.2. Hence, this case does not exist when $F = \mathbb{Q}_3$.

Then we turn to the case 5.3. We know that $\pi_\sigma = \pi_1 \otimes \eta\pi_1 \otimes \pi_2 \otimes \eta\pi_2$. We have $W^{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z}$. Analysis case by case:

- $\gamma = (abcd) = 1$, $E^\gamma/W^{\mathfrak{s}} = E/W^{\mathfrak{s}}$.

- $\gamma = ((badc), E^\gamma = \{(a,a,c,c),(a,-a,c,-c) : a,c \in T\}/T \cong \{(1,1,z,z),(1,-1,z,-z) : z \in T\} \cong \mathbb{T} \sqcup \mathbb{T}$.
$$E^\gamma/W^\gamma \cong \mathbb{T} \sqcup \mathbb{T}$$

Hence, we have the decomposition

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T} \sqcup \mathbb{T} \tag{11}$$

From now, we try to exhaust the tempered dual with respect to $\mathfrak{s}$. First of all, we consider t in the form of $(1,1,z,z)$ and $(1,-1,z,-z)$. For the point $t = (1,1,z,z)$, χ_t corresponds to the character $\chi_t = (1,1,z^{\mathrm{val}},z^{\mathrm{val}})$. After twisting ,we have the representation $\chi_t \otimes \sigma$. Now we compute the R-group for this representation. This implies we can consider $\chi_t \otimes \pi_\sigma \cong \pi_1 \otimes \eta\pi_1 \otimes z^{\mathrm{val}}\pi_2 \otimes z^{\mathrm{val}}\eta\pi_2$. By computation, we know that the trivial character and η are also contained in $\bar{L}(\chi_t \otimes \pi_\sigma)$. Then we have $R(\chi_t \otimes \sigma) =< 1, \eta >$ and $X(\chi_t \otimes \pi_\sigma) = 1$. We note that the character η is ramified and of order 2. This leads that $R(\sigma) \cong \mathbb{Z}/2\mathbb{Z}$. This implies the representation $\lambda(t)$ induced by $\chi_t \otimes \sigma$ is reducible and can be decomposed as two parts:

$$\lambda(t) = \lambda^+ \oplus \lambda^-$$

Indeed, λ^+ and λ^- are tempered.

Similarly, we consider the point in the form of $t = (1,-1,z,-z)$ and χ_t corresponds to the character $\chi_t = (1,(-1)^{val},z^{\mathrm{val}},(-z)^{\mathrm{val}})$. After twisting, we have the representation $\chi_t \otimes \sigma$. Then we can consider $\chi_t \otimes \pi_\sigma \cong \pi \otimes (-1)^{\mathrm{val}}\eta\pi_1 \otimes z^{\mathrm{val}}\pi_2 \otimes (-z)^{\mathrm{val}}\eta\pi_2$. We obtain $R(\chi_t \otimes \sigma) =< (-1)^{\mathrm{val}}\eta >= \mathbb{Z}/2\mathbb{Z}$. This implies that the representation $\rho(t)$ induced by $\chi_t \otimes \sigma$ has two irreducible constituents:

$$\rho(t) = \rho^+ \oplus \rho^-$$

where ρ^+ and ρ^- are tempered representations of G. The definition of the map

$$\mu : E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T} \sqcup \mathbb{T} \longrightarrow \mathrm{Irr}^{\mathrm{t}}(G)^{\mathfrak{s}}$$

is as follows:

$$\begin{array}{rcl}
T & \longmapsto & \lambda^+ \hookrightarrow \mathrm{Ind}_M^G(\pi_1 \otimes \eta\pi_1 \otimes z^{\mathrm{val}}\pi_2 \otimes z^{\mathrm{val}}\eta\pi_2) \\
T & \longmapsto & \rho^+ \hookrightarrow \mathrm{Ind}_M^G(\pi \otimes (-1)^{\mathrm{val}}\eta\pi_1 \otimes z^{\mathrm{val}}\pi_2 \otimes (-z)^{val \circ det}\eta\pi_2) \\
t & \longmapsto & \mathrm{Ind}_M^G(\chi_t \otimes \sigma)
\end{array}$$

It is easy to verify that $inf.ch. \circ \mu = \pi_{\sqrt{q}}$. Hence, we have

Lemma 2.7. *Part (3) of the conjecture is true for* $\mathfrak{s} = [T, \pi \otimes \eta\pi \otimes \eta^2\pi \otimes \eta^3\pi]$.

