On the positive region of $\pi(x) - li(x)$

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ON THE POSITIVE REGION OF $\pi(X) - LI(X)$

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The University of Manchester

Stefanie Zegowitz

Master of Science

On The Positive Region Of

$\pi(x) - li(x)$

December 5, 2010

In this project, we study the positive region of $\pi(x) - li(x)$. We provide several new theorems based on Saouter-Demichel's article [6] which was published in 2010. In the second chapter, we give a new theorem with a better estimate for the error term in Lehman's Theorem. The third chapter makes further improvements to the error term. Chapter four provides numerical results with a new theorem for the smallest interval such that $\pi(x) - li(x)$ is positive. In the fifth chapter, we sharpen the interval with new theorems, and chapter six improves the estimates for regions of positivity with new theorems.

Declaration

No portion of the work referred to in this project has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

This chapter covers previous work and background material for my project and introduces the notation that I will use throughout.

Previous Work

The Riemann Zeta function $\zeta(s)$ is a function of a complex variable $s = \sigma + it$. It is an infinite series which converges for all s such that $\Re(s) = \sigma > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots \ \Re(s) = \sigma > 1\,.$$

$\rho = \beta + i\gamma$ denotes the complex zeros of the Riemann Zeta function. We have trivial zeros at $s = -2, -4, -6, -8, \ldots$. In 1859, Riemann established a relationship between the zeros of the Riemann Zeta function and the distribution of the prime numbers in his memoir "On The Number of Primes Less Than a Given Magnitude". He conjectured that the non-trivial zeros lie in the critical strip $(0 < \sigma < 1)$ at $\sigma = \frac{1}{2}$. This is called the Riemann Hypothesis.

The function counting the primes numbers is classically denoted by $\pi(x)$:

$$\pi(x) = \sum_{p \le x} 1\,.$$

Riemann's prime counting function is denoted by $\Pi(x)$:

$$\Pi(x) = \sum_{n=1}^{\infty} \frac{1}{n}\,\pi(x^{1/n}) = \pi(x) + \frac{1}{2}\,\pi(x^{1/2}) + \frac{1}{3}\,\pi(x^{1/3}) + \ldots\,.$$

In 1791, Gauss conjectured that $\pi(x) \sim \frac{x}{\log x}$. This was proven by Hadamard and de la Vallée-Poussin in 1896. Then in 1849, Gauss suggested that the log-integral function gives a better approximation for $\pi(x)$. This function is denoted by $li(x)$:

$$li(x) = \lim_{\varepsilon \to 0} \left\{ \int_0^{1-\varepsilon} \frac{1}{\log t}\, dt + \int_{1+\varepsilon}^{x} \frac{1}{\log t}\, dt \right\}.$$

In 1859, Riemann established his Explicit Formula as a relationship between $\pi(x)$ and $li(x)$:

$$\Pi(x) = li(x) - \sum_{\rho} li(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{1}{t\,(t^2-1)\,\log t}\, dt$$

where ρ are the complex zeros of $\zeta(s)$ in the critical strip. Gauss further noted that the inequality $\pi(x) < li(x)$ holds for the first hundred thousand x. Since then, this property has been checked up to 10^{14}.

On the other hand, in 1914, Littlewood proved that the difference of $\pi(x) - li(x)$ changes signs infinitely many often. In 1933, Skewes proved that $\pi(x) > li(x)$ holds at least once for a value $x < 10^{10^{10^{34}}}$ when assuming the Riemann Hypothesis. A considerable improvement to this was given by Lehman in 1966. He established that there exists a region near 1.65×10^{1165} where the difference of $\pi(x) - li(x)$ is positive. In 1987, te Riele discovered a region near 6.65×10^{370}, and Bays and Hudson found a region near 1.40×10^{316} in 1999. In 2006, Chao and Plymen improved the error term in Lehman's theorem. This enabled them to further sharpen Bays and Hudson's region to 1.398×10^{316}.

We will show that the error term of Lehman's theorem as well as the lower bound can be further improved.

Background Information

Estimation of Area

For the estimation of area, we consider three cases.

In the first case, we consider a continuous function. Two examples are shown below:

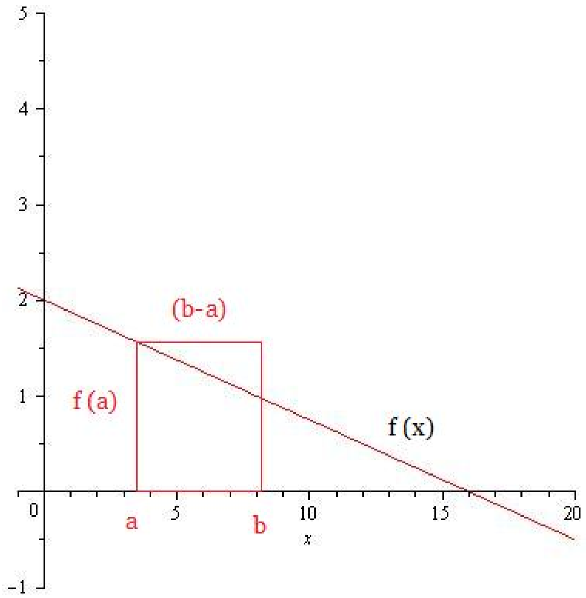

Figure 1.1: Estimation of Area For Continuous Functions

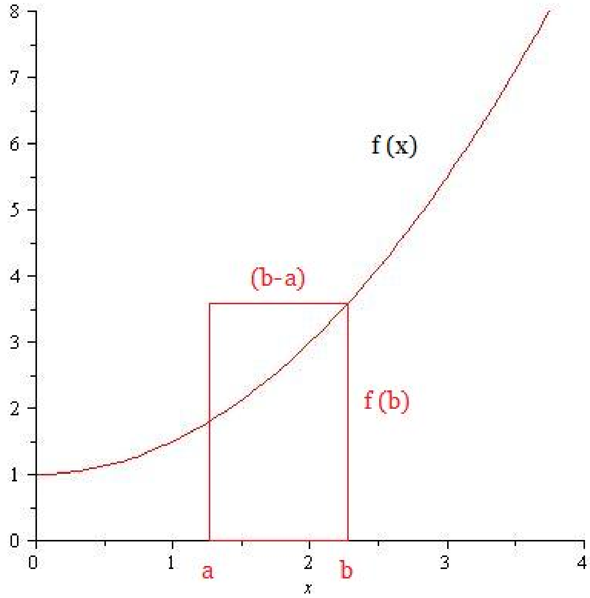

Figure 1.2: Estimation of Area For Continuous Functions

Then

$$\int_a^b f(x)\,dx \leq \text{length} \times \text{height of the rectangle}\,.$$

Hence in Figure 1.1 , we have

$$\int_a^b f(x)\,dx \leq (b-a) \times f(a)\,,$$

and in Figure 1.2 , we have

$$\int_a^b f(x)\,dx \leq (b-a) \times f(b)\,.$$

For the second case, we consider a bell shaped curve whose total area is equal to 1. First, we look at the area around the center of the curve. Consider the two graphs below:

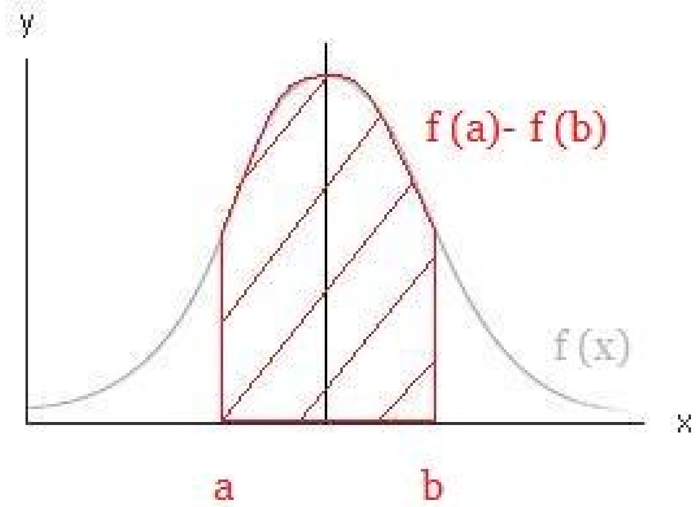

Figure 1.3: Estimation of Area For a Bell-Shaped Curve

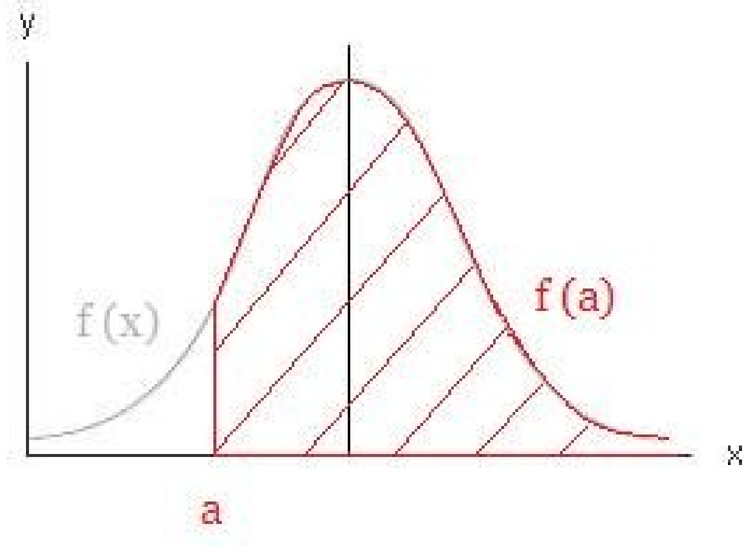

Figure 1.4: Estimation of Area For a Bell-Shaped Curve

The red shaded region in Figure 1.3 shows $f(a) - f(b)$. The red shaded region in Figure 1.4 shows $f(a)$. Hence for $a < b$ the area around the center of a bell-shaped curve can be estimated by

$$\int_a^b f(x)\, dx \leq f(a)\,.$$

Now if we consider the area around the tails towards either the left side or the right side of the center, then the first case applies.

In the third case, we have a complex valued continuous function on a contour C. Then if $|f(z)|$ is bounded by a constant M for all z on C and $l(C)$ denotes the arc length of C, we have

$$\left| \int_C f(z)\, dz \right| \leq M\, l(C)\,.$$

In particular we may take the maximum

$$M = \max_{z \in C} |f(z)|\,.$$

This is called the Estimate Lemma.

Big-Oh-Notation

Let $f(x)$ and $g(x)$ be two functions defined on some subset of the real numbers. Then

$$f(x) = O(g(x)) \text{ as } x \to \infty$$

if and only if there exists a positive real number M and a real number x_0 such that

$$|f(x)| \leq M|g(x)| \text{ for } x > x_0 \,.$$

Some Final Remarks

Throughout the paper, $\log x$ denotes the natural logarithm.

Any numerical values that differ from Saouter-Demichel's numerical results were computed using Maple 12. These computations can be found in the Appendix.

Chapter 2

Lehman's Theorem

This chapter is based on Lehman's Theorem. It gives fundamental knowledge for understanding the following chapters.

Theorem 2.0.1 (Lehman's Theorem[3]) *Let A be a positive number such that $\beta = \frac{1}{2}$ for all complex zeros $\rho = \beta + i\gamma$ of the Riemann Zeta function $\zeta(s)$ for $0 < \gamma \leq A$. Let α, η, and ω be positive values such that $\omega - \eta > 1$, $\frac{4A}{\omega} \leq \alpha \leq A^2$, and $\frac{2A}{\alpha} \leq \eta < \frac{\omega}{2}$.*
Let $K(y) = \sqrt{\frac{\alpha}{2\pi}}\, e^{-\alpha y^2/2}$.
Let $I(\omega, \eta) = \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\, u\, e^{-u/2}[\pi(e^u) - li(e^u)]\, du$.
Then for $2\pi e < T < A$, we have
$I(\omega, \eta) = -1 - \sum\limits_{0<|\gamma|\leq T} \frac{e^{i\gamma\omega}}{\rho}\, e^{\gamma^2/2\alpha} + R$ where $|R| < S_1 + S_2 + S_3 + S_4 + S_5 + S_6$ with
$S_1 = \frac{3}{\omega-\eta} + 4\,(\omega+\eta)\, e^{-(\omega-\eta)/6}$,
$S_2 = \frac{2\, e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}\,\eta}$,
$S_3 = 0.08\, \sqrt{\alpha}\, e^{-\alpha\eta^2/2}$
$S_4 = e^{-T^2/2\alpha}\left(\frac{\alpha}{\pi T^2}\, \log\frac{T}{2\pi} + 8\,\frac{\log T}{T} + \frac{4\alpha}{T^3}\right)$,
$S_5 = \frac{0.05}{\omega-\eta}$,
$S_6 = A \log A\, e^{-A^2/2\alpha + (\omega+\eta)/2}\left(4\,\alpha^{-1/2} + 15\,\eta\right)$.
If the Riemann Hypothesis holds, conditions $\frac{4A}{\omega} \leq \alpha \leq A^2$ and $\frac{2A}{\alpha} \leq \eta \leq \dfrac{\omega}{2}$ may be omitted as well as the term S_6 which may be omitted in the upper bound for R.

