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An Exploration of the Relationship between Mathematics and Music

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Between Mathematics and Music

MATH30000, 3rd Year Project

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Preface

Music has always been an important part of my life. I have listened to, studied, played and enjoyed different genres of music in different settings. Music is instrumental in my practice of yoga, meditation and chanting. In these instances, it serves as a form of expression, therapy and prayer. I have played the piano throughout my life, and was a viola player in a community orchestra for several years. In my childhood, *Saturday at the Symphony* was on my weekly schedule. Throughout my study of music, I learnt about European music theory and music history. Music has become one of my passions.

Mathematics is a discipline I have always found challenging and interesting. I fully began to appreciate the subject in high school when I did a project about fractals. It was here that I saw mathematics is a beautiful, complex subject, involving far more than what we learn in school. The presence of mathematics is everywhere! It is in nature, we use it daily, and its applications reach far into other disciplines.

While my university mathematics courses expanded my knowledge on the many different types of mathematics, I was failing to understand its greater significance or importance. We did not learn about the rich history behind the mathematics. As a result, I decided that I wanted to undertake a third year mathematics project to expand my knowledge and appreciation for this complex subject. I had a slight understanding at this stage, that mathematics and music were linked. I knew that mathematics has influenced music, but beyond that I knew little else. This project would allow me to explore this connection.

When I started my research, I was shocked and amazed to discover how much information was available about the relationship between mathematics and music, and how much controversy and difference of opinion was involved in classifying some of these

relationships. I was fascinated further when I discovered the relationship between mathematics and music is steeped in history; I loved reading about its Ancient Greek origins. This connection was at least two thousand years old, and spans different cultures and civilizations. Studying music as a part of mathematics was once part of mathematics education. This made me think that it truly is an important relationship to study! I quickly discovered that the connection between mathematics and music is huge, with a wealth of information. In this project, I have simply given a snapshot on some many areas I think are interesting and important. I have also approached my research with a Western view; I have analyzed Western musicians, composers, mathematicians and ideas. Similar research can be done for the connection between mathematics and music from other cultures, but this is not the focus of this project.

The more I read and researched, the more I thought how important it is to study and understand how mathematics relates to other disciplines, and to bring mathematics into as many fields as possible. I want to eventually teach primary school (I begin my Post Graduate Certificate in Education in September 2010). My area of focus and interest is mathematics education for very young children. I think that when a child is young, they need to learn mathematics in new and exciting ways. Children need to be shown mathematics has many applications to real life, and that it can be a challenging, exciting and fascinating subject. With this project, if I can expand my knowledge and interest in mathematics, and improve my understanding of how it's used in a greater contest, then perhaps when I teach children, I can show children how exciting this subject can be. Perhaps I can bring music into my mathematical teaching, making the subject more relevant and enjoyable for those in early childhood. This project has two fold meaning for me: to increase my knowledge and excitement for mathematics and music education and study, so that eventually, I can increase the excitement of others for these two beautiful subjects.

I would finally like to take this opportunity to thank Prof. Plymen for his guidance throughout this project. I enjoyed our discussions on this subject, and have a great deal of respect for your knowledge. I also want to thank my parents. It's because of them and the opportunities and guidance they have provided for me throughout my life that has ignited my love of mathematics and music, and really my love of learning and discovery.

Saloni Shah

May 2010

1.0 Music and Mathematics: An Introduction to their Relationship

Mathematics, in some form, has been in existence since ancient civilizations. The Inca, Egyptians and Babylonians all used mathematics, yet it was not studied for its own sake until Greek Antiquity (600-300 BC) [1]. Mathematics is a vast subject that has been approached, used and studied in different ways and forms for hundreds of years, by different cultures and civilizations. It is a subject that constantly changes, and is thus difficult to define. In the twenty-first century, a western view of mathematics is that it is the abstract science of shape, space, change, number, structure and quantity [2]. Mathematicians seek out new patterns and new conjecture using rigorous deduction. They use abstract thinking, logic and reasoning to problem solve. Mathematics can be studied for its own pleasure, or can be applied to explain phenomena in other disciplines. Physicists, for example, use mathematical language to describe the natural world.

In comparison, music is the art or science of combining vocal or instrumental (or both) sounds to produce beauty of form and harmony [3]. It is an intrinsic aspect of human existence. Like mathematics, music has been an integral aspect of cultures throughout history. Music is an artistic way of expressing emotions and ideas, and is often used to express and portray one's self and identity. Different forms of music are studied, performed, played and listened to.

Music theory is a beautiful subject that has been studied for thousands of years. Music theory is simply the study of how music works and the properties of music. It may include the analysis of any statement, belief or conception of or about music. Often music theorists will study the language and notation of music. They seek to identify patterns and structures found in composers techniques, across or within genres, and of historical periods.

Comparing the basic general definitions of mathematics and music implies that they are two very distinct disciplines. Mathematics is a scientific study, full of order, countability and calculability. Music, on the other hand, is thought to be artistic and expressive. The study of these two disciplines, though seemingly different, however, are linked and have been for over two thousand years. Music itself is indeed very mathematical, and mathematics is inherent to many basic ideas in music theory. Music theorists, like experts in other disciplines, use mathematics to develop, express and communicate their ideas.

Mathematics can describe many phenomena and concepts in music. Mathematics explains how strings vibrate at certain frequencies, and sound waves are used to describe these mathematical frequencies. Instruments are mathematical; cellos have a particular shape to resonate with their strings in a mathematical fashion. Modern technology used to make recordings on a compact disc (CD) or a digital video disc (DVD) also rely on mathematics. The relationship between mathematics and music is complex and constantly expanding, as illustrated by these examples.

This report aims to give an overview of this intricate relationship between mathematics and music by examining its different aspects. The history of the study of mathematics and music is intertwined, so it is only natural to begin this report by briefly outlining this relationship. Questions and problems arising in music theory have often been solved by investigations into mathematics and physics throughout history. The second section will discuss some of the mathematics of sound and music. Conversely, mathematical ideas and language have often directly influenced concepts of music theory. There are many examples of composers who use mathematical techniques throughout their work. The mathematical techniques of Olivier Messiaen's "musical language" will be discussed in the third section. The fourth section will discuss how music often has a religious connotation

and message, and religious composers often use music to express their ideas and beliefs. This idea will be supported with an analysis of the work of Bach and the techniques of Messiaen. Finally, the report will conclude with analysis of an argument by an American academic Jim Henle who analyze artistic aspects of mathematics, a subject traditionally deemed to be a science. He presents an argument to explain mathematicians fascination with music by claiming the two subjects are profoundly similar

2.0 Historical Connections Between Mathematics and Music

This section briefly explains the historical connection between mathematics and music. The two disciplines have been interlinked throughout history since Ancient Greek academics began their theoretical study; since antiquity, mathematicians have often been music theorists. The fascination that mathematicians have with music will then be discussed.

2.1 Music Theorists and Mathematicians: Are they one in the same?

For about a millennium, from 600 BC, Ancient Greece was one of the world's leading civilizations. The ideas and knowledge produced at this time have had a lasting influence on modern western civilizations. The "Golden Age" in Greek antiquity was approximately 450 BC, and much of what constitutes western culture today began its invention then [1]. Brilliant Greek academics contributed a wealth of knowledge about music, philosophy, biology, chemistry, physics, architecture and many other disciplines.

With the Ancient Greeks came the dawn of serious mathematics. Before their time, mathematics was a craft [1]. It was studied and used to solve everyday problems. For example, farmers might implement mathematical tools to help them lay their fields in the most economical way possible. In Greek antiquity, mathematics became an art. It was studied purely for the sake of knowledge and enjoyment [1]. Philosophers and mathematicians questioned the fundamental ideas of mathematics.

Figure 1: *Pythagoreans, followers of an Ancient Greek religion which worships numbers celebrate the early morning sunrise in a painting by Fyodor Bronnikoy.*

Pythagoras, Plato and Aristotle were three very clever academics, and very influential figures when detailing the historic connection between mathematics and music [4]. Pythagoras was born in the Classical Greek period (approximately 600 BC to 300 BC) when Greece was made of individual city-states. A dictator governed the island on which he lived, so he fled to Italy. It was there that he founded a religion (often called a cult) of mathematics. Pythagoreans, the followers of his religion, believed mathematical structures were mystical. They had elaborate rituals and rules based on mathematical ideas. To the followers, the numbers 1, 2, 3 and 4 were divine and sacred. They believed reality was constructed out of these numbers and 1, 2, 3 and 4 were deemed the building blocks of life [1]. Pythagoras was instrumental in the origin of mathematics as purely a theoretical science. In fact, the theories and results that were developed by Pythagoreans were not intended for practical use or for applications. It was forbidden for members of the Pythagorean school of thought to even earn money from teaching mathematics [1]. Throughout history, numbers have always been the building block of mathematics [2].

Plato was a Pythagorean who lived after the Golden Age of Ancient Greece. Plato believed that mathematics was the core of education [1]. He founded the first university in Greece, the Academy. Mathematics was so central to the curriculum, that above the doors of the university, the words "Let no man enter through these doors if ignorant of geometry" were written [1]. From antiquity, many famous Greek mathematicians attended Plato's university.
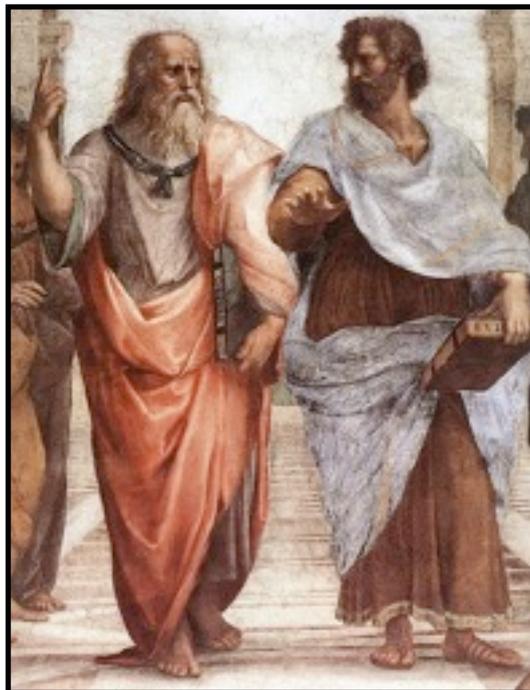

Figure 2: *A fresco from 1509 by Raphael depicting the School of Athens. Aristotle (right) gestures down to the earth, representing his belief in knowledge through empirical observation and experience. He holds a book of ethics in his hand. Plato (left) gestures to the heavens, representing his belief in the Forms.*

Aristotle, the teacher of Alexander the Great, is an example of a famous student of Plato. Aristotle was a man of great genius and the father of his own school. He studied every subject possible at the time. His writings had vast subject matter, including music, physics, poetry, theatre, logic, rhetoric, government, politics, ethics and zoology. Together with Plato and Socrates (Plato's teacher), Aristotle was one of the most important founding figures in western philosophy. He was one of the first to create a comprehensive system detailing ideas of morality, philosophy, aesthetics, logic, science, politics and metaphysics [2].

A natural question now arises: why are these ancient figures so important in understanding the relationship between mathematics and music? The answer is simple. It was these early Greek teachers and their schools of thought (the schools of Pythagoras, Plato, and Aristotle) who not only began to study mathematics and music, but considered music to be a part of mathematics [4]. Ancient Greek mathematics education was comprised of four sections: number theory, geometry, music and astronomy; this division of mathematics into four sub-topics is called a quadrivium [4]. It's been previously stated that the ideas and works of the Ancient Greeks were influential and had had a lasting effect throughout history. Those of music and mathematics were no different. The four way division of mathematics, which detailed music should be studied as part of mathematics, lasted until the end of the middle ages (approximately 1500 AD) in European culture [4].
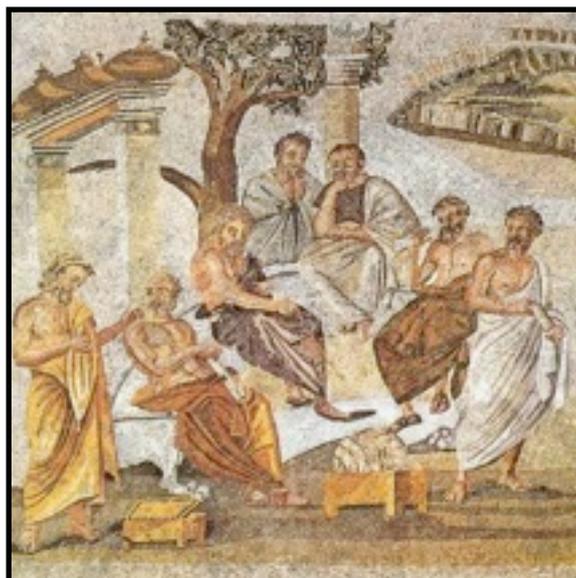
Figure 3: *A mosaic from Pompeii detailing a scene at Plato's Academy.*

The Renaissance (meaning rebirth), a period from about the fourteenth to seventeenth centuries, began in Florence in the late middle ages and spread throughout Europe. The Renaissance was a cultural movement, characterized by the resurgence of learning based on classical sources, and a gradual but widespread educational reform. Education became heavily focused rediscovering Ancient Greek classical writing about cultural knowledge and literature [1]. Music was no longer studied as a field of mathematics. Instead, theoretical music became an independent field, yet strong links with mathematics were maintained [4].

It is interesting to note that during and after the Renaissance, musicians were music theorists, not performers. Music research and teaching were occupations considered more prestigious than music composing or performing [4]. This contrasts earlier times in history. Pythagoras, for example, was a geometer, number theorist and musicologist, but also a performer who played many different instruments.

In the seventeenth and eighteenth centuries, several of the most prominent and significant mathematicians were also music theorists [4]. René Descartes, for example, had many mathematical achievements include creating the field of analytic geometry, and developing Cartesian geometry. His first book, *Compendium Musicale* (1618) was about music theory [4]. Marin Mersenne, a mathematician, philosopher and music theorist is often called the father of acoustics. He authored several treaties on music, including *Harmonicorum Libri* (1635) and *Traité de l'Harmonie Universelle* (1636) [4]. Mersenne also corresponded on the subject with many other important mathematicians including Descartes, Isaac Beekman and Constantijn Huygens [4].
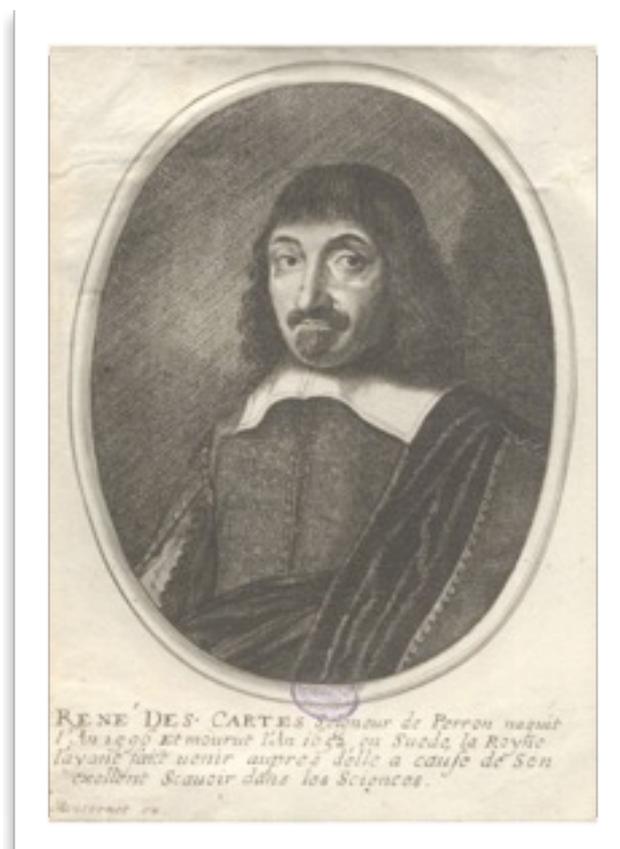

Figure 4: *René Descartes, a brilliant mathematician.*

John Wallis, an English mathematician in the fifteenth and sixteenth centuries, published editions of the works of Ancient Greeks and other academics, especially those about music and mathematics [4]. His works include fundamental works of Ptolemy (2 AD), of Porhyrius (3 AD), and of Bryennius who was a fourteenth century Byzantine musicologist [4]. Leonhard Euler was the preeminent mathematician of the eighteenth century and one of the greatest mathematicians of all time. While he contributed greatly to the field of mathematics, he also was a music theorist. In 1731, Euler published *Tentamen Novae Theoriae Musicae Excertissimis Harmoniae Princiliis Dilucide Expositae* [4]*.* In 1752, Jean d'Alembert published works on music including *Eléments de Musique Théorique et Pratique Suivant les Principes de M. Rameau* and in 1754, *Réflexions sur la Musique* [4]. D'Alembert was a French mathematician, physicist and philosopher who was instrumental in studying wave equations [4].