Case 5.4 is next. The isotropy group $W^{\mathfrak{s}}$ is trivial. Indeed each induced representation by unitary twist with σ is irreducible. This implies every induced representation is irreducible. From the side of extended quotient $E^{\mathfrak{s}}//W^{\mathfrak{s}}$, we know that $E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}}$ because $W^{\mathfrak{s}} = 1$. The bijection μ between $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ and $\mathrm{Irr}^{\mathrm{t}}(G)^{\mathfrak{s}}$

$$\mu : E^{\mathfrak{s}}//W^{\mathfrak{s}} \to \mathrm{Irr}^{\mathrm{t}}(G)^{\mathfrak{s}}$$

is given by

$$t \in E^{\mathfrak{s}}/W^{\mathfrak{s}} \mapsto \mathrm{Ind}(\chi_t \otimes \sigma)$$

Lemma 2.8. *Part 3 of the conjecture is true for* $\mathfrak{s} = [T, \sigma]$ *where* $\pi_\sigma = \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$

Theorem 2.9. *Part (3) of the geometric conjecture is true for* $\mathfrak{s} = [T, \sigma]$.

Proof. Combine 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7 and 2.8 above. $\square$

3 A tetrahedron of reducibility

We exhibit a *tetrahedron of reducibility* in the tempered dual of $\mathrm{SL}_4(\mathbb{Q}_2)$ which does not occur in the tempered dual of $\mathrm{SL}_4(\mathbb{Q}_p)$ when $p > 2$. This confirms a special case of the recent conjecture in [1, 2, 3], and also has independent interest. Let F denote the p-adic field $\mathbb{Q}_p$, and let $U_F^n := 1 + p^n\mathbb{Z}_p$, $n \geq 1$ denote the standard congruence unit groups. Let $U_F = \mathfrak{o}_F^\times$. For $p = 2$ define homomorphisms $\eta, \chi : U/U^3 \to \mathbb{Z}/2\mathbb{Z}$ as in [11, p. 18]:

$$\eta(x) = 0, x \equiv 1 \mod 4$$

$$\eta(x) = 1, x \equiv -1 \mod 4$$

$$\chi(x) = 0, x \equiv \pm 1 \mod 8$$

$$\chi(x) = 1, x \equiv \pm 5 \mod 8$$

The map η defines an isomorphism of U/U^2 onto $\mathbb{Z}/2\mathbb{Z}$ and the map χ defines an isomorphism of U^2/U^3 onto $\mathbb{Z}/2\mathbb{Z}$. The *level* of a character ψ of $F^\times$ is the least integer $n \geq 0$ such that ψ is trivial on U_F^{n+1}. Then we have η is level 1 and χ is level 2. The product $\eta \cdot \chi$ is also level 2.

The three ramified quadratic characters of $\mathbb{Q}_2^\times$ create a unitary character of the standard Borel subgroup in $\mathrm{SL}_4(\mathbb{Q}_2)$:

$$\tau : \begin{bmatrix} x_1 & * & * & * \\ 0 & x_2 & * & * \\ 0 & 0 & x_3 & * \\ 0 & 0 & 0 & x_4 \end{bmatrix} \mapsto \eta(x_2)\chi(x_3)(\eta \cdot \chi)(x_4)$$