We will prove the theorem below which is Lehman's Theorem but with an improvement of term S_6.

Theorem 2.0.2 *Let A be a positive number such that $\beta = \frac{1}{2}$ for all complex zeros $\rho = \beta + i\gamma$ of the Riemann Zeta function $\zeta(s)$ for $0 < \gamma \le A$. Let α, η, and ω be positive values such that $\omega - \eta > 1$, $\frac{4A}{\omega} \le \alpha \le A^2$, and $\frac{2A}{\alpha} \le \eta < \frac{\omega}{2}$.*
Let $K(y) = \sqrt{\frac{\alpha}{2\pi}}\, e^{-\alpha y^2/2}$.
Let $I(\omega, \eta) = \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\, u\, e^{-u/2}[\pi(e^u) - li(e^u)]\, du$.
Then for $2\pi e < T < A$, we have
$I(\omega, \eta) = -1 - \sum_{0<|\gamma|\le T} \frac{e^{i\gamma\omega}}{\rho}\, e^{\gamma^2/2\alpha} + R$ *where* $|R| < S_1 + S_2 + S_3 + S_4 + S_5 + S_6$ *with*
$S_1 = \frac{3}{\omega-\eta} + 4\,(\omega+\eta)\, e^{-(\omega-\eta)/6}$,
$S_2 = \frac{2\, e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}\,\eta}$,
$S_3 = 0.08\,\sqrt{\alpha}\, e^{-\alpha\eta^2/2}$
$S_4 = e^{-T^2/2\alpha}\left(\frac{\alpha}{\pi T^2} log \frac{T}{2\pi} + 8\,\frac{\log T}{T} + \frac{4\alpha}{T^3}\right)$,
$S_5 = \frac{0.05}{\omega-\eta}$,
$S_6' = A\log A\, e^{-A^2/2\alpha + (\omega+\eta)/2}\left(3.2\,\alpha^{-1/2} + 14.4\,\eta\right)$.
If the Riemann Hypothesis holds, conditions $\frac{4A}{\omega} \le \alpha \le A^2$ and $\frac{2A}{\alpha} \le \eta \le \dfrac{\omega}{2}$ may be omitted as well as the term S_6' which may be omitted in the upper bound for R.

We will use the following results and definitions to prove above theorem.

Proposition 2.0.3 ([3, page 400]) *Let $N(T)$ be the number of zeros for which $0 < \gamma \le T$. Then for $T \ge 2\pi e$, $N(T) = \frac{1}{2\pi}\int_{2\pi e}^{T} \log\frac{t}{2\pi}\, dt + \frac{7}{8} + 2\,\vartheta \log T$.*

Proposition 2.0.4 ([3, Lemma 1]) *If $\varphi(t)$ is a continuous function which is positive and monotone decreasing for $2\pi e \le T_1 \le t \le T_2$, then*
$\sum_{T_1 \le \gamma \le T_2} \varphi(\gamma) = \frac{1}{2\pi}\int_{T_1}^{T_2} \varphi(t) \log\frac{t}{2\pi}\, dt + \vartheta\left[4\,\varphi(T_1)\log T_1 + 2\int_{T_1}^{T_2} \frac{\varphi(t)}{t}\, dt\right]$.

Proposition 2.0.5 ([3, Lemma 2]) *If $T \ge 2\pi e$, then $\sum_{\gamma > T} \frac{1}{\gamma^n} < T^{1-n}\log T$ for n = 2, 3, ...*

Proposition 2.0.6 ([5, page 28]) $\sum_{0<\gamma<\infty} \frac{1}{\gamma^2} < 0.025$.

Proposition 2.0.7 ([3, Lemma 4]) *If* $\alpha > 0$ *and* $\varphi(t)$ *is positive and monotone decreasing for* $t \geq T > 0$*, then* $\int_T^\infty \varphi(t)\, e^{-t^2/2\alpha}\, dt < \frac{\alpha}{T}\, \varphi(T)\, e^{-T^2/2\alpha}$.

Proposition 2.0.8 ([3, page 398]) $\pi(x) =$
$li(x) - \frac{x^{1/2}}{\log x} - \sum\limits_\rho li(x^\rho + \vartheta\left(\frac{3\,x^{1/2}}{\log^2 x} + 4\,x^{1/3}\right).$

Definition For $w = u + iv$, $v \neq 0$, $li(e^w) = \int_{-\infty+iv}^{u+iv} \frac{e^z}{z}\, dz$.

Since the proof for above theorem has considerable length, we split it into several steps.

In **Step 1** (page 19), we show that

$$\int_{-\infty}^{+\infty} K(y)\, dy = 1\,.$$

Step 2 (page 21) uses Proposition 2.0.8 to show that

$$\begin{aligned} I(\omega,\eta) &= \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\,\vartheta\left(\frac{3}{u} + 4\,u\,e^{-u/6}\right) du \\ &\leq \frac{3}{\omega+\eta} + 4\,(\omega+\eta)\,e^{-(\omega-\eta)/6}\,. \end{aligned}$$

In **Step 3** (page 21), we use the fact that due to the property that

$$\int_{-\infty}^{+\infty} K(y)\, dy = 1\,,$$

we have

$$\int_{-\infty}^{\omega-\eta} K(u-\omega)\, du = \int_{\omega+\eta}^{+\infty} K(u-\omega)\, du\,.$$

Using Proposition 2.0.7, we then show

$$\int_{-\infty}^{\omega-\eta} K(u-\omega)\, du < \frac{e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}\,\eta}\,.$$

Step 4 (page 23) combines Step 2 and Step 3 to show that

$$\begin{aligned} I(\omega,\eta) = -1 - \sum_\rho \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\,u\,e^{-u/2}\,li(e^{u\rho})\,du \\ + \vartheta\left(\frac{3}{\omega-\eta} + 4\,(\omega+\eta)\,e^{-(\omega-\eta)/6} + \frac{2\,e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}\,\eta}\right). \end{aligned}$$

This gives us term S_1 and S_2.

In **Step 5** (page 23), we assume the Riemann Hypothesis. Then through integration by parts we get

$$\begin{aligned}-\sum_{\rho}\int_{\omega-\eta}^{\omega+\eta}K(u-\omega)\,u\,e^{-u/2}\,li(e^{\rho u})\,du = &-\sum_{0<|\gamma|\le A}\frac{1}{\rho}\int_{\omega-\eta}^{\omega+\eta}K(u-\omega)\,e^{i\gamma u}\,du\\ &-\sum_{0<|\gamma|\le A}\int_{\omega-\eta}^{\omega+\eta}K(u-\omega)\,\frac{\vartheta}{u(\gamma)^2}\,du\\ &-\sum_{|\gamma|>A}\int_{\omega-\eta}^{\omega+\eta}K(u-\omega)\,u\,e^{-u/2}\,li(e^{\rho u})\,du\end{aligned}$$

Step 6 (page 25)uses Proposition 2.0.6 to evaluate the sum

$$-\sum_{0<|\gamma|\le A}\frac{1}{\rho}\int_{\omega-\eta}^{\omega+\eta}K(u-\omega)\,e^{i\gamma u}\,du$$

to get

$$-\sum_{0<|\gamma|\le A}\frac{e^{i\gamma\omega}}{\rho}\,e^{-\gamma^2/2\alpha}+0.08\,\vartheta\,\sqrt{\alpha}\,e^{-\alpha\eta^2/2}\,.$$

To be able to take the above sum over just the zeros, we add another error term. Using Proposition 2.0.4 and Proposition 2.0.7, we then have for $2\pi\le T\le A$

$$\begin{aligned}-\sum_{0<|\gamma|\le A}\frac{1}{\rho}\int_{\omega-\eta}^{\omega+\eta}K(u-\omega)\,e^{i\gamma u}\,du = e^{-T^2/2\alpha}&\left(\frac{\alpha}{\pi T^2}\,\log\frac{T}{2\pi}+\frac{8\log T}{T}+\frac{4\,\alpha}{T^3}\right)\\ &+0.08\,\vartheta\,\sqrt{\alpha}\,e^{-\alpha\eta^2/2}\,.\end{aligned}$$

This gives us term S_3 and S_4.

In **Step 7** (page 27), we use Proposition 2.0.6 to prove that

$$-\sum_{0<|\gamma|\le A}\int_{\omega-\eta}^{\omega+\eta}K(u-\omega)\,\frac{\vartheta}{u(\gamma)^2}\,du\le\frac{0.5}{\omega-\eta}\,.$$

This gives us term S_5.

Step 8 (page 28) combines all the previous steps. Letting $A\to+\infty$, we then have

$$I(\omega,\eta)=-1-\sum_{0\le|\gamma|\le T}\frac{e^{i\gamma\omega}}{\rho}\,e^{-\gamma^2/2\alpha}+R$$

where

$$\begin{aligned}|R|<\frac{3.05}{\omega+\eta}&+4\,(\omega+\eta)\,e^{-(\omega-\eta)/6}+\frac{2\,e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}\,\eta}+0.08\,\vartheta\,\sqrt{\alpha}\,e^{-\alpha\eta^2/2}\\ &+e^{-T^2/2\alpha}\left(\frac{\alpha}{\pi T^2}\,\log\frac{T}{2\pi}+\frac{8\log T}{T}+\frac{4\,\alpha}{T^3}\right).\end{aligned}$$

This is the conclusion of the theorem when we assume the Riemann Hypothesis.

For **Step 9** (page 28), we consider the case where we do not assume the Riemann Hypothesis. We use the function

$$f_\rho(s) = \rho\, s\, e^{-\rho s}\, li(e^{\rho s})\, e^{-\alpha(s-\omega)^2/2}$$

to estimate

$$-\sum_{|\gamma|>A} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\, u\, e^{-u/2}\, li(e^{\rho u})\, du = -\sqrt{\frac{\alpha}{2\pi}} \sum_{|\gamma|>A} \frac{1}{\rho} \int_{\omega-\eta}^{\omega+\eta} e^{(\rho-1/2)u}\, f_\rho(u)\, du\,.$$

Using integration by parts and the Estimation Lemma, we have for $1 \le N \le \frac{\alpha\omega^2}{16}$

$$\begin{aligned} & -\sum_{|\gamma|>A} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\, u\, e^{-u/2}\, li(e^{\rho u})\, du \\ & \le 2\sqrt{\frac{\alpha}{2\pi}}\, e^{(\omega+\eta)/2} \sum_{\gamma>A} \left[\frac{4\, e^{-\alpha\eta^2/8}}{\gamma^2} \sum_{n=0}^{N-1} \frac{n!}{(\gamma\eta/2)^n} + \frac{4\,\eta, N!}{\gamma^{N+1}} \left(\frac{\alpha e}{N}\right)^{N/2} \right]. \end{aligned}$$

Then we use Proposition 2.0.5 to show that

$$\sum_{\gamma>A} \left[\frac{4\, e^{-\alpha\eta^2/8}}{\gamma^2} \sum_{n=0}^{N-1} \frac{n!}{(\gamma\eta/2)^n} + \frac{4\,\eta\, N!}{\gamma^{N+1}} \left(\frac{\alpha e}{N}\right)^{N/2} \right] < 4\, e^{3/2}\, \eta\, e^{-A^2/2\alpha} A\, \alpha^{-1/2} \log A\,.$$

Hence through combining above results, we proof that

$$\left| -\sum_{|\gamma|>A} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\, u\, e^{-u/2}\, li(e^{\rho u})\, du \right| < A \log A\, e^{-A^2/2\alpha+(\omega+\eta)/2} (3.2\, \alpha^{-1/2} + 14.4\, \eta)\,.$$

This gives us term S_6 and is the conclusion of the theorem when we do not assume the Riemann Hypothesis.

Proof For this proof we closely follow the proof of Lehman's Theorem [3].

Let α, ω, and η be positive numbers such that $\omega - \eta > 1$. Let $0 < \beta < 1$. Let $|\vartheta| \leq 1$.

