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Geometric structure in the representation theory of reductive p-adic groups II

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1. Introduction

This expository note will state the ABP (Aubert-Baum-Plymen) conjecture [2, 3, 4]. The conjecture can be stated at four levels:

- K-theory of C^* algebras
- Periodic cyclic homology of finite type algebras
- Geometric equivalence of finite type algebras
- Representation theory

The emphasis in this note will be on representation theory. The first two items in the above list are topological, and the third item is algebraic. Validity for the two topological items is quite plausible, and thus gives some credibility to the representation theory version of the conjecture.

A recent result of M. Solleveld [51], when combined with results of [13, 14, 18, 19, 20, 22, 24, 31, 32, 40, 42, 43, 44, 45, 46, 48] proves a very substantial part of the conjecture for many examples. See Section 9 below for a summary of this development.

See Section 10 below for an indication of the apparent connection between L-packets and ABP.

2. Bernstein Components

Let G be a reductive p-adic group. Examples are $\mathrm{GL}(n, F), \mathrm{SL}(n, F)$ etc. where F is a local nonarchimedean field, that is, F is a finite extension of the p-adic numbers $\mathbb{Q}_p$, or the local function field $\mathbb{F}_q((x))$.

Definition. A *representation* of G is a group homomorphism

$$\phi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

where V is a vector space over the complex numbers $\mathbb{C}$.

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The local field F in its natural topology is a locally compact and totally disconnected topological field. Hence G (in its p-adic topology) is a locally compact and totally disconnected topological group.

Definition. A representation

$$\phi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

of G is *smooth* if for every $v \in V$,

$$G_v = \{g \in G \mid \phi(g)v = v\}$$

is an open subgroup of G.

The smooth (or admissible) dual of G, denoted $\widehat{G}$, is the set of equivalence classes of smooth irreducible representations of G.

$$\widehat{G} = \{\text{Smooth irreducible representations of } G\}/\sim$$

Problem. Describe $\widehat{G}$.

Remark. A smooth representation of G is *admissible* if when restricted to any compact open subgroup H of G each irreducible representation of H appears (in the restricted representation) with at most a finite multiplicity. According to a result of Jacquet [26] any smooth irreducible representation of G is admissible. Thus the smooth dual $\widehat{G}$ and the admissible dual (i.e. the set of equivalence classes of smooth irreducible admissible representations of G) are the same.

Since G is locally compact we may fix a (left-invariant) Haar measure dg for G. The Hecke algebra of G, denoted $\mathcal{H}G$, is then the convolution algebra of all locally-constant compactly-supported complex-valued functions $f : G \to \mathbb{C}$.

$$\begin{array}{rcl} (f+h)(g) & = & f(g)+h(g) \\ (f*h)(g_0) & = & \displaystyle\int_G f(g)h(g^{-1}g_0)dg \end{array} \quad \left\{ \begin{array}{l} g \in G \\ g_0 \in G \\ f \in \mathcal{H}G \\ h \in \mathcal{H}G \end{array} \right.$$

Definition. A *representation* of the Hecke algebra $\mathcal{H}G$ is a homomorphism of $\mathbb{C}$ algebras

$$\psi : \mathcal{H}G \to \mathrm{End}_{\mathbb{C}}(V)$$

where V is a vector space over the complex numbers $\mathbb{C}$.

Definition. A representation

$$\psi : \mathcal{H}G \to \mathrm{End}_{\mathbb{C}}(V)$$

of the Hecke algebra $\mathcal{H}G$ is *irreducible* if ψ is not the zero map and there does not exist a vector subspace W of V such that W is preserved by the action of $\mathcal{H}G$ and W is neither the zero subspace nor all of V.

Definition. A *primitive ideal* I in $\mathcal{H}G$ is the null space of an irreducible representation of $\mathcal{H}G$. Note that $\mathcal{H}G$ itself is not a primitive ideal.

Thus whenever I is a primitive ideal in $\mathcal{H}G$ there is an irreducible representation of $\mathcal{H}G$, $\psi : \mathcal{H}G \to \mathrm{End}_{\mathbb{C}}(V)$ such that

$$0 \longrightarrow I \hookrightarrow \mathcal{H}G \xrightarrow{\psi} \mathrm{End}_{\mathbb{C}}(V)$$

is an exact sequence of $\mathbb{C}$ algebras.

A bijection of sets (i.e. a bijection in the category of sets)

$$\widehat{G} \longleftrightarrow \mathrm{Prim}(\mathcal{H}G)$$

where $\mathrm{Prim}(\mathcal{H}G)$ is the set of primitive ideals in $\mathcal{H}G$ is defined as follows. Let

$$\phi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

be an irreducible smooth representation of G. Consider the representation of $\mathcal{H}G$ given by

$$f \mapsto \int_G f(g)\phi(g)dg \qquad\qquad f \in \mathcal{H}G$$

This is an irreducible representation of $\mathcal{H}G$, and the bijection sends ϕ to its null space.

What has been gained from this bijection?

On $\mathrm{Prim}(\mathcal{H}G)$ there is a topology—the Jacobson topology. If S is a subset of $\mathrm{Prim}(\mathcal{H}G)$ then the closure $\overline{S}$ (in the Jacobson toplogy) of S is

$$\overline{S} = \{J \in \mathrm{Prim}(\mathcal{H}G) \ | \ J \supset \bigcap_{I \in S} I\}$$

$\mathrm{Prim}(\mathcal{H}G)$ (with the Jacobson topology) is the disjoint union of its connected components.

$\pi_o\mathrm{Prim}(\mathcal{H}G)$ denotes the set of connected components of $\mathrm{Prim}(\mathcal{H}G)$.

$\pi_o\mathrm{Prim}(\mathcal{H}G)$ is a countable set and has no further structure.

$\pi_o\mathrm{Prim}(\mathcal{H}G)$ is also known as the *Bernstein spectrum* of G.

$\pi_o\mathrm{Prim}(\mathcal{H}G) = \{(M, \sigma)\}/\sim$ where (M, σ) is a cuspidal pair i.e. M is a Levi factor of a parabolic subgroup P of G and σ is an irreducible super-cuspidal representation of M. $\sim$ is the conjugation action of G, combined with tensoring by unramified characters of M. Thus $(M, \sigma) \sim (M', \sigma')$ iff there exists an unramified character $\psi\colon M \to \mathbb{C} - \{0\}$ of M and an element g of G, $g \in G$, with

$$g(M, \psi \otimes \sigma) = (M', \sigma')$$

The meaning of this equality is:

- $gMg^{-1} = M'$
- $g_*(\psi \otimes \sigma)$ and σ' are equivalent smooth irreducible representations of M'.

For each $\mathfrak{s} \in \pi_o\mathrm{Prim}(\mathcal{H}G)$, $\widehat{G}_{\mathfrak{s}}$ denotes the subset of $\widehat{G}$ which is mapped to the $\mathfrak{s}$-th connected component of $\mathrm{Prim}(\mathcal{H}G)$ under the bijection $\widehat{G} \longleftrightarrow \mathrm{Prim}(\mathcal{H}G)$.

Using the bijections

$$\widehat{G} \longleftrightarrow \mathrm{Prim}(\mathcal{H}G) \qquad\qquad \pi_o\mathrm{Prim}(\mathcal{H}G) \longleftrightarrow \{(M, \sigma)\}/\sim$$

$\widehat{G}_{\mathfrak{s}}$ is obtained by fixing (M, σ) and then taking the irreducible constituents of $Ind_M^G(\psi \otimes \sigma)$ where Ind_M^G is (smooth) parabolic induction and ψ can be any unramified character of M.

The subsets $\widehat{G}_{\mathfrak{s}}$ of $\widehat{G}$ are known as the Bernstein components of $\widehat{G}$. The problem of describing $\widehat{G}$ now breaks up into two problems.

Problem 1: Describe the Bernstein spectrum $\pi_o\mathrm{Prim}(\mathcal{H}G) = \{(M, \sigma)\}/\sim$.

Problem 2: For each $\mathfrak{s} \in \pi_o\mathrm{Prim}(\mathcal{H}G) = \{(M, \sigma)\}/\sim$, describe the Bernstein component $\widehat{G}_{\mathfrak{s}}$.