We will twist this quadratic character by an unramified unitary character ψ and form the induced representation $\mathrm{Ind}_B^G(\psi\tau)$. Let D be the irreducible component in the Bernstein variety containing τ, let $E \subset D$ be the corresponding compact manifold. The subgroup of the Weyl group which fixes E is the finite group $W := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We have the standard projection

$$\pi : E//W \to E/W$$

of the extended quotient onto the ordinary quotient. The extended quotient $E//W$ is the disjoint union of 6 unit intervals a, b, c, d, e, f and the ordinary quotient E/W. In the projection π, these 6 intervals assemble themselves into the 6 edges of a tetrahedron in E/W. The cardinality of each fibre of π creates a perfect model of reducibility. The locus of reducibility is the 1-skeleton $\mathfrak{R}$ of a tetrahedron, and we have

$$|\pi^{-1}(\psi\tau)| = |Ind_B^G(\psi\tau)|$$

for all unramified unitary characters ψ of T. On the interior of each edge $\pi(a), \ldots, \pi(f)$ of $\mathfrak{R}$, each induced representation admits 2 distinct irreducible constituents; on each vertex of $\mathfrak{R}$, each induced representation admits 4 distinct irreducible components.

In this section, we intend to discuss a special case when $F = \mathbb{Q}_2$. Recall that the case 5.2 for $\mathrm{SL}_4(\mathbb{Q}_p)$, $p \geq 3$ does not exists since it cannot admit two ramified quadratic characters which it is not the twist with an unramified character each other. Recall that the representation π_σ is given by $\pi \otimes \chi\pi \otimes \eta\pi \otimes \chi\eta\pi$. In this case, the R-group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and we also know that the isotropy group is $W^{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

- $\gamma = (abcd) = 1$, $E^\gamma/W^{\mathfrak{s}} = E/W^{\mathfrak{s}}$.
- $\gamma = (badc)$, $E^\gamma = \{(1, 1, z, z), (1, -1, z, -z) : z \in \mathbb{T}\}$.

$$E^\gamma/W^\gamma \cong \mathbb{I} \sqcup \mathbb{I}$$

- $\gamma = (cdab)$, $E^\gamma = \{(1, z, 1, z), (1, z, -1, -z) : z \in \mathbb{T}\}$.

$$E^\gamma/W^\gamma \cong \mathbb{I} \sqcup \mathbb{I}$$

- $\gamma = (dcab)$, $E^\gamma = \{(1,z,z,1),(1,z,-z,-1) : z \in \mathbb{T}\}$.

$$E^\gamma/W^\gamma \cong \mathbb{I} \sqcup \mathbb{I}$$

Therefore, we can decompose the extended quotient as follows:

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} \cong E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup \mathbb{I}$$

where $\mathbb{I}$ is the unit interval in complex plane.

We start from the two disjoint varieties $(1,1,z,z)$ and $(1,-1,z,-z)$. We assume $z \in \{1,-1\}$. Firstly, we check the character χ_t generated by $t = (1,1,z,z)$. Then we have $\chi_t = (1,1,z^{\mathrm{val}},z^{\mathrm{val}})$ and $\chi_t \otimes \pi_\sigma \cong \pi \otimes \chi\pi \otimes \eta z^{\mathrm{val}}\pi \otimes \chi\eta z^{\mathrm{val}}\pi$. In fact, χ is the element of $\bar{L}(\chi_t \otimes \pi_\sigma)$ except the trivial character. Roughly speaking, the reason is, in this component, the variety fixes the terms which twists with character z^{val}. Easily, we have $R(\chi_t \otimes \pi_\sigma) =<\chi>= \mathbb{Z}/2\mathbb{Z}$. This implies for each $t = (1,1,z,z)$, the representation induced $\delta_1(z)$ by $\chi_t \otimes \pi_\sigma$ is reducible.

Similarly, the character χ_t generated by $t = (1,-1,z,-z)$ is given by

$$\chi_t = (1,(-1)^{\mathrm{val}},z^{\mathrm{val}},(-z)^{\mathrm{val}}).$$

We have $\chi_t \otimes \pi_\sigma \cong \pi \otimes (-1)^{\mathrm{val}}\chi\pi \otimes \eta z^{\mathrm{val}}\pi \otimes \chi\eta(-z)^{\mathrm{val}}\pi$. Then we know $\bar{L}(\chi_t \otimes \pi_\sigma) = \{1,(-1)^{\mathrm{val}}\chi\}$. Indeed, $R(\chi_t \otimes \sigma) = \mathbb{Z}/2\mathbb{Z}$. This means for each $t = (1,-1,z,-z)$ the representation induced by $\chi_t \otimes \pi_\sigma$ is reducible.
$(1,z,1,z)$ and $(1,z,-1,-z)$ are two disjoint varieties. Following the similar method, we know

$$t = (1,z,1,z) \to \chi_t = (1,z^{\mathrm{val}},1,z^{\mathrm{val}})$$

and we have $R(\chi_t \otimes \sigma) =<\eta>= \mathbb{Z}/2\mathbb{Z}$.