Step1

Let

$$K(y) = \sqrt{\frac{\alpha}{2\pi}}\, e^{-\alpha y^2/2}.$$

Then for any $\gamma \in \Re$, we have

$$\begin{aligned}
\int_{-\infty}^{+\infty} K(y)\, e^{i\gamma y}\, dy &= \int_{-\infty}^{+\infty} \sqrt{\frac{\alpha}{2\pi}}\, e^{-\alpha y^2/2}\, e^{i\gamma y}\, dy \\
&= \sqrt{\frac{\alpha}{2\pi}} \int_{-\infty}^{+\infty} e^{-\alpha y^2/2 + i\gamma y}\, dy \\
&= \sqrt{\frac{\alpha}{2\pi}} \int_{-\infty}^{+\infty} e^{-\alpha y^2/2 + i\gamma y + \gamma^2/2\alpha - \gamma^2/2\alpha}\, dy \\
&= \sqrt{\frac{\alpha}{2\pi}} \int_{-\infty}^{+\infty} e^{-\alpha(y^2 - 2i\gamma y/\alpha - \gamma^2/\alpha^2)/2 - \gamma^2/2\alpha}\, dy \\
&= e^{-\gamma^2/2\alpha} \sqrt{\frac{\alpha}{2\pi}} \int_{-\infty}^{+\infty} e^{-\alpha(y - i\gamma/\alpha)^2/2}\, dy \\
&= \frac{e^{-\gamma^2/2\alpha}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sqrt{\alpha}\, e^{-\alpha(y - i\gamma/\alpha)^2/2}\, dy.
\end{aligned}$$

Let $t = \sqrt{\alpha}\,(y - i\gamma/\alpha)$. Then

$$\begin{aligned}
\int_{-\infty}^{+\infty} K(y)\, e^{i\gamma y}\, dy &= \frac{e^{-\gamma^2/2\alpha}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sqrt{\alpha}\, e^{-\alpha(y - i\gamma/\alpha)^2/2}\, dy \\
&= \frac{e^{-\gamma^2/2\alpha}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-t^2/2}\, dt.
\end{aligned}$$

To integrate $\int_{-\infty}^{+\infty} e^{-t^2/2}\, dt$, we consider $\left(\int_{-\infty}^{+\infty} e^{-t^2/2}\, dt\right)^2$ and use polar coordinates.

Then

$$\begin{aligned}\left(\int_{-\infty}^{+\infty} e^{-t^2/2}\,dt\right)^2 &= \left(\int_{-\infty}^{+\infty} e^{-x^2/2}\,dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2/2}\,dy\right)\\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-1(x^2+y^2)/2}\,dx\,dy\\ &= \int_{0}^{2\pi}\int_{0}^{+\infty} r\,e^{-r^2/2}\,dr\,d\theta.\end{aligned}$$

Letting $u = r^2/2$, we get

$$\begin{aligned}\int_{0}^{2\pi}\int_{0}^{+\infty} r\,e^{-r^2/2}\,dr\,d\theta &= \int_{0}^{2\pi}\int_{0}^{+\infty} e^{-u}\,du\,d\theta\\ &= -\int_{0}^{2\pi}\int_{0}^{+\infty}\left(e^{-u}\Big|_{0}^{+\infty}\right)d\theta\\ &= -\int_{0}^{2\pi}\int_{0}^{+\infty} 1\,d\theta\\ &= \theta\Big|_{0}^{2\pi}\\ &= 2\,\pi.\end{aligned}$$

Hence

$$\int_{-\infty}^{+\infty} e^{-t^2/2}\,dt = \sqrt{2\pi}.$$

Then

$$\begin{aligned}\int_{-\infty}^{+\infty} K(y)\,dy &= \sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{+\infty} e^{-\alpha y^2/2}\,dy\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\sqrt{\alpha}\,e^{-\alpha y^2/2}\,dy\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-t^2/2}\,dt\\ &= \frac{1}{\sqrt{2\pi}}\left(\sqrt{2\pi}\right)\\ &= 1.\end{aligned}$$

Step 2

Consider

$$I(\omega,\eta) = \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\, u\, e^{-u/2}[\pi(e^u) - li(e^u)]\, du.$$

By Proposition 2.0.8, for $u > 1$ we have

$$\pi(e^u) - li(e^u) = \frac{-e^{u/2}}{u} - \sum_\rho li(e^{u\rho}) + \vartheta\left(\frac{3\,e^{u/2}}{u^2} + 4\,e^{u/3}\right).$$

Then

$$\begin{aligned} u\, e^{-u/2}[\pi(e^u) - li(e^u)] &= \frac{-u\, e^{-u/2}\, e^{u/2}}{u} - \sum_\rho u\, e^{-u/2}\, li(e^{u\rho}) \\ &\quad + \vartheta\left(\frac{3\,u\, e^{-u/2} e^{u/2}}{u^2} + 4\,u\, e^{-u/2}\, e^{u/3}\right) \\ &= -1 - \sum_\rho u\, e^{-u/2}\, li(e^{u\rho}) + \vartheta\left(\frac{3}{u} + 4\,u\, e^{-u/6}\right). \end{aligned}$$

Hence due to the property that

$$\int_{-\infty}^{+\infty} K(y)\, dy = 1,$$

we use the second case for the estimation of area. Then

$$\left| \int_{\omega-\eta}^{\omega+\eta} \vartheta\,(\frac{3}{u} + 4\,u\, e^{-u/6})\, K(u-\omega)\, du \right| \le \frac{3}{\omega-\eta} + 4\,(\omega+\eta)\, e^{-(\omega-\eta)/6}\,.$$

Step 3

Note that

$$\int_{-\infty}^{\omega-\eta} K(u-\omega)\, du = \int_{\omega+\eta}^{+\infty} K(u-\omega)\, du.$$

Let $y = u - \omega$. Then

$$\begin{aligned} \int_{\omega+\eta}^{+\infty} K(u-\omega)\, du &= \int_{\eta}^{+\infty} K(y)\, dy \\ &= \int_{\eta}^{+\infty} \sqrt{\frac{\alpha}{2\pi}}\, e^{-\alpha y^2/2}\, dy \\ &= \frac{1}{\sqrt{2\pi\alpha}} \int_{\eta}^{+\infty} \alpha\, e^{-\alpha y^2/2}\, dy. \end{aligned}$$

Let $t = \alpha y$. Then

$$\begin{aligned}\int_{-\infty}^{\omega-\eta} K(u-\omega)\,du &= \int_{\omega+\eta}^{+\infty} K(u-\omega)\,du \\ &= \frac{1}{\sqrt{2\pi\alpha}} \int_{\eta}^{+\infty} \alpha\, e^{-\alpha y^2/2}\,dy \\ &= \frac{1}{\sqrt{2\pi\alpha}} \int_{\eta\alpha}^{+\infty} e^{-t^2/2\alpha}\,dt.\end{aligned}$$

Using Proposition 2.0.7, we have

$$\begin{aligned}\frac{1}{\sqrt{2\pi\alpha}} \int_{\eta\alpha}^{+\infty} e^{-t^2/2\alpha}\,dt &< \frac{1}{\sqrt{2\pi\alpha}}\,\frac{\alpha}{\eta\,\alpha}\, e^{-\eta^2\alpha^2/2\alpha} \\ &= \frac{e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}\,\eta}.\end{aligned}$$

Hence

$$\begin{aligned}\int_{-\infty}^{\omega-\eta} K(u-\omega)\,du &= \int_{\omega+\eta}^{+\infty} K(u-\omega)\,du \\ &< \frac{e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}\,\eta}.\end{aligned}$$

Step 4

As a consequence of Step 2 and Step 3, we then have

$$\begin{aligned}
I(\omega,\eta) &= \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\, u\, e^{-u/2}[\pi(e^u) - li(e^u)]\, du \\
&= \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\Big[-1 - \sum_\rho u\, e^{-u/2}\, li(e^{u\rho}) + \vartheta\Big(\frac{3}{u} + 4\, u\, e^{-u/6}\Big)\Big]\, du \\
&= -\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\, du - \int_{\omega-\eta}^{\omega+\eta} \sum_\rho u\, e^{-u/2}\, li(e^{u\rho})\, K(u-\omega)\, du \\
&\quad + \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\, \vartheta\Big(\frac{3}{u} + 4\, u\, e^{-u/6}\Big)\, du \\
&= -\int_{-\infty}^{+\infty} K(u-\omega)\, du + \int_{-\infty}^{\omega-\eta} K(u-\omega)\, du + \int_{\omega+\eta}^{+\infty} K(u-\omega)\, du \\
&\quad - \int_{\omega-\eta}^{\omega+\eta} \sum_\rho u\, e^{-u/2}\, li(e^{u\rho})\, K(u-\omega)\, du \\
&\quad + \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\, \vartheta\Big(\frac{3}{u} + 4\, u\, e^{-u/6}\Big)\, du \\
&= -1 + 2\,\frac{e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}\,\eta} - \int_{\omega-\eta}^{\omega+\eta} \sum_\rho u\, e^{-u/2}\, li(e^{u\rho})\, K(u-\omega)\, du \\
&\quad + \vartheta\Big(\frac{3}{\omega-\eta} + 4\,(\omega+\eta)\, e^{-(\omega-\eta)/6}\Big)\, du \\
&= -1 - \sum_\rho \int_{\omega+\eta}^{\omega-\eta} K(u-\omega) u e^{-u/2} li(e^{u\rho})\, du \\
&\quad + \vartheta\Big(\frac{3}{\omega-\eta} + 4(\omega+\eta)e^{-(\omega-\eta)/6} + \frac{2e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}\eta}\Big).
\end{aligned}$$

The interchange of summation is justified because

$$-1 - \sum_\rho u\, e^{-u/2}\, li(e^{u\rho}) + \vartheta\Big(\frac{3}{u} + 4\, u\, e^{-u/6}\Big)$$

converges boundedly in the interval $\omega - \eta \le u \le \omega + \eta$.

Step 5

From the definition of $li(x)$, it follows that

$$li(e^{u\rho}) = \int_{-\infty+iu\gamma}^{\rho u} \frac{e^z}{z}\, dz.$$

Let $t = \rho u - z$. Then

$$\int_{-\infty+iu\gamma}^{\rho u} \frac{e^z}{z}\,dz = -\int_{+\infty}^{0} \frac{e^{\rho u-t}}{\rho u - t}\,dt$$
$$= \int_{0}^{+\infty} \frac{e^{\rho u-t}}{\rho u - t}\,dt.$$

Through integration by parts, we then get

$$\begin{aligned} li(e^{\rho u}) &= \int_0^{+\infty} \frac{e^{\rho u-t}}{\rho u - t}\,dt \\ &= \frac{e^{\rho u}}{\rho u} + \int_0^{+\infty} \frac{e^{\rho u-t}}{(\rho u - t)^2}\,dt \\ &= \frac{e^{\rho u}}{\rho u} + \int_0^{+\infty} \frac{\vartheta\, e^{\rho u-t}}{(u\gamma)^2}\,dt \\ &= \frac{e^{\rho u}}{\rho u} + \frac{\vartheta\, e^{\beta u-t}}{(u\gamma)^2}\Bigg|_0^{+\infty} \\ &= \frac{e^{\rho u}}{\rho u} + \frac{\vartheta\, e^{\beta u}}{(u\gamma)^2}. \end{aligned}$$

Using this result and assuming that for a positive number A such that $|\gamma| \leq A$ the Riemann Hypothesis holds, i.e. $\beta = \frac{1}{2}$, we have

$$\begin{aligned} -\sum_{\rho}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\,u\,e^{-u/2}\,li(e^{\rho u})\,du = &-\sum_{0<|\gamma|\leq A}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\,u\,e^{-u/2}\,li(e^{\rho u})\,du \\ &-\sum_{|\gamma|>A}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\,u\,e^{-u/2}\,li(e^{\rho u})\,du. \end{aligned}$$

Further

$$\begin{aligned} &-\sum_{0<|\gamma|\leq A}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)ue^{-u/2}li(e^{\rho u})\,du \\ &= -\sum_{0<|\gamma|\leq A}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\left[\frac{u\,e^{-u/2}\,e^{u/2+i\gamma u}}{\rho u} + \frac{\vartheta\,u\,e^{-u/2}\,e^{u/2}}{(u\gamma)^2}\right]du \\ &= -\sum_{0<|\gamma|\leq A}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\left[\frac{e^{i\gamma u}}{\rho} + \frac{\vartheta}{u(\gamma)^2}\right]du \\ &= -\sum_{0<|\gamma|\leq A}\frac{1}{\rho}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\,e^{i\gamma u}\,du - \sum_{0<|\gamma|\leq A}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\frac{\vartheta}{u(\gamma)^2}\,du. \end{aligned}$$

Hence

$$
\begin{aligned}
&-\sum_{\rho}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\, u\, e^{-u/2}\, li(e^{\rho u})\, du \\
&= -\sum_{0<|\gamma|\le A}\frac{1}{\rho}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\, e^{i\gamma u}\, du - \sum_{0<|\gamma|\le A}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\,\frac{\vartheta}{u(\gamma)^2}\, du \\
&\quad -\sum_{|\gamma|>A}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\, u\, e^{-u/2}\, li(e^{\rho u})\, du.
\end{aligned}
$$