In this note we shall be concerned with Problem 2. Problem 1 involves describing the irreducible super-cuspidal representations of Levi subgroups of G. The basic conjecture on this issue is that if M is a reductive p-adic group (e.g. M is a Levi factor of a parabolic subgroup of G) then any irreducible super-cuspidal representation of M is obtained by smooth induction from an irreducible representation of a subgroup of M which is compact modulo the center of M. This basic conjecture is now known to be true to a very great extent [30] [52]. For Problem 2 the ABP conjecture proposes that each Bernstein component $\widehat{G}_{\mathfrak{s}}$ has a very simple geometric structure.

3. Infinitesimal Character

Notation. $\mathbb{C}^\times$ denotes the (complex) affine variety $\mathbb{C} - \{0\}$.

Definition. A *complex torus* is a (complex) affine variety T such that there exists an isomorphism of affine varieties

$$T \cong \mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times.$$

Bernstein [11] assigns to each $\mathfrak{s} \in \pi_o\mathrm{Prim}(\mathcal{H}G)$ a complex torus $T_{\mathfrak{s}}$ and a finite group $W_{\mathfrak{s}}$ acting on $T_{\mathfrak{s}}$. Bernstein's construction can be recalled as follows. First, fix (M, σ). M^0 denotes the subgroup of M consisting of all $g \in M$ such that:

$$\text{whenever } \varphi\colon M \to F^\times \text{ is an algebraic character, } \varphi(g) \in \mathcal{O}_F^\times.$$

Here F is the p-adic field over which G is defined, $\mathcal{O}_F$ is the integers in F and $\mathcal{O}_F^\times$ is the invertible elements of the ring $\mathcal{O}_F$. Equivalently, M^0 is the (closed) normal subgroup of M generated by all the compact subgroups of M. The quotient group M/M^0 is discrete and is a free abelian group of finite rank. Therefore $\mathrm{Hom}_{\mathbb{Z}}(M/M^0, \mathbb{C}^\times)$ is a complex torus. The points of this complex torus are (by definition) the unramified characters of M. Within this complex torus consider all the unramified characters φ of M such that:

$$\varphi \otimes \sigma \text{ is equivalent (as an irrreducible smooth representation of M) to } \sigma.$$

Denote this set of characters by I_σ. I_σ is a finite subgroup of $\mathrm{Hom}_{\mathbb{Z}}(M/M^0, \mathbb{C}^\times)$ so the quotient group $\mathrm{Hom}_{\mathbb{Z}}(M/M^0, \mathbb{C}^\times)/I_\sigma$ is a complex torus, and this is Bernstein's torus $T_{\mathfrak{s}}$.

$$T_{\mathfrak{s}} = \mathrm{Hom}_{\mathbb{Z}}(M/M^0, \mathbb{C}^\times)/I_\sigma$$

Denote $\mathrm{Hom}_{\mathbb{Z}}(M/M^0, \mathbb{C}^\times)$ by $T'_{\mathfrak{s}}$. Denote the quotient map $T'_{\mathfrak{s}} \to T_{\mathfrak{s}}$ by

$$\eta\colon T'_{\mathfrak{s}} \longrightarrow T_{\mathfrak{s}}$$

The Weyl group W_M of M is $N_G(M)/M$, where $N_G(M)$ is the normalizer of M in G. $W_{\mathfrak{s}}$ is the subgroup of W_M consisting of all $w \in W_M$ such that:

$$\text{given } w,\ \exists \varphi_w \in T'_{\mathfrak{s}} = \mathrm{Hom}_{\mathbb{Z}}(M/M^0, \mathbb{C}^\times) \text{ with } \varphi_w \otimes \sigma \text{ equivalent to } w_*(\sigma)$$

($\varphi_w \otimes \sigma$ is equivalent as an irrreducible smooth representation of M to $w_*(\sigma)$). If I_σ is not the trivial one-element group, then $w \mapsto \varphi_w$ is not well-defined as a map from $W_{\mathfrak{s}}$ to $T'_{\mathfrak{s}}$. However, when composed with the quotient map $T'_{\mathfrak{s}} \to T_{\mathfrak{s}}$, a well-defined group homomorphism $W_{\mathfrak{s}} \to T_{\mathfrak{s}}$ is obtained:

$$\begin{aligned} W_{\mathfrak{s}} &\longrightarrow T_{\mathfrak{s}} \\ w &\longmapsto \eta(\varphi_w) \end{aligned}$$

The evident conjugation action of W_M on M gives an action of W_M on $T'_{\mathfrak{s}} = \mathrm{Hom}_{\mathbb{Z}}(M/M^0, \mathbb{C}^\times)$. When restricted to $W_{\mathfrak{s}}$ this gives an action of $W_{\mathfrak{s}}$ on $T'_{\mathfrak{s}}$ which preserves I_σ and hence $W_{\mathfrak{s}}$ acts on the quotient torus $T_{\mathfrak{s}}$. Note that at this point $W_{\mathfrak{s}}$ is acting as automorphisms of the algebraic group $T_{\mathfrak{s}}$. Denote this action by:

$$wx = w_*(x) \qquad\quad w \in W_{\mathfrak{s}} \quad x \in T_{\mathfrak{s}}$$

In Bernstein's action of $W_{\mathfrak{s}}$ on $T_{\mathfrak{s}}$, $w \in W_{\mathfrak{s}}$ acts by

$$x \longmapsto w_*(x)[\eta(\varphi_w)] \qquad\quad x \in T_{\mathfrak{s}}$$

i.e. when $w \in W_{\mathfrak{s}}$ is applied to $x \in T_{\mathfrak{s}}$, the result is the product (using the group structure of $T_{\mathfrak{s}}$) of $w_*(x)$ and $\eta(\varphi_w)$. Hence $W_{\mathfrak{s}}$ is acting as automorphisms of the affine variety $T_{\mathfrak{s}}$ — but not necessarily as automorphisms of the algebraic group $T_{\mathfrak{s}}$. For an example where $W_{\mathfrak{s}}$ is not acting as automorphisms of the algebraic group $T_{\mathfrak{s}}$, see Section 4 of [49]. This same example is also used in [22]. Examples of this kind cannot occur within the principal series — i.e. within the principal series $W_{\mathfrak{s}}$ does act as automorphisms of the algebraic group $T_{\mathfrak{s}}$.

Consider the quotient variety $T_{\mathfrak{s}}/W_{\mathfrak{s}}$. Denote the coordinate algebra of $T_{\mathfrak{s}}$ by $\mathcal{O}(T_{\mathfrak{s}})$. $W_{\mathfrak{s}}$ acts on $\mathcal{O}(T_{\mathfrak{s}})$, and the coordinate algebra of $T_{\mathfrak{s}}/W_{\mathfrak{s}}$ is the subalgebra of invariant elements.

$$\mathcal{O}(T_{\mathfrak{s}}/W_{\mathfrak{s}}) = \mathcal{O}(T_{\mathfrak{s}})^{W_{\mathfrak{s}}}$$

Define a surjective map $\pi_{\mathfrak{s}}$ mapping $\widehat{G}_{\mathfrak{s}}$ onto $T_{\mathfrak{s}}/W_{\mathfrak{s}}$

$$\begin{array}{c} \widehat{G}_{\mathfrak{s}} \\ \big\downarrow{\scriptstyle \pi_{\mathfrak{s}}} \\ T_{\mathfrak{s}}/W_{\mathfrak{s}} \end{array}$$

by:

given $\zeta \in \widehat{G}_{\mathfrak{s}}$ select $\varphi \in T'_{\mathfrak{s}}$ such that ζ is an irreducible constituent of $Ind_M^G(\varphi \otimes \sigma)$.

Then set $\pi_{\mathfrak{s}}(\zeta) = \eta(\varphi)$.

This map $\pi_{\mathfrak{s}}$ is referred to as the *infinitesimal character* or the *central character*. In Bernstein's work $\widehat{G}_{\mathfrak{s}}$ is a set (i.e. is only a set) so $\pi_{\mathfrak{s}}$

$$\begin{array}{c} \widehat{G}_{\mathfrak{s}} \\ \big\downarrow{\scriptstyle \pi_{\mathfrak{s}}} \\ T_{\mathfrak{s}}/W_{\mathfrak{s}} \end{array}$$

is a map of sets.