For $t = (1,z,-1,-z)$, we get the corresponding unitary character χ_t as follows

$$t = (1,z,-1,-z) \to \chi_t = (1,z^{\mathrm{val}},(-1)^{\mathrm{val}},(-z)^{\mathrm{val}})$$

and we will have $R(\chi_t \otimes \sigma) =<(-1)^{\mathrm{val}}\eta>= \mathbb{Z}/2\mathbb{Z}$.

In this component, $(1,z,z,1)$ and $(1,z,-z,-1)$ are two disjoint varieties. Following the similar method, we know

$$t = (1,z,z,1) \to \chi_t = (1,z^{\mathrm{val}},z^{\mathrm{val}},1)$$

and we have $R(\chi_t \otimes \sigma) =<\chi\eta>= \mathbb{Z}/2\mathbb{Z}$.

For $t = (1,z,-z,-1)$, we get the corresponding unitary character χ_t as follows

$$t = (1,z,-z,-1) \to \chi_t = (1,z^{\mathrm{val}},(-z)^{\mathrm{val}},(-1)^{\mathrm{val}})$$

and we will have $R(\chi_t \otimes \sigma) =< (-1)^{\mathrm{val}}\eta\chi >= \mathbb{Z}/2\mathbb{Z}$.

For convenience, we denote

$$\begin{aligned} (1,1,z,z) &- (a) \\ (1,-1,z,-z) &- (b) \\ (1,z,1,z) &- (c) \\ (1,z,-1,-z) &- (d) \\ (1,z,z,1) &- (e) \\ (1,z,-z,-1) &- (f) \end{aligned}$$

Now, we investigate the points

$$\begin{aligned} (1,1,1,1) \\ (1,-1,1,-1) \\ (1,1,-1,-1) \\ (1,-1,-1,1) \end{aligned}$$

It is easy to check that

$$\begin{aligned} (1,1,1,1) &\in (a),(c),(e) \\ (1,-1,1,-1) &\in (b),(c),(f) \\ (1,1,-1,-1) &\in (a),(d),(f) \\ (1,-1,-1,1) &\in (b),(d),(e) \end{aligned}$$

In fact, for such points, the R-group $R(\chi_t \otimes \sigma)$ is given by $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. This implies, for each $\mathrm{Ind}_M^G(\chi_t \otimes \sigma)$, there are 4 irreducible constituents. The extended quotient is the disjoint union of the ordinary quotient and six unit intervals. The six intervals are sent to the edges of a tetrahedron by the canonical projection

$$\pi : E^{\mathfrak{s}}//W^{\mathfrak{s}} \to E^{\mathfrak{s}}/W^{\mathfrak{s}}$$

The preimage of the interior of one edge is the union of two open intervals (the one corresponding to the given edge and one in the ordinary quotient), replicating the fact that the R-group has order 2, while the preimage of a vertex is the union of three endpoints of intervals and one point in the ordinary quotient, replicating the fact that the R-group has order 4 here. The 1-skeleton of the tetrahedron is perfect model of reducibility and confirms the ABP-conjecture in this case.