Step 6

Let $y = u - \omega$. Then

$$
\begin{aligned}
&-\sum_{0<|\gamma|\le A}\frac{1}{\rho}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\, e^{i\gamma u}\, du \\
&= -\sum_{0<|\gamma|\le A}\frac{e^{i\gamma\omega}}{\rho}\int_{-\eta}^{+\eta} K(y)\, e^{i\gamma y}\, dy \\
&= -\sum_{0<|\gamma|\le A}\frac{e^{i\gamma\omega}}{\rho}\int_{-\infty}^{+\infty} K(y)\, e^{i\gamma y}\, dy + \sum_{0<|\gamma|\le A}\frac{e^{i\gamma\omega}}{\rho}\int_{-\infty}^{-\eta} K(y)\, e^{i\gamma y}\, dy \\
&\quad + \sum_{0<|\gamma|\le A}\frac{e^{i\gamma\omega}}{\rho}\int_{+\eta}^{+\infty} K(y)\, e^{i\gamma y}\, dy \\
&= -\sum_{0<|\gamma|\le A}\frac{e^{i\gamma\omega}}{\rho}\int_{-\infty}^{+\infty} K(y)\, e^{i\gamma y}\, dy + 2\sum_{0<\gamma\le A}\left|\frac{e^{i\gamma\omega}}{\frac{1}{2}+i\gamma}\int_{+\eta}^{+\infty} K(y)\, e^{i\gamma y}\, dy\right| \\
&= -\sum_{0<|\gamma|\le A}\frac{e^{i\gamma\omega}}{\rho}\, e^{-\gamma^2/2\alpha} + 4\,\vartheta\sum_{0<\gamma\le A}\frac{1}{\gamma}\left|\int_{+\eta}^{+\infty} K(y)\, e^{i\gamma y}\, dy\right|
\end{aligned}
$$

since

$$
\int_{-\infty}^{+\infty} K(y)\, e^{i\gamma y}\, dy = e^{-\gamma^2/2\alpha}
$$

by previous result. Using integration by parts, we then get

$$
\begin{aligned}
\int_{+\eta}^{+\infty} K(y)\, e^{i\gamma y}\, dy &= \int_{+\eta}^{+\infty}\left[\frac{K'(y)\, e^{i\gamma\eta}}{i\gamma} - \frac{K'(y)\, e^{i\gamma y}}{i\gamma}\right] dy \\
&= \int_{+\eta}^{+\infty}\frac{K'(y)(e^{i\gamma\eta} - e^{i\gamma y})}{i\gamma}\, dy.
\end{aligned}
$$

Thus because $K(y)$ is monotone decreasing for $y > 0$, we get

$$
\begin{aligned}
\left|\int_{+\eta}^{+\infty} K(y)\, e^{i\gamma y}\, dy\right| &= \left|\int_{+\eta}^{+\infty} \frac{K'(y)(e^{i\gamma\eta} - e^{i\gamma y})}{i\gamma}\, dy\right| \\
&\leq \int_{+\eta}^{+\infty} \left|K'(y)\right| \left|\frac{e^{i\gamma\eta} - e^{i\gamma y}}{i\gamma}\right| dy \\
&\leq \frac{2}{\gamma} \int_{+\eta}^{+\infty} \left|K'(y)\right| dy \\
&= \frac{2}{\gamma}\, K(y)\Big|_{+\eta}^{+\infty} \\
&= \frac{2}{\gamma}\, K(\eta) \\
&= \frac{2}{\gamma} \sqrt{\frac{\alpha}{2\pi}}\, e^{-\alpha\eta^2/2}.
\end{aligned}
$$

Next, we use Proposition 2.0.6 and the fact that $\frac{1}{\sqrt{2\pi}} < 0.4$ to get

$$
\begin{aligned}
&- \sum_{0<|\gamma|\leq A} \frac{1}{\rho} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\, e^{i\gamma u}\, du \\
&= - \sum_{0<|\gamma|\leq A} \frac{e^{i\gamma\omega}}{\rho}\, e^{-\gamma^2/2\alpha} + 4\,\vartheta \sum_{0<\gamma\leq A} \frac{1}{\gamma} \left|\int_{+\eta}^{+\infty} K(y)\, e^{i\gamma y}\, dy\right| \\
&= - \sum_{0<|\gamma|\leq A} \frac{e^{i\gamma\omega}}{\rho}\, e^{-\gamma^2/2\alpha} + 4\,\vartheta \sum_{0<\gamma\leq A} \frac{2}{\gamma^2} \sqrt{\frac{\alpha}{2\pi}} e^{-\alpha\eta^2/2} \\
&= - \sum_{0<|\gamma|\leq A} \frac{e^{i\gamma\omega}}{\rho}\, e^{-\gamma^2/2\alpha} + 8\,\vartheta \sqrt{\frac{\alpha}{2\pi}}\, e^{-\alpha\eta^2/2} \sum_{0<\gamma\leq A} \frac{1}{\gamma^2} \\
&= - \sum_{0<|\gamma|\leq A} \frac{e^{i\gamma\omega}}{\rho}\, e^{-\gamma^2/2\alpha} + 0.08\,\vartheta\,\sqrt{\alpha}\, e^{-\alpha\eta^2/2}.
\end{aligned}
$$

This sum can be taken over just the zeros if we add another error term. Given $T \geq 2\pi e$, by Proposition 2.0.4 we have

$$\begin{aligned}
&\left| - \sum_{T<|\gamma|\leq A} \frac{e^{i\gamma\omega}}{\rho}\, e^{-\gamma^2/2\alpha} \right| \\
&\leq \sum_{T<\gamma\leq A} \left| \frac{e^{i\gamma\omega}}{\rho} \right| \left| e^{-\gamma^2/2\alpha} \right| \\
&\leq 2 \sum_{T<\gamma\leq A} \frac{e^{-\gamma^2/2\alpha}}{\gamma} \\
&\leq \frac{1}{\pi} \int_T^{+\infty} \frac{e^{-t^2/2\alpha}}{t} \log \frac{t}{2\pi}\, dt + 8\, \frac{e^{-T^2/2\alpha}}{\gamma} \log T + 4 \int_T^{+\infty} \frac{e^{-t^2/2\alpha}}{t^2}\, dt.
\end{aligned}$$

Using Proposition 2.0.7 to estimate the integrals, we get

$$\begin{aligned}
\int_T^{+\infty} \frac{e^{-t/2\alpha}}{t}\, dt &< \frac{\alpha}{T}\, \frac{1}{T}\, e^{-T^2/2\alpha} \\
&= \frac{\alpha}{T^2}\, e^{-T^2/2\alpha}\, .
\end{aligned}$$

Hence for $2\pi \leq T \leq A$, we have

$$\begin{aligned}
\left| - \sum_{T<|\gamma|\leq A} \frac{e^{i\gamma\omega}}{\rho} e^{-\gamma^2/2\alpha} \right| &< \frac{\alpha}{\pi\, T^2}\, \log \frac{T}{2\pi}\, e^{-T^2/2\alpha} + \frac{8\, e^{-T^2/2\alpha}\, \log T}{T} + \frac{4\,\alpha}{T^3} e^{-T^2/2\alpha} \\
&= e^{-T^2/2\alpha} \left(\frac{\alpha}{\pi\, T^2}\, \log \frac{T}{2\pi} + \frac{8 \log T}{T} + \frac{4\,\alpha}{T^3} \right).
\end{aligned}$$

Combining above results, we then get

$$\begin{aligned}
- \sum_{0\leq|\gamma|\leq A} \frac{1}{\rho} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\, e^{i\gamma u}\, du \leq e^{-T^2/2\alpha} &\left(\frac{\alpha}{\pi\, T^2}\, \log \frac{T}{2\pi} + \frac{8 \log T}{T} + \frac{4\,\alpha}{T^3} \right) \\
&+ 0.08\, \vartheta\, \sqrt{\alpha}\, e^{-\alpha\eta^2/2}\, .
\end{aligned}$$

Step 7

The sum

$$- \sum_{0\leq|\gamma|\leq A} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\, \frac{\vartheta}{u\gamma^2}\, du$$

can be estimated using Proposition 2.0.6. Let $y = u - \omega$. Then

$$\begin{aligned}\left| - \sum_{0 \le |\gamma| \le A} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) \, \frac{\vartheta}{u\gamma^2} \, du \right| &= \left| - \sum_{0 \le |\gamma| \le A} \int_{-\eta}^{+\eta} K(y) \, \frac{\vartheta}{y+\omega} \, dy \right| \\ &\le -\sum_{\rho} \frac{1}{\gamma^2} \int_{-\eta}^{+\eta} \frac{|K(y)|}{y+\omega} \, dy \\ &\le \frac{0.5}{\omega - \eta} \, .\end{aligned}$$

Step 8

Note that as of now we have not made use of the conditions $\frac{4A}{\omega} \le \alpha \le A^2$, and $\frac{2A}{\alpha} \le \eta < \frac{\omega}{2}$.

Let $A \to +\infty$. If we assume the Riemann Hypothesis, then we can combine

$$I(\omega, \eta) = -1 - \sum_{0 < |\gamma| \le T} \frac{e^{i\gamma\omega}}{\rho} \, e^{-\gamma^2/2\alpha} + R$$

where

$$\begin{aligned}|R| < & \frac{3.05}{\omega-\eta} + 4(\omega+\eta)e^{-(\omega-\eta)/6} + \frac{2e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}\eta} + 0.08\sqrt{\alpha}e^{-\alpha\eta^2/2} \\ & + e^{-T^2/2\alpha}\left(\frac{\alpha}{\pi T^2} log\frac{T}{2\pi} + \frac{8\log T}{T} + \frac{4\alpha}{T^3}\right) .\end{aligned}$$

Since $A \to +\infty$, we do not need to consider the last term in the estimate of R. Hence if the Riemann Hypothesis holds, we obtain the conclusion of the theorem with the last term in the estimate for R being omitted.

Step 9

To complete the proof, it is sufficient to show that

$$\left| - \sum_{|\gamma| > A} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) \, u \, e^{-u/2} \, li(e^{\rho u}) \, du \right| \le A \log A \, e^{-A^2/2\alpha + (\omega+\eta)/2} (3.2 \, \alpha^{-1/2} + 14.4 \, \eta)$$

when A, α, ω, and η satisfy $\frac{4A}{\omega} \le \alpha \le A^2$ and $\frac{2A}{\alpha} \le \eta < \frac{\omega}{2}$.

Consider the function

$$f_\rho(s) = \rho \, s \, e^{-\rho s} \, li(e^{\rho s}) \, e^{-\alpha(s-\omega)^2/2}$$

in the sector $\frac{-\pi}{4}\pi \le arg(s) \le \frac{\pi}{4}$. The inequality $\frac{5}{12}\pi < |arg(\rho)| < \frac{\pi}{2}$ holds because $0 < \beta < 1$ and $|\gamma| > 14$ for every complex zero ρ. It follows from the definition of

$li(e^w)$ that $f_\rho^{'}(s)$ exists for all s in the sector since for $arg(\rho s)$ we have $arg(\rho s) = arg(\rho) + arg(s)$. Then $\frac{5}{12}\pi - \frac{\pi}{4} \leq |arg(\rho s)| \leq \frac{\pi}{2} + \frac{\pi}{4}$. Hence $\frac{\pi}{6} < |arg(\rho s)| < \frac{3}{4}\pi$. Also

$$\begin{aligned}
|f_\rho(s)| &= \left|\rho\, s\, e^{-\rho s}\, li(e^{\rho s})\, e^{-\alpha(s-\omega)^2/2}\right| \\
&= \left|\rho\, s\, e^{-\rho s}\, e^{\rho s} \int_0^{+\infty} \frac{e^{-t}}{\rho s - t}\, dt\, e^{-\alpha(s-\omega)^2/2}\right| \\
&= \left|\rho\, s\, e^{-\alpha(s-\omega)^2/2} \int_0^{+\infty} \frac{e^{-t}}{\rho s - t}\, dt\right| \\
&\leq \frac{|\rho s||e^{-\alpha(s-\omega)^2/2}|}{|\Im(\rho s)|} \int_0^{+\infty} e^{-t}\, dt \\
&\leq 2|e^{-\alpha(s-\omega)^2/2}|
\end{aligned}$$

since

$$li(e^{\rho s}) = e^{\rho s} \int_0^{+\infty} \frac{e^{-t}}{\rho s - t}\, dt$$

by previous result. Further

$$\begin{aligned}
&- \sum_{|\gamma|>A} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\, u\, e^{-u/2}\, li(e^{\rho u})\, du \\
&= -\sqrt{\frac{\alpha}{2\pi}} \sum_{|\gamma|>A} \int_{\omega-\eta}^{\omega+\eta} e^{-\alpha(u-\omega)^2/2}\, u\, e^{-u/2}\, li(e^{\rho u})\, du \\
&= -\sqrt{\frac{\alpha}{2\pi}} \sum_{|\gamma|>A} \int_{\omega-\eta}^{\omega+\eta} u\, e^{-\rho u}\, e^{(\rho-1/2)u}\, li(e^{\rho u})\, e^{-\alpha(u-\omega)^2/2}\, du \\
&= -\sqrt{\frac{\alpha}{2\pi}} \sum_{|\gamma|>A} \frac{1}{\rho} \int_{\omega-\eta}^{\omega+\eta} e^{(\rho-1/2)u}\, f_\rho(u)\, du.
\end{aligned}$$