$\pi_{\mathfrak{s}}$ is surjective, finite-to-one and generically one-to-one; *generically one-to-one* means that there is a sub-variety $\mathfrak{R}_{\mathfrak{s}}$ of $T_{\mathfrak{s}}/W_{\mathfrak{s}}$ such that the pre-image of each point in $T_{\mathfrak{s}}/W_{\mathfrak{s}} - \mathfrak{R}_{\mathfrak{s}}$ consists of just one point. $\mathfrak{R}_{\mathfrak{s}}$ is the subvariety of reducibility, i.e. $\mathfrak{R}_{\mathfrak{s}}$ is given by those unramified twists of σ such that there is reducibility when parabolically induced to G.

Remark. Let $\Delta_{\mathfrak{s}}$ be the maximal compact subgroup of $T_{\mathfrak{s}}$. As above, the Bernstein action of $w \in W_{\mathfrak{s}}$ on $T_{\mathfrak{s}}$ is given by:

$$x \longmapsto w_*(x)[\theta(\varphi_w)] \qquad\quad x \in T_{\mathfrak{s}}.$$

This implies that $\Delta_{\mathfrak{s}}$ is preserved by the action.

4. Extended Quotient

Let Γ be a finite group acting on an affine variety X as automorphisms of the affine variety

$$\Gamma \times X \to X.$$

The quotient variety X/Γ is obtained by collapsing each orbit to a point. More precisely, recall (e.g. see [21]) that the category of affine varieties over $\mathbb{C}$ is equivalent to the opposite of the category of commutative unital finitely generated nilpotent-free $\mathbb{C}$ algebras.

$$\left(\text{affine } \mathbb{C} \text{ varieties}\right) \sim \left(\begin{array}{c}\text{commutative unital finitely generated}\\ \text{nilpotent}-\text{free } \mathbb{C} \text{ algebras}\end{array}\right)^{\text{op}}$$

The functor which gives the equivalence assigns to an affine variety X its coordinate algebra $\mathcal{O}(X)$. With X, Γ as above, Γ acts on $\mathcal{O}(X)$ and the coordinate algebra of X/Γ is the subalgebra of invariant elements

$$\mathcal{O}(X/\Gamma) = \mathcal{O}(X)^{\Gamma}$$

This determines X/Γ as an affine variety.

For $x \in X$, Γ_x denotes the stabilizer group of x:

$$\Gamma_x = \{\gamma \in \Gamma \mid \gamma x = x\}.$$

$c(\Gamma_x)$ denotes the set of conjugacy classes of Γ_x. The extended quotient is obtained by replacing the orbit of x by $c(\Gamma_x)$. This is done as follows:

Set $\widetilde{X} = \{(\gamma, x) \in \Gamma \times X \mid \gamma x = x\} \subset \Gamma \times X$. $\widetilde{X}$ is an affine variety and is a sub-variety of $\Gamma \times X$. The coordinate algebra of $\widetilde{X}$ is $\mathcal{O}(\Gamma \times X)/\mathcal{I}$ where $\mathcal{I}$ is the ideal in $\mathcal{O}(\Gamma \times X)$ consisting of all $f \in \mathcal{O}(\Gamma \times X)$ such that $f(\gamma, x) = 0$ whenever $\gamma x = x$.

$$\mathcal{O}(\widetilde{X}) = \mathcal{O}(\Gamma \times X)/\mathcal{I}$$

Γ acts on $\widetilde{X}$:

$$\Gamma \times \widetilde{X} \to \widetilde{X} \qquad \alpha(\gamma, x) = (\alpha\gamma\alpha^{-1}, \alpha x), \qquad \alpha \in \Gamma, \quad (\gamma, x) \in \widetilde{X}.$$

The extended quotient, denoted $X/\!/\Gamma$, is $\widetilde{X}/\Gamma$. Thus the extended quotient $X/\!/\Gamma$ is the ordinary quotient for the action of Γ on $\widetilde{X}$. Γ acts on $\mathcal{O}(\widetilde{X})$ and the coordinate algebra of $X/\!/\Gamma$ is the subalgebra of invariant elements.

$$\mathcal{O}(X/\!/\Gamma) = \mathcal{O}(\widetilde{X})^{\Gamma}$$

The projection $\widetilde{X} \to X$, $(\gamma, x) \mapsto x$ passes to quotient spaces to give a morphism of affine varieties

$$\rho \colon X/\!/\Gamma \to X/\Gamma.$$

This map will be referred to as the projection of the extended quotient onto the ordinary quotient.

The inclusion

$$\begin{aligned} X &\hookrightarrow \widetilde{X} \\ x &\mapsto (e, x) \qquad e = \text{identity element of } \Gamma \end{aligned}$$

passes to quotient spaces to give an inclusion $X/\Gamma \hookrightarrow X//\Gamma$. This will be referred to as the inclusion of the ordinary quotient in the extended quotient. Using this inclusion, $X//\Gamma - X/\Gamma$ denotes $X//\Gamma$ with X/Γ removed.

5. ABP Conjecture Part 1

As above, G is a reductive p-adic group and $\mathfrak{s}$ is a point in the Bernstein spectrum of G.

Consider the two maps indicated by vertical arrows:

$$\begin{tikzcd} T_{\mathfrak{s}}//W_{\mathfrak{s}} \arrow[d, "\rho_{\mathfrak{s}}"'] & \text{and} & \widehat{G}_{\mathfrak{s}} \arrow[d, "\pi_{\mathfrak{s}}"] \\ T_{\mathfrak{s}}/W_{\mathfrak{s}} & & T_{\mathfrak{s}}/W_{\mathfrak{s}} \end{tikzcd}$$

$\pi_{\mathfrak{s}}$ is the infinitesimal character and $\rho_{\mathfrak{s}}$ is the projection of the extended quotient on the ordinary quotient. In practice, $T_{\mathfrak{s}}//W_{\mathfrak{s}}$ and $\rho_{\mathfrak{s}}$ are much easier to calculate than $\widehat{G}_{\mathfrak{s}}$ and $\pi_{\mathfrak{s}}$.

The maps $\rho_{\mathfrak{s}}$ and $\pi_{\mathfrak{s}}$ are conceptually quite different; nevertheless, we conjecture that one can pass from one to the other, via a simple algebraic correction, and, in so doing, predict the number of inequivalent irreducible constituents in a given parabolically induced representation of G. The precise conjecture (ABP Part 1) consists of two statements.

Conjecture.

(1) *The infinitesimal character*

$$\pi_{\mathfrak{s}} : \widehat{G}_{\mathfrak{s}} \to T_{\mathfrak{s}}/W_{\mathfrak{s}}$$

is one-to-one if and only if the action of $W_{\mathfrak{s}}$ on $T_{\mathfrak{s}}$ is free.

(2) *There exists a bijection*

$$\mu_{\mathfrak{s}} : T_{\mathfrak{s}}//W_{\mathfrak{s}} \longleftrightarrow \widehat{G}_{\mathfrak{s}}$$

with the following five properties:

Notation for Property 1:

$\mathfrak{s} \in \pi_o \mathrm{Prim}(\mathcal{H}G)$

Within the smooth dual $\widehat{G}$ have the tempered dual $\widehat{G}_{tempered}$.

$\widehat{G}_{tempered}$ = {*smooth tempered irreducible representations of G*}$/\sim$

$\Delta_{\mathfrak{s}}$ = *maximal compact subgroup of $T_{\mathfrak{s}}$.*

$\Delta_{\mathfrak{s}}$ is a compact torus. The action of $W_{\mathfrak{s}}$ on $T_{\mathfrak{s}}$ preserves the maximal compact subgroup $\Delta_{\mathfrak{s}}$, so can form the compact orbifold $\Delta_{\mathfrak{s}}//W_{\mathfrak{s}}$.