Case	π_σ	$W^0_{\mathfrak{s}}$	$C_{\mathfrak{s}}$	$H^0_{\mathfrak{s}}$
1	$\pi \otimes \pi \otimes \pi \otimes \pi$	$\mathfrak{S}_4$	1	$\mathrm{SL}_4(F)$
2	$\pi_1 \otimes \pi_1 \otimes \pi_1 \otimes \pi_2$	$\mathfrak{S}_3$	1	$(\mathrm{GL}_3(F) \times F^\times) \cap G$
3.1	$\pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_2$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	1	$(\mathrm{GL}_2(F) \times \mathrm{GL}_2(F)) \cap G$
3.2	$\pi \otimes \pi \otimes \eta\pi \otimes \eta\pi$ $\eta^2 = 1$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathrm{GL}_2(F) \times \mathrm{GL}_2(F)) \cap G$
4	$\pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_3$	$\mathbb{Z}/2\mathbb{Z}$	1	$(\mathrm{GL}_2(F) \times F^\times \times F^\times) \cap G$
5.1	$\pi \otimes \eta\pi \otimes \eta^2\pi \otimes \eta^3\pi$ $\eta^4 = 1$	1	$\mathbb{Z}/4\mathbb{Z}$	T
5.2	$\pi \otimes \chi\pi \otimes \eta\pi \otimes \eta\chi\pi$ $\chi^2 = 1, \eta^2 = 1 (p = 2)$	1	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	T
5.3	$\pi_1 \otimes \eta\pi_1 \otimes \pi_2 \otimes \eta\pi_2$ $\eta^2 = 1$	1	$\mathbb{Z}/2\mathbb{Z}$	T
5.4	$\pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$	1	1	T

Table 1: Table of the groups $W_{\mathfrak{s}} = W^0_{\mathfrak{s}} \rtimes C_{\mathfrak{s}}$ and $H_{\mathfrak{s}} = H^0_{\mathfrak{s}} \rtimes C_{\mathfrak{s}}$.

4 Decomposition according to cells

Let $G = \mathrm{SL}_4(F)$. Let $T = (F^\times \times F^\times \times F^\times \times F^\times) \cap G$ be the standard maximal torus in G. Let $\mathfrak{s} = [T, \sigma]_G$ be a Bernstein component with respect to a character σ of T. In this section, we denote by $W_{\mathfrak{s}}$ the isotropy group of $\mathfrak{s}$. We let $\pi_\sigma|_T = \sigma$ where π_σ is a (unitary) character of the standard maximal torus $\tilde{T} = F^\times \times F^\times \times F^\times \times F^\times$ of $\tilde{G} = \mathrm{GL}_4(F)$. We write $\pi_\sigma = \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$.

We denote by $W^0_{\mathfrak{s}}$ the isotropy of $\tilde{\mathfrak{s}} = [\tilde{T}, \pi_\sigma]_{\tilde{G}}$. The group $W^0_{\mathfrak{s}}$ is a finite Weyl group. Let $\Phi_{\mathfrak{s}}$ denote a root system for $W^0_{\mathfrak{s}}$, and let $\Phi^+_{\mathfrak{s}} = \Phi_{\mathfrak{s}} \cap \Phi^+$, where Φ^+ is a positive root system for the Weyl group of G. Then $\Phi^+_{\mathfrak{s}}$ is a positive system in $\Phi_{\mathfrak{s}}$. The group $W_{\mathfrak{s}}$ is not a Weyl group in general. However, we have the following relation (see for instance [7, Prop. 2.3]):

$$W_{\mathfrak{s}} = W^0_{\mathfrak{s}} \rtimes C_{\mathfrak{s}}, \tag{12}$$

where

$$C_{\mathfrak{s}} = \left\{ w \in W^0_{\mathfrak{s}} \ : \ w \cdot \Phi^+_{\mathfrak{s}} = \Phi^+_{\mathfrak{s}} \right\}.$$

In the table we list all the possibilities for π_σ due to the relations among the π_i. The conditions for the π_i in the first column of the table are that π_i is not equivalent to a twist of π_j by any unramified character of $F^\times$ if $i \neq j$. The determination of the group $W^0_{\mathfrak{s}}$ is obvious, and $W_{\mathfrak{s}}$ was computed in §2. From (12), we know that $C_{\mathfrak{s}}$ is isomorphic to the quotient $W_{\mathfrak{s}}/W^0_{\mathfrak{s}}$. The results are listed in the fourth column of Table 1.