2.2 WHY ARE MATHEMATICIANS SO FASCINATED BY MUSIC THEORY?

Mathematicians fascination with music theory are explained clearly and precisely by Jean Philippe Rameau in *Traité de l'Harmonie Réduite à ses Principes Naturels* (1722). Some musicologists and academics argue that Rameau was the greatest French music theorist of the eighteenth century [4]. Rameau said:

"Music is a science which must have determined rules. These rules must be drawn from a principle which should be evident, and this principle cannot be known without the help of mathematics. I must confess that in spite of all the experience I have acquired in music by practicing it for a fairly long period, it is nevertheless only with the help of mathematics that my ideas became disentangled and that light has succeeded to a certain darkness of which I was not aware before." [1] [4]
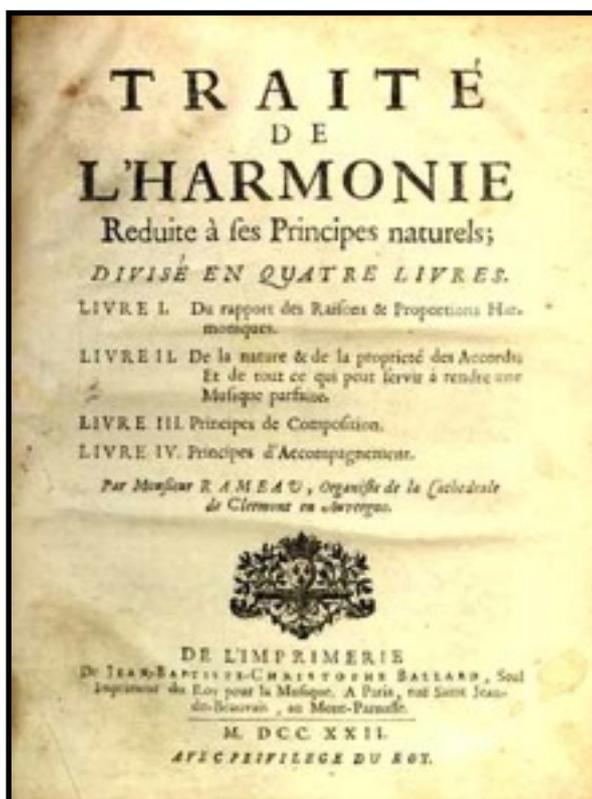
Figure 5: The title page of Rameau's work Traité de l'Harmonie

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[1] "La musique est une science qui doit avoir des règles certaines; ces règles doivent être tirées d'un principe évident, et ce principe ne peut guère nous être connu sans le secours des mathématiques. Aussi dois-je avouer que, nonobstant toute l'expérience que je pouvais m'être acquise dans la musique pour l'avour pratiquée pendant une assez longue suite de temps, ce n'est cependant que par le secours de mathématiques que mes idées se sont débrouillées, et que la lumière y a succédé à une certaine obscurité dont je ne m'apercevais pas auparavant."

Mathematicians have been attracted to the study of music theory since the Ancient Greeks, because music theory and composition require an abstract way of thinking and contemplation [4],[5]. This method of thinking is similar to that required for pure mathematical thought [4],[5]. Milton Babbitt, a composer who also taught mathematics and music theory at Princeton University, wrote that "a musical theory should be statable as connected set of axions, definitions and theorems, the proofs of which are derived by means of an appropriate logic" [4].

Those who create music use symbolic language as well as a rich system of notation, including diagrams [4]. In the case of European music, from the eleventh century, the diagrams used in music are similar to mathematical graphs of discrete functions in two-dimensional Cartesian coordinates [5]. The x-axis represents time, while the y-axis represents pitch. See *Figure 6*.
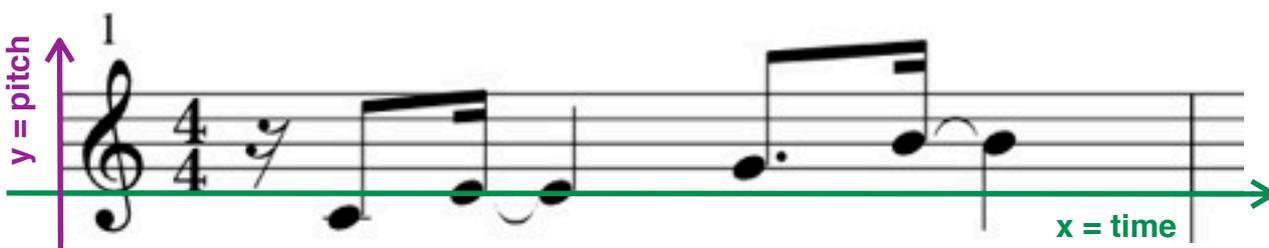
Figure 6: *A musical graph. The time that has elapsed as the music is played is represented by the x-axis. The pitch of the notes are given by the y-axis, with extra information being provided by the key signature. The notes themselves represent the coordinates.*

The Cartesian graph used to represent music was used by music theorists before they were introduced into geometry [4]. In fact, many musical scores of twentieth century musicians have many forms that are similar to mathematical diagrams.

At the beginning of a piece of music, after the clef is marked, the time signature is marked by a fraction on the music staff [5]. Common time signatures include 2/4, 3/4, 4/4. and 6/8. The denominator of the fraction, is the unit of measure, and used to denote pulse. The numerator indicates the number of these units or their equivalent included in the division of a measure[2]. Groups of stressed and relaxed pulses in music are called meters. The meter is also given in the numerator of the time signature [5]. Common meters are 2, 3, 4, 6, 9, 12 which denote the number of beats or pulses in the measure [5]. For example, take the time signature 3/4. Each measure is equivalent to three (information from the numerator) quarter notes (information from the denominator). The count in each measure would be: **1**, 2, 3. The **1** is the stressed pulse, while the 2 and 3 are relaxed. The time signature 3/4 is common in waltzes [5].

Besides abstract language and notation, mathematics concepts such as symmetry, periodicity, proportion, discreteness, and continuity make up a piece of music [4]. Numbers are also very instrumental, and influence the length of a musical interval, rhythm, duration, tempo and several other notations [4]. The two fields have been studied in such unison, that musical words have been applied to mathematics. For example, harmonic is a word that is used throughout mathematics (harmonic series, harmonic analysis), yet its origin is in music theory [4].

It's been discussed that throughout history, mathematicians have long been fascinated with music theory. This concept will be further developed in the final section of this report, which suggest mathematics is, like music, a form of art.

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[2] Measures are separated by a vertical line on the staff.

3.0 The Mathematics of Music

Questions and problems arising in music theory have constituted, at several points in history, strong motivation for investigations in mathematics and physics. This section will explore the use of mathematics to explain the phenomena in music.

Initially, Pythagorean scales will be discussed. Before the introduction of the tempered scale, different scales existed and were used for different kinds of music. From the perspective of European music, Pythagoras is referred to as the first music theorist, so it is fitting to discuss his Pythagorean scale. The move away from Pythagorean scales and tuning will then be discussed. Finally, compositional techniques that are steeped in mathematics (the golden ratio and the circle of fifths) will be discussed.

3.1 Pythagoras and the Theory of Music Intervals

When human ears hear a note, they are really perceiving a periodic sequence of vibrations; sound enters our ears as a sine wave, which compresses the air in a period pattern [6]. The frequency of this sine wave is defined by the frequency at which maximum and minimum air pressure alternate per second [6]. Sounds, including notes played by instruments, do not reach our ears in their pure, basic sound wave. Instead, the note's sound wave is accompanied with overtones. An overtone is a note whose frequency is an exact multiple of the fundamental [6]. Ancient Greeks were not aware of the power of overtones, which were discovered in 1636 by the French mathematician Marian Mersenne [6]. Then, in 1702, Joseph Sauveur studied overtones in great detail. In 1878, the physical properties of overtones were exhaustively discussed by John Strutt, 3rd Baron Rayleigh (1842-1919), in his book (a classic in the field of acoustics even today) *Theory of Sound* [6]. He discovered that the degree to which overtones enrich their fundamentals is

responsible for the specific timbre and quality of sound produced by a musical instrument, which includes the voice [6].

A musical interval is the ratio of the frequency of the sound waves of two tones, a fundamental and a second tone that is either a step lower or higher in pitch [6]. These two notes would be sounded together, or immediately after each other. The most basic musical interval is the prime, where the fundamental note is played in comparison to itself [7]. The ratio of this frequency obviously 1:1.

The next interval (second most basic), is the octave, where the fundamental relates to a second note that has double the frequency of the fundamental. The ratio of the fundamental and second note when they differ by an octave is 1:2. This second note, is an overtone. The higher note of the octave is now the new fundamental note. All overtones related to this new fundamental, would still be the overtones of the original fundamental [6]. After the prime interval, the octave is the second most consonant (pleasant sounding) interval, because our human ears hear all sounds generated by these two tones as belonging together [7]. When sounded together or right after each other, the two tones of an octave sound the same to our ears; the two notes are heard to be equivalent, if the frequency of one is double the frequency of the other [7]. From any fundamental, the second note that makes a musical interval must sound at least as high as this first tone, but sound lower than its octave [6].

In [2], it is explained that the interval of a fifth corresponds to the numerical ration 3:2. This can be calculated by beginning with the overtone series of D. The overtone that follows after the octave is A, which is three times the frequency of D. If this A is played one octave lower, then the resulting interval D-A corresponds to the numerical ration 3:2.

Pythagoras, the first real music theorist[3], and his school of thought, were the first to made this important discovery [4]. Pythagoras found the relation of musical intervals with ratios of integers, by using the interval of the fifth to create further intervals. Described by a Masonic[4] biographer of Pythagoras, Jamblichus in his writing:

> "[Pythagoras was] reasoning with himself, whether it would be possible to devise instrumental assistance to the hearing, which could be firm and unerring, such as the sight obtains through the compass and rule." [4]

How did Pythagoras make this discovery two thousand years ago, when the theory of overtones was not known? He used experimentation and mathematics. Walking through the shop of a man who works with bronze, Pythagoras heard different sounds produced by hammers hitting an anvil [4]. He implemented his notion of consonance and dissonance, the fact that two notes don't always necessarily sound good together. He noticed that the pitch of the musical note that was produced by a particular hammer depended not on the magnitude of the stroke or place the anvil was hit, but rather on the weight of the hammer [4]. The musical interval between two notes that were produced by two different hammers, depended only on the weights of the hammers, and in particular the consonant musical intervals (which, in Ancient Greek music, was the intervals of the octave, the fifth, and fourth), corresponded with weights to fractions, 2/1, 3/2, and 4/3 respectively [4]. Pythagoras conducted a series of experiments, as explained in [4], using different instruments to confirm the relationship between musical intervals and fractions.

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[3] From the perspective of European music.

[4] The Freemasons had great respect for Pythagoras and his teachings.

For example:

- He listened to the pitch produced by the vibration of strings that have the same length. Pythagoras suspended these strings from one end and attached weights to the other lose ends.
- He listened to the pitch of strings, all of different lengths, that were stretched end to end then like an instrument.
- He listened to the pitch of notes played on popes and wind instruments.
- Pythagoras considered a collection of vases, each partly filled with different quantities of the same liquid. He observed them on "rapidity and slowness of movements of air vibrations" [4]. Then, he hit the vases in pairs and listened to the harmonies produced. He associated numbers to consonances. Pythagoras concluded that the octave, fifth and fourth correspond respectively to the ratios 2/1, 3/2, 4/3 in terms of quotients of levels of liquid.

All these experiments agreed with Pythagoras' hypothesis, that musical intervals correspond to defined ratios of integers in an immutable way, whether the integers were the length of pipes, strings or weights. These experiments conducted by Pythagoras had results so accurate, that when his experiments were repeated and reinterpreted by acousticians in the seventeenth century, his results held true [4]. The ideas and observations by Pythagoras and his school established the relationship between music intervals and ratios of intervals.

Once Pythagoras established the ratio of the octave and the fifth, he used these relationships and simple mathematics to obtain further intervals. An explanation of the calculation of such intervals was explained in [6], and is summarized as follows.

<u>The second:</u>

The interval D-A is the fifth, with D being the fundamental. When A is the fundamental, the interval A-E is the fifth. By a factor of 3/2, E is higher than A, and A is higher than the original fundamental D. Thus, to comparing the frequencies of D and E, all that is required is multiplication.

∴ the frequency ratio of E-D

= (3:2) × (3:2)

= 9:4

E must now be transposed down one octave. Recall that the frequency ratio of the note one octave below the fundamental and the fundamental itself is the ratio 1:2. Multiplying again gives the required ratio.

∴ the frequency ratio of D-E

= (9:4) × (1:2)

= 9:8

⇒ any interval with the ratio 9:8 is a <u>second</u>

<u>The sixth:</u>

The interval E-B is the fifth, with E being the fundamental. By a factor of 3/2, the frequency of B is higher than E.

∴ the frequency ratio of D-B

= (E-B) × (D-E)

= (3:2) × (9:8)

= 27:16

⇒ any interval with the ratio 27:16 is a <u>sixth</u>

The fourth:

When G is the fundamental, D forms a fifth and the frequency ration G-D has the ratio 3:2. Inverting this ration, D-G becomes 2:3 (the reciprocal was taken). Transposing G up an octave, together with the original D a fourth is formed:

$$\therefore (2:3) \times (2:1)$$

$$= 4:3$$

$$\Rightarrow \text{ any interval with the ratio 4:3 is a \underline{fourth}}$$

The seventh:

The fundamental C with G forms a fifth. The note C frequency ratio of 2:3 with G, which is higher than the original fundamental D by a factor of 4:3 when transposed up an octave.

$$\therefore \text{ the frequency ratio of C and D}$$

$$= (2:3) \times (4:3)$$

$$= 8:9$$

$$\Rightarrow \text{ any interval with the ratio 8:9 is a \underline{seventh}}$$

The third:

The fundamental F with C forms a fifth. The frequency of F is lower than C by a factor of 2:3.

$$\therefore \text{ the frequency ratio F-D}$$

$$= (2:3) \times (16:9)$$

$$= 32:27$$

$$\Rightarrow \text{ any interval with the ratio 32:27 is a \underline{third}}$$
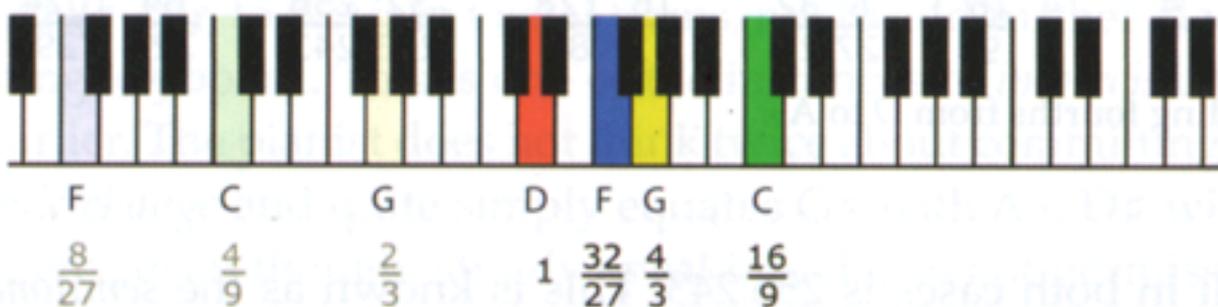
Figure 7: *Some of the frequency ratios created by the Pythagorean method of stacking fifths.*

Pythagoras used these intervals to create an octave scale of whole tones [6]. On a piano, this scale would be each white note in the octave. Creating a Pythagorean scale is an iterative process, where pure fifths are essentially built on top of one another [4]. This process can give an infinite number of notes, but it is reasonable to stop after one octave has been divided into seven intervals.

Table 1: *The frequency ratios and corresponding notes to make a Pythagorean whole tone scale, with the note D as the fundamental.*

Note	D	E	F	G	A	B	C	D
Freq. Ratio	1	9/8	32/27	4/3	3/2	27/16	16/9	2
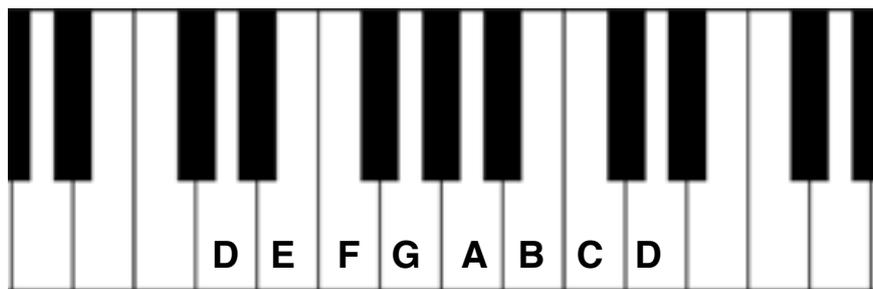
Figure 8: *The notes of one octave of the Pythagorean whole tone scale labeled on the piano.*

The semitone step (the minor second):

Looking at the piano keyboard above, evidently the smallest difference between whole tones exists between the notes E and F, and B and C.

The interval D-E is a second, so multiplying the frequency of E by 8:9 gives the frequency of D. The interval D-F is a third, so multiplying the frequency of D by 32:27 gives the frequency of F.

∴ the frequency ratio E-F

= (8:9) × (32:27)

= 256:243

The inversion of the interval B-D (a sixth) multiplied by the seventh D-C gives the frequency ratio for B-C.

∴ the frequency ratio B-C

= (16:27) × (16:9)

= 256:243

In both of these cases, the resulting frequency ratio is 256:243, and is known either as the semitone step, or the minor second. This ratio tells us that every 256th overtone of the lower tones (E and B) coincides with every 243rd overtone of the higher tones (F and C)! [6]

Now, the rest of the semitones can be added to the established whole tones, changing the diatonic scale to a chromatic one [6]. This means that the white keys on the piano are supplemented by the black ones. To do this, fifths continue to be added or moved downwards in intervals of fourths and transpose the tones obtained this way until they lie within the necessary octave [4].

The Pythagorean scale has many beautiful properties. Fourths and fifths, the building blocks of all other intervals, are all pure sounds [4]. For example, the value of the interval between a second and a fifth note is:

$$(3/2) \times (9/8) = 4/3$$

Pure intervals were so highly valued, that providing a scale with the maximum number of pure intervals because a huge area of research in early music theory [4]. It is, however, impossible to have only pure intervals in a scale, unless it is short [4]. This is the main problem with the Pythagorean scale. On instruments whose ranges cover several octaves (such as the guitar or harpsichord), perfect fifths must be built on top of each other to create a Pythagorean scale that spans more than one octave [4]. For example, it would be expected that catenation of twelve perfect fifths (or $(3/2)^{12}$) gives the same numerical value as a seven octaves (or $(2/1)^7$). Analyzing this mathematically, it is evident that the two do not equate:

$$(3/2)^{12} \neq (2/1)^7$$

$$129.7463379 \neq 128$$

In fact,

$$(3/2)^{12} \; > \; (2/1)^{7}$$

$$129.7463379 > 128$$

The difference between the two intervals is called the Pythagorean comma, and is calculated as follows:

$$(3/2)^{12} \; / \; (2/1)^{7} \simeq 1.013643$$

The Pythagorean comma is a very small difference, and on most instruments such as the piano, it cannot be noted [5]. The Pythagorean comma describes, for example, the difference between the notes G♯ and A. On a piano keyboard, they share the same black key. Instrument creators have decided not to enrich the scale, and have not stacked more fifths on top of each other [4] (more black keys have not been added to the piano keyboard). On instruments such as the violin, for example, such a discrepancy can be heard. The make-up of the instrument, however is such that this problem can be avoided. The strings allow for greater precision so a musician cannot commit the sin of enharmonic change and play a G♯ as an A.