Property 1 of the bijection $\mu_{\mathfrak{s}}$

- The bijection $\mu_{\mathfrak{s}} : T_{\mathfrak{s}}//W_{\mathfrak{s}} \longleftrightarrow \widehat{G}_{\mathfrak{s}}$ maps
  $\Delta_{\mathfrak{s}}//W_{\mathfrak{s}}$ onto $\widehat{G}_{\mathfrak{s}} \cap \widehat{G}_{tempered}$
  $\Delta_{\mathfrak{s}}//W_{\mathfrak{s}} \longleftrightarrow \widehat{G}_{\mathfrak{s}} \cap \widehat{G}_{tempered}$

Property 2 of the bijection $\mu_{\mathfrak{s}}$

- For many $\mathfrak{s}$ the diagram

$$\begin{tikzcd} T_{\mathfrak{s}}/\!/W_{\mathfrak{s}} \arrow[r, "\mu_{\mathfrak{s}}"] \arrow[d, "\rho_{\mathfrak{s}}"'] & \widehat{G}_{\mathfrak{s}} \arrow[d, "\pi_{\mathfrak{s}}"] \\ T_{\mathfrak{s}}/W_{\mathfrak{s}} \arrow[r, "I"'] & T_{\mathfrak{s}}/W_{\mathfrak{s}} \end{tikzcd}$$

does not commute. ($I =$ the identity map of $T_{\mathfrak{s}}/W_{\mathfrak{s}}$.)

Property 3 of the bijection $\mu_{\mathfrak{s}}$

- In the possibly non-commutative diagram

$$\begin{tikzcd} T_{\mathfrak{s}}/\!/W_{\mathfrak{s}} \arrow[r, "\mu_{\mathfrak{s}}"] \arrow[d, "\rho_{\mathfrak{s}}"'] & \widehat{G}_{\mathfrak{s}} \arrow[d, "\pi_{\mathfrak{s}}"] \\ T_{\mathfrak{s}}/W_{\mathfrak{s}} \arrow[r, "I"'] & T_{\mathfrak{s}}/W_{\mathfrak{s}} \end{tikzcd}$$

the bijection $\mu_{\mathfrak{s}} : T_{\mathfrak{s}}/\!/W_{\mathfrak{s}} \longrightarrow \widehat{G}_{\mathfrak{s}}$ is continuous where $T_{\mathfrak{s}}/\!/W_{\mathfrak{s}}$ has the Zariski topology and $\widehat{G}_{\mathfrak{s}}$ has the Jacobson topology

AND the composition

$$\pi_{\mathfrak{s}} \circ \mu_{\mathfrak{s}} : T_{\mathfrak{s}}/\!/W_{\mathfrak{s}} \longrightarrow T_{\mathfrak{s}}/W_{\mathfrak{s}}$$

is a morphism of algebraic varieties.

Property 4 of the bijection $\mu_{\mathfrak{s}}$

- For each $\mathfrak{s} \in \pi_o\mathrm{Prim}(\mathcal{H}G)$ there is an algebraic family

$$\theta_\tau : T_{\mathfrak{s}}/\!/W_{\mathfrak{s}} \longrightarrow T_{\mathfrak{s}}/W_{\mathfrak{s}}$$

of finite morphisms of algebraic varieties, with $\tau \in \mathbb{C}^\times$, such that

$$\theta_1 = \rho_{\mathfrak{s}}, \quad \theta_{\sqrt{q}} = \pi_{\mathfrak{s}} \circ \mu_{\mathfrak{s}}, \quad \text{and} \quad \theta_{\sqrt{q}}(T_{\mathfrak{s}}/\!/W_{\mathfrak{s}} - T_{\mathfrak{s}}/W_{\mathfrak{s}}) = \mathfrak{R}_{\mathfrak{s}}.$$

Here q is the order of the residue field of the p-adic field F over which G is defined and $\mathfrak{R}_{\mathfrak{s}} \subset T_{\mathfrak{s}}/W_{\mathfrak{s}}$ is the sub-variety of reducibility.

Property 5 of the bijection $\mu_{\mathfrak{s}}$ (Correcting cocharacters)

- Fix $\mathfrak{s} \in \pi_o\mathrm{Prim}(\mathcal{H}G)$. For each irreducible component $\mathbf{c}$ of the affine variety $T_{\mathfrak{s}}/\!/W_{\mathfrak{s}}$ there is a cocharacter (i.e. a homomorphism of algebraic groups)

$$h_c : \mathbb{C}^\times \longrightarrow T_{\mathfrak{s}}$$

such that

$$\theta_\tau(w, t) = \lambda(h_c(\tau) \cdot t)$$

for all $(w, t) \in \mathbf{c}$.

λ: $T_{\mathfrak{s}} \longrightarrow T_{\mathfrak{s}}/W_{\mathfrak{s}}$ is the quotient map from $T_{\mathfrak{s}}$ to $T_{\mathfrak{s}}/W_{\mathfrak{s}}$.

Remark. The equality
$$\theta_\tau(w,t) = \lambda(h_c(\tau)\cdot t)$$
is to be interpreted thus:
Let $Z_1, Z_2, \ldots, Z_r$ be the irreducible components of the affine variety $T_\mathfrak{s}//W_\mathfrak{s}$ and let $h_1, h_2, \ldots, h_r$ be the cocharacters as in the statement of Property 5. Let
$$\nu_\mathfrak{s}\colon \widetilde{T_\mathfrak{s}} \longrightarrow T_\mathfrak{s}//W_\mathfrak{s}$$
be the quotient map.
Then irreducible components $X_1, X_2, \ldots, X_r$ of the affine variety $\widetilde{T_\mathfrak{s}}$ can be chosen with

- $\nu_\mathfrak{s}(X_j) = Z_j$ for $j = 1, 2, \ldots, r$
- For each $\tau \in \mathbb{C}^\times$, the map $m_\tau\colon X_j \to T_\mathfrak{s}/W_\mathfrak{s}$ which is the composition
$$\begin{array}{ccccc} X_j & \longrightarrow & T_\mathfrak{s} & \longrightarrow & T_\mathfrak{s}/W_\mathfrak{s} \\ (w,t) & \longmapsto & h_j(\tau)t & \longmapsto & \lambda(h_j(\tau)t) \end{array}$$
makes the diagram
$$\begin{tikzcd} X_j \arrow[r, "\nu_\mathfrak{s}"] \arrow[d, "m_\tau"'] & Z_j \arrow[d, "\theta_\tau"] \\ T_\mathfrak{s}/W_\mathfrak{s} \arrow[r, "I"'] & T_\mathfrak{s}/W_\mathfrak{s} \end{tikzcd}$$
commutative.
Note that $h_j(\tau)t$ is the product in the algebraic group $T_\mathfrak{s}$ of $h_j(\tau)$ and t.

Remark. The conjecture asserts that to calculate
$$\pi_\mathfrak{s}\colon \widehat{G}_\mathfrak{s} \longrightarrow T_\mathfrak{s}/W_\mathfrak{s}$$
two steps suffice:
• Step 1: Calculate $\rho_\mathfrak{s}\colon T_\mathfrak{s}//W_\mathfrak{s} \longrightarrow T_\mathfrak{s}/W_\mathfrak{s}$.
• Step 2: Determine the correcting cocharacters.

6. Where are the correcting cocharacters coming from?

In this section, G is a split reductive group defined over F. $G^\vee$ denotes the Langlands dual group of G (a complex Lie group with root datum dual to that of G, for instance $\mathrm{GL}(n,F)^\vee = \mathrm{GL}(n,\mathbb{C})$, $\mathrm{SL}(n,F)^\vee = \mathrm{PGL}(n,\mathbb{C})$, $\mathrm{PGL}(n,F)^\vee = \mathrm{SL}(n,F)$). W_F is the Weil group attached to F.
By a "Langlands parameter" for G we mean a homomorphism of topological groups
$$W_F \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow G^\vee.$$
such that:

- When restricted to $\mathrm{SL}(2,\mathbb{C})$ the homomorphism of topological groups is algebraic.
- When restricted to W_F, the homomorphism of topological groups maps the Frobenius element of W_F to a semi-simple element of $G^\vee$ (see [33; p. 278], [47]).

Remark. For an earlier definition of Langlands parameter based on the Weil-Deligne group see [15; § 8.2] and [12; Ch. 11]. The $W_F \times \mathrm{SL}(2, \mathbb{C})$ definition is better suited to connecting ABP and local Langlands.