Using integration by parts, we get

$$
\begin{aligned}
&\int_{\omega-\eta}^{\omega+\eta} e^{(\rho-1/2)u}\, f_\rho(u)\, du \\
&= \frac{e^{(\rho-1/2)u}}{\rho-1/2}\, f_\rho(u)\Big|_{\omega-\eta}^{\omega+\eta} - \int_{\omega-\eta}^{\omega+\eta} \frac{e^{(\rho-1/2)u}}{\rho-1/2}\, f_\rho^{(1)}(u)\, du \\
&= \frac{e^{(\rho-1/2)(\omega+\eta)}}{\rho-1/2}\, f_\rho(\omega+\eta) - \frac{e^{(\rho-1/2)(\omega-\eta)}}{\rho-1/2}\, f_\rho(\omega-\eta)\Big|_{\omega-\eta}^{\omega+\eta} - \int_{\omega-\eta}^{\omega+\eta} \frac{e^{(\rho-1/2)u}}{\rho-1/2}\, f_\rho^{(1)}(u)\, du \\
&= \frac{e^{(\rho-1/2)\omega}}{\rho-1/2}\left[e^{(\rho-1/2)\eta}\, f_\rho(\omega+\eta) - e^{-(\rho-1/2)\eta}\, f_\rho(\omega-\eta)\right] - \frac{e^{(\rho-1/2)u}}{(\rho-1/2)^2}\, f_\rho^{(1)}(u)\Big|_{\omega-\eta}^{\omega+\eta} \\
&\quad + \int_{\omega-\eta}^{\omega+\eta} \frac{e^{(\rho-1/2)u}}{(\rho-1/2)^2}\, f_\rho^{(2)}(u)\, du \\
&= \frac{e^{(\rho-1/2)\omega}}{\rho-1/2}\left[e^{(\rho-1/2)\eta}\, f_\rho(\omega+\eta) - e^{-(\rho-1/2)\eta}\, f_\rho(\omega-\eta)\right] \\
&\quad - \frac{e^{(\rho-1/2)\omega}}{(\rho-1/2)^2}\left[e^{(\rho-1/2)\eta}\, f_\rho^{(1)}(\omega+\eta) - e^{-(\rho-1/2)\eta}\, f_\rho^{(1)}(\omega-\eta)\right] \\
&\quad + \int_{\omega-\eta}^{\omega+\eta} \frac{e^{(\rho-1/2)u}}{(\rho-1/2)^2}\, f_\rho^{(2)}(u)\, du \\
&= \sum_{n=0}^{N-1} \frac{(-1)^n\, e^{(\rho-1/2)\omega}}{(\rho-1/2)^{n+1}}\left[e^{(\rho-1/2)\eta}\, f_\rho^{(n)}(\omega+\eta) - e^{-(\rho-1/2)\eta}\, f_\rho^{(n)}(\omega-\eta)\right] \\
&\quad + (-1)^N \int_{\omega-\eta}^{\omega+\eta} \frac{e^{(\rho-1/2)u}}{(\rho-1/2)^N}\, f_\rho^{(N)}(u)\, du
\end{aligned}
$$

where N is a positive integer which we will fix later.

Next, we estimate $f_\rho^{(n)}(u)$ for $\omega-\eta \le u \le \omega+\eta$ by using a contour integral around a circle of radius $r \le \frac{\omega}{4}$ about the point u. If s is on this circle, then $\Re(s) \ge \omega-\eta-\frac{\omega}{4} > \frac{\omega}{4}$ because $\eta < \frac{\omega}{2}$, and $|\Im(s)| \le \frac{\omega}{4}$. Hence the circle lies in the sector $|arg(s)| \le \frac{\pi}{4}$ where $|f_\rho(s)| \le 2|e^{-\alpha(s-\omega)^2/2}|$. Consequently, for $\omega-\eta \le u \le \omega+\eta$ we have

$$
\begin{aligned}
f_\rho(u) &= \frac{1}{2\pi i}\oint \frac{f_\rho(s)}{s-u}\, ds, \\
f_\rho^{(1)}(u) &= \frac{1}{2\pi i}\oint \frac{f_\rho(s)}{(s-u)^2}\, ds, \\
f_\rho^{(2)}(u) &= \frac{2!}{2\pi i}\oint \frac{f_\rho(s)}{(s-u)^3}\, ds, \\
f_\rho^{(3)}(u) &= \frac{3!}{2\pi i}\oint \frac{f_\rho(s)}{(s-u)^4}\, ds, \\
&\quad ....
\end{aligned}
$$

Thus it follows that

$$f_\rho^{(n)}(u) = \frac{n!}{2\pi i} \oint \frac{f_\rho(s)}{(s-u)^{n+1}}\, ds.$$

Therefore

$$\begin{aligned}
|f_\rho^{(n)}(u)| &= \left| \frac{n!}{2\pi i} \oint \frac{f_\rho(s)}{(s-u)^{n+1}}\, ds \right| \\
&\le \frac{n!}{2\pi} \oint \frac{|f_\rho(s)|}{|(s-u)^{n+1}|}\, ds \\
&\le \frac{n!}{2\pi} \frac{2|e^{-\alpha(s-\omega)^2/2}|}{r^{n+1}} 2\pi r \\
&\le \frac{2n!}{r^n} \max_{|s-u|=r} |e^{-\alpha(s-\omega)^2/2}|
\end{aligned}$$

by the Estimate Lemma.

If $s = \sigma + it$, then on the circle $(\sigma - u)^2 + t^2 = r^2$ we have

$$\begin{aligned}
|e^{-\alpha(s-\omega)^2/2}| &= |e^{-\alpha(\sigma+it-\omega)^2/2}| \\
&= |e^{\alpha(-\sigma^2+2\sigma\omega-\omega^2+2\sigma it+2\omega it+t^2)/2}| \\
&= |e^{\alpha[t^2-(\sigma-\omega)^2]/2} e^{\alpha(\sigma it+\omega it)}| \\
&= e^{\alpha[t^2-(\sigma-\omega)^2]/2} \\
&= e^{\alpha[r^2-(\sigma-u)^2-(\sigma-\omega)^2]/2} \\
&\le e^{\alpha r^2/2}.
\end{aligned}$$

If $N \le \frac{\alpha\omega^2}{16}$, we can fix $r = \frac{N}{\alpha}$ since $r \le \frac{\omega}{4}$. Then we get for $\omega - \eta \le u \le \omega + \eta$

$$\begin{aligned}
|f_\rho^{(N)}(u)| &\le \frac{2N!}{r^n} \max_{|s-u|=r} |e^{-\alpha(s-\omega)^2/2}| \\
&\le \frac{2N!}{r^N}\, e^{\alpha r^2/2} \\
&= 2N! \left(\frac{\alpha}{N}\right)^{N/2} e^{\alpha(N/\alpha)/2} \\
&= 2N! \left(\frac{\alpha}{N}\right)^{N/2} e^{N/2} \\
&= 2N! \left(\frac{\alpha e}{N}\right)^{N/2} e^{N/2}.
\end{aligned}$$

To estimate the derivative at $\omega \pm \eta$, we let $r = \frac{\eta}{2}$. Since $\eta < \frac{\omega}{2}$, we have $r < \frac{\omega}{4}$. Then on the circle $|s - (\omega \pm \eta)| = r$ we have

$$\begin{aligned} |e^{-\alpha(s-\omega)^2/2}| &\leq e^{\alpha[r^2-(\sigma-(\omega\pm\eta))^2-(\sigma-\omega)^2]/2} \\ &\leq e^{\alpha r^2/2} \\ &= e^{\alpha(\eta/2)^2/2} \\ &= e^{\alpha\eta^2/2}. \end{aligned}$$

Note that since $0 < \beta < 1$ we have $|\beta - \frac{1}{2}| < \frac{1}{2}$. Hence

$$\begin{aligned} |e^{(\beta-1/2)(\omega\pm\eta)}| &\leq e^{(\omega\pm\eta)/2} \\ &\leq e^{(\omega+\eta)/2}. \end{aligned}$$

Then

$$\begin{aligned} |e^{(\rho-1/2)(\omega+\eta)} - e^{(\rho-1/2)(\omega-\eta)}| &= |e^{(\beta+i\gamma-1/2)(\omega+\eta)} - e^{(\beta+i\gamma-1/2)(\omega-\eta)}| \\ &= |e^{(\beta-1/2)(\omega+\eta)}\, e^{i\gamma(\omega+\eta)} - e^{(\beta-1/2)(\omega-\eta)}\, e^{i\gamma(\omega-\eta)}| \\ &\leq 2\, e^{(\omega+\eta)/2}. \end{aligned}$$

Combining all of above results, we then get

$$\begin{aligned}
&\left|-\sum_{|\gamma|>A}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\, u\, e^{-u/2}\, li(e^{\rho u})\, du\right| \\
&= \left|-\sqrt{\frac{\alpha}{2\pi}}\sum_{|\gamma|>A}\frac{1}{\rho}\int_{\omega-\eta}^{\omega+\eta} e^{(\rho-1/2)u}\, f_\rho(u)\, du\right| \\
&= \left|\sqrt{\frac{\alpha}{2\pi}}\sum_{|\gamma|>A}\frac{1}{\rho}\left[-\sum_{n=0}^{N-1}\frac{(-1)^n\, e^{(\rho-1/2)\omega}}{(\rho-1/2)^{n+1}}\left(e^{(\rho-1/2)\eta}\, f_\rho^{(n)}(\omega+\eta)\right)\right.\right. \\
&\quad - \sum_{n=0}^{N-1}\frac{(-1)^n\, e^{(\rho-1/2)\omega}}{(\rho-1/2)^{n+1}}\left(-e^{-(\rho-1/2)\eta}\, f_\rho^{(n)}(\omega-\eta)\right) \\
&\quad \left.\left. - \frac{(-1)^N}{(\rho-1/2)^N}\int_{\omega-\eta}^{\omega+\eta} e^{(\rho-1/2)u}\, f_\rho^{(N)}(u)\, du\right]\right| \\
&\le \left|\sqrt{\frac{\alpha}{2\pi}}\sum_{|\gamma|>A}\frac{1}{\rho}\left[-\sum_{n=0}^{N-1}\frac{(-1)^n}{(\rho-1/2)^{n+1}}\left(e^{(\rho-1/2)(\omega+\eta)}\, 2n!\,(\eta/2)^{-n}\, e^{-\alpha\eta^2/8}\right)\right.\right. \\
&\quad - \sum_{n=0}^{N-1}\frac{(-1)^n}{(\rho-1/2)^{n+1}}\left(-e^{(\rho-1/2)(\omega-\eta)}\, 2n!\,(\eta/2)^{-n}\, e^{-\alpha\eta^2/8}\right) \\
&\quad \left.\left. - \frac{(-1)^N}{(\rho-1/2)^N}\int_{\omega-\eta}^{\omega+\eta} e^{(\rho-1/2)u}\, 2N!\left(\frac{\alpha e}{N}\right)^{N/2} du\right]\right| \\
&\le 2\sqrt{\frac{\alpha}{2\pi}}\sum_{\gamma>A}\frac{1}{\gamma}\left[\sum_{n=0}^{N-1}\frac{2n!\, e^{\alpha\eta^2/8}}{\gamma^{n+1}(\eta/2)^n}\left|e^{(\rho-1/2)(\omega+\eta)} - e^{(\rho-1/2)(\omega-\eta)}\right|\right. \\
&\quad \left. + \frac{2N!}{\gamma^N}\left(\frac{\alpha e}{N}\right)^{N/2}\int_{\omega-\eta}^{\omega+\eta}|e^{(\rho-1/2)u}|\, du\right] \\
&\le 2\sqrt{\frac{\alpha}{2\pi}}\sum_{\gamma>A}\left[\sum_{n=0}^{N-1}\frac{2n!\, e^{-\alpha\eta^2/8}}{\gamma^{n+2}(\eta/2)^n}2e^{(\omega+\eta)/2} + \frac{2N!}{\gamma^{N+1}}\left(\frac{\alpha e}{N}\right)^{N/2} e^{(\omega+\eta)/2}\, 2\eta\right] \\
&= 2\sqrt{\frac{\alpha}{2\pi}}\, e^{(\omega+\eta)/2}\sum_{\gamma>A}\left[\frac{4\, e^{-\alpha\eta^2/8}}{\gamma^2}\sum_{n=0}^{N-1}\frac{n!}{(\gamma\eta/2)^n} + \frac{4\,\eta N!}{\gamma^{N+1}}\left(\frac{\alpha e}{N}\right)^{N/2}\right]
\end{aligned}$$

provided $1 \le N \le \frac{\alpha\omega^2}{16}$.