The Pythagorean scale is one example of a scale from Ancient Greece. It is founded using fourths and fifths, the interval its creators deemed to be the most pure. Defining and creating a scale evidently involves mathematical calculations, but also relies on arbitrary thought, such as which interval is believed to be the purest. The Ancient Greeks had many scales, as each were adopted to different melodies and different types of instruments [4]. The choice of scale determined the character and psychological effect of the music on the listener [4]. This subtle dependence of a piece of music on the scale chosen, lasted until the adoption of the tempered scale in European music.

3.2 The Move Away From Pythagorean Scales

Pre-renaissance music, such as that of classical Greece, included complicated systems of scales. Greek mathematical treatise contain descriptions of scales in terms of fractions, and the logic behind each definition is clearly discussed [4]. The Pythagorean scale is one such example that has been discussed in great detail in the previous section. An example of another scale, is one made by Aristoxenus (4 BC). Aristoxenus was an Ancient Greek philosopher and a student of Aristotle. He wrote about philosophy, ethics and music. While much of his work has been lost, parts of one musical treatise *Elements of Harmony* have been found. Aristoxenus created a systematic theory of scales that consist of tetrachords. Tetrachords are scales that are made up of four notes, which correspond to different divisions of fourths by tones and semitones [4]. These were short scales, but longer ones could easily be created by concatenating tetrachords.

While the composition and creation of scales has differed, scales have always been considered the building blocks of musical composition (at least in tonal music, pre-twentieth century European music) [4]. It's been shown that the creation of scales is a very arithmetic process, yet scales are also a musical language. Many old music compositions are based on scales, and often contain fragments of scales in various forms. After the Renaissance, however, Western European classical music began to use a very limited number of scales [4]. Since the eighteenth century, there's been general acceptance of the tempered scale [8]. There are two forms of the tempered scale: major and minor. The tempered scale divides the octave into twelve equal intervals. Two semitones make up a tone, and the distance between any two tones is a semitone. Each unit in a tempered scale is a tempered semitone, with a value of $\sqrt[12]{2}$. Any two major scales (or two minor scales) are simply transpositions of each other on the set of pitches. The piano is an instrument which uses equal tempered scales.

The eight intervals of the octave have frequency ratios [5]:

$$1, f, f^2, f^3, f^4, f^5, f^6, f^7, f^8, f^9, f^{10}, f^{11}, f^{12}$$

$$\text{where } f^{12} = 12 \ (\Rightarrow f = \sqrt[12]{2})$$

The interval, in semitones, between any two tones of the tempered scale is [5]:

$$12 \times \log_2 (\text{frequency ration})$$

Table 2: *Intervals and frequencies (in cents) of the modern equal tempered scale.*

where: 1 semitone = 100 cents, 1 cent = 1200 $\log_2 (f_1 / f_0)$

Interval	Cent (from starting point)
unison	0
semitone or minor 2nd	100
whole tone or major 2nd	200
minor 3rd	300
major 3rd	400
perfect 4th	500
augmented 4th	600
diminished 5th	
perfect fifth	700
minor 6th	800
major 6th	900
minor 7th	1000
major 7th	1100
octave	1200

An octave of a major tempered scale consists of the following pattern of whole tones (W) and semitones (S):

W, W, S, W, W, W, S

This scale can begin on any of the twelve frequencies in the octave.

A minor scale has two forms. The harmonic minor scale:

W, S, W, W, S, (W + S = 3S), S

The melodic minor scale differs on the ascent (↑) and descent (↓):

↑: W, S, W, W, W, W, S

↓: W, S, W, W, S, W, W

Equal temperament is a controversial tuning system. On the one hand, it is very advantageous which is evident since it has dominated Western music for two hundred years. An equal tempered scale is perfectly suited to the design of a keyboard. These scales follow the same pattern, regardless of key allowing composers the freedom to modulate and transpose up or down without a change in the musical intervals [7]. In comparison, Pythagorean scales (and others before the introduction of equal temperament), maintained exact integer-ratio proportions to different intervals.

Another argument against equal temperament exists, on the other hand. Professor Ross W. Duffin eloquently argues against equal temperament in his book *How equal temperament ruined harmony and why you should care* [9]. He claims that equal temperament was a technique that began to be used two hundred years ago to attract people to play an instrument, yet in using the tempered scale, quality and depth in the music is lost [8]. A composition will sound flat. Prof Duffin argues in fact, that equal temperament was created to supply an expanding middle class population with instruments simple and easy enough that they could play themselves [8], [9]. Furthermore,

the compositional geniuses of two centuries ago, did not support this move in tuning

system [8]. Bach wrote *The Well-Tempered Clavier* for irregular temperaments, working in

a wide variety of keys [6]. Instruments in this work would not need to be retuned when you

change keys. The mystical charm of many keys give this masterpiece depth far beyond

that of compositions using equal temperament. In 1766, Bach was still having split key

pianos (with seventeen keys) imported [8]. Haydn is another example of a seventeenth

century composer who shunned equal temperament, since in 1802, he made an explicit

note in the score of *Op 77 No 2 Quartet* that the cello's E♭ should be played as a D♯ [8].

3.3 RAMEAU ADDS TO THE DISCOVERY OF PYTHAGORAS

Two centuries after Pythagoras, French the composer and theoretician Jean-Philippe

Rameau made an important connection between music as an expressive, creative art, and

mathematics as a rigorous, deductive science [4]. Rameau used Pythagoras' discovery

about relationship between musical intervals and pairs of integers and enhanced it. He

gave a musical context to the entire sequence of positive integers [7].
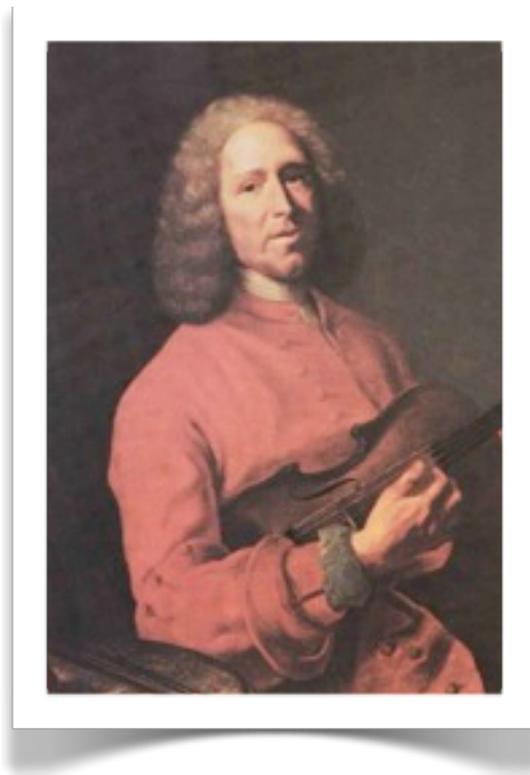
Figure 9: *A painting of Jean-Philippe Rameau, by Jacques André Joseph Aved, 1728.*

Rameau believe that the infinite sequence of integers is elegantly contained in nature, masked as a series of frequencies [4]. When a rich body such as a voice or instrument vibrates, a long periodic variation of air pressure is created. The vibration, which is an acoustic wave, increases and multiplies. When this acoustic sound wave hits our eardrums, we hear a musical note. When a musical note, for example one that is produced by a vibrating string, is usually a superposition of a fundamental tones and overtones. The frequency of overtones is called the harmonic frequency [4]. Rameau discovered that the harmonic frequencies are multiples of the frequency of fundamental tones, and these multiples are given by the positive integers [7].
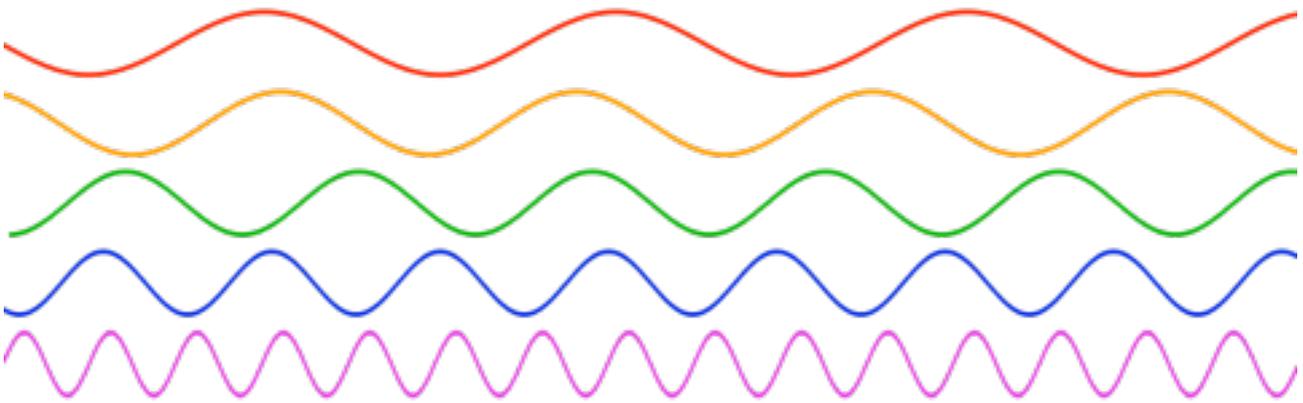
Figure 10: *An illustration of sound waves with different frequencies as time passes. The bottom, more condensed waves with greater oscillations have higher frequencies than those above.*

For example, take the note C_1, which corresponds to the lowest C key on the piano keyboard. The frequency of C_1 is approximately:

$$f_1 = 33Hz \text{ (cycles per second)}$$

The frequency of the corresponding overtones are multiples of f_1, namely:

$$f_1, 2 f_1, 3 f_1, 4 f_1, 5 f_1, 6 f_1 \dots$$

Whose values in Hz are:

$$33, 66, 99, 132, 165, 198 \dots$$

and corresponding sequence notes are:

$$C_1, C_2, G_2, C_3, E_3, G_3 \dots$$

Human hears can (in theory) hear the first four or five overtones on an instrument such as the organ. Astoundingly, Mersenne in *Harmonie Universelle* claimed he could hear the first nine [4]!

Rameau's theoretical work in sound theory is extremely important, but he could not have made such discoveries without the work of previous and other academics. In particular, he used the work of French mathematician Joseph Sauveur. Sauveur was deeply interested in music and acoustic theory, and has been credited with coining the term *acoustique* (he derived it from the Ancient Greek word ακουστός, which means "able to hear" [4]. Sauveur researched the correlation between frequency and musical pitch. Sauveur understood the phenomen of harmonics in music before Rameau, but it was Rameau who used it as the basis of his music teaching in his *Traité de l'Harmonie Réduite à Ses Principes Naturels* [7]. All the theories Rameau developed and detailed in his writings, are based on simple rules which are derived from existence and the properties of the harmonic sequence.

In his later work, Rameau argued that since the fundamental objects in mathematics are derived from sequences of positive integers, and since this sequence continues in music, mathematics is itself part of music [4]. Rameau's ideas unsettled other eighteenth century mathematicians, such as Castel and d'Alembert, and a rift between these mathematicians developed [4].

Rameau's work, like that of Pythagoras, shed new light on the music theory and provided a foundation which others could use for their own research. Jacques Chailley, a famous musicologist and professor of music at the University of Paris, said of Rameau and Pythagoras:

> "In 2500 years of written history, music has perhaps only known two genuine theoreticians and what the others did was only repackage or patch up their propositions. The first one in the VIst century before our era, was the fabulous Pythagoras. The other one died in Paris in 1764: this was Jean-Philippe Rameau." [4]

3.4 MUSIC AND FIBONACCI

Music is evidently more than a collection of notes which create harmony. It is about rhythm and melody, and the changing of notes in relation to time. Interestingly, arithmetic and geometric patterns can be fond in music and its compositions if examined closely.

Figure11: *A nineteenth century statue of Fibonacci in Camposanto, Pisa.*

Leonardo Fibonacci (also known as Leonardo de Pisa, or simply Fibonacci), was a mathematician from Pisa, Italy. In 1201, he developed a mathematical theory which constructs an infinite series of integers. A Fibonacci sequence begins with the numbers 1 followed by a 1. Each successive term is constructed by adding the two previous terms.

For example, the first ten Fibonacci numbers are:

$$\Rightarrow 1, 1, 2, 3, 5, 8, 13, 21, 34, 55$$

A sequence of Fibonacci ratios is the series of numbers produced when each Fibonacci number is divided by the number that precedes it.
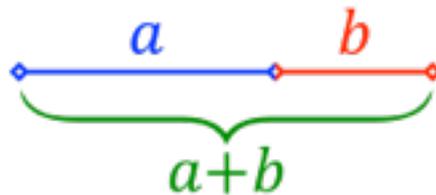
Figure12: *Dividing a line into segments according to the Fibonacci ratio, implies that the ratio of the length of a to a+b is the same as the ratio of the length of b to a.*

The first seven ratios are:

$$r(1) = 1/1 = 1$$
$$r(2) = 2/1 = 2$$
$$r(3) = 3/2 = 1.5$$
$$r(4) = 5/3 = 1.67$$
$$r(5) = 8/5 = 1.6$$
$$r(6) = 13/8 = 1.625$$
$$r(7) = 21/13 = 1.6125$$

These ratios converge to a constant limit which is called the golden ratio (also called the golden proportion, or golden section). The golden ratio is an irrational number which is defined as:

$$\psi = 1.61803398\ldots$$

One can observe that the odd terms of the Fibonacci ratio (the 1st, 3rd, 5th… terms) are all less than the golden ratio, while the even terms of the Fibonacci ration (the 2nd, 4th, 6th … terms) are all above the golden ration.

The golden ratio is a powerful tool as it has a geometric interpretation. Dividing a line into two unequal parts follows the geometric application of this ratio if the proportion of the length of the whole line to the larger line segment is equal to the proportion of the bigger line segment to the smaller line segment.

Figure 13: *Ancient architectural marvels, such as the Greek Parthenon, used the power of the golden ration.*

The golden ratio is meant to make objects aesthetically pleasing. It is found in geometric forms, such as in the length of the diagonal in relation to the length of the side of a pentagon. It is found in abundance in nature, such as in the length of a tree trunk in relation to the diameter of a tree, or in the physical properties of starfish and pinecones. When used, the golden ration makes works of art appear balanced and beautiful. It is found: throughout architecture, such as in mosques and the Acropolis; in book design; photographs; and paintings. Artists do not always consciously use the golden ratio, but sometimes its use is a result of impression of beauty and harmony.

The golden ratio is a concept that is also found in music. The golden section is often used to generate rhythmic change or to develop a melody line, and is found in the musical timing of compositions. The climax of a song, for example, is often found at the point of the golden ration (approximately 61.8% of the way through a composition). This is often also the place where significant changes in key or chord structure are placed [10]. A thirty-two bar song for example, would have its climax at bar twenty. Deliberate application of the golden ration can be seen in *Schillenger System of Musical Composition*. It can also be seen in the first movement of Béla Bartók's piece *Music for Strings, Percussion and Caleste* where the climax is at the fifty-fifth bar of an eighty-nine bar composition. Many of Chopin's works (his Nocturnes and Études) are also based on the golden ration. The greatest musical expression and technical difficulty is in the last third of these works. Finally, there is much debate on whether Mozart used the golden section in his work. Mozart was a musical genius, yet no one knows how he created his music. Did he use inspiration from daily events, or did he compose measures of music from mathematical equations? [10]

Figure 14: *A watercolour painting by Carmontelle ca. 1763, of the Mozart family on tour. From left to right: Leopold, Wolfgang, and Nannerl.*

Mozart was a child prodigy, and evidence exists that he was interested in mathematics. His sister claimed that "Wolfgang talked of nothing, thought of nothing but figures" during his school days [10]. In fact, in the margins of some of his compositions such as *Fantasia* and *Fugue in C Major*, he made a note of mathematical equations [10]. Studies have been performed to see if Mozart did indeed use this golden ratio [10]. Results indicate that he did, but only in some of his compositions. While this does not prove that Mozart purposefully employed the golden ration, it does imply that in addition to a great interest in music, Mozart was a genius who also enjoyed playing with numbers [10]. Analyzing the works of Beethoven, Debussy and other musical innovators in different musical periods shows the presence of Fibonacci sequence.

Use of the golden ratio is also seen in the design of different instruments, including string instruments such as the violin. The piano is designed using the golden ratio as well. Modern music tools, such as speaker wires, are also designed using the golden ratio.

Finally, musical scales are based on Fibonacci numbers [5]. Disregarding the first one of the Fibonacci sequence, the next six Fibonacci numbers are: 1, 2, 3, 5, 8, 13.

1st note ➔ root tone of the scale

2nd note ➔ whole tone two steps away from the root tone

3rd and 5th ➔ make the basic foundation of chords; based on the whole tone

8 ➔ eight notes make up a scale

13 ➔ number of notes in span of any note through its octave

The dominant note of a major scale is the fifth note. This is the eighth note of all thirteen notes that make up the scale, and is related to the golden ratio.

$$\Rightarrow \quad 8/13 \simeq \psi$$

This section outlined a the use and definition of the golden ratio. It is a mystical irrational number, that appears throughout nature, art, architecture, music, etc. Artists and musicians alike exploit its beauty, either intentionally or unintentionally, to create the most aesthetically pleasing work possible.