The correcting co-characters of Property 5 above seem to be produced by the $\mathrm{SL}(2, \mathbb{C})$ part of the Langlands parameters— i.e. the standard maximal torus of $\mathrm{SL}(2, \mathbb{C})$ identifies with $\mathbb{C}^\times$:

$$\zeta \longleftrightarrow \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}, \qquad \zeta \in \mathbb{C}^\times$$

Using this identification, when a Langlands parameter is restricted to the maximal torus of $\mathrm{SL}(2, \mathbb{C})$, a cocharacter of $G^\vee$

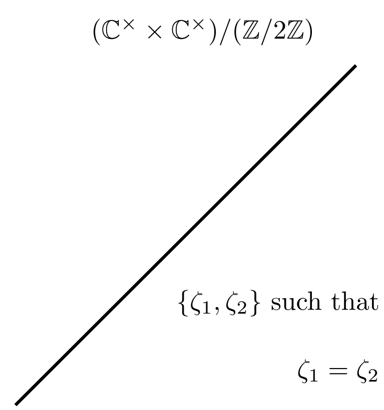
$$\mathbb{C}^\times \longrightarrow G^\vee$$

is obtained, and in examples all the correcting cocharacters arise this way.

Example (The Iwahori-spherical component of $\mathrm{GL}(2, F)$)**.** Let $G = \mathrm{GL}(2, F)$.

- $\widehat{G}_{\mathfrak{s}}$ = {Smooth irreducible representations of $\mathrm{GL}(2, F)$ having a non-zero Iwahori fixed vector}.
- $T_{\mathfrak{s}}$ = {unramified characters of the maximal torus of $\mathrm{GL}(2, F) = \mathbb{C}^\times \times \mathbb{C}^\times$.
- $W_{\mathfrak{s}}$ = the Weyl group of $\mathrm{GL}(2, F) = \mathbb{Z}/2\mathbb{Z}$.
- $0 \neq \gamma \in \mathbb{Z}/2\mathbb{Z} \qquad \gamma(\zeta_1, \zeta_2) = (\zeta_2, \zeta_1) \quad (\zeta_1, \zeta_2) \in \mathbb{C}^\times \times \mathbb{C}^\times$

The extended quotient $(\mathbb{C}^\times \times \mathbb{C}^\times)//(\mathbb{Z}/2\mathbb{Z})$ is the disjoint union of the ordinary quotient $(\mathbb{C}^\times \times \mathbb{C}^\times)/(\mathbb{Z}/2\mathbb{Z})$ and $\mathbb{C}^\times$. The ordinary quotient $(\mathbb{C}^\times \times \mathbb{C}^\times)/(\mathbb{Z}/2\mathbb{Z})$ consists of unordered pairs of non-zero complex numbers. Such an unordered pair will be denoted $\{\zeta_1, \zeta_2\}$. The projection of the extended quotient onto the ordinary quotient is the identity map when restricted to the copy of the ordinary quotient contained in the extended quotient— and when restricted to $\mathbb{C}^\times$ maps ζ to $\{\zeta, \zeta\}$. Hence the picture for the projection of the extended quotient onto the ordinary quotient is:

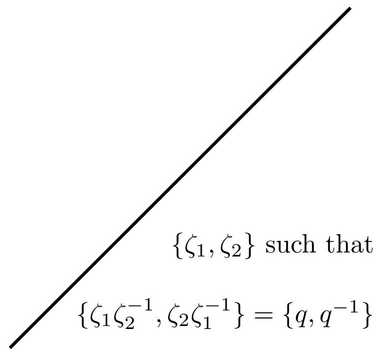


In this picture the ambient variety is the ordinary quotient $(\mathbb{C}^\times \times \mathbb{C}^\times)/(\mathbb{Z}/2\mathbb{Z})$. The locus of points in the ordinary quotient whose pre-image consists of more than one point is $\{\zeta_1, \zeta_2\}$ such that $\zeta_1 = \zeta_2$. This locus is a sub-variety indicated by the slanted line.

For the bijection $\mu_{\mathfrak{s}}$ composed with the infinitesimal character, the picture is:

$$(\mathbb{C}^\times \times \mathbb{C}^\times)//(\mathbb{Z}/2\mathbb{Z}) = (\mathbb{C}^\times \times \mathbb{C}^\times)/(\mathbb{Z}/2\mathbb{Z}) \coprod \mathbb{C}^\times$$

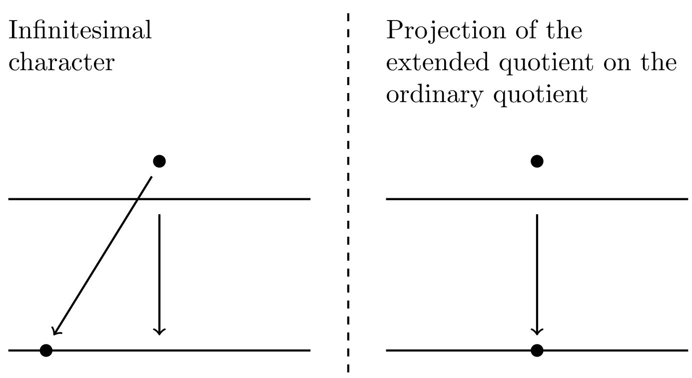
$(\mathbb{C}^\times \times \mathbb{C}^\times)/(\mathbb{Z}/2\mathbb{Z})$



In this picture the ambient variety is the ordinary quotient $(\mathbb{C}^\times \times \mathbb{C}^\times)/(\mathbb{Z}/2\mathbb{Z})$. The locus of points in the ordinary quotient whose pre-image consists of more than one point is $\{\zeta_1, \zeta_2\}$ such that $\{\zeta_1\zeta_2^{-1}, \zeta_2\zeta_1^{-1}\} = \{q, q^{-1}\}$, where q is the order of the residue field of F. This locus is a sub-variety indicated by the slanted line. On the copy of the ordinary quotient contained within the extended quotient, the map is the identity map. On $\mathbb{C}^\times$ the map is

$$\zeta \longmapsto \{q^{1/2}\zeta, q^{-1/2}\zeta\}$$

The correcting cocharacter $\mathbb{C}^\times \longrightarrow \mathbb{C}^\times \times \mathbb{C}^\times$ is $\tau \mapsto (\tau, \tau^{-1})$



The picture for ρ_5 is obtained by taking the picture for π_5 and setting $q = 1$.

7. Interaction with Baum-Connes and Local Langlands

ABP can be viewed as providing a link between LL (local Langlands conjecture) and BC (Baum-Connes conjecture).

$$\begin{array}{c}\text{Baum-Connes}\\ \updownarrow\\ \text{ABP}\\ \updownarrow\\ \text{Local Langlands}\end{array}$$

This picture is intended to make the point that LL and BC taken by themselves do not appear to have much interaction — but ABP interacts with both LL and BC. LL together with a precise determination of the L-packets should imply validity for ABP. See [5]. Does ABP imply LL? This is unclear at the present time. See [5].

As indicated above, the correcting cocharacters of ABP appear to be coming from the $SL(2,\mathbb{C})$ part of Langlands parameters. Thus ABP interacts with LL. An intriguing question is "In the ABP view of $\widehat{G}$, what are the L-packets?". See section 10 below for a possible answer to this question.

For a reductive p-adic group G, BC [7] asserts that the Baum-Connes map

$$K_j^G(\beta G) \longrightarrow K_j(C_r^*G) \qquad j=0,1$$

is an isomorphism of abelian groups. $K_j^G(\beta G)$ is the Kasparov equivariant K-homology of the (extended) affine Bruhat-Tits building βG of G.

$$K_j^G(\beta G) := KK_G^j(C_0(\beta G),\mathbb{C}) \qquad j=0,1$$

C_r^*G is the reduced C^* algebra of G. This is the C^* algebra obtained by completing $\mathcal{H}G$ via the (left) regular representation of G. $K_*(C_r^*G)$ is the K-theory— in the sense of C^* algebra K-theory— of C_r^*G. The Hecke algebra $\mathcal{H}G$ of G decomposes (canonically) into a direct sum of two-sided ideals:

$$\mathcal{H}(G) = \bigoplus_{\mathfrak{s}\in\pi_o\mathrm{Prim}(\mathcal{H}G)} \mathcal{I}_{\mathfrak{s}}$$