Let $T^\vee$ denote the dual torus of T in the Langlands dual $G^\vee = \mathrm{PGL}_4(\mathbb{C})$ of G. Let $X(T^\vee)$ be the group of characters of $T^\vee$. We have

$$X(T^\vee) \simeq \left\{(l_1, l_2, l_3, l_4) \in \mathbb{Z}^4 \ : \ l_1 + l_2 + l_3 + l_4 = 0\right\}.$$

We set

$$W_{\mathfrak{s}}^{\mathrm{e}} = W_{\mathfrak{s}}^0 \rtimes X(T^\vee).$$

Then $W_{\mathfrak{s}}^{\mathrm{e}}$ is the extended affine Weyl group of the p-adic group $H_{\mathfrak{s}}^0$ described in the fifth column of the table The group $H_{\mathfrak{s}}^0$ arises from [10, § 8]. Let $\Phi_{\mathfrak{s}}^\vee$ denote the set of coroots of the root system $\Phi_{\mathfrak{s}}$. The quadruple $(X(T), \Phi_{\mathfrak{s}}, X(T^\vee), \Phi_{\mathfrak{s}}^\vee)$ is the root datum of $H_{\mathfrak{s}}^0$. We set $H_{\mathfrak{s}} := H_{\mathfrak{s}}^0 \rtimes C_{\mathfrak{s}}$. The unipotent classes of $H_{\mathfrak{s}}^0$ will be easy to figure out. We will attach a unipotent class to each cocharacter. In particular, the minimal (for the usual order) unipotent class (this is the trivial unipotent class) should correspond to the trivial cocharacter. In other words, all the connected components of the compact extended quotient which are attached in §2 to a trivial cocharacter should correspond to the minimal unipotent class. When the group $W_{\mathfrak{s}}^0 = \{1\}$, there is only one unipotent class and all the cocharacters are trivial as proved in §2.

Case 1. The Langlands dual group of $\mathrm{SL}_4(F)$ is the complex Lie group $\mathrm{PGL}_4(\mathbb{C})$. There are five unipotent classes in $\mathrm{PGL}_4(\mathbb{C})$:

$$\mathbf{u}_0 \le \mathbf{u}_3 \le \mathbf{u}_2 \le \mathbf{u}_1 \le \mathbf{u}_e,$$

which are respectively parametrized by the following partitions of 4:

$$(1^4) \le (2, 1^2) \le (2^2) \le (3, 1) \le (4).$$

They correspond (see for instance [12]) to the two-sided cells

$$\mathbf{c}_0 \le \mathbf{c}_3 \le \mathbf{c}_2 \le \mathbf{c}_1 \le \mathbf{c}_e.$$

We write

$$\begin{array}{cc} \mathrm{pt}_1 = (1,1,1,1) & \mathrm{pt}_2 = (1,-1,1,-1) \\ \mathrm{pt}_3 = (1,i,1,-i) & \mathrm{pt}_4 = (1,-i,-1,i) \end{array} \tag{13}$$

and define

$$(E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_0 := E_{\mathfrak{s}}/W_{\mathfrak{s}} \sqcup \{(z,-z,1,-1) : z \in \mathbb{T}\} \sqcup \mathrm{pt}_3 \sqcup \mathrm{pt}_4 \simeq E_{\mathfrak{s}}/W_{\mathfrak{s}} \sqcup \mathbb{T} \sqcup \mathrm{pt}_3 \sqcup \mathrm{pt}_4$$

$$(E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_3 := \{(z_1, z_1, z_2, 1) \ : \ z_1, z_2 \in \mathbb{T}\} \simeq \mathbb{T}^2$$

$$(E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_2 := \{(z, z, 1, 1) \ : \ z \in \mathbb{T}\} \sqcup \mathrm{pt}_1 \simeq \mathbb{T} \sqcup \mathrm{pt}_1$$

$$(E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_1 := \{(z, z, z, 1) \ : \ z \in \mathbb{T}\} \simeq \mathbb{T}$$

$$(E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_e := \mathrm{pt}_2.$$

From Eqn.(3), we get the following cell-decomposition of $E_{\mathfrak{s}}/\!/W_{\mathfrak{s}}$:

$$E_{\mathfrak{s}}/\!/W_{\mathfrak{s}} = (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_0 \sqcup (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_3 \sqcup (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_2 \sqcup (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_1 \sqcup (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_e,$$

with cocharacters

$$\begin{array}{cc} h_0 = 1, & h_3(t) = h_2(t) = (t, t^{-1}, 1, 1), \\ h_1(t) = (t^2, 1, t^2, 1), & h_e(t) = (t^3, t, t^{-1}, t^{-3}). \end{array}$$