3.5 Circle of Fifths

The circle of fifths is a concept in music theory which geometrically describes the relationship of the twelve tones of the chromatic scale with key signatures in major and minor keys [7]. It illustrates pitch classes of chromatic scales. A circle of fifths is a useful tool for composers when creating harmonizing their work, creating melodies, building chords and moving to different keys in compositions [7]. A perfect fifth is a distance of five steps within a scale, this concept applies to both major and minor scales.

Figure 15: *The circle of fifths has been a tool to help composers for hundreds of years. Nikolay Diletsky's circle of fifths in* Ideal Grammatiki Musikiyskoy*, Moscow 1679.*

The circle begins at the top with C major and A minor, and no sharps or flats. Moving clockwise from the top, the notes ascend by fifths and a sharp in the key signature is gained until the maximum seven sharps is reached. Moving anti-clockwise, the notes descend by fourths and a flat is gained until the maximum seven flats is reached. At the bottom of the circle, six sharps and six flats overlap. This is the enharmonic key signatures.
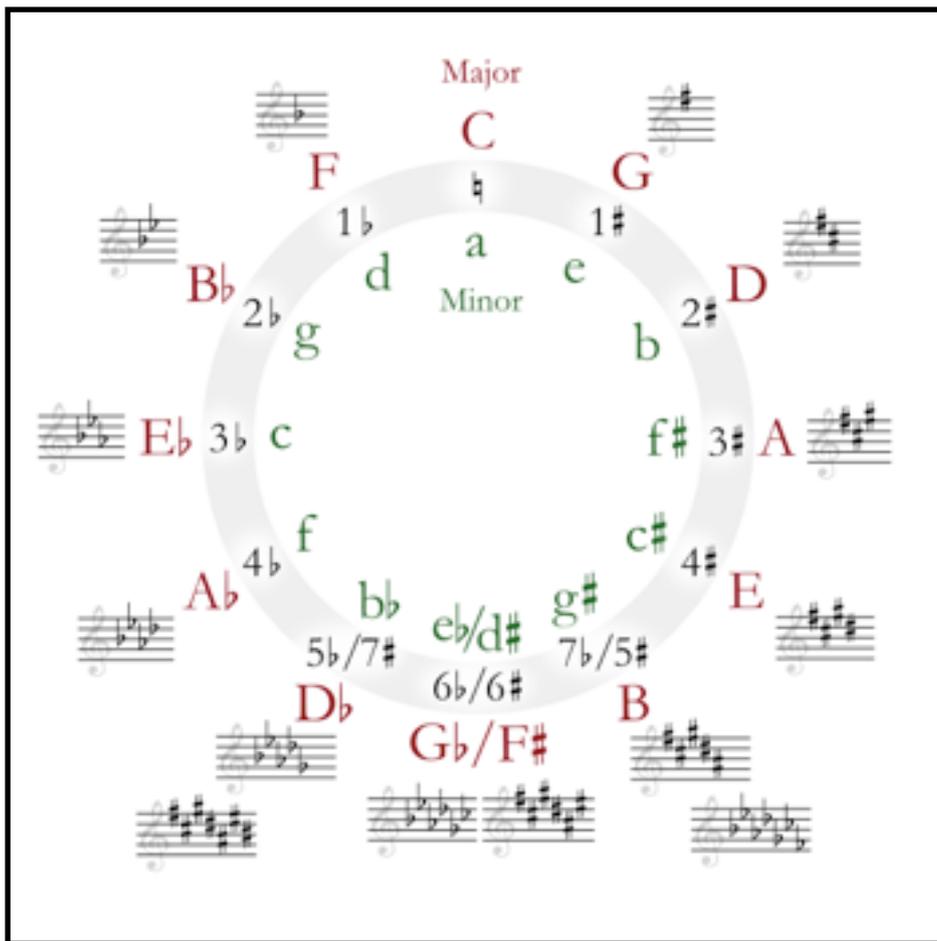
Figure 16: *A circle of fourths and fifths diagram for major and minor keys of a diatonic scale. Moving clockwise around the circle gives a circle of fifths. Moving anti-clockwise gives a circle of fourths.*

Additionally, beginning at any pitch on the circle of fifths, one passes all twelve tones and returns to the beginning pitch of ascents are by the interval of an equally tempered perfect fifth. Ascending by just tuned perfect fifths, results in the circle not being completely closed by the amount of the Pythagorean comma. Reversing direction, tones can be separated by a perfect fourth.

The circle of fifths is typically used in the composition of classical music, while the circle of fourths is used in the analysis of jazz music [7]. The circle of fifths and fourths represents diatonic scales, or scales which are made up of seven notes, five of which are whole steps, and two are half steps with the half steps being maximally separated. The outside circle represents the major diatonic scales. Rotating this outside circle three spots to the left creates the inner circle. This circle shows the minor diatonic scales.

Music often modulates, or changes from one key, tonic or tonal centre to another [10]. Modulations articulate or create structure and form in many pieces. They also add depth and interest to a composition, are are instrumental in keeping the audience captivated in a musical performance. The circle of fifths is a vital tool for composers, as music often modulates by moving between adjacent scales on the circle of fifths [10]. A diatonic scale has seven pitch classes, each being a perfect fifth apart from it's adjacent class on the circle of fifths. Adjacent classes share six of their seven notes in a diatonic scale, and the uncommon note differs only by a semitone. Modulating by a perfect fifth is therefore discrete and easy, as only one note would change by a difference of a semitone [10]. This modulation does not necessarily need to include a change in the key signature [10]. For example, a piece in A major may modulate to E or D major, the two scales adjacent to A major on the circle of fifths. Moving to E major, the note D would become sharp. Moving to D major, the G sharp from the A major scale would no longer be a sharp.

4.0 Messiaen: The Mathematics of his Musical Language

Olivier Messiaen (1908-1992), a French composer and organist, was a great contributor to contemporary music and thinking. Messiaen's long musical career includes thirty seven years of teaching at the Paris Conservatoire (1941-1978), and a lifetime of research in fields of music analysis, composition, rhythms (ancient and modern), birdsongs and theology [11]. Although Messiaen's techniques make his music distinctive and original, his brilliance extended beyond his technique and theory, but deeply affects universal questions of creativity and inspiration [11].
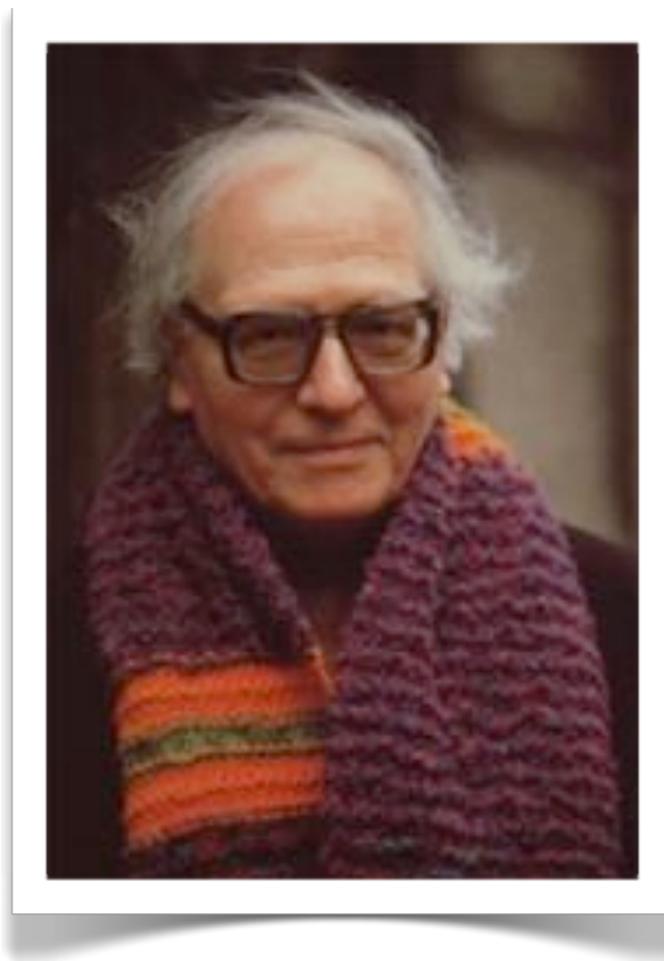
Figure 17: *Olivier Messiaen*

Messiaen's relationship with music stems from his childhood. After he had taught himself to play the piano, he began formal lessons [12]. At the age of eleven, he attended the Paris Conservatoire where he made excellent academic progress [12]. His vast repertoire of studied works include the orchestral works by Heitor Villa-Lobos, Jean-Phillippe Rameau, Isaac Albeniz, Chopin's Études, Mozart's instrumental works and operas, Claude le Jeune's *Le Printemps.* Messiaen's greatest inspiration, however, came from the works (especially operas) of Claude Debussy, who he said had "probably the most decisive influence on me" [12]. Both Chopin and Debussy used some of what Messiaen called MOLT throughout their work, which shows the great affect they had on Messiaen's style and technique.

Figure 18: *Debussy performing at the piano, 1893.*

In 1940 when the Nazis occupied Germany, Messiaen was made a prisoner of war. Throughout his captivity, he met with small groups of prisoners to discuss his creative ideas, especially his new symmetric scales (modes of limited transposition, or MOLT) and Ancient Greek rhythmic patterns [11]. It was in this camp that Messiaen composed the remarkable *Quatuor pour la fin du temps (Quartet for the End of Time)* for the four

available instruments: the piano, violin, cello and clarinet. After his release, Messiaen became a professor at the Paris Conservatoire until his retirement in 1978. He has an extensive list of distinguished pupils, including Quincy Jones, Robert Sherlaw-Johnson, Yvonne Loriod[5] and Iannis Xenakis. Greek Xenakis, for example, was a famous composer who Messiaen provided with encouragement to take exploit and utilize his mathematical and architecture background in his music [13].

Figure 19: *Photography of Quincy Jones, an American music conductor, record producer, musical arranger, and musician. Quincy Jones attended the Paris Conservatoire and was influenced by the teaching of Messiaen. Jones spent five decades in the entertainment industry, earning 75 Grammy Award nominations, 27 Gramm Awards including the Grammy Legend Award in 1991. He produced Michael Jackson's album* Thriller*, which sold +110million copies world wide. Jones also conducted the hit charity song* We Are the World.

---

[5] Yvonne Loriod, a distinguished pianist, eventually became Messiaen's second wife.

Messiaen was fundamentally a harmonist who was interested in rhythm [11]. His instrumental music is best known for his slow pace and strange modal melodic contours, sensuous harmonies, unheard of timbres and registrations, and exotic rhythmic formulae [11]. He was very well read and travelled, and paid attention to the works and ideas of both past and contemporary composers. Additionally, he absorbed many exotic musical influences, which is evident in his compositions. Messiaen's work can be described as being outside of tradition, but greatly influenced by it [11]. Throughout his work he denies western conventions, but the creators of the western masterpieces are who he is most influenced by [11]. In 1942, his unique techniques which constantly evolved were summarized by himself in a treatise called *Technique de Mmn Language Musical (Technique of my Musical Language).* This work influenced modern musicians because of his investigations into Greek meters from antiquity, modality, and palindromic rhythmic techniques.

Messiaen's knowledge on realms beyond his expertise of music was plentiful. He was curious about astronomy, and the imagery and symbolic meaning of the stars, planets and constellations were evoked in his work [11]. This is evident, for example, in *Amen des Étoiles* and *de la Planète à l'Anneau* which are in *Les Visions de l'Amen* which is composed for the piano. His Roman Catholic faith and nature also both affected his work [14]. Birds he believed were the greatest musicians of all and had the most beautiful musical language [12]. As a result, Messiaen wrote several works resembling bird songs, but also incorporated transcriptions of bird songs in most of his music.

Messiaen gave much thought to every parameter of sound, including pitch, dynamics, duration, and timbre [11]. He was fascinated with time and rhythm, and it is his contributions regarding time and rhythm that Messiaen made his work so unique [12]. Messiaen compared how composers organize time to how sculptors shape matter. The ponder the overall rhythm of the work, and then relate the rhythm to a larger form [11]. It is these ideas and influences which he tried to portray in his unique and evolving compositional techniques, which formed the basis for his music. His special techniques were integrated into his musical style, yet he also found and absorbed foreign techniques, such as Ancient Greek and Hindu rhythms (Śārṅgadeva's list of one hundred and twenty rhythmic units called the deçî-tâlas).

Of his music in *The Technique of My Musical Language*, Messiaen said:

> "One point will attract our attention at the outset: the charm of impossibilities. It is a glistening music we seek, giving to the aural sense voluptuously refined pleasures. At the same time, this music should be able to express some noble sentiments (and especially the most noble of all, the religious sentiments exalted by the theology and the truths of our Catholic faith). This charm, at once voluptuous and contemplative, resides particularly in certain mathematical impossibilities of the modal and rhythmic domains."[6] [15]

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[6] "Un point fixera d'abord notre attention: le charme des impossibilités. C'est une musique chatoyante que nous cherchons, donnant, au sense auditif des plaisirs voluptueusement raffinés. En même temps, cette musique doit pouvoir exprimer des sentiments nobles (et spécialement les plus nobles de tous, les sentiments religieux exaltés par la théologie et les vérités de notre foi catholique). Ce charme, à la fois voluptueux et contemplatif, réside particulièrement dans certaines impossiblités mathématique des domaines modal et rythmique" (Messiaen, *Technique de mon langage musical* [Paris: Alphonse Leduc, 1944], p.5)

Messiaen's belief that "a technical process had all the more power when it came up, in its very essence, against an insuperable obstacle" is the foundation for his technique, which he called the "charm of impossibilities" [15]. He uses the term "charm of impossibilities" to explain how in his musical language, "certain mathematical impossibilities, certain closed circuits, possesses a strength of bewitchment, a magic strength, a charm" [15]. His three principle innovations, modes of limited transposition (MOLT), non-retrogradable rhythms, and symmetric permutations describe this power.

"Magical enchantment" in the musical impossibilities of his compositions, according to Messiaen, is a result of using mathematics to create structural symmetries in his musical language [15]. The MOLT are created by dividing an octave into symmetrical groups. It is impossible to transpose them to all twelve notes of the octave without returning to the original note of the first transposition [15]. Non-retrogradable rhythms are palindromically structured rhythms [15]. A palindrome is a pattern that is read the same backwards of forwards. It is impossible to play rhythm of this form without repeating the original order of values. It is not possible to generate an astronomical number of permutations without returning to the original one [15]. Each of these innovative impossibilities form a closed circuit, bringing each musical form back to the origin [15]. The power of Messiaen's charm was that it challenged an "insuperable obstacle" - the compositional limitation established in each of the three innovations. This effect gave his music an added dimension beyond time and sound[7]. Messiaen was using the language and tools of mathematics to overcome impossibilities in his musical language. These two of Messiaen's techniques, which are mathematical in their nature, will now be discussed: modes of limited transposition, and non-retrogradable rhythms.

---

[7] The dimension beyond time and sound that Messiaen was trying to reach will be discussed further in Section 5.3: Messiaen's Use of Mathematical Ideas to Convey Religious Ones.

4.1 Modes of Limited Transposition

> "My passion for the sound-colour relationship drove me to work with these modes of limited transposition, which people did not understand either, because they thought it had to be an arithmetical problem. But first and foremost it is a colour phenomenon. Each mode has a precisely definable colour, which changes every time it is transposed." [15]

Sound-colour and Messiaen's perception of colour is a structural element that is fundamental to his music. In fact, some of Messiaen's scores*, Coulerus de la Cité Célestes* and *Des Canyons aux Étoiles* for example, noted the colour used in the music [11]. He did this not to specify which colours should be heard, but to help the conductor direct and interpret the music. Messiaen's sources of inspiration, Claudio Monteverdi, Mozart, Chopin, Richard Wagner, Mussorgsky and Stravinsky all wrote strongly coloured music [12]. He claimed that only two types of music existed: music with colour, or music without colour [12], and at the heart of his music, there must be colour. Messiaen had an "inner vision", and saw colour in his minds eye when he heard or imagined music [15]. Messiaen credits sound-colour relationship to his childhood, where he developed a strong imagination that was nurtured by fairy tales, poetry and Shakespeare [15]. His perspective of colour is also largely related to his experiences with dazzlement. As a ten year old child, he felt dazzled and overwhelmed with beautiful colours when he first saw the stained glass windows of the Sainte Chapelle in Paris [15]. This was a shining revolution for him, and he began to link colours with emotions [15].
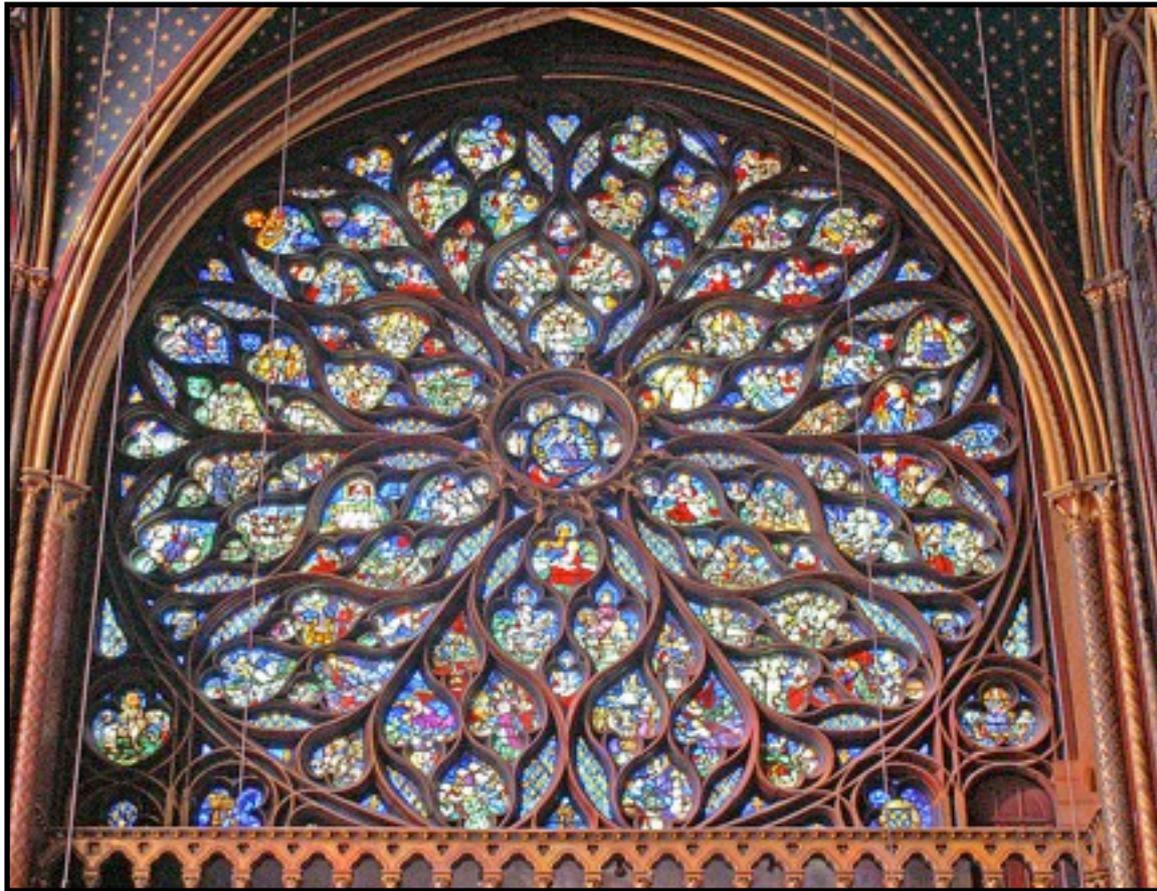
Figure 20: *The Rose window in Sainte Chapelle, a Gothic Cathedral in the heart of Paris where Messiaen discovered the power of colour.*

Messiaen's MOLT, which appeared from his earliest compositions, are themselves very mathematical, but his inspiration in their creation came from his sound-colour perception. They are a systematized description of Messiaen's use of sound-colour in his musical language [15]. Messiaen began to use these modes instinctively, and he was guided by the colours each provoked [15]. Each mode possessed its own characteristic colours, which change with each transposition [15]. The colours are formed by symmetrical formulas in the modal domain, and it is this aspect of symmetry that Messiaen emphasizes [15].