Passing to the C^* algebra completion yields a direct sum decomposition—in the sense of C^* algebras—

$$C_r^*G = \bigoplus_{\mathfrak{s}\in\pi_o\mathrm{Prim}(\mathcal{H}G)} \overline{\mathcal{I}}_{\mathfrak{s}}$$

and this gives a direct sum decomposition of $K_*C_r^*G$

$$K_jC_r^*G = \bigoplus_{\mathfrak{s}\in\pi_o\mathrm{Prim}(\mathcal{H}G)} K_j(\overline{\mathcal{I}}_{\mathfrak{s}}) \qquad j=0,1$$

ABP at the level of C^*algebra K-theory is:

Conjecture. *Let G be a reductive p-adic group. Then for each $\mathfrak{s}\in\pi_o\mathrm{Prim}(\mathcal{H}G)$*

$$K_j(\overline{\mathcal{I}}_{\mathfrak{s}}) = K^j_{W_{\mathfrak{s}}}(\Delta_{\mathfrak{s}}) \qquad j=0,1$$

Here $K^j_{W_{\mathfrak{s}}}(\Delta_{\mathfrak{s}})$ is Atiyah-Hirzebruch-Segal topological equivariant K-theory [1] for the finite group $W_{\mathfrak{s}}$ acting on the compact Hausdorff space $\Delta_{\mathfrak{s}}$. Note that the group structure of $\Delta_{\mathfrak{s}}$ is not being used. Applying the Chern character [8] gives a map

$$K^j_{W_{\mathfrak{s}}}(\Delta_{\mathfrak{s}}) \longrightarrow \bigoplus_l H^{j+2l}(\Delta_{\mathfrak{s}}//W_{\mathfrak{s}};\mathbb{C})$$

which becomes an isomorphism when $K^j_{W_{\mathfrak{s}}}(\Delta_{\mathfrak{s}})$ is tensored with $\mathbb{C}$. Hence ABP at the level of C^* algebra K-theory gives a much finer and more precise formula for $K_*C^*_rG$ than BC alone provides. For an explicitly computed example see [27].

Theorem 1 (V. Lafforgue [34])**.** *Baum-Connes is valid for any reductive p-adic group G.*

Theorem 2 ([25, 23, 35])**.** *Local Langlands is valid for* $\mathrm{GL}(n,F)$.

Theorem 3 ([2, 16, 17])**.** *ABP is valid for* $\mathrm{GL}(n,F)$.

8. Where is the bijection $\mu_{\mathfrak{s}}\colon T_{\mathfrak{s}}//W_{\mathfrak{s}} \longleftrightarrow \widehat{G}_{\mathfrak{s}}$ coming from? (ABP Part 2)

Notation. If X is a (complex) affine variety, $\mathcal{O}(X)$ denotes the co-ordinate algebra of X.

As above, the Hecke algebra of G decomposes (canonically) into a direct sum of two-sided ideals:

$$\mathcal{H}(G) = \bigoplus_{\mathfrak{s}\in\pi_o\mathrm{Prim}(\mathcal{H}G)} \mathcal{I}_{\mathfrak{s}}$$

Each ideal $\mathcal{I}_{\mathfrak{s}}$ is canonically Morita equivalent to a unital finite-type $k_{\mathfrak{s}}$-algebra where

$$k_{\mathfrak{s}} = \mathcal{O}(T_{\mathfrak{s}}/W\mathfrak{s})$$

This unital finite-type $k_{\mathfrak{s}}$-algebra will be denoted $\mathcal{H}(G)_{\mathfrak{s}}$. The set of (isomorphism classes of) simple $\mathcal{H}(G)_{\mathfrak{s}}$ modules (equivalently, the set of primitive ideals in $\mathcal{H}(G)_{\mathfrak{s}}$) is canonically in bijection with $\widehat{G}_{\mathfrak{s}}$.

$$\mathrm{Prim}(\mathcal{H}(G)_{\mathfrak{s}}) \longleftrightarrow \widehat{G}_{\mathfrak{s}}$$

Fix an affine variety X and consider the category of all finite-type k-algebras, where $k = \mathcal{O}(X)$. These k-algebras are required to be of finite-type (i.e. are required to be finitely generated as k-modules), but are not required to be unital. In [3] the authors of this note introduced an equivalence relation called *geometric equivalence* for such algebras. This equivalence relation is a weakening of Morita equivalence— if two unital finite-type k-algebras are Morita equivalent, then they are geometrically equivalent. A detailed exposition of geometric equivalence will be given in [6]. If A_1 and A_2 are two finite-type k-algebras which are geometrically equivalent, then there is an isomorphism of periodic cyclic homology [9, 10]

$$\mathrm{HP}_*(A_1) \cong \mathrm{HP}_*(A_2)$$

and a bijection of sets

$$\mathrm{Prim}(A_1) \longleftrightarrow \mathrm{Prim}(A_2).$$

Part 2 of the ABP conjecture is the assertion that:

Conjecture.

- *The finite-type $\mathcal{O}(T_{\mathfrak{s}}/W_{\mathfrak{s}})$-algebras $\mathcal{H}(G)_{\mathfrak{s}}$ and $\mathcal{O}(T_{\mathfrak{s}}//W_{\mathfrak{s}})$ are geometrically equivalent.*

- $\mathrm{HP}_j(\mathcal{H}(G)_\mathfrak{s}) = \bigoplus_l H^{j+2l}(T_\mathfrak{s}/\!/W_\mathfrak{s};\mathbb{C})$
- *A bijection* $\mu_\mathfrak{s} : T_\mathfrak{s}/\!/W_\mathfrak{s} \longleftrightarrow \widehat{G}_\mathfrak{s}$ *having properties 1–5 listed above can be constructed by choosing a suitable geometric equivalence between* $\mathcal{H}(G)_\mathfrak{s}$ *and* $\mathcal{O}(T_\mathfrak{s}/\!/W_\mathfrak{s})$.

9. Recent Developments

Maarten Solleveld [51] has recently proved a result which implies validity for Part 1 of the ABP Conjecture as stated in section 5 whenever $\mathcal{H}(G)_\mathfrak{s}$ is Morita equivalent(as a $k_\mathfrak{s}$-algebra) to an extended affine Hecke algebra, perhaps with unequal parameters. For Part 2 of ABP Solleveld does prove the isomorphism of periodic cyclic homology

$$\mathrm{HP}_j(\mathcal{H}(G)_\mathfrak{s}) = \bigoplus_l H^{j+2l}(T_\mathfrak{s}/\!/W_\mathfrak{s};\mathbb{C}).$$

Although he does not prove that $\mathcal{H}(G)_\mathfrak{s}$ is geometrically equivalent to $\mathcal{O}(T_\mathfrak{s}/\!/W_\mathfrak{s})$, he does obtain a result of this kind for appropriate topological completions of $\mathcal{H}(G)_\mathfrak{s}$ and $\mathcal{O}(T_\mathfrak{s}/\!/W_\mathfrak{s})$.

The unital finite-type $k_\mathfrak{s}$-algebra $\mathcal{H}(G)_\mathfrak{s}$ has been proved to be Morita equivalent (as a $k_\mathfrak{s}$-algebra) to an extended affine Hecke algebra in the following cases:

- G split, $\mathfrak{s}$ the Iwahori-spherical component [14];
- G split (some restriction on the residual characteristic of F), $\mathfrak{s}$ in the principal series [48];
- G arbitrary, σ of level 0 [42, 43, 46, 40].
- $G = \mathrm{GL}_n(F)$, $\mathfrak{s}$ arbitrary, and $\mathrm{SL}_n(F)$, many $\mathfrak{s}$ - from the work of Bushnell and Kutzko on types [18, 19, 20] and Goldberg and Roche [22].
- $G = \mathrm{SO}_n(F)$, $G = \mathrm{Sp}_{2n}(F)$ or G an inner form of GL_n, $\mathfrak{s}$ arbitrary, see [24], see also Kim [31, 32].
- $G = \mathrm{GSp}_4(F)$ or $G = \mathrm{U}(2,1)$, $\mathfrak{s}$ arbitrary [44, 45, 13];

One of these cases is the Iwahori component. This raises the question of reconciling ABP with the Kazhdan-Lusztig parametrization of the Iwahori component. This reconciliation will be given in [5]. More generally, the reconciliation with Reeder's result [47] on the principal series will also be given in [5]. For Reeder, G is assumed to be split and to have connected center (in the Zariski topology).

Detailed calculations which verify ABP completely for the principal series of the p-adic group G_2 ($p \neq 2,3,5$) will appear in [4].