We have included pt_1 in the subset $(E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_2$ in order to attach the two elements ρ^+ and ρ^- (defined in §2) to the same unipotent class. It should be a general fact that all the elements in a given L-packet are attached to the same unipotent class.

Case 2. The Langlands dual group of $H_{\mathfrak{s}}^0$ is $(\mathrm{GL}_3(\mathbb{C}) \times \mathbb{C}^\times)/\mathbb{C}^\times$. There are three unipotent classes in it:

$$\mathbf{u}_0 \le \mathbf{u}_1 \le \mathbf{u}_e,$$

which are respectively parametrized by the following partitions of 3:

$$(1^3) \le (2, 1) \le (3).$$

They correspond to the two-sided cells

$$\mathbf{c}_0 \le \mathbf{c}_1 \le \mathbf{c}_e.$$

We define

$$\begin{aligned} (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_0 &:= E_{\mathfrak{s}}/W_{\mathfrak{s}} \\ (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_1 &:= \{(z, z, z, 1) \,:\, z \in \mathbb{T}\} \simeq \mathbb{T} \\ (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_e &:= \{(z_1, z_1, z_2, 1) \,:\, z_1, z_2 \in \mathbb{T}\} \simeq \mathbb{T}^2. \end{aligned}$$

From Eqn.(4), we get the following cell-decomposition of $E_{\mathfrak{s}}/\!/W_{\mathfrak{s}}$:

$$E_{\mathfrak{s}}/\!/W_{\mathfrak{s}} = (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_0 \sqcup (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_1 \sqcup (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_e,$$

with cocharacters

$$h_0 = 1, \quad h_1(t) = (t^2, 1, t^{-2}, 1), \quad h_e(t) = (t, t^{-1}, 1, 1).$$

Case 3. The Langlands dual group of $H_{\mathfrak{s}}^0$ is $(\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}))/\mathbb{C}^\times$. There are four unipotent classes in it: $\mathbf{u}_0 \leftrightarrow (2, 2)$, $\mathbf{u}_1 \leftrightarrow (2, 1^2)$, $\mathbf{u}_1' \leftrightarrow (1^2, 2)$, $\mathbf{u}_e \leftrightarrow (1^2, 1^2)$. The closure order on unipotent classes is the following:

$$\begin{array}{ccc} & \mathbf{u}_e & \\ \diagup & & \diagdown \\ \mathbf{u}_1 & & \mathbf{u}_1' \, . \\ \diagdown & & \diagup \\ & \mathbf{u}_0 & \end{array}$$

Case 3.1. We define

$$(E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_0 := E_{\mathfrak{s}}/W_{\mathfrak{s}} \sqcup \{(1,-1,z,-z) \,:\, z\in\mathbb{T}\} \simeq E_{\mathfrak{s}}/W_{\mathfrak{s}} \sqcup \mathbb{T}$$

$$(E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_1 := \{(1,1,z_1,z_2) \,:\, z_1,z_2\in\mathbb{T}\} \simeq \mathbb{T}^2$$

$$(E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_1' := \{(z_1,z_2,1,1) \,:\, z_1,z_2\in\mathbb{T}\} \simeq \mathbb{T}^2$$

$$(E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_e := \{(1,1,z,z) \,:\, z\in\mathbb{T}\} \simeq \mathbb{T}.$$