Of his MOLT, Messiaen says:

> "Based on our present chromatic system, a tempered system of twelve sounds,
> these modes are formed of several symmetric groups, the last note in each
> group always common with the first of the following group. At the end of a
> certain number of chromatic transpositions which varies with each mode, they
> are no longer transposable, giving exactly the same notes as the first." [15]

Using the fact that one octave is made up of twelve semitones, and the number twelve is divisible by various numbers, Messiaen formed the modes by dividing the octave into different recurring groups, each being a tiny transposition [15]. Each group has an identical order of intervals, and the last pitch of one group serves as the first pitch of the next [15]. The original form of each mode is called the first transposition, and always begins on the note C. Each transposition thereafter, begins on subsequent chromatic steps [15]. Each group within a mode is constructed in the same way, so only a limited number of transpositions would result in new modes [15]. Thus Messiaen created the term "modes of _limited_ transposition". Messiaen's seven modes can be seen and characterized in the following **Figure 21** and **Table 3**.

Table 3: *Characteristics of Messiaen's modes of limited transposition.*

Mode	Alternative Name	Number of Groups	Number of Notes per Group	Number of Transpositions	Number of Modes
1	whole-tone scale	6	2	2	1
2	octatonic, diminished, semitone-tone	4	3	3	2
3	-	3	4	4	3
4	-	2	5	6	4
5	-	2	4	6	3
6	-	2	5	6	4
7	-	2	6	6	5
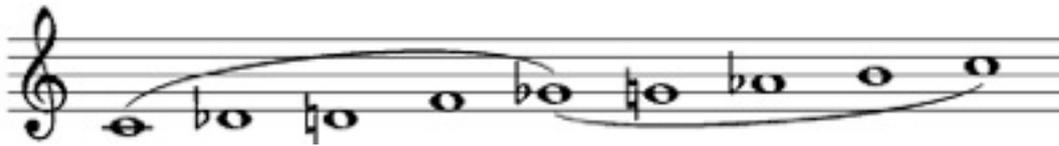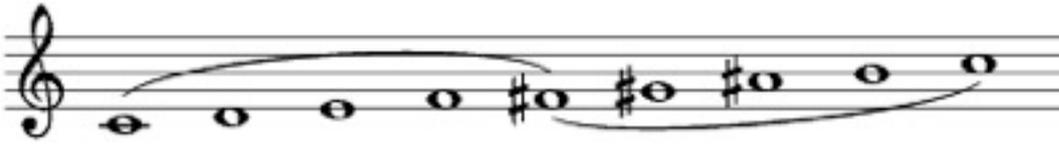
Figure 21: *Messiaen's seven modes of limited transposition.*

The transposition of Messiaen's modes can be defined by chromatic transformation.A diatonic scale is a scale with seven tones, five of which are whole steps, and two are semitone (or half) steps, with the half steps being maximally separated [12]. That is, between each half step there is two or three whole tines. Taking a major diatonic scale and transposing each note up a semitone results in each transposition giving a new note. The first MOLT, the whole-tone scale, has the following notes: C - D - E - F♯ - G♯ - A♯ -C.

Transposing this scale up a semi-tone twice results in the same combination of notes: D - E - F♯ - G♯ - A♯ - C - D. This is illustrated in ***Figure 22***. The first mode therefore has only two transpositions. This process is repeated for each of the other modes.
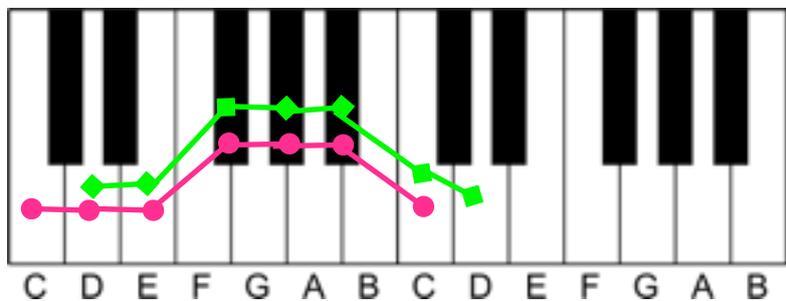
Figure 22: *A piano keyboard which illustrates the chromatic transposition from Messiaen's first mode of limited transpositions. Begin on the lowest C where the pink circle is. Each consecutive whole tone step is shown, and connected with a* *pink* *line. Transposing up two semitones, results in the combination of notes shown by in* *green*. *In Messiaen's first MOLT, two transpositions are all that's needed to return to the original set of notes.*

Messiaen found and defined seven modes, but primarily used colours of four of them: modes 2, 3, 4 and 6 [15]. Additionally, mode 2 occurs most frequently in his music, and its first transposition evoked shades of violet, his favourite colour [15]. Looking at ***Table 3,*** mode 2 only allows three transpositions, which possessed for Messiaen a strong sense of the power of impossibilities. Mode 2 is made up of four groups of three notes (4 × 3 = 12), and each group contains a half step and then a whole step. The first transposition of mode 2 begins on C, the second on D♯, the third on D, and the fourth on E♭. It is the fourth transposition that results in the original set of notes. Messiaen's colour descriptions of mode 2 are surreal and mystical, sentiments he wished to convey to the listener [15]. Interestingly, Messiaen found that his immediate predecessors used mode 2 in their work. In his treatise *The Technique of My Musical Language*, he points out its use by Rimsky-Korsakov, Scriabin, Ravel and Stravinsky [15].

Messiaen's set of seven modes is a group that cannot be expanded or altered. Messiaen himself says that no more modes can be found. "Their series is closed, it is mathematically impossible to find others of them, at least in our tempered system of twelve semitones" [12]. There is an impossibility of further transpositions, and it was this limitation that fascinated Messiaen [15]. It is a huge limitation, that's been created by the tiny transpositions used to construct the mode [15]. The charm of the modes, for Messiaen, lay in the impossibility of further transpositions: their power was "from the impossibility of transpositions and also from the colour liked to this impossibility" [15].

4.2 Non-retrogradable Rhythms

"It is one of my favourite discoveries. As in the case of many discoveries, I

simply found something that already existed, potentially, if not in fact. However,

in spite of the very ancient Hindu "dhenkî … and the antique Greek

"amphimacer" … which are, in date, the first known non-retrogradable rhythms,

no one thought of establishing a musical theory of these rhythms and even less

of putting them into practice."[8] [15]

Messiaen's second creator of the "charm of impossibilities" makes use of the power of
time. As a student, Messiaen discovered non-retrogradable rhythmic patterns in the study
of ancient Hindu rhythms. More specifically, he found a list of one hundred and twenty
deçî-tâlas from Sharngadeva's thirteenth century treatise, the Samgitaratnakara [15]. He
studied these ancient rhythms from North India from "every possible angle" [15]. When he
applied the technique of retrogradation, Messiaen discovered something astonishing; this
discovery was, for him, the "primordial element of these ancient Hindu rhythms" [15]. He
found the existence of a rhythmic palindrome, that is, a special rhythmic form that is the
same whether it is read backwards or forwards. Messiaen thus named gave the name the
"non-retrogradable rhythm" [15].

---

[8] C'est une de mes découvertes préférées. Comme cela se passe dans beaucoup de découvertes, je n'ai fait
que retrouver une chose qui existant déjà, en puissance sinon en fait. Cependant, malgré le très ancien
"dkenkî" hindou … et l'antique "amphimacre" grec … qui sont, en date, les premiers rythmes non
rétrogradables connus - personne ne pensait à établir une théorie musicale de ces rythmes et encore moins
à les mettre en pratique" (Messiaen, *Traité de rythme, de couleur, et d'ornithologie (1949-1992),* vol. II [Paris:
Alphonse Leduc, 1995], p.7)

Among the list of the deçî-tâlas, Messiaen identified what he considered to be the first known retro-gradable rhythm: the dhenkî, deçî-tâla number 58: SIS. Messiaen translated the Hindu notation into the rhythm:

According to Messiaen,

S = Hindu time value *guru* = ♩ = quarter note

I = Hindu time value *laghu* = ♪ = eighth note

The entire rhythm contains five *mâtras*, the Hindu nit for counting these rhythms, and which corresponds to five eighth notes [15].

In *Traité de rythme, de couleur, et d'ornithologie*, Messiaen describes this rhythm.

"Dhenkî is a Bengali word designating a devide for the shelling of rice. This device is generally naeuvered by two women, the one on the right, the other on the left, the device between them, just as here the laghu is placed between the two gurus. Our tâla maybe also reproduces the movement imparted to the device by the two women, during the shelling… It is without doubt very old, like all the rhythms based on the number five, the number of fingers of the hand. The Dhenkî (I emphasize this) is the oldest, the simples and the most natural of the non-retrogradable rhythms."[9] [15]

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[9] "Dhenkî est un mot bengali désignant un appareil pour le décorticage du riz. Cet appareil est généralement manuvré par 2 femmes, l'une à droite, l'autre à gauche, l'appareil entre les deux comme ici le laghu est placé entre les 2 guru. Notre tâla reproduit peut-être aussi le mouvement imprimé à l'appareil par les 2 femmes, pendant le décorticage… Il est sans doute très ancien, comme tous les rythmes basés sur le chiffre 5, nombre des doigts de la main. Le Dhenkî (je le répète avec force) est le plus ancien, le plus simple et le plus naturel des rythmes non rétrogradables."

Interestingly, Messiaen found the same rhythmic pattern in the Amphimacer or Cretic rhythm of Ancient Greece [15]. The rhythmic pattern in this case was:

In his music, Messiaen represented this rhythmic pattern as:

Upon further study of these Hindu rhythmic patterns, Messiaen created a principle for non-retrogradable rhythms. For simple rhythms which have only three values (such as the ones previously discussed), this rhythmic pattern holds if the outer two values are identical, and surround what Messiaen called a "free central value" [15].

free central value

Figure 23: *The placement of the free central value.*

When rhythms are more complex and contain more than three values, Messiaen extends his principle: "all rhythms divisible into two groups, one of which is the retrograde of the other, with a common central value, are non-retrogradable" [15].

These rhythms are simple mathematical patterns, yet Messiaen believed they held philosophic and symbolic importance: they "drew their strength from a temporal impossibility" just as the modes "drew their strength from a resonant impossibility" [15]. In his academic writing, Messiaen detailed the three main strength of these patterns [15]:

(i) Because of the identical relationship of the two outer groups of values (which are retrogrades of each other), closed circuitry is formed. When the two outer values are linked by a common central value a non-retrogradable rhythm is formed. A retrograde can no longer exist, and the pattern is the same whether read from left or right.

(ii) Because the patterns are rhythmic palindromes, they don't change whether played backwards or forwards, and simply repeat themselves. This creates an irreversibility of time; whether time moves forward or backwards, the events are the same.

(iii) Messiaen believed this powerful rhythm could be liked to our temporal life. The two outer groups in his analogy are the past and the future. The middle and free central value is the present. The rhythm links past to future with the present in between. Messiaen claims that we can't distinguish the past and future without the freedom of the present, so it is impossible to distinguish the outer groups of the non-retrogradable rhythm without the freedom of a common and central value.

Figure 24: *From Messiaen's* Oiseaus Exotiques. *This example illustrates Messiaen's use of ancient and exotic rhythms. The percussion shows use of Ancient Greek rhythms, and examples of Messiaen's interpretation of the decî-tâla from Śārṅgadeva are also present. This example also illustrates the accuracy and skill Messiaen had and used when detailing the bird song. He identifies the exact instruments in the music who immitate certain birds. The brass and wind instruments, for example, mimic the crested laughing thrush, while the xylophone mimics the orchard oriole.*

Messiaen found this power not only in Hindu rhythms, but everywhere around him. Non-retrogradable rhythms were present throughout life and could be found in: architecture and the arts; patterns in nature such as leaves, seashells, and butterfly wings; ancient magic spells which used palindromic words; and even in the human body [15].

Messiaen was a brilliant man. He held new and original ideas about rhythm, orchestration, modal harmony, melodic writing and form. His versatility and liberalism in sharing his knowledge and experience made him one of the most eminent music teachers of the twentieth century.

5.0 Religious Symbolism and Mathematics in Music

Throughout history, music has complemented nearly every religion, and has been composed for religious use. For example, music is an integral part of Christian services. Early Christians sung songs such as *Phos Hilaron* (Greek for Gladsome Light) during early morning prayers. Christian church services and special ceremonies such as baptisms, include singing of hymns. Gospel choirs add a contemporary element to music with religious meanings. Sikhs listen to and sing sacred hymns from Guru Granth Sahib often. Native Japanese have ceremonial music called Shintō music. Rastafarian music, Nyabinghi, which connections religion and music has been popularized by artists such as Bob Marley. The music is played at worship ceremonies, to complement drumming, chanting, dancing, prayer and ganja smoking. Buddhists recite the sutra throughout meditation or Buddhist ceremonies. Evidently different religions use different forms and types of music to share religious messages or speak to God.

Conversely, throughout history, composers from different cultures and civilizations have drawn inspiration from their religion. Popular musicians who currently use religious ideas to influence their work include: the rock band Kings of Leon, a pop act called the Jonas Brothers, and different rap artists including Mase.

The connection of mathematics to religious symbolism in music will be analyzed in this section of the report. The mystical and religious symbolism of numbers will first be explored. Then, two religious composers who used elements of mathematics to convey spiritual messages will be discussed. First, J. S. Bach was one of the most important and influential European classical composers, and wrote music for the Lutheran Church. An argument will be presented regarding his use of numbers in his work. Second, Olivier Messiaen, who's technique has been discussed in section four, was a deeply religious

man. He used a mathematical layout in his work to convey his spiritual messages. Both of these composers use mathematical ideas and techniques to communicate with God, and represent their intensely religious philosophies.

5.1 NUMBERS ARE GOD'S TOOLS

Since Greek antiquity, numbers have always held religious meaning, and have been used to communicate and explain the world. In Ancient Greece, the Pythagoreans believed that the numbers 1, 2, 3 and 4 were God's "playing cards" [6]. These four numbers were the building blocks of life; everything on earth was created using numbers. Throughout history, this concept has remained in the minds and ideas of academics. Galileo expressed his ideas as follows:

> "The whole of Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to understand the language and to decipher the characters in which it is expressed. This language is mathematics, and its characters are triangles, circles, and other geometric figures. Without knowledge of these it is humanly impossible to understand a single word of this book and we are condemned to traipse around aimlessly, lost in a dark labyrinth." [6]

Galileo claimed that mathematics (numbers in particular) is the language needed to understand the world around us. Galileo believed that mathematics was founded in geometry [6]. Geometry, which included lines, circles and points of intersection, can be detected by our senses. We can see the elements of geometry in our natural world. Numbers and arithmetic, in comparison, have no direct link to our senses [6]. We can't see, taste, smell or touch the number five for example. We can see the elements we count up to make five, and mentally connect it to the symbol 5.
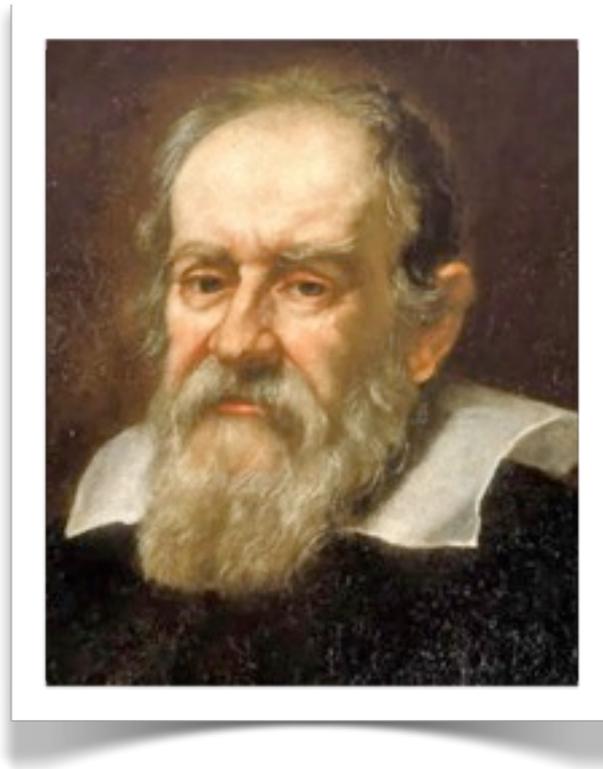
Figure 25: *A portrait of Galileo Galilei, by Giusto Sustermans.*

David Hilbert, a brilliant mathematician, provided great insight into geometry: all of geometry can be explained and incorporated by numbers [6]. All the insights and conclusions provided by geometry, can be deduced from arithmetic without the tangible element of sight and touch previously required to detail geometry. This idea has been extended. Academics now believe that all of intelligible reality can be explained by numbers [6]. Numbers are the materials God used to create our reality; they are his "playing cards". Mathematics is a religious tool of communication, which is also used to explain and understand the world around us.