Fix $N = \lfloor \frac{A^2}{\alpha} \rfloor$.

Because $\frac{4A}{\omega} \leq \alpha \leq A^2$, it follows that $\frac{4A}{\omega\alpha} \leq \frac{\alpha}{\alpha} \leq \frac{A^2}{\alpha}$. Hence $1 \leq \frac{A^2}{\alpha} = N$.

Since $\frac{2A}{\alpha} \leq \eta < \frac{\omega}{2}$, we have

$$\begin{aligned} 2A \leq \eta\alpha &\leq \frac{\omega\alpha}{2} \\ A \leq \frac{\eta\alpha}{2} &< \frac{\omega\alpha}{4} \\ A^2 \leq \frac{\eta^2\alpha^2}{4} &< \frac{\omega^2\alpha^2}{16} \\ \frac{A^2}{\alpha} \leq \frac{\eta^2\alpha}{4} &< \frac{\omega^2\alpha}{16}. \end{aligned}$$

Hence $1 \leq N \leq \frac{\omega^2\alpha}{16}$ as required.

Also since $\frac{2A}{\alpha} \leq \eta < \frac{\omega}{2}$, we have $\frac{2A}{\alpha} = \frac{A^2}{\alpha}\frac{2}{A} = \frac{2N}{A}$. Hence $\eta \geq \frac{2N}{A}$.

Note that $\frac{N}{A} \leq \frac{(A^2/\alpha)}{A} = \frac{A}{\alpha}$ since $N \leq \frac{A^2}{\alpha}$. Also note that

$$\sum_{n=0}^{N-1} n! \leq \sum_{n=0}^{N-1} N^n.$$

Further by Proposition 2.0.5, we have

$$\begin{aligned} \sum_{\gamma>A} \frac{1}{\gamma^2} \sum_{n=0}^{N-1} \frac{1}{\gamma^n} &= \sum_{\gamma>A} \frac{1}{\gamma}\left(1 + \frac{1}{\gamma} + \frac{1}{\gamma^2} + \frac{1}{\gamma^3} + ... + \frac{1}{\gamma^{N-1}}\right) \\ &= \sum_{\gamma>A} \frac{1}{\gamma^2} + \sum_{\gamma>A} \frac{1}{\gamma^3} + \sum_{\gamma>A} \frac{1}{\gamma^4} + \sum_{\gamma>A} \frac{1}{\gamma^5} + ... + \sum_{\gamma>A} \frac{1}{\gamma^{N+1}} \\ &< A^{-1}\log A + A^{-2}\log A + A^{-3}\log A + A^{-4}\log A + ... + A^{-N}\log A \\ &= \log A\big(A^{-1} + A^{-2} + A^{-3} + A^{-4} + ... + A^{-N}\big) \\ &= \log A \sum_{n=0}^{N-1} \frac{1}{A^{n+1}}. \end{aligned}$$

Hence

$$\begin{aligned}
\sum_{\gamma>A} \frac{4\,e^{-\alpha\eta^2/8}}{\gamma^2} \sum_{n=0}^{N-1} \frac{n!}{(\gamma\eta/2)^n} &= \sum_{\gamma>A} \frac{4e^{-\alpha\eta^2/8}}{\gamma^2} \sum_{n=0}^{N-1} \frac{1}{\gamma^n} \frac{n!}{(\eta/2)^n} \\
&< 4\,e^{-\alpha\eta^2/8} \log A \sum_{n=0}^{N-1} \frac{n!}{(\eta/2)^n\, A^{n+1}} \\
&\leq 4\,e^{-\alpha\eta^2/8} \log A \sum_{n=0}^{N-1} \frac{N^n}{\eta^n (A/2)^n\, A} \\
&\leq 4\,e^{-\alpha\eta^2/8} \log A \sum_{n=0}^{N-1} \frac{N^n}{(2N/A)^n (A/2)^n A} \\
&= 4\,e^{-\alpha\eta^2/8} \log A \sum_{n=0}^{N-1} \frac{N^n}{N^n A} \\
&= 4\,e^{-\alpha\eta^2/8} \log A \sum_{n=0}^{N-1} \frac{1}{A} \\
&= 4\,e^{-\alpha\eta^2/8} N A^{-1} \log A \\
&\leq 4\,e^{-\alpha\eta^2/8} \alpha^{-1} A \log A.
\end{aligned}$$

Note that since $N = \left\lfloor \frac{A^2}{\alpha} \right\rfloor$, we have $\frac{A^2}{\alpha} - 1 \leq N \leq \frac{A^2}{\alpha}$. Also note that $N! \leq e^{1-N}\, N^{N+1/2}$.

Hence

$$\begin{aligned}
\sum_{\gamma>A} \frac{4\eta\, N!}{\gamma^{N+1}}\left(\frac{\alpha e}{N}\right)^{N/2} &< 4\eta\, N!\left(\frac{\alpha e}{N}\right)^{N/2} A^{-N}\log A \\
&\le 4\eta\, e^{1-N}\, N^{N+1/2}\left(\frac{\alpha e}{N}\right)^{N/2} A^{-N}\log A \\
&= 4\eta\, e^{1-N+N/2}\, N^{1/2}\, N^{N}\left(\frac{\alpha}{N}\right)^{N/2} A^{-N}\log A \\
&= 4\eta\, e^{1-N/2}\, N^{1/2}(N^2)^{N/2}\left(\frac{\alpha}{N}\right)^{N/2} (A^{-2})^{N/2}\log A \\
&= 4\eta\, e^{1-N/2}\, N^{1/2}\left(\frac{\alpha N^2}{NA^2}\right)^{N/2} \log A \\
&= 4\eta\, e^{1-N/2}\, N^{1/2}\left(\frac{\alpha N}{A^2}\right)^{N/2} \log A \\
&\le 4\eta\, e^{1-N/2}\, N^{1/2}\left(\frac{\alpha}{A^2}\,\frac{A^2}{\alpha}\right)^{N/2} \log A \\
&= 4\eta\, e^{1-N/2}\, N^{1/2}\log A \\
&\le 4\eta\, e^{1-(A^2/2\alpha-1/2)}(A/\sqrt{\alpha})\log A \\
&= 4\, e^{3/2}\,\eta\, e^{-A^2/2\alpha} A\,\alpha^{-1/2}\log A.
\end{aligned}$$

Combining, we get

$$\begin{aligned}
&2\sqrt{\frac{\alpha}{2\pi}}\, e^{(\omega+\eta)/2}\sum_{\gamma>A}\left[\frac{4\, e^{-\alpha\eta^2/8}}{\gamma^2}\sum_{n=0}^{N-1}\frac{n!}{(\gamma\eta/2)^n} + \frac{4\eta\, N!}{\gamma^{N+1}}\left(\frac{\alpha e}{N}\right)^{N/2}\right] \\
&< 2\sqrt{\frac{\alpha}{2\pi}}\, e^{(\omega+\eta)/2}\left[4\, e^{-\alpha\eta^2/8}\,\alpha^{-1}A\log A + 4\, e^{3/2}\,\eta\, e^{-A^2/2\alpha}A\,\alpha^{-1/2}\log A\right].
\end{aligned}$$

Since $\frac{1}{\sqrt{2\pi}} < 0.4$, $e^{3/2} < 4.5$, and $\frac{2A}{\alpha} \le \eta < \frac{\omega}{2}$, we then get

$$\begin{aligned}
&2\sqrt{\frac{\alpha}{2\pi}}\, e^{(\omega+\eta)/2}\left[4\, e^{-\alpha\eta^2/8}\,\alpha^{-1}A\log A + 4\, e^{3/2}\,\eta\, e^{-A^2/2\alpha}A\,\alpha^{-1/2}\log A\right] \\
&< 3.2\,\alpha^{-1/2}A\log A\, e^{-\alpha\eta^2/8+(\omega+\eta)/2} + 14.4\,\eta\, A\log A\, e^{-A^2/2\alpha+(\omega+\eta)/2}A\,\alpha^{-1/2}\log A \\
&\le 3.2\,\alpha^{-1/2}A\log A\, e^{-A^2/2\alpha+(\omega+\eta)/2} + 14.4\,\eta\, A\log A\, e^{-A^2/2\alpha+(\omega+\eta)/2}A\,\alpha^{-1/2}\log A \\
&= A\log A\, e^{-A^2/2\alpha+(\omega+\eta)/2}\big(3.2\,\alpha^{-1/2} + 14.4\,\eta\big).
\end{aligned}$$

Hence

$$\left|-\sum_{|\gamma|>A}\int_{\omega-\eta}^{\omega+\eta}K(u-\omega)\,u\,e^{-u/2}\,li(e^{\rho u})\,du\right|< A\log A\,e^{-A^2/2\alpha+(\omega+\eta)/2}\big(3.2\,\alpha^{-1/2}+14.4\,\eta\big).$$

□

The application of Lehman's Theorem makes two essential assumptions. The first assumption is that the Riemann Hypothesis has to be checked up to a height A. The second assumption is that explicit values for the complex zeros of the Riemann Zeta function $\zeta(s)$ have to be known up to a height T. If both assumptions are met, then we can estimate the integral

$$I(\omega,\eta)=\int\limits_{\omega-\eta}^{\omega+\eta}K(u-\omega)\,u\,e^{-u/2}[\pi(e^u)-li(e^u)]\,du$$

using the equation

$$I(\omega,\eta)=-1-\sum_{0\le|\gamma|\le T}\frac{e^{i\gamma\omega}}{\rho}\,e^{-\gamma^2/2\alpha}+R$$

where R is as previously defined in Theorem 2.0.2. Next, we find suitable values for α and ω such that the first two terms on the right-hand side of the equation

$$I(\omega,\eta)=-1-\sum_{0\le|\gamma|\le T}\frac{e^{i\gamma\omega}}{\rho}\,e^{-\gamma^2/2\alpha}+R$$

sum up to be a positive value larger than the associated error term $|R|$. Then the integral

$$I(\omega,\eta)=\int\limits_{\omega-\eta}^{\omega+\eta}K(u-\omega)\,u\,e^{-u/2}[\pi(e^u)-li(e^u)]\,du$$

is established to be positive and therefore the term $[\pi(e^u)-li(e^u)]$ must admit some positive values for u in the interval $[\omega-\eta,\omega+\eta]$.

Chapter 3

Improvements

In this chapter, we will further improve the error term R in the equation

$$I(\omega, \eta) = -1 - \sum_{0<|\gamma|\leq T} \frac{e^{i\gamma\omega}}{\rho} e^{\gamma^2/2\alpha} + R$$

where R is as previously defined in Theorem 2.0.2. In fact, the dominating term for R is S_1.

Theorem 3.0.9 (Dusart's Theorem [2, Theorem 1.10]) *If $x \geq 32299$, we have $\frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{1.8}{\log^2 x}\right) \leq \pi(x)$. If $x \geq 355991$, we have $\pi(x) \leq \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2.51}{\log^2 x}\right)$.*

We are using above result to prove following theorem:

Theorem 3.0.10 ([6, Theorem 3.2]) *Under the hypothesis of Lehman's Theorem and if ω - $\eta > 25.57$, the equation $I(\omega, \eta) = -1 - \sum_{0<|\gamma|\leq T} \frac{e^{i\gamma\omega}}{\rho} e^{\gamma^2/2\alpha} + R$ still holds if S_1 is replaced by $S_1' = \frac{2}{\omega-\eta} + \frac{10.04}{(\omega-\eta)^2} + \log 2\,(\omega+\eta)\, e^{-(\omega-\eta)/2} + \frac{2}{\log 2}\,(\omega+\eta)\, e^{-(\omega-\eta)/6}$. R is as previously defined in Lehman's Theorem.*

We will split the proof of above theorem into two steps.

In **Step 1** (page 39), we use Dusart's Theorem and the Riemann Explicit Formula to show that

$$\pi(e^u) - li(e^u) \geq -\sum_{\rho} li(e^{u\rho}) - \log 2 - \frac{e^{u/2}}{u}\left(1 + \frac{2}{u} + \frac{10.04}{u^2}\right) - 2\left(\frac{e^{u/3}}{\log 2}\right).$$

Step 2 (page 41) then uses the result of Step 1 to show that the upper bound of the expression

$$J := \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\left(u\, e^{-u/2}\log 2 + \frac{2}{u} + \frac{10.04}{u^2} + \frac{2\, u\, e^{-u/6}}{\log 2}\right) du$$

can be estimated by

$$\frac{2}{\omega-\eta} + \frac{10.04}{(\omega-\eta)^2} + \frac{2\,(\omega+\eta)}{\log 2}\, e^{-(\omega-\eta)/6} + \log 2\,(\omega+\eta)\, e^{-(\omega-\eta)/2} = S_1'$$

which gives us the conclusion of the theorem.