10. L-packets

In this section we give our proposed answer to the question "In the ABP view of $\widehat{G}$, what are the L-packets?". This answer is based on calculations of examples — and on the connection between cells (in the appropriate extended Coxeter group) and correcting cocharacters described in the appendix below. The main point is that in the list of correcting cocharacters $h_1, h_2, \ldots, h_r$ (one for each irreducible component of the affine variety $T_\mathfrak{s}/\!/W_\mathfrak{s}$) there may be repetitions, i.e. it might happen that for some i, j with $1 \le i < j \le r$, $\quad h_i = h_j$, and these repetitions give rise to L-packets.

So (as in the statement of ABP above) assume that $\mathfrak{s} \in \pi_o\mathrm{Prim}(\mathcal{H}G)$ has been fixed. Let

$$\mu_{\mathfrak{s}} : T_{\mathfrak{s}}//W_{\mathfrak{s}} \longrightarrow \widehat{G}_{\mathfrak{s}}$$

be the bijection of ABP, and let $h_1, h_2, \ldots, h_r$ be the correcting cocharacters. As in section 5 above, for $j = 1, 2, \ldots, r$ consider the commutative diagram

$$\begin{tikzcd}
X_j \arrow[r, "\nu_{\mathfrak{s}}"] \arrow[d, "m_\tau"'] & Z_j \arrow[d, "\theta_\tau"] \\
T_{\mathfrak{s}}/W_{\mathfrak{s}} \arrow[r, "I"'] & T_{\mathfrak{s}}/W_{\mathfrak{s}}
\end{tikzcd}$$

For each $\tau \in \mathbb{C}^\times$ there is then the map of affine varieties

$$\theta_\tau : T_{\mathfrak{s}}//W_{\mathfrak{s}} \longrightarrow T_{\mathfrak{s}}/W_{\mathfrak{s}}$$

Conjecture. *Two points (w, t) and (w', t') in $T_{\mathfrak{s}}//W_{\mathfrak{s}}$ have*

$$\mu_{\mathfrak{s}}(w,t) \textit{ and } \mu_{\mathfrak{s}}(w',t') \textit{ in the same } L\textit{-packet}$$

if and only

$$h_i = h_j \qquad \textit{where } (w,t) \in Z_i \quad \textit{and } (w',t') \in Z_j$$

and

$$\textit{For all } \tau \in \mathbb{C}^\times,\ \theta_\tau(w,t) = \theta_\tau(w',t')$$

Remark. An L-packet can have non-empty intersection with more than one Bernstein component of $\widehat{G}$. This conjecture does not address that issue. The conjecture only describes the intersections of L-packets with $\widehat{G}_{\mathfrak{s}}$ once $\mathfrak{s} \in \pi_o\mathrm{Prim}(\mathcal{H}G)$ has been fixed.

The following appendix indicates how repetitions can occur among correcting cocharacters.

11. Appendix

By *extended Coxeter group* we will mean below a semidirect product of a Coxeter group by a finite abelian group. The finite abelian group is assumed to act as automorphisms of the Coxeter system. We give below a construction which in many examples assigns to a Bernstein component an extended (infinite) Coxeter group. In this setting (in examples) a significant simplification is achieved in the correcting cocharacters because the cocharacters are indexed by the cells of the associated extended Coxeter group. In particular, this reveals repetitions among the cocharacters. So, if the conjecture of the preceding section is valid, this creates L-packets.

The group $W_{\mathfrak{s}}$ admits the structure of a finite extended Coxeter group.

Let A_M denote the identity component of the center of M and let Φ_M denote the set of roots of A_M. For each $\alpha \in \Phi_M$, we write U_α for the corresponding root group and set $G_\alpha := \langle M, U_\alpha, U_{-\alpha}\rangle$. Then $G_\alpha = G_{-\alpha}$ is a reductive group. It has two parabolic subgroups with Levi component M, $P_\alpha = MU_\alpha$ and $P_{-\alpha} = MU_{-\alpha}$. Let ψ be an unramified character of M. By normalized parabolic induction from

P_α and $P_{-\alpha}$, we obtain the representations $\mathrm{Ind}_{P_\alpha}^{G_\alpha}(\psi\otimes\sigma)$ and $\mathrm{Ind}_{P_{-\alpha}}^{G_\alpha}(\psi\otimes\sigma)$. These are related by standard intertwining operators

$$J_{-\alpha,\alpha}\colon \mathrm{Ind}_{P_\alpha}^{G_\alpha}(\psi\otimes\sigma) \to \mathrm{Ind}_{P_{-\alpha}}^{G_\alpha}(\psi\otimes\sigma) \quad \text{and} \quad J_{\alpha,-\alpha}\colon \mathrm{Ind}_{P_{-\alpha}}^{G_\alpha}(\psi\otimes\sigma) \to \mathrm{Ind}_{P_\alpha}^{G_\alpha}(\psi\otimes\sigma).$$

There is a rational function $j_\alpha(\psi,\sigma)$ on the complex torus $\widetilde{T}_\mathfrak{s}$ such that

$$J_{\alpha,-\alpha} \circ J_{-\alpha,\alpha} = j_\alpha(\psi,\sigma)\,\mathrm{id}$$

(and $j_\alpha(\psi,\sigma) = j_{-\alpha}(\psi,\sigma)$).

Let

$$\Phi_\mathfrak{s} := \{\alpha \in \Phi_M \;:\; j(\psi,\sigma) \text{ has a pole}\}\,.$$

The set $\Phi_\mathfrak{s}$ is a root system. It is preserved by $W_\mathfrak{s}$. Let $W_\mathfrak{s}^0$ denote the finite Weyl group associated to $\Phi_\mathfrak{s}$. Fix a positive system $\Phi_\mathfrak{s}^+$ in $\Phi_\mathfrak{s}$ and set

$$C_\mathfrak{s} := \left\{w \in W_\mathfrak{s} \;:\; w(\Phi_\mathfrak{s}^+) \subset \Phi_\mathfrak{s}^+\right\}.$$

Then the group $W_\mathfrak{s}$ is the semi-direct product

$$W_\mathfrak{s} = W_\mathfrak{s}^0 \rtimes C_\mathfrak{s}.$$

Hence $W_\mathfrak{s}$ occurs to be a finite extended Coxeter group.

An extended (infinite) Coxeter group attached to $\mathfrak{s}$.

Set

$$M_\mathfrak{s} := \bigcap_{\varphi \in I_\sigma} \mathrm{Ker}(\varphi) \quad \text{and} \quad \Lambda_\mathfrak{s} := M_\mathfrak{s}/M^0.$$

The group $\Lambda_\mathfrak{s}$ is free abelian of the same rank as M/M^0. Conjugation by $W_\mathfrak{s}$ preserves $M_\mathfrak{s}$. There is therefore an induced action of $W_\mathfrak{s}$ on $\Lambda_\mathfrak{s}$. Now $\Lambda_\mathfrak{s}$ is isomorphic to the group of characters $X(T_\mathfrak{s})$ of the complex torus $T_\mathfrak{s}$.

Then we set

$$\widetilde{W}_\mathfrak{s} := X(T_\mathfrak{s}) \rtimes W_\mathfrak{s}.$$

We have

$$\widetilde{W}_\mathfrak{s} \simeq (X(T_\mathfrak{s}) \rtimes W_\mathfrak{s}^0) \rtimes C_\mathfrak{s}.$$

In many examples the group $X(T_\mathfrak{s}) \rtimes W_\mathfrak{s}^0$ is an affine Weyl group. Hence $\widetilde{W}_\mathfrak{s}$ in many examples is an extended Coxeter group.