5.2 RELIGIOUS SYMBOLISM AND NUMBERS IN BACH'S MUSIC

Johann Sebastian Bach (1685-1750) was an exceptional German composer, organist, harpsichordist, violist and violinist. Bach is considered to be the supreme composer of the Baroque period of music. His unique musical style was influenced by his improvisation at the keyboard, exposure to music from different parts of Europe (North Germany, South Germany, Italy and France), and his devotion to the Lutheran liturgy. Sacred music is at the centre of his repertoire. He wrote violin concertos, suites, six Brandenburg Concertos, sacred cantatas, and large scale choral works. Some of his masterpieces include: *Goldenberg Variations, The Well-Tempered Clavier, The Art of Fugue* and *Mass in B Minor.*
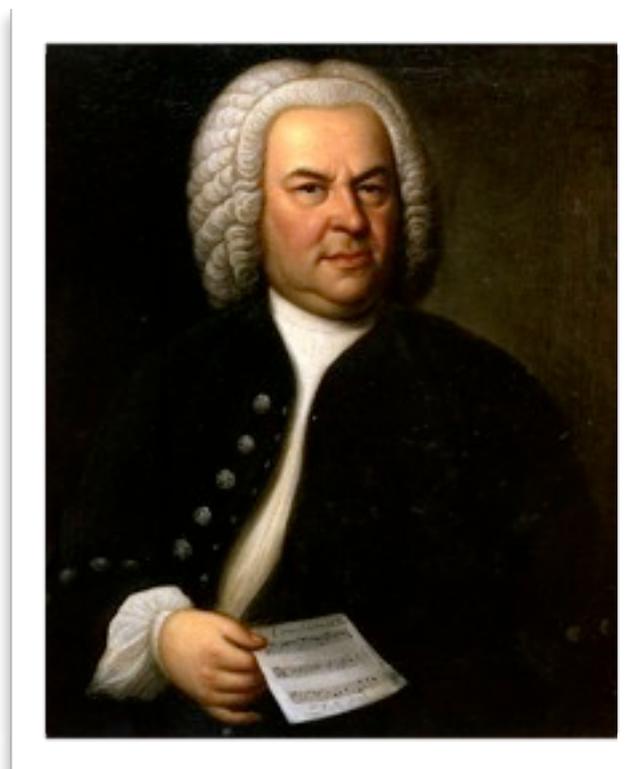
Figure 26: *A portrait by Haussmann of J.S. Bach, 1748.*

Bach's music, and that of other geniuses including Haydn, Beethoven and Schubert, displays intellectual depth, technical command and artistic beauty [6]. His work achieves an internal balance. Each phrase has a purpose, making the piece sound incomplete without it. His musical compositions achieve aesthetic perfection. A mathematical approach to Bach's music would question: can numbers explain the building blocks Bach used (either consciously or unconsciously) to construct his very religious work [6]? The Baroque era, of which Bach was the master, was very aware of the symbolic significance of numbers concerning religion. It soon became obvious that numerical relationships were significant in Bach's work [6]. Every note in Bach's work is purposeful. Did Bach use numerical relationships between sequences of notes to conceal religious messages throughout his composition?

Evidence exists that Bach was greatly influenced by his Martin Luther translation of the Bible [6]. He created a numerological-symbolist disguise for his religious ideas using knowledge from this translation of the Bible. In his Bible copy[10], Bach underlined all passages concerning people or events that featured numbers [6]. In this interpretation, God is seen as a guiding figure, who details and executes his wishes using the tool of numbers, and this is evident in his resulting product. This inspired and motivated Bach, who concluded that God was placing numbers at his disposal [6]. With the numbers throughout this Bible translation, Bach was to construct his own religious music. These guiding points must then be evident in the work of Back.

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[10] Bach's Bible copy included a commentary from the influential Abraham Calovius (1612-1686). He was a Lutheran theologian, who held strong views on Lutheran versus Catholic ideology.

The background knowledge required to present Bach's religious ideas through numbers also came from the philosophy and rationalist thinking of Gottfried Wilhelm Leibniz (1646-1716) [6]. Leibniz was a mathematical genius, who possessed great originality in his thoughts. His influence on Bach's work can be seen in the composition's permutation of notes. In this instance, Bach derived guidance from Leibniz' work *De Arte Combinatoria (On the Art of Combination),* which Leibniz published in 1666 (when he was barely twenty years old). One such permutation that is often found in Bach's music is that of A, B ♭ , C and B. This permutation occurs in several forms, particularly the ascent A - B ♭ - B - C, the decent C - B - B ♭ - A, and the cruciform B ♭ - A - C - B. These permutations are especially important, because in German notation, B ♭ is represented by B, and B by H. Therefore, Bach interpreted these set of notes as A, B, C, H, and not as A, B ♭ , C, B. Notice that his interpretation of the set of notes form his name: B, A, C, H. It is Bach's personal musical signature, which is illustrated in **Figure 27**, that can be heard throughout his work [6].
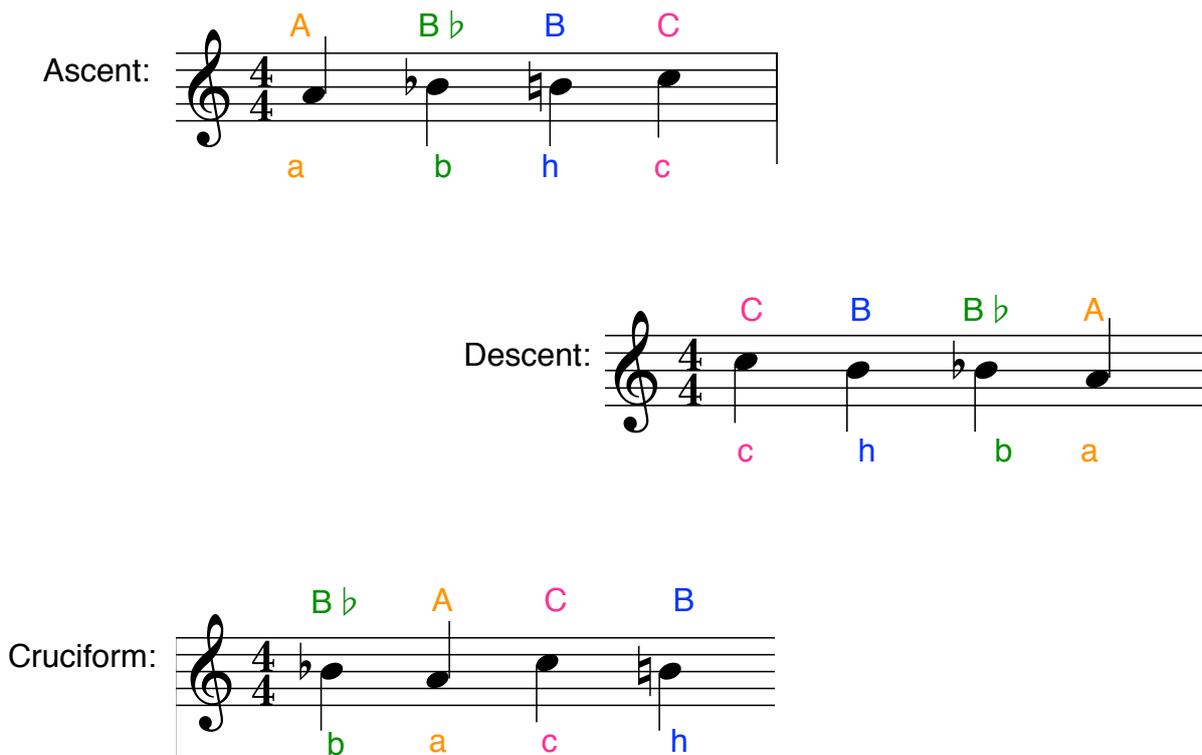
Figure 27: *Bach's musical signature.*

A famous instance of Bach's musical signature is seen in the final counterpoint of *Die Kunst der Fuge (The Art of the Fugue),* in which the final theme is B♭- A - C - B. The theme is then concluded with the notes C♯- D, which are the next two notes of Bach's ascending signature. The combination of notes in the final counterpart hints at the notion of exaltation, or the notion of highly praising someone or something. In his Bible, Bach has underlined the phrase: "Humble yourselves therefore under the mighty hand of God, that he may exalt you in due time" [6].

Bach's musical signature can also be seen in the first bar of the A minor prelude from Part II of *Das Wohltemperierte Klavier (The Well-Tempered Clavier)* [6]. This is seen in ***Figure 28.***

Figure 28: *Bach's name signature in the first bar of the A minor prelude from Part II of* Das Wohltemperierte Klavier.

When played on the piano, the right hand treble begins with a sequence of expressive pain, incorporating the four signature notes in a descending sequence of semitones. The first pair of notes, C - B, differ by just a semitone, and the second pair of notes, B♭ - A, also differ by a semitone. In the bass, six descending notes are added, each differing from the previous note by a semitone. Interestingly, there are ten notes in all, each of which corresponds to one of the ten commandments [6].

The final theme to be discussed which uses numbers symbolically, is in the last fugue from Part I of *The Well-Tempered Clavier*, which is written in B minor. This theme [11] is seen below in **Figure 29**. Another similar theme is found in the prelude before *"Es ist vollbracht" (It is finished)* in *The Passion According to St John,* which is also a prelude in B minor [6].

---

[11] Listening to this theme and "Kyrie" theme of *The Mass in B Minor*, similarities are evident. This theme begins "with an arpeggiated B minor triad. The next twelve notes sigh their way through a series of six stepped minor seconds until the theme ends with an arpeggiated F♯ triad and the return to the dominant of F♯."

Figure 29: *Theme of the last discussion.*

Numerological connections exist in the lengths of these themes, as Bach's musical signature. The prelude is exactly nineteen bars long, and the fugue in B minor from *The Well-Tempered Clavier* is seventy-six bars long (4×19 = 76). In addition, the fugue is composed of fourteen repetitions of the theme. Remarkably, if each letter of the alphabet was assigned a number in consecutive order, 14 = 2 + 1 + 3 + 8 = B + A + C + H. Bach's signature appears right through to the end of the final fugue of the masterpiece that is *The Well-Tempered Clavier* [6].

Numerological significance and representation is apparent in this fugue in another form: the fugue uses each of the twelve notes from the chromatic scale. In numerology, the number twelve is very symbolic [6]:

- 12 notes in the chromatic scale
- 12 zodiac signs
- 12 months in a year
- 12 = 3 × 4

> ⇒ 12 = 3 persons of the trinity × 4 points on the compass

> ⇒ 12 = 3 spatial dimensions × 4 elements of antiquity (earth, air, water, fire)

In Bach's number symbolism, inverting the order of numbers results in a reversal of meaning [6]. Back used the number 12 to represent God's perfection of Creation, so the number 21 would thus mean the yearning for redemption. Interestingly, there are twenty-one notes in the theme of the fugue [6]!

Furthermore, in the theme of the B minor fugue, the twelve tones number of occurrences of each note is different. The dominant of B (the tonic) is F♯, occurs five times and this which is the most frequent of all the notes. The remaining tones occur five times below the F♯, and eleven times above it. In numerology, five and eleven occur in tragic contexts. [6]

It's been shown how Bach's work reflects his deeply spiritual ideas and values. His work is seeped with numerical symbolism, yet academics argue whether these symbolic messages are planted by Bach (consciously or unconsciously), or are simply ideas that are "forced" out of the work.

5.3 MESSIAEN'S USE OF MATHEMATICAL IDEAS TO CONVEY RELIGIOUS ONES

Olivier Messiaen and his unique musical techniques have been previously been investigated. This section analyzes his use of mathematical structures to represent his religious ideas in his compositions.

Messiaen drew his strength and energy to both live and compose from three sources: his strong and intense faith in Roman Catholicism; his love of nature; and the myth of Tristan and Isolde [11]. All three sources of inspiration complemented each other. In particular, Messiaen aimed to depict what he called "the marvellous aspects of the [Roman Catholic] faith" in his work [16]. Messiaen's life long endeavour to "hi-light the theological truths of the Catholic faith" was achieved through his compositions [12] Through his work, he depicted aspects of theology such as sin, but also more joyous ideas such as divine love and redemption. He composed works to express Christ's nativity, crucifixion, resurrection, ascension, transfiguration, and apocalypse [12]. Messiaen brought modern religious work out of the church and into the concert hall [11].

The power of Messiaen's musical charm comes from the impossibilities of his three techniques: modes of limited transposition, non-retrogradable rhythms, and symmetric permutations. The power of this charm is harnessed by challenging the obstacle of compositional limitation which exists because of each innovation [15]. Each innovation formed a complete group, and a closed circuit which would always go back to the beginning. This was Messiaen's way of describing his religious beliefs: with Catholic faith, you will always return to the truth of eternity [15].
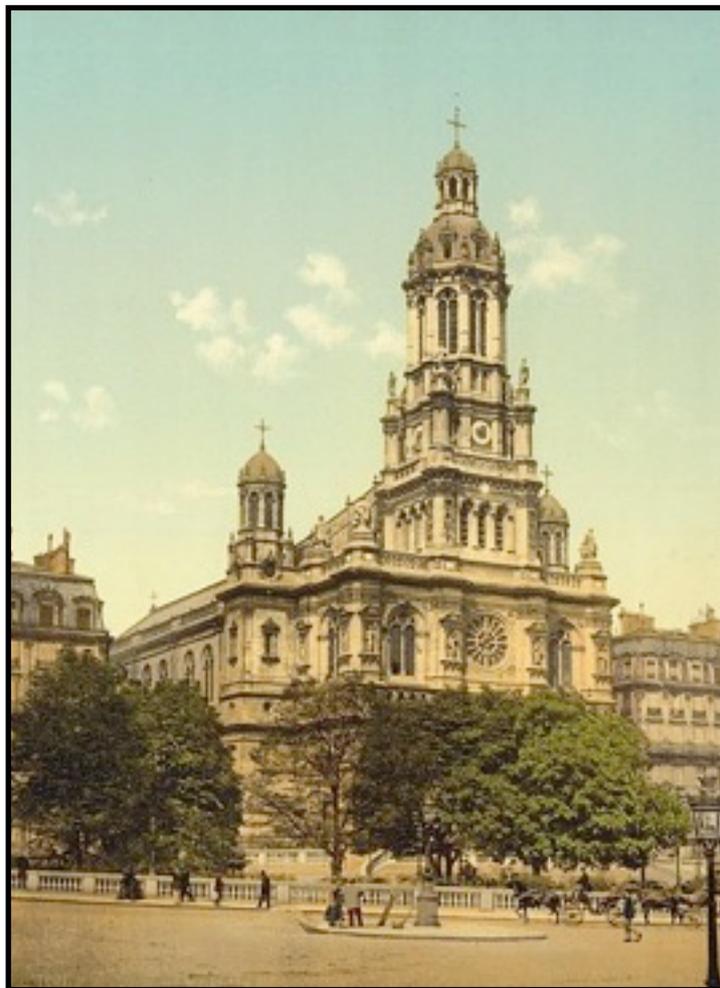

Figure 30: *Église de la Sainte-Trinité, ca. 1890-1900. Messiaen was the organist at this church from 1931 to his death in 1992.*

Analyzing his work further, in his paper *The Spiritual Layout in Messiaen's Conteplations of the Manger,* Siglind Bruhn claims that by examining Messian's *Vingt regards sur l'Enfant-Jesus,* Messiaen used musical symbols to represent spiritual messages. The layout of this piece was carefully planned, as it can be divided further into mathematical cycles, which portray different ideas. [17]

As a devout Catholic composer, Messiaen was faced with a sever limitation: it was impossible while still on earth, to express the truths of his faith [15]. Messiaen used the mathematical techniques of his musical language, to transcend the temporal limitations of music, and express his faith [15]. Each technique reflects his belief that "a technical process had all the more power when it came up, inits very essence, against an insuperable obstacle" [15]. The foundation of the mystical power of his music was his innovations, Messiaen's "charm of impossibilities". He overcame the impossibilities and limitations using mathematics.

6.0 Musical Mathematics: The Artistic Aspect of Mathematics

This section analyzes Jim Henle's argument that while music contains many mathematical elements, it is the fact that mathematics is musical which has attracted mathematicians to the study of music for throughout history [18]. The focus of this section will be specifically on Henle's study, yet other studies exist that draw parallels between the arts and mathematics. Henle's argument has been summarized, and supported (and therefore strengthened) with other examples.

For thousands of years throughout history, mathematicians and philosophers have been fascinated and attracted to music. For example, ancient Greek scholars including Pythagoras, Aristoxenus, and Boethius established the discipline of music as a branch of mathematics, a notion which lasted until the end of the middle ages.