From Eqn.(5), we get the following cell-decomposition of $E_{\mathfrak{s}}/\!/W_{\mathfrak{s}}$:

$$E_{\mathfrak{s}}/\!/W_{\mathfrak{s}} = (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_0 \sqcup (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_1 \sqcup (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_1' \sqcup (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_e,$$

with cocharacters

$$h_0 = 1, \quad h_1(t) = (t,t^{-1},1,1), \quad h_1'(t) = (1,1,t,t^{-1}), \quad h_e(t) = (t,t^{-1},t,t^{-1}).$$

Case 3.2. In this case, $H_{\mathfrak{s}}$ is disconnected, and it seems that one has to consider unipotent classes in the disconnected complex Lie group

$$((\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}))/\mathbb{C}^\times) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

This group can be considered as "the Langlands dual group of $H_{\mathfrak{s}}$" (see [10, top of page 395]).

This should give only three unipotent classes

$$\mathbf{u}_0 \le (\mathbf{u}_1 \cup \mathbf{u}_1') \le \mathbf{u}_e.$$

We set

$$\mathbb{T}_0 := \{(z,1,-z,-1) \,:\, z\in\mathbb{T}\} \simeq \mathbb{T}$$

$$\mathbb{T}_0' := \{(z,1,z,1) \,:\, z\in\mathbb{T}\} \simeq \mathbb{T}$$

$$\mathbb{T}_0'' := \{(1,-1,z,-z) \,:\, z\in\mathbb{T}\} \simeq \mathbb{T}$$

We define

$$(E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_0 := E_{\mathfrak{s}}/W_{\mathfrak{s}} \sqcup \mathbb{T}_0 \sqcup \mathbb{T}_0' \sqcup \mathbb{T}_0'' \sqcup \mathrm{pt}_1 \sqcup \mathrm{pt}_2 \sqcup \mathrm{pt}_3 \sqcup \mathrm{pt}_4,$$

where pt_1, pt_2, pt_3, pt_4 are defined as in Eqn. (13),

$$(E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_1 := \{(z_1,z_1,z_2,1) \,:\, z_1,z_2\in\mathbb{T}\} \simeq \mathbb{T}^2$$

$$(E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_e := \{(1,1,z,z) \,:\, z\in\mathbb{T}\} \simeq \mathbb{T}.$$

From Eqn.(8), we get the following cell-decomposition of $E_{\mathfrak{s}}/\!/W_{\mathfrak{s}}$:

$$E_{\mathfrak{s}}/\!/W_{\mathfrak{s}} = (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_0 \sqcup (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_1 \sqcup (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_e,$$

with cocharacters

$$h_0 = 1, \quad h_1(t) = (t, t^{-1}, 1, 1), \quad h_e(t) = (t, t^{-1}, t, t^{-1}).$$

CASE 4. The Langlands dual group of $H_{\mathfrak{s}}^0$ is $(\mathrm{GL}_2(\mathbb{C}) \times \mathbb{C}^\times \times \mathbb{C}^\times)/\mathbb{C}^\times$. This group admits two unipotent classes:

$$\mathbf{u}_0 \leftrightarrow (1^2, 1, 1) \ \leq \ \mathbf{u}_e \leftrightarrow (2, 1, 1).$$

We define

$$(E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_0 := E_{\mathfrak{s}}/W_{\mathfrak{s}}$$
$$(E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_e := \{(z_1, z_1, z_2, 1) \, : \, z_1, z_2 \in \mathbb{T}\} \simeq \mathbb{T}^2.$$

From Eqn.(9), we have the following cell-decomposition:

$$(E_{\mathfrak{s}}/\!/W_{\mathfrak{s}} = (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_0 \sqcup (E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_e$$

with cocharacters

$$h_0 = 1, \quad h_e(t) = (t, t^{-1}, 1, 1).$$

CASES 5.1, 5.2, 5.3, 5.4. The Langlands dual group of $H_{\mathfrak{s}}^0 = T$ is the complex torus $T^\vee$ in $\mathrm{PGL}_4(\mathbb{C})$. There is only one unipotent class in $T^\vee$: the trivial class $\mathbf{u}_0$. Hence we set

$$(E_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_0 = E_{\mathfrak{s}}/\!/W_{\mathfrak{s}}.$$

There only one cocharacter, the trivial cocharacter.

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