A weight function attached to $\mathfrak{s}$

Let $M^\vee$ denote the identity component of the Langlands dual group of M, let $\alpha \in \Phi_M$, and let r_α denote the adjoint representation of $M^\vee$ on the Lie algebra of $U_\alpha^\vee$. We are assuming here that σ is generic so that the corresponding local L-functions $L(s,\sigma,r_\alpha)$ are defined by Shahidi. Then the definition of $\Phi_\mathfrak{s}$ can be rephrased as follows (thanks to a formula by Shahidi for $j_\alpha(\psi,\sigma)$ [50]):

$$\Phi_\mathfrak{s} = \{\alpha \in \Phi_M \;:\; L(s,\sigma,r_\alpha) \neq 1\}\,.$$

For each $\alpha \in \Phi_M$, we denote by q_α the degree of the L-function $L(s,\sigma,r_\alpha)$, that is, q_α is the degree of $P(T)$ where $P(T)$ is the polynomial such that $L(s,\sigma,r_\alpha) = P(q^{-s})^{-1}$. We have $q_{w\alpha} = q_\alpha$ for all $w \in W_\mathfrak{s}$. From this we define a *weight function* $\mathbf{q}_\mathfrak{s}\colon \Phi_M \to \mathbf{N}$ by

$$\mathbf{q}_\mathfrak{s}(\alpha) := q_\alpha \quad \alpha \in \Phi_M.$$

The collection of two-sided cells attached to $\mathfrak{s}$

In the case when the function $\mathbf{q}_\mathfrak{s}$ is constant, we will attach to $\mathfrak{s}$ the collection of all the *two-sided cells* of the group $\widetilde{W}_\mathfrak{s}$ as originally defined by Lusztig in [36], [37].

In the general case, Lusztig stated in [41] (and proved in several cases) a list of conjectures, and, assuming their validity, defined partitions of any extended Coxeter group equipped with a weight function as above into left cells, right cells and two-sided cells, which extend the theory previously developped by Kazhdan and him in the case of "equal parameters" (that is, when $\mathbf{q}_\mathfrak{s}$ is constant). Hence we can attach (at least conjecturally) a collection of two-sided cells to $(\widetilde{W}_\mathfrak{s}, \mathbf{q}_\mathfrak{s})$. We will denote this collection by $\mathrm{Cell}(\widetilde{W}_\mathfrak{s}, \mathbf{q}_\mathfrak{s})$.

It is part of ABP Conjecture that each cocharacter should be attached to a two-sided cell of $(\widetilde{W}_\mathfrak{s}, \mathbf{q}_\mathfrak{s})$. This part of ABP Conjecture is proved for principal series of G_2 (when $p \neq 2, 3, 5$) in [4].

More precisely, for each Bernstein component $\mathfrak{s}$ attached to a principal series of G_2, the weight function $\mathbf{q}_\mathfrak{s}$ is the constant function, and there is a decomposition of the extended quotient

$$T_\mathfrak{s}//W_\mathfrak{s} = \bigsqcup_{\mathbf{c} \in \mathrm{Cell}(\widetilde{W}_\mathfrak{s}, \mathbf{q}_\mathfrak{s})} (T_\mathfrak{s}//W_\mathfrak{s})_\mathbf{c},$$

such that each $(T_\mathfrak{s}//W_\mathfrak{s})_\mathbf{c}$ is a union of irreducible components c of the affine variety $T_\mathfrak{s}//W_\mathfrak{s}$ and

$$h_c = h_{c'} \quad \text{for all } c, c' \subset (T_\mathfrak{s}//W_\mathfrak{s})_\mathbf{c}.$$

Moreover, the partition can be chosen so that the following property holds:

$$T_\mathfrak{s}/W_\mathfrak{s} \subset (T_\mathfrak{s}//W_\mathfrak{s})_{\mathbf{c}_0},$$

where $\mathbf{c}_0$ denotes the *lowest* two-sided cell in $\widetilde{W}_\mathfrak{s}$ in the natural partial ordering on $\mathrm{Cell}(\widetilde{W}_\mathfrak{s}, \mathbf{q}_\mathfrak{s})$. Note the inclusion above is not an equality in general.

Where is the geometric equivalence $\mathcal{H}(G)_\mathfrak{s} \asymp \mathcal{O}(T_\mathfrak{s}//W_\mathfrak{s})$ coming from?

Let $\mathcal{J}^\mathfrak{s}$ denote the *Lusztig asymptotic algebra* attached to the group $(\widetilde{W}_\mathfrak{s}, \mathbf{q}_\mathfrak{s})$ (in case the weight function $\mathbf{q}_\mathfrak{s}$ is constant, $\mathcal{J}^\mathfrak{s}$ is defined in [38; §1.3], in general, the definition of $\mathcal{J}^\mathfrak{s}$ is given in [41] up to a list of conjectural properties). The algebra $\mathcal{J}^\mathfrak{s}$ admits a canonical decomposition into finitely many two-sided ideals

$$\mathcal{J}^\mathfrak{s} = \bigoplus_{\mathbf{c} \in \mathrm{Cell}(\widetilde{W}_\mathfrak{s}, \mathbf{q}_\mathfrak{s})} \mathcal{J}^\mathfrak{s}_\mathbf{c}.$$

One can provide $\mathcal{J}^\mathfrak{s}$ (and also each $\mathcal{J}^\mathfrak{s}_\mathbf{c}$) with a structure of $k_\mathfrak{s}$-module algebra (see [3; § 9]). Then $\mathcal{J}_\mathfrak{s}$ is a finite-type $k_\mathfrak{s}$ algebra.

It is also part of ABP Conjecture that the conjectural geometric equivalence $\mathcal{H}(G)_\mathfrak{s} \asymp \mathcal{O}(T_\mathfrak{s}//W_\mathfrak{s})$ comes from the combination of the following two geometric equivalences:

$$\mathcal{H}(G)_\mathfrak{s} \asymp \mathcal{J}^\mathfrak{s} \quad \text{and} \quad \mathcal{J}^\mathfrak{s} \asymp \mathcal{O}(T_\mathfrak{s}//W_\mathfrak{s}).$$

The first geometric equivalence is proved for G arbitrary, $\mathfrak{s}$ the Iwahori component in [9, 10], and G split (with mild restriction on the residual characteristic) in the principal series case, by combining [9, 10] with [48].

The second geometric equivalence is proved for $G = \mathrm{GL}(n, F)$, $\mathfrak{s}$ arbitrary in [16] [17], for $G = \mathrm{SL}(2, F)$, $\mathfrak{s}$ arbitrary in [3; § 7], for $G = \mathrm{G}_2$, principal series case in [4], for $G = \mathrm{PGL}(n, F)$, $\mathfrak{s}$ the Iwahori component in [3; § 12], for $G = \mathrm{SO}(5, F)$,

$\mathfrak{s}$ the Iwahori component in [3; § 13], for $G = \mathrm{SO}(4, F)$, $\mathfrak{s}$ the Iwahori component in [4; § 8].

Geometric equivalence respects direct sums. For $G = \mathrm{G}_2$, principal series case, and for $G = \mathrm{SO}(4, F)$, $\mathfrak{s}$ the Iwahori component, the second geometric equivalence comes from a finite collection of geometric equivalences:

$$\mathcal{J}_{\mathbf{c}}^{\mathfrak{s}} \asymp \mathcal{O}(T_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_{\mathbf{c}}, \quad \text{for any } \mathbf{c} \in \mathrm{Cell}(\widetilde{W}_{\mathfrak{s}}, \mathbf{q}_{\mathfrak{s}}).$$

Beyond the p-adic world.

Thanks to the work of Solleveld [51], the ABP Conjecture (at least Part 1) still makes sense (and is partly proved) even if there is no p-adic group in the picture. In the situation considered by Solleveld, no field F is given. He is working with an (extended) Hecke algebra with parameters — which essentially means that the weight function $\mathbf{q}_{\mathfrak{s}}$ is here replaced by a more general parameter function

$$\mathbf{q} \colon \Phi_{\mathrm{nr}}^{\vee} \to \mathbb{R}_{>0},$$

where $\mathcal{R} = (X, \Phi, X^{\vee}, \Phi^{\vee})$ is a root datum, Φ a reduced root system, and $\Phi_{\mathrm{nr}}^{\vee}$ the non-reduced root system:

$$\Phi_{\mathrm{nr}}^{\vee} := \Phi \cup \{2\alpha \;:\; \alpha^{\vee} \in X^{\vee}\}.$$

The function $\mathbf{q}$ is assumed to be W_0-invariant, with W_0 the Weyl group of $\mathcal{R}$. Solleved extends in [51] the previous notion of cocharacters to this setting, and states (and to a great extent proves) a version of the ABP conjecture in the context of extended Hecke algebras with parameters. His work includes in particular the case of exotic Kazhdan-Lusztig parameters as defined in [28].

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