Figure 31: *Boethius' academic work c.480. Boethius was a Roman philosopher who lived from Antiquity to the Middle Ages. He translated Greek authors such as Aristotle into Latin. He created Encyclopedic books of knowledge about arithmetic, geometry, astronomy and the theory of music for the quadrivium. Boethius wrote his own works for study as well. Among other things, he wrote about the relationship between music and science: the pitch of a note heard by human ears is related to the frequency of sound.*

After that, many prominent mathematicians, such as René Descartes, in the seventeenth and eighteenth centuries were also music theorists and wrote extensively not only on mathematics, but music as well. Discounting the few exceptions of academic composers whose compositions are based on mathematics, these feelings are not reciprocated and musicians often do not share the same enthusiasm for mathematics [18]. Why are mathematicians then so infatuated with the study of music?

In theory, as previously discussed in this report, music and musical techniques can often be explained by mathematics. The physics of sound, arithmetic of rhythm, and algebra of scales are examples of such a relationship and have been researched extensively by academics. Is this as far as the argument goes? Do mathematicians simply see music as an intellectual mathematically based discipline? One American academic, Jim Henle [18] argues that this is not the case.

> "I would argue, in fact, that cause and effect have been confused here. The existence of countless mathematical analyses of music is merely evidence that mathematicians have been around, picking at the corpus of music in an attempt to understand its appeal." [18]

Jim Henle claims that the affinity mathematicians have with music is not because music is mathematical, rather because mathematics is musical [18]. He claims that there is something profoundly similar about mathematics and music, which he deducted by the way the two fields respond to the intellectual currents in society. By analyzing the patterns in their growth over centuries, he found remarkable similarities which support his claim.[12] His claim can be supported with three arguments: mathematics can be defined as an art

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[12] It is important to note that this study is not a mathematical study of music. Instead, it aims to explain the real affinity between mathematics and music. Henle claims that this affinity is emotional and spiritual, not procedural and intellectual, with importance placed on the cultural context over abstract principles.

as it shares many of the same characteristics; as in other forms of art, the history of mathematics contains different artistic periods; and these mathematical art periods share many of the same characteristics as corresponding musical periods, but differ from painting or literature periods.

6.1 Mathematics as Art

Mathematics like other disciplines (science, art, religion, etc.) is complex and comprised of many different sub-disciplines. What exactly is mathematics? This is a convoluted question, with no one correct answer. The definition of mathematics, rather, depends on personal views and historical context [2]. Throughout history, civilizations have been defining and developing mathematics differently, to be used for their relevant purposes [1]. In Ancient Babylon, for example, scribes held the mathematical knowledge. For seven-hundred years, they kept sophisticated astronomical records on clay tablets which were stored in huge libraries. Sophisticated mathematics was used to record superficial patterns of celestial activity. Pre-Greek mathematicians were craft mathematicians who focused on methods and concrete objets, and disregarded abstract thinking and deep theoretical knowledge of how things work. Mathematics was purposeful, and used to solve particular everyday problems. They, like European civilizations until the Renaissance, used ordinary language to detail their work, not symbols. The Ancient Greeks, however, were theorists who studied theoretical aspects of mathematics for it's beauty. Pythagoras, for example, founded a religion of mathematics based on the numbers 1, 2, 3 and 4. Pythagoreanism stated that mathematical structures were mystical, and they followed elaborate rituals and rules. Reality for Pythagoreans was constructed out of the four sacred numbers. In the history of mathematics, the 1600s and 1700s brought about a symbolic revolution. The first mathematical law of physics was invented by Descartes through his study of optics to explain rainbows. Descartes is also the inventor of analytic geometry. Whereas the pre-Greek and Greek idea of a number was a concrete group of individual objects, Descartes

begins to see numbers symbolically, and places greater importance on equations. In the 19th century, "new" mathematics begins to emerge, including negative numbers, irrationals, and vectors. In the current 20th century, three main views exist to define mathematics: formalism (believed by Hilbert), platonism (believed by Gödel), and physicolism (believed by Mills, Kitcher) [1].

Evidently, throughout history, the idea of what constitutes mathematics has changed, creating "phases" of mathematics. Today, the core of western civilization is believed to be science, and the core of science is mathematics [19]. In this case, mathematics is deemed to be rooted in the real world, to explain and hold absolute truth. Mathematics may not hold absolute truth, however, as in recent centuries many different types of mathematics have been discovered. For example, Euclid's infamous book *The Elements*, was believed to contain the only viable form of geometry until the 19th century [20].
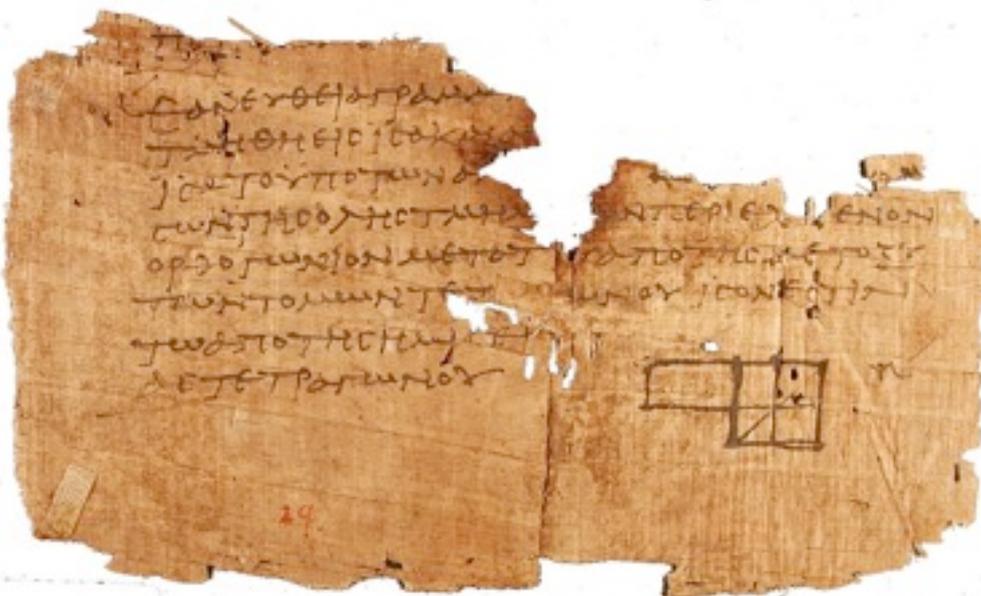

Figure 32: *A fragment of Euclid's elements, found at Oxyrhynchus, dated to c.100 AD.*

In the 1800s, however, Gauss and his contemporaries went beyond Euclid's geometry. They stated the existence of other forms of geometry such as the geometry of spheres, which compared to Euclid's flat geometry, abides by different rules. Another example of where mathematics does not hold absolute truth is in the definition and importance of a number, which has varied greatly throughout history. Hilary Putnam, an important philosopher of mathematics, claims that we know the power of mathematics, but where it resides and comes from there's no agreed view [21]. Mathematics, especially modern mathematics, is a compound subject that contains elements of philosophy, theology, etc. Henle claims [18] that analytic examination shows similarities between mathematics and art.

Firstly, like artists, mathematicians are creators. Some branches of mathematics explain real-world phenomena, and are thus "forced into being" [18]. In comparison, Henle claims that similar to artistic inventions, other types of math are created [18]. For example, the famous mathematician Sir William Rowan Hamilton contributed to 4-dimensional spatial thinking when he first described quaternions in 1843. Quaternions are the quotient of two vectors. It is a type of multiplication which neglects a fundamental rule of normal multiplication: the commutative property. Through this mathematical development, Hamilton demonstrated that multiplication is more abstract than previously thought. In essence, he created a new type of mathematics, using a combination of rules. Quaternions are the artistic masterpiece which Hamilton created using the artistic tools of mathematical knowledge and ideas.

Secondly, both art and mathematics are concerned with expressing ideas [18]. Form, means, channels and presentation are important and valued. Mathematicians seek harmony and elegance in the offering of their and ideas. Proofs are described by mathematicians as being beautiful. However, like artists, mathematicians tastes and

passions vary, each being interested in different fields of mathematics. Different ideas exist on what is deemed to be aesthetically pleasing, as well as varied ideas on what makes different works of mathematics beautiful (one mathematics structure may be admired for symmetry, while the other for singularity) [18]. According to Henle, this invites emotion into the study of mathematics.

6.2 MATHEMATICAL PERIODS

In his argument, Henle now applies periods in the history of mathematics to four artistic periods: the Renaissance, Baroque, Classical and Romantic. In his study, Henle states that the meaning of the terms Renaissance, Baroque, Classical and Romantic have evolved over centuries. Popular definitions were used, from contemporary sources from the same cultural context. Standard texts were used in music, art, literature, and mathematics: [3], [22], [23], and [19] respectively.

Renaissance Mathematics

Henle states that the Renaissance mathematics period is marked by the recovery of Ancient Greek mathematics, and this revival was fuelled by the rise of commerce. The main characteristic of Renaissance mathematics is that the work moves beyond annotation and summarization of the classics from antiquity [18]. To illustrate this point, pre-Renaissance, Pacioi in his work *Summa de Arithmetica, Geometria, Proportioni et Proportionalita* (1494) organized and collected work and ideals already known and respected. He stated that future progress of mathematical ideas was unlikely. In the Renaissance period, Cardano in *Ars Magna* (1545) on the other hand, focuses on the solution of third and fourth degree equations. These were new ideas not previously discussed by classical mathematicians.
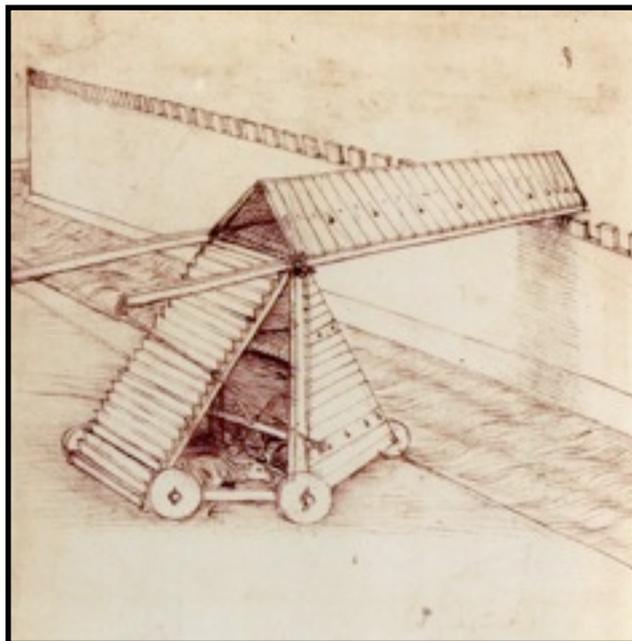
Figure 33: *Leonardo da Vinci used perspective throughout his work, and this is evident in his sketch of this siege machine c.1480.*

Renaissance mathematics at this time was mainly developed for artists and painters, to help their two-dimensional art appear to be three dimensional and full of depth [18]. One of the main results was the geometry of perspective. These ideas were new, and independent of those of Ancient Greek academics. Key contributors to the thoughts and ideas of Renaissance mathematics were: Leone Battista Alberti (1440-1472), Piero della Francesca (c. 1410-1492), Leonardo (1452-1519), and Albrecht Dürer (1471-1528).

Figure 34: *Albrecht Dürer was a German print maker who made important contributions to the polyhedral literature in his book* Underweysung der Messung, *1525. This is one of Durer's masterpieces, the engraving Melancholia I. A frustrated character is sitting by an uncommon polyhedron.*

Baroque Mathematics

New means of expression, according to music historian Grout in *History of Music* characterizes the musical Baroque period. He writes:

> "Just as seventeenth-century philosophers were discarding outmoded ways of thinking about the world and establishing other more fruitful rationales, the contemporary musicians were seeking out other realms of emotions and an expanded language in which to cope with the new needs of expression." [18]

The mathematical Baroque period also is a time of new expression and of new mathematics [18]. Before this era, much of mathematical focus had been on geometry. Fermat (1601-1665) and Descartes (1596-1650) discovered, at this time, that geometric forms and ideas could be expressed algebraically. Thus algebra, a new mathematical language, and the field of analytic geometry developed, leading to advances in mathematics [18].

Grout also characterizes music of the Baroque period conflicting and having a tense relationship with music from previous eras.

> "Baroque music show conflict and tension between the centrifugal forces of freedom of expression and the centripetal forces of discipline and order in a musical composition. This tension, always latent in any work of art, was eventually made overt and consciously exploited by Baroque musicians; and this acknowledged dualism is the most important single principle which distinguishes between the music of this period and that of the Renaissance." [18]

With the introduction of algebra, mathematicians witnessed a dual between the new field of algebra, and the historically established field of geometry [18]. As a disciple of mathematics, geometry had been studied in the same form from the time of the Ancient Greeks to the sixteenth century. It studied concrete ideas and objects, and followed well established rules. Algebra, in comparison, was for "freedom of expression" [18]. It encouraged the use of infinities, a notion that was banned by the Greeks. Geniuses such as Leibniz and Euler proved many fundamental algebraic results. Despite this, the gap between the certainty of geometry, and the perceived "lack of substance" of algebra divided mathematicians and philosophers [18].
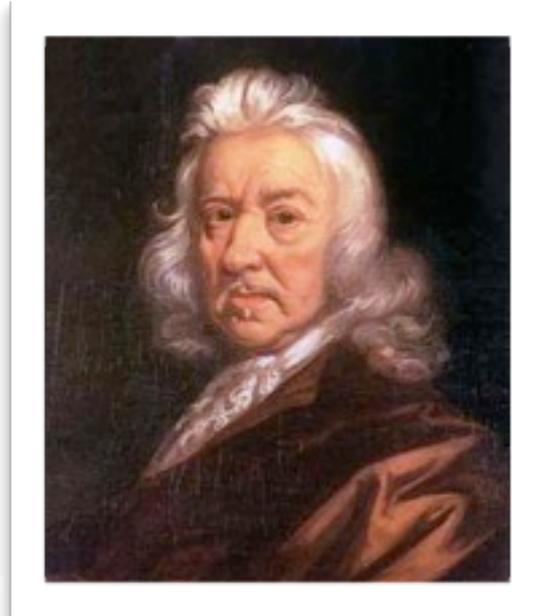
Figure 35: *Thomas Hobbes.*

Thomas Hobbes (1588-1679), for example, felt the excess of algebra and its ideas was totally unjustified. In fact, he referred to the topic as "a scab of symbols" [18].

Classical Mathematics

Grout describes the music of the classical period as follows:

> "The ideal of music of the middle and later eighteenth century, then, might be described as follows: its language should be universal, not limited by national boundaries; it should be noble as well as entertaining; it should be expressive within the bounds of decorum; it should be 'natural', in the sense of being free of needless technical complications and capable of immediately pleasing any normally sensitive listener." [18]

Classical music should connect with the listener at once, communicating easily and directly. The listener should have an immediate understanding and appreciation for a piece. Henle argues that Classical mathematics has the same effect [18]. He characterizes classical mathematics as not being concerned with theory or philosophy, but simply motivated by the real world. To support his argument, Henle calls on the work of Kline, who in his book *Mathematical Thought from Ancient to Modern* says:

> "Far more than in any other century, the mathematical work of the eighteenth was directly inspired by physical problems. In fact, one can say that the goal of the work was not mathematics, but rather the solution of physical problems; mathematics was just a means to physical ends." [19]

Mathematicians, according to Kline, "dared merely to apply the rules and yet assert the reliability of their conclusions" [19]. Mathematicians were content with their results as long as when applied to problems, physically verifiable solutions resulted. Mathematicians of this era has a sense that formal inadequacies in their methods existed. Henle asserts that employment of mathematics, like enjoyment of music, was free of "needless technical complications" [18].

Romantic Mathematics

Romantic art, in comparison to Classical art, places considerable importance on feelings of remoteness, strangeness and boundlessness. Grout characterizes romanticism with regards to art as follows:

> "…Romanticism cherishes freedom, movement, passion, and endless pursuit of the unattainable. Just because its goal can never be attained, romantic art is haunted by a spirit of longing, of yearning after an impossible fulfillment." [18]

For example, these feelings are evident in the painting by Caspar David Friedrich,[13]

Wanderer above the Sea of Fog, **Figure 36**. This painting is a well known Romantic

masterpiece, painted with the unique content and style of Friedrich. John Lewis Gaddis, a

writer, claims this painting leaves the viewer with a contradictory impression, "suggesting

at once mastery over a landscape and the insignificance of the individual within it" [22]. In

the painting, the audience is faced with the subject's back. Facial expressions are not

visible, so it's not clear what the young man was feeling.

Figure 36: *Painting by the German Caspar David Friedrich entitled* Wanderer Above the Sea of Fog*, 1818*

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[13] Casper David Friedrich was an important nineteenth century German Romantic painter. He is best known for his allegorical landscapes, featuring contemplative figures placed against night skies, morning mists, barren trees, or Gothic ruins. Friedrich's work aims to convey a subjective, emotional response to the natural world, and his paintings are often symbolic.

In a similar fashion to Romantic art, mathematics of this era expressed two main and important ideas: the infinite and the impossible [18]. Throughout mathematics history before the nineteenth century, people alternately shunned and embraced the notion of absolute infinity. Paradoxes, such as those by Xeno, about infinity existed since the time of the Ancient Greeks, when philosophers would seriously contemplate such ideas [2]. While the validity of arguments about infinity have been discussed for over two thousand years, its concepts never rose above philosophy or religion. The first steps to mathematical success only came in the early nineteenth century with Augustin Cauchy. Great advancements regarding the infinite happened throughout the 1800s, and by 1900, George Cantor had laid the foundation for the theory of infinite numbers.