Proof For this proof, we closely follow Saouter-Demichel's proof of Theorem 3.0.10.

Step 1

Let

$$\Pi(x) = \pi(x) + \frac{1}{2}\,\pi(x^{1/2}) + \frac{1}{3}\,\pi(x^{1/3}) + \ldots .$$

This is a finite function because $\pi(x^{1/k}) = 0$ for $x^{1/k} < 2$. $x^{1/k} < 2$ when $k > \frac{logx}{log2}$. Hence we have $\left\lfloor \frac{\log x}{\log 2} \right\rfloor$ number of terms.

Let

$$\Pi_0(x) = \lim_{\varepsilon\to 0} \frac{1}{2}\left(\Pi(x+\varepsilon) + \Pi(x-\varepsilon)\right).$$

For $x > 1$, the Riemann Explicit Formula is

$$\Pi_0(x) = li(x) - \sum_{\rho} li(x^{\rho}) + \int_x^{+\infty} \frac{1}{(u^2-1)u\,\log u}\, du - \log 2$$

where ρ are the complex zeros of the Riemann Zeta function ζ in the critical strip.

Then

$$\frac{1}{2}\,\pi(x^{1/2}) + \frac{1}{3}\,\pi(x^{1/3}) + \ldots \leq \frac{1}{2}\,\pi(x^{1/2}) + \frac{1}{3}\,\pi(x^{1/3})\left\lfloor \frac{\log x}{\log 2} \right\rfloor .$$

Using Dusart's Theorem and the classic bound $\pi(x) \leqslant \frac{2x}{\log x}$, for x ≥ 355991, we then get

$$\begin{aligned}
&\frac{1}{2}\,\pi(x^{1/2}) + \frac{1}{3}\,\pi(x^{1/3}) + ... \\
&\le \frac{1}{2}\left[\frac{x^{1/2}}{\log x^{1/2}}\left(1 + \frac{1}{\log x^{1/2}} + \frac{2.51}{\log^2 x^{1/2}}\right)\right] + \frac{1}{3}\,\frac{2\,x^{1/3}}{\log x^{1/3}}\left\lfloor\frac{\log x}{\log 2}\right\rfloor \\
&\le \frac{1}{2}\left[\frac{x^{1/2}}{1/2\log x}\left(1 + \frac{1}{1/2\log x} + \frac{2.51}{1/4\log^2 x}\right)\right] + \frac{1}{3}\left(\frac{2x^{1/3}}{1/3\log x}\right)\left\lfloor\frac{\log x}{\log 2}\right\rfloor \\
&\le \frac{x^{1/2}}{\log x}\left(1 + \frac{1}{1/2\log x} + \frac{2.51}{1/4\log^2 x}\right) + 2\left(\frac{x^{1/3}}{\log x}\right)\left\lfloor\frac{\log x}{\log 2}\right\rfloor \\
&\le \frac{x^{1/2}}{\log x}\left(1 + \frac{2}{\log x} + \frac{10.04}{\log^2 x}\right) + 2\left(\frac{x^{1/3}}{\log x}\right)\left\lfloor\frac{\log x}{\log 2}\right\rfloor .
\end{aligned}$$

From this, we have

$$\frac{1}{2}\,\pi(x^{1/2}) + \frac{1}{3}\,\pi(x^{1/3}) + ... \le \frac{x^{1/2}}{\log x}\left(1 + \frac{2}{\log x} + \frac{10.04}{\log^2 x}\right) + 2\left(\frac{x^{1/3}}{log 2}\right).$$

Combining this result with the Riemann Explicit Formula $\Pi_0(x)$, we get

$$\begin{aligned}
&\pi(x) + \frac{1}{2}\,\pi(x^{1/2}) + \frac{1}{3}\,\pi(x^{1/3}) + ... \\
&= li(x) - \sum_{\rho} li(x^{\rho}) + \int_x^{+\infty} \frac{1}{(u^2-1)u\,\log u}\,du - \log 2 .
\end{aligned}$$

Then

$$\begin{aligned}
&\pi(x) + \frac{x^{1/2}}{\log x}\left(1 + \frac{2}{\log x} + \frac{10.04}{\log^2 x}\right) + 2\left(\frac{x^{1/3}}{\log 2}\right) \\
&\ge li(x) - \sum_{\rho} li(x^{\rho}) + \int_x^{+\infty} \frac{1}{(u^2-1)u\,\log u}\,du - \log 2 .
\end{aligned}$$

Hence we have

$$\begin{aligned}
\pi(x) &\ge li(x) - \sum_{\rho} li(x^{\rho}) + \int_x^{+\infty} \frac{1}{(u^2-1)u\,\log u}\,du - \log 2 \\
&\quad - \left[\frac{x^{1/2}}{\log x}\left(1 + \frac{2}{\log x} + \frac{10.04}{\log^2 x}\right) + 2\left(\frac{x^{1/3}}{\log 2}\right)\right] \\
&\ge li(x) - \sum_{\rho} li(x^{\rho}) - \log 2 - \frac{x^{1/2}}{\log x}\left(1 + \frac{2}{\log x} + \frac{10.04}{\log^2 x}\right) - 2\left(\frac{x^{1/3}}{\log 2}\right).
\end{aligned}$$

Substituting $x = e^u$ for u > 25.57, we have

$$\pi(e^u) \geq li(e^u) - \sum_{\rho} li(e^{u\rho}) - \log 2 - \frac{e^{u/2}}{u}\left(1 + \frac{2}{u} + \frac{10.04}{u^2}\right) - 2\left(\frac{e^{u/3}}{\log 2}\right)$$

$$\pi(e^u) - li(e^u) \geq -\sum_{\rho} li(e^{u\rho}) - \log 2 - \frac{e^{u/2}}{u}\left(1 + \frac{2}{u} + \frac{10.04}{u^2}\right) - 2\left(\frac{e^{u/3}}{\log 2}\right).$$

Step 2

Multiplying the inequality from Step 1 by $u\,e^{-u/2}$, we get

$$\begin{aligned}
&u\,e^{-u/2}[\pi(e^u) - li(e^u)] \\
&\geq u\,e^{-u/2}\left[-\sum_{\rho} li(e^{u\rho}) - \log 2 - \frac{e^{u/2}}{u}\left(1 + \frac{2}{u} + \frac{10.04}{u^2}\right) - 2\left(\frac{e^{u/3}}{\log 2}\right)\right. \\
&= -\sum_{\rho} u\,e^{-u/2}\, li(e^{u\rho}) - u\,e^{-u/2}\log 2 - \frac{u\,e^{-u/2}\,e^{u/2}}{u}\left(1 + \frac{2}{u} + \frac{10.04}{u^2}\right) \\
&\quad - 2\left(\frac{u\,e^{-u/2}\,e^{u/3}}{\log 2}\right) \\
&= -\sum_{\rho} u\,e^{-u/2}\, li(e^{u\rho}) - u\,e^{-u/2}\log 2 - \left(1 + \frac{2}{u} + \frac{10.04}{u^2}\right) - 2\left(\frac{u\,e^{-u/6}}{\log 2}\right) \\
&= -\sum_{\rho} u\,e^{-u/2}\, li(e^{u\rho}) - u\,e^{-u/2}\log 2 - 1 - \frac{2}{u} - \frac{10.04}{u^2} - \frac{2\,u\,e^{-u/6}}{\log 2}.
\end{aligned}$$

This can be used to improve term S_1 in the equation

$$I(\omega, \eta) = -1 - \sum_{0<|\gamma|\leq T} \frac{e^{i\gamma\omega}}{\rho} e^{\gamma^2/2\alpha} + R$$

where R is as defined in Theorem 3.0.10 . Following the proof of Lehman's Theorem [3], we have the same bounding terms S_2, S_3, S_4, S_5, and S_6. We derive term S_1 from bounding the expression

$$J := \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\left(u\,e^{-u/2}\log 2 + \frac{2}{u} + \frac{10.04}{u^2} + \frac{2\,u\,e^{-u/6}}{\log 2}\right) du.$$

Then due to the property that

$$\int_{-\infty}^{+\infty} K(y)\,dy = 1$$

we can use case 2 for the estimation of area to get

$$\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\big(u\, e^{-u/2}\log 2\big)\, du \leq (\omega+\eta)e^{-(\omega-\eta)/2}\log 2\,,$$

$$\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\left(\frac{2}{u}\right) du \leq \frac{2}{(\omega-\eta}\,,$$

$$\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\left(\frac{10.04}{u^2}\right) du \leq \frac{10.04}{(\omega-\eta)^2}\,, \text{ and}$$

$$\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\big(\frac{2\,u\,e^{-u/6}}{\log 2}\big)\, du \leq \frac{2\,(\omega+\eta)\,e^{-(\omega-\eta)/6}}{\log 2}\,.$$

Hence

$$J \leq \frac{2}{\omega-\eta} + \frac{10.04}{(\omega-\eta)^2} + \frac{2\,(\omega+\eta)}{\log 2}\, e^{-(\omega-\eta)/6} + \log 2\,(\omega+\eta)\, e^{-(\omega-\eta)/2}.$$

Let this expression be equal to $S_1^{'}$. □

We have seen in above proof that the value 2 in the term $\frac{2}{\omega-\eta}$ derives from the term $\frac{1}{\log x}$ in Dusart's Theorem. This value cannot be further improved since the term $\frac{1}{\log x}$ is fixed for the given values of x.

Due to the improvement of term S_6 in Lehman's Theorem in chapter 2, we can then claim

Theorem 3.0.11 *Under the hypothesis of Lehman's Theorem and if* ω - η > *25.57, the equation* $I(\omega,\eta) = -1 - \sum_{0<|\gamma|\leq T} \frac{e^{i\gamma\omega}}{\rho}\, e^{\gamma^2/2\alpha} + R$ *still holds if* S_1 *is replaced by* $S_1^{'} = \frac{2}{\omega-\eta} + \frac{10.04}{(\omega-\eta)^2} + \log 2\,(\omega+\eta)\, e^{-(\omega-\eta)/2} + \frac{2}{\log 2}(\omega+\eta)e^{-(\omega-\eta)/6}$. *R is as previously defined in Theorem 2.0.2 .*

Chapter 4

Numerical Results

This chapter is based on Saouter-Demichel's numerical values with some changes due to the improvement of term S_6 in Lehman's Theorem.

As previously stated, Theorem 2.0.2 requires numerical verification of the Riemann Hypothesis up to a height A. Then for $0 < T \leq A$, the complex zeros $\rho = \beta + i\gamma$ such that $|\gamma| < T$ have real part $\beta = \frac{1}{2}$. Since ρ occurs in conjugate pairs in the critical strip of the Riemann Zeta function $\zeta(s)$, the sum from Theorem 2.0.2 to evaluate is

$$\sum_{0<|\gamma|\leq T} \frac{e^{i\gamma\omega}}{\rho} e^{-\gamma^2/2\alpha} = \sum_{0<\gamma\leq T} \left[\frac{e^{i\gamma\omega}}{\beta + i\gamma} + \frac{e^{-i\gamma\omega}}{\beta - i\gamma}\right] e^{-\gamma^2/2\alpha}\,.$$

Then

$$\begin{aligned}
\sum_{0<|\gamma|\leq T} \frac{e^{i\gamma\omega}}{\rho} e^{-\gamma^2/2\alpha} &= \sum_{0<\gamma\leq T} \left[\frac{e^{i\gamma\omega}}{\beta+i\gamma} + \frac{e^{-i\gamma\omega}}{\beta-i\gamma}\right] e^{-\gamma^2/2\alpha} \\
&= \sum_{0<\gamma\leq T} \left[\frac{e^0\,[cos(\gamma\omega)+i\,sin(\gamma\omega)]}{\beta+i\gamma}\right. \\
&\quad \left. + \frac{e^0\,[cos(-\gamma\omega)+i\,sin(-\gamma\omega)]}{\beta-i\gamma}\right] e^{-\gamma^2/2\alpha} \\
&= \sum_{0<\gamma\leq T} \left[\frac{(\beta-i\gamma)[cos(\gamma\omega)+i\,sin(\gamma\omega)]}{(\beta+i\gamma)(\beta-i\gamma)}\right. \\
&\quad \left. + \frac{(\beta+i\gamma)[cos(\gamma\omega)-i\,sin(\gamma\omega)]}{(\beta-i\gamma)(\beta+i\gamma)}\right] e^{-\gamma^2/2\alpha} \\
&= \sum_{0<\gamma\leq T} \left[\frac{\beta cos(\gamma\omega)+i\,\beta\,sin(\gamma\omega)-i\,\gamma\,cos(\gamma\omega)+\gamma\,sin(\gamma\omega)}{(\beta+i\gamma)(\beta-i\gamma)}\right. \\
&\quad \left. + \frac{\beta cos(\gamma\omega)-i\,\beta\,sin(\gamma\omega)+i\,\gamma\,cos(\gamma\omega)+\gamma\,sin(\gamma\omega)}{(\beta+i\gamma)(\beta-i\gamma)}\right] e^{-\gamma^2/2\alpha} \\
&= \sum_{0<\gamma\leq T} \left[\frac{2\,\beta\,cos(\gamma\omega)+2\,\gamma\,sin(\gamma\omega)}{\beta^2+\gamma^2}\right] e^{-\gamma^2/2\alpha} \\
&= \sum_{0<\gamma\leq T} \left[\frac{2(\frac{1}{2})cos(\gamma\omega)+2\,\gamma\,sin(\gamma\omega)}{\frac{1}{2}^2+\gamma^2}\right] e^{-\gamma^2/2\alpha} \\
&= \sum_{0<\gamma\leq T} \left[\frac{cos(\gamma\omega)+2\,\gamma\,sin(\gamma\omega)}{\frac{1}{4}+\gamma^2}\right] e^{-\gamma^2/2\alpha}.
\end{aligned}$$

For this, Saouter and Demichel computed the first 22 million zeros of $\zeta(s)$. From that, we have $T = 10379599.727431060$.