The notion of impossibility was another main focus of Romantic mathematics. Since ancient times, a series of problems which had not been solved had plagued mathematicians [20]. For centuries, mathematicians had attempted to solve problems such as the representation of π with radicals, and proof of Euclid's fifth postulate. The early nineteenth century marked a crucial turning point in the way mathematicians regarded such problems.
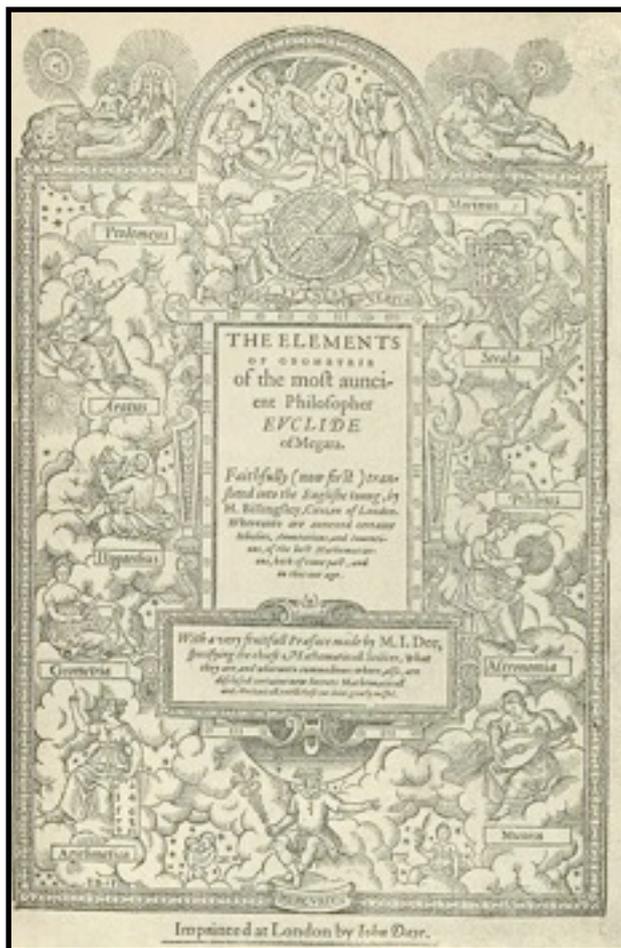
Figure 37: *The cover of Sir Henry Billingsley's first English version of Euclid's* Elements,
1570.

Euclid's *Elements* was written in Alexandria in c.300 BC. It was comprised of 13 books and
465 postulates from plane and solid geometry, and from number theory. Little of this work
was Euclid's own invention. Rather he synthesized two hundred years of mathematical
research, or all the known Greek mathematics, to create a superbly organized treatise
[20]. His work was a self-contained system, that obliterated all preceding works of its type.

The first book of the treatise began by stating five postulates. They were deemed to be obvious, simple statements which required no proof. The first four postulates are [20]:

1. Between two points there is a line

2. Any finite line can be extended

3. Around any point there is a circle

4. All right angles are equal

Controversy arose regarding Euclid's fifth postulate, however. It lacked the simplicity and compelling nature of the others, and thus mathematicians felt it could and should be proved [20]. Euclid himself was unsettled by this postulate, and avoided using it in his work until absolutely necessary. Postulate 5 states:

> "If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles." [20]
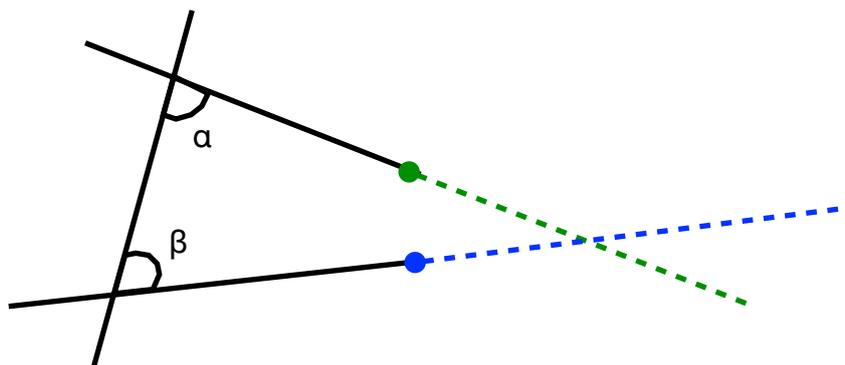

Figure 38: *Illustration of Euclid's fifth postulate. If two lines cross a third such that the angles are <180º, then they intersect.*

The problem of solving this ugly postulate festered for 2000 years, and only towards the end of eighteenth century mathematicians began to believe it was impossible to solve. This leave in thought brought logical implications. Henle claims that if one believes 5th postulate can't be proved, then one must imagine the existence of a geometry in which first 4 axioms are true, the fifth false [18]. If the fifth postulate can't be proven, then there must exist a geometry unlike any previously known.

> "Mathematicians could accept that the axiom was not probable, yet they could not make the logical step and imagine a different geometry. This step was taken in the early nineteenth century." [18]

This example is vitally important in illustrating how thinking of mathematicians changed in the Romantic era [18]. All the pieces required to solve this puzzle were in the hands of mathematicians for hundreds of years. The problem, however, remained unsolved. It was eventually solved by independent mathematicians: Gauss (c.1813), Bolyai (c.1823), Lobachevsky (1827), Schweikart (c.1812), and Young (1860). How come all the solutions to this problem arose in the Romantic era of the 1800s? The answer, according to Henle, is that the environment and the intellectual climate of the nineteenth century was very different from other eras [18]. While all the requisite knowledge was already there, the requisite imagination needed to find a solution was not present before. Einstein said "imagination is more important than knowledge" [18]. This statement proved to be true by the mathematics produced during romanticism.

After Romanticism

In his argument, Henle does not describe any era after Romanticism.

> "We are still too close to the twentieth century to understand it, especially when
> we are trying to grasp a phenomenon as delicate as the artistic milieu of
> mathematics." [18]

He claims categorizing this era at this stage in history would be premature [18].
Additionally, after the romantic period, there was an explosion of different forms of music,
mathematics, art and literature. In the twentieth-century, no one genre of any of the listed
artistic forms of expression is clearly most popular. Thus categorization and generalization
for comparison purposes would be difficult.

6.3 MATHEMATICS PERIODS VS. MUSICAL PERIODS

The previous section set the context of mathematical history in an artistic context.
Definitions and characteristics of musical eras inspired the search of mathematics that had
the same characteristics. For example, Grout's characteristics of the Baroque era were
used, when Henle selected the mathematics that belonged to a corresponding
mathematical Baroque era. When analyzing the times of each era, Henle used the
expertise of music, mathematics, art and literature historians who used the main results
and contributions of each era to date them [18]. Mathematical and music eras occurred at
the same times, while visual art and literature eras occurred together. This is illustrated in
Table 4. It is also noteworthy that this music and mathematic periods began after those of
visual art and literature.

Table 4: *Artistic Periods*

	1400		1500		1600		1700		1800		1900
MUSIC				R			B		C	Ro	
MATH.				R			B		C	Ro	
ART		R				B		C	Ro		
LIT.		R				B		C	Ro		

(Key: R = Renaissance, B = Baroque, C = Classical, Ro = Romantic)[14]

This study does not disqualify any connections between mathematics and other arts such as literature and visual arts, nor is the only study that analyzes the cultural context of mathematics [18]. Other writers have discussed the connection between mathematics and other art forms such as poetry. Scott Buchanan, for example, wrote a book entitled *Poetry and Mathematics.* In addition, different writers have discussed the connection between mathematics and music. Yves Hellegouarch, for example, has written several papers which detail connections between mathematics and music. One such example is *Le Romantisme des les Mathematiques* which examines the romantic features of mathematics in the nineteenth century. On a similar note, much literature exists on the topic of Romanticism in the general sciences.

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[14] Further information regarding specific period dates can be read in Henle's study.

6.4 Is further analysis needed?

Henle's argument draws interesting parallels between mathematics and music. He shows that music and mathematics share many similar characteristics. Each can be categorized into four periods: Renaissance, Baroque, Classical and Romantic by the works and ideas of each time. Furthermore, art and literature periods occur together before mathematics and music periods, which also occur at the same time. The study argues that mathematics can be an artistic invention, and need not always be thought of in a scientific context.

As previously stated, many mathematicians throughout history have been interested in music, so much so that they were also considered music theorists. Mathematicians contributed to the wealth of knowledge on music theory, often by writing books and sharing their ideas. Do mathematical and music periods coincide because mathematicians were also music theorists? Did the published and shared work of mathematicians influence musicians? To enhance the argument in this study, these extra area of research can be undertaken.

7.0 Conclusion

The report began by setting the stage by describing the historical context of the beginning of this relationship. Mathematicians are often music theorists, and some basic ideas to explain this were given. The mathematics in music was then discussed, by detailing the contributions of Pythagoras and J.P. Rameau, two of the greatest contributors to this field. Pythagoras and the Ancient Greeks are among the most critical and significant characters when detailing the relationship of mathematics and music. They were the first to understand how music can and should be studied as a part of mathematics. Rameau, unlike others in his generation, continued this thought pattern. The report then outlined Fibonacci's golden ratio and the circle of fifths, two great mathematical tools that are used by composers to create beautiful music which is as aesthetically pleasing to the ear as possible. Messiaen, a modern composer from the twentieth century, and his mathematical techniques which he uses to compose his work are then analyzed. In the first technique, he created his own set of seven modes. The second technique, non-retrogradable rhythms, are a rediscovery of Ancient Greek and Hindu rhythmic palindromic patterns. Many composers, including Messiaen and Bach, were very religious. Their use of mathematics and numbers to convey religious ideas throughout their work was the topic of the next section. Bach used numbers from a copy of his Bible to weave his religious message throughout his work. Messiaen's mathematical techniques, which he called his "charm of impossibilities", were the tool he used to convey his religious ideas, and be closer to God. Finally, the similarities between thought and ideas of musical and mathematics periods were considered. Renaissance, Baroque, Classical and Romantic periods happened in all genres of the arts (including visual art, music, literature and mathematics), yet mathematics and music went through similar stages of revolution at the same time in history (later than that of visual art and literature).

The relationship between mathematics and music is immense. It spans over two thousand years of history, and involves hundreds of people ranging from mathematicians, to musicians, to music theorists. Research and literature has been published on the different characters, eras and contributions involved. It would be impossible to discuss every aspect of the this complex relationship in a report of this nature. This report has thus provided a "snap shot" of this relationship. I've tried to include the topics I found most interesting, and that I could best relate to and understand given my mathematics and music training. Individuals vary in their views on which connections between mathematics and music are valid, and which are most consequential and significant. I've also discussed the main events and people, who I think have made a great contribution to this field, all the while striving to give as broad an overview of this subject as possible. The relationship between mathematics and music is incredibly interesting, and this exploration is one that could last a lifetime!

SOURCES OF FIGURES AND TABLES

The following list details the website addresses or books where figures and tables were found. Details for figures of tables not listed below were created by the author of this report.

Figures:

Figure 1: http://en.wikipedia.org/wiki/Pythagoreanism

Figure 2: http://en.wikipedia.org/wiki/File:Sanzio_01_Plato_Aristotle.jpg

Figure 3: http://en.wikipedia.org/wiki/File:Plato%27s_Academy_mosaic_from_Pompeii.jpg

Figure 4: http://en.wikipedia.org/wiki/File:Descartes-moncornet.jpg

Figure 5: http://en.wikipedia.org/wiki/File:Rameau_Traite_de_l'harmonie.jpg

Figure 7: Taschner, R., 2007. *Numbers at work: a cultural perspective.* Translated from German by O. Binder & D. Sinclair-Jones. Massachusetts: A K Peters Ltd.

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Figure 9: http://en.wikipedia.org/wiki/Jean-Philippe_Rameau

Figure 10: http://en.wikipedia.org/wiki/Sound_wave#Longitudinal_and_transverse_waves

Figure 11: http://en.wikipedia.org/wiki/Fibonacci

Figure 12: http://en.wikipedia.org/wiki/Golden_ratio

Figure 13: http://en.wikipedia.org/wiki/File:Acropolis_of_Athens_01361.JPG

Figure 14: http://en.wikipedia.org/wiki/Wolfgang_Amadeus_Mozart

Figure 15: http://en.wikipedia.org/wiki/Circle_of_fifths

Figure 16: http://www.music-theory-for-musicians.com/circle-of-fifths.html

Figure 17: http://andante.com/article/article.cfmid=17397&highlight=1&highlightterms=&IstKeywords=

Figure 18: http://commons.wikimedia.org/wiki/File:Debussy_1893.jpg

Figure 19: http://www.achievement.org/autodoc/printmember/jon0bio-1

Figure 20: http://en.wikipedia.org/wiki/Sainte-Chapelle

Figure 23: Created by the author of this report, but ideas and information came from: Wu, Jean M., 1998. Mystical symbols of faith: Olivier Messiaen's charm of impossibilities. In: S. Bruhn. *Messiaen's language of mystical love.* New York: Garland Publishing Inc.

Figure 24: http://en.wikipedia.org/wiki/Messiaen

Figure 25: http://en.wikipedia.org/wiki/File:Galileo.arp.300pix.jpg

Figure 26: http://en.wikipedia.org/wiki/Johann_Sebastian_Bach

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Figure 28: Taschner, R., 2007. *Numbers at work: a cultural perspective.* Translated from German by O. Binder & D. Sinclair-Jones. Massachusetts: A K Peters Ltd.

Figure 29: Taschner, R., 2007. *Numbers at work: a cultural perspective.* Translated from German by O. Binder & D. Sinclair-Jones. Massachusetts: A K Peters Ltd.

Figure 30: http://en.wikipedia.org/wiki/Église_de_la_Ste.-Trinité,_Paris

Figure 31: http://www.moneymuseum.com/frontend/library/pictures/tour.jsp?lang=en&ix=36

Figure 32: http://www.math.ubc.ca/~cass/Euclid/papyrus/papyrus.html

Figure 33: http://www.drawingsofleonardo.org/

Figure 34: http://www.georgehart.com/virtual-polyhedra/durer.html

Figure 35: http://en.wikipedia.org/wiki/Thomas_Hobbes

Figure 36: http://en.wikipedia.org/wiki/Wanderer_Above_the_Sea_of_Fog

Figure 37: http://en.wikipedia.org/wiki/Euclid's_Elements

Tables:

Table 1: Created by the author of this report, but ideas and information came from: Taschner, R., 2007. *Numbers at work: a cultural perspective.* Translated from German by O. Binder & D. Sinclair-Jones. Massachusetts: A K Peters Ltd.

Table 2: Created by the author of this report, but ideas and information came from: Olson, Harry F., 1967. *Music, physics and engineering.* New York: Dover Publications Inc.

Table 4: Created by the author of this report, but ideas and information came from: Henle, J., 1996. Classical mathematics. *The American Mathematical Monthly,* [online]. Vol 103 (No 1), pp. 18-29.
Available at: http://www.jstor.org/stable/2975210
[Accessed 19 February 2010]

REFERENCES

[1] Joseph, G. G., 1991. *The crest of the peacock: non-European roots in mathematics.* New York: Penguin books.

[2] Restivo, S., 1992. *Mathematics in society and history: sociological inquiries.* The Netherlands: Kulwer Academic Publishers.

[3] Grout, D. J., 1973. *A history of western music.* London: J.M. Dent.

[4] Papadopoulos, A., 2002. Mathematics and music theory: from Pythagoras to Rameau. *The Mathematical Intelligencer.* Vol 24 (No 1), pp. 65-73.

[5] Olson, Harry F., 1967. *Music, physics and engineering.* New York: Dover Publications Inc.

[6] Taschner, R., 2007. *Numbers at work: a cultural perspective.* Translated from German by O. Binder & D. Sinclair-Jones. Massachusetts: A K Peters Ltd.

[7] Pierce, John R., 1992. *The science of musical sound.* 3rd ed. New York: W H Freeman and Company.

[8] Lezard, N., 2008. Why hitting the wrong note matters. *The Guardian,* 22 November.

[9] Duffin, R. W., 2008. *How equal temperament ruined harmony: and why you should care.* New York: W.W. Norton & Co.

[10] May, M., Did Mozart use the golden section? *American Scientist* [online]
Available at: http://www.americanscientist.org/issues/pub/did-mozart-use-the-golden-section
[Accessed 19 February 2010]

[11] Boivin, J., 1998. Messiaen's teaching at the Paris Conservatoire: a humanist's legacy. In: S. Bruhn. *Messiaen's language of mystical love.* New York: Garland Publishing Inc.

[12] Sherlaw-Johnson, R, 1975. *Messiaen.* London: J M Dent & Sons Ltd.

[13] Xenakis, I., 1971. *Formalized music: thought and mathematics in composition.* New York: Indiana University Press.

[14] Fabbi, R., 1998. Theological implications of restrictions in Messiaen's compositional process. In: S. Bruhn. *Messiaen's language of mystical love.* New York: Garland Publishing Inc.

[15] Wu, Jean M., 1998. Mystical symbols of faith: Olivier Messiaen's charm of impossibilities. In: S. Bruhn. *Messiaen's language of mystical love.* New York: Garland Publishing Inc.

[16] Sherlaw-Johnson, R., 1998. Rhythmic technique and symbolism in the music of Olivier Messiaen. In: S. Bruhn. *Messiaen's language of mystical love.* New York: Garland Publishing Inc.

[17] Bruhn, S., 1998. The spiritual layout in Messiaen's Contemplations of the Manger. In: S. Bruhn. *Messiaen's language of mystical love.* New York: Garland Publishing Inc.

[18] Henle, J., 1996. Classical mathematics. *The American Mathematical Monthly,* [online]. Vol 103 (No 1), pp. 18-29.
Available at: http://www.jstor.org/stable/2975210
[Accessed 19 February 2010]

[19] Kline, M., 1972. *Mathematical thought from ancient to modern times.* New York: Oxford University Press.

[20] Dunham, W., 1991. *Journey through the genius: the great theorems of mathematics.* London: Penguin Books.

[21] Ben-Menahem, Y. ed., 2005. *Hilary Putnam.* New York: Cambridge University Press.

[22] Gardner, H., 1948. *Art through the ages.* London: G. Bell & Sons.

[23] Benét, R., 1948. *The reader's encyclopedia: an encyclopedia of world literature and the arts.* London: Harrap.