Then the relative precision that occurs due to rounding when computing the right-hand side of the above equation is bounded by $\Delta I = |\gamma^* - \gamma||t'(\gamma)|$. In his work [4] , te Riele gives the approximation $|\gamma^* - \gamma| < 10^{-9}$. Hence we have

$$\Delta I = 10^{-9} \sum_{0<\gamma\leq T} \frac{d}{d\,\gamma}\left(\left[\frac{cos(\gamma\omega)+2\,\gamma\,sin(\gamma\omega)}{\frac{1}{4}+\gamma^2}\right] e^{-\gamma^2/2\alpha}\right).$$

Numerically, the least known value for ω such that $I(\omega, \eta)$ is positive is $\omega = 727.951335792$. For the width α, we have $\alpha = 6 \times 10^{12}$. The value for A which minimizes the interval length is $A = 6.85 \times 10^7$. Note that even though we were able to improve term S_6 in Lehman's Theorem, it does not change the optimal value for A. Since we want the smallest possible value for η, we let $\eta = \frac{2A}{\alpha}$. Then $\eta = 0.00002283333334$.

One should note that all of the above values satisfy the conditions $\frac{4A}{\omega} \leq \alpha \leq A^2$ and $\frac{2A}{\alpha} \leq \eta < \frac{\omega}{2}$ of Lehman's Theorem. Further $\omega - \eta > 25.57$ which satisfies the condition for Theorem 3.0.11.

Through computation, we obtain

$$\sum_{0<|\gamma|\leq T} \frac{e^{i\gamma\omega}}{\rho} e^{-\gamma^2/2\alpha} = -0.002906086981405.$$

Hence we have $I^*(\omega, \eta) = 0.002906086981405$ as an estimate for

$$I(\omega, \eta) = \sum_{0<|\gamma|\leq T} \frac{ei\gamma\omega}{\rho} e^{-\gamma^2/2\alpha} + R$$

where $|R| < S_1^{'} + S_2 + S_3 + S_4 + S_5 + S_6^{'}$, and

$$I(\omega, \eta) \geq I^*(\omega, \eta) - \Delta I - S_1^{'} - S_2 - S_3 - S_4 - S_5 - S_6^{'} .$$

Then we have

$$S_1^{'} = 0.002766382992\,,$$

$$S_2 = 7.612616054 \times 10^{-682}\,,$$

$$S_3 = 1.045693526 \times 10^{-674}\,,$$

$$S_4 = 0.00003202055302\,,$$

$$S_5 = 0.00006868591225\,, \text{ and}$$

$$S_6^{'} = 7.329532854 \times 10^{-7}\,.$$

Hence

$$I(\omega, \eta) \geq 0.00003754811147\,.$$

Above numerical results then show that $I(\omega, \eta)$ is positive, and it follows that there exists a value x in the interval $[\omega - \eta, \omega + \eta] =$ $[\exp(727.951312959), \exp(727.951358625)]$ for which $\pi(x) - li(x) > 0$ holds.

Further Chao and Plymen's work [1, page 689] shows that for u in some interval $(\omega - \eta, \omega + \eta)$ where $u\, e^{-u/2}[\pi(e^u) - li(e^u)] > \delta$, we have $\pi(e^u) - li(e^u) > u^{-1}\, e^{u/2}\, \delta$. Hence for our interval $[\exp(727.951312959), \exp(727.951358625)]$, we have

$$\pi(e^u) - li(e^u) > 6.096911165 \times 10^{150}$$

where $u = \omega$ and $\delta = 0.00003754811147$.

Thus we can claim

Theorem 4.0.12 *There exists at least one value x in the interval* $[\exp(727.951312959), \exp(727.951358625)]$ *for which* $\pi(x) - li(x) > 0$*. Further, there are more than* $6.096911165 \times 10^{150}$ *successive integers in the vicinity of* $\exp(727.951335792)$ *where the inequality holds.*

This improves Saouter and Demichel's original theorem:

Theorem 4.0.13 ([6, Theorem 4.1]) *There exists at least one value x in the interval* $[\exp(727.9513130), \exp(727.9513586)]$ *for which* $\pi(x) > li(x)$ *holds. Moreover, there are more than* 6.09×10^{150} *successive integers in the vicinity of* $\exp(727.951335792)$ *where the inequality holds.*

Even though Saouter-Demichel's interval appears to be smaller, this is due to rounding and exhibits the same interval as in Theorem 4.0.12. The improvement lies with the number of successive integers.

Chapter 5

Sharpening the Interval

In this chapter, we further sharpen the interval of Theorem 4.0.12. For this, we take a look at the growth of $\pi(x) - li(x)$.

First, we will consider the general case for which we do not assume the Riemann Hypothesis. We will use the following theorem:

Theorem 5.0.14 ([6, Theorem 5.1]) *If* $x \geq e^8$*, we have*
$0 \leq li(x) - \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) - C_1 \leq \frac{12x}{\log^4 x} + C_2$ *with*
$C_1 = li(2) - \frac{2}{\log 2}\left(1 + \frac{1}{\log 2} + \frac{2}{\log^2 2}\right)$ *and* $C_2 = \int_2^{e^8} \frac{48}{\log^5 t}\, dt - \frac{24}{\log^4 2}$.

We will split the proof of above theorem into two steps.

In **Step 1** (page 48), we use integration by parts to show that

$$\begin{aligned} li(x) - \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) - C_1 &= \int_2^x \frac{6}{\log^4 t}\, dt \\ &\geq 0\,. \end{aligned}$$

Step 2 (page 49) proves that for $t \geq e^8$, we have

$$\int_2^x \frac{6}{\log^4 t}\, dt \leq \frac{12\,x}{\log^4 x} + C_2\,.$$

Combining Step 1 and Step 2, we get the conclusion of the theorem.

Proof We will closely follow Saouter-Demichel's proof of the above theorem.

Step 1

From the definition of $li(x)$ given in chapter 2, we have for $x \geq 2$

$$\begin{aligned} li(x) &= \int_0^x \frac{1}{\log t}\,dt \\ &= \int_0^2 \frac{1}{\log t}\,dt + \int_2^x \frac{1}{\log t}\,dt \\ &= li(2) + \int_2^x \frac{1}{\log t}\,dt\,. \end{aligned}$$

Using integration by parts, we then get

$$\begin{aligned} \int_2^x \frac{1}{\log t}\,dt &= \frac{t}{\log t}\Big|_2^x + \int_2^x \frac{1}{\log^2 t}\,dt \\ &= \frac{t}{\log t}\Big|_2^x + \frac{t}{\log^2 t}\Big|_2^x + \int_2^x \frac{2}{\log^3 t}\,dt \\ &= \frac{t}{\log t}\Big|_2^x + \frac{t}{\log^2 t}\Big|_2^x + \frac{2\,t}{\log^3 t}\Big|_2^x + \int_2^x \frac{6}{\log^4 t}\,dt \\ &= \left[\frac{t}{\log t}\left(1 + \frac{1}{\log t} + \frac{2}{\log^2 t}\right)\right]_2^x + \int_2^x \frac{6}{\log^4 t}\,dt\,. \end{aligned}$$

Hence

$$li(x) = li(2) + \left[\frac{t}{\log t}\left(1 + \frac{1}{\log t} + \frac{2}{\log^2 t}\right)\right]_2^x + \int_2^x \frac{6}{\log^4 t}\,dt\,.$$

Then

$$\begin{aligned} li(x) - &\frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) - C_1 \\ &= li(x) - \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) - \left[li(2) - \frac{2}{\log 2}\left(1 + \frac{1}{\log 2} + \frac{2}{\log^2 2}\right)\right] \\ &= li(x) - li(2) - \left[\frac{t}{\log t}\left(1 + \frac{1}{\log t} + \frac{2}{\log^2 t}\right)\right]_2^x \\ &= li(2) + \left[\frac{t}{\log t}\left(1 + \frac{1}{\log t} + \frac{2}{\log^2 t}\right)\right]_2^x + \int_2^x \frac{6}{\log^4 t}\,dt \\ &\quad - li(2) - \left[\frac{t}{\log t}\left(1 + \frac{1}{\log t} + \frac{2}{\log^2 t}\right)\right]_2^x \\ &= \int_2^x \frac{6}{\log^4 t}\,dt \end{aligned}$$

and

$$\int_2^x \frac{6}{\log^4 t}\, dt \geq 0\,.$$

Hence

$$0 \leq li(x) - \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) - C_1\,.$$

Step 2

Using integration by parts, we have for $x \geq 2$

$$\int_2^x \frac{6}{\log^4 t}\, dt = \frac{6\,t}{\log^4 t}\bigg|_2^x + \int_2^x \frac{24}{\log^5 t}\, dt\,.$$

Then

$$\begin{aligned}\int_{e^8}^x \frac{24}{\log^5 t}\, dt &= \int_2^x \frac{24}{\log^5 t}\, dt - \int_2^{e^8} \frac{24}{\log^5 t}\, dt \\ &= \int_2^x \frac{6}{\log^4 t}\, dt - \frac{6\,t}{\log^4 t}\bigg|_2^x - \int_2^{e^8} \frac{24}{\log^5 t}\, dt\,.\end{aligned}$$

For $t = e^8$, we have

$$\begin{aligned}\frac{24}{\log^5 t} &= \frac{24}{(\log e^8)^5} \\ &= \frac{24}{(8)^5} \\ &= \frac{24}{8(8)^4} \\ &= \frac{3}{(8)^4} \\ &= \frac{3}{(\log e^8)^4} \\ &= \frac{3}{\log^4 t} \\ &= \frac{1}{2}\,\frac{6}{\log^4 t}\,.\end{aligned}$$

Then for $t \geq e^8$, we have

$$\frac{24}{\log^5 t} \leq \frac{1}{2}\,\frac{6}{log^4 t}\,.$$

Hence for $t \geq e^8$, we get

$$\begin{aligned}\int_2^x \frac{6}{\log^4 t}\,dt - \frac{6t}{\log^4 t}\Big|_2^x - \int_2^{e^8} \frac{24}{\log^5 t}\,dt &= \int_{e^8}^x \frac{24}{\log^5 t}\,dt \\ &\leq \frac{1}{2}\int_{e^8}^x \frac{6}{\log^4 t}\,dt \\ &\leq \frac{1}{2}\int_2^x \frac{6}{\log^4 t}\,dt\,,\end{aligned}$$

so we have

$$\begin{aligned}\int_2^x \frac{6}{\log^4 t}\,dt - \frac{6\,t}{\log^4 t}\Big|_2^x - \int_2^{e^8} \frac{24}{\log^5 t}\,dt &\leq \frac{1}{2}\int_2^x \frac{6}{\log^4 t}\,dt \\ \frac{1}{2}\int_2^x \frac{6}{\log^4 t}\,dt - \frac{6\,t}{\log^4 t}\Big|_2^x - \int_2^{e^8} \frac{24}{\log^5 t}\,dt &\leq 0 \\ \frac{1}{2}\int_2^x \frac{6}{\log^4 t}\,dt &\leq \frac{6\,t}{\log^4 t}\Big|_2^x + \int_2^{e^8} \frac{24}{\log^5 t}\,dt \\ \int_2^x \frac{6}{\log^4 t}\,dt &\leq \frac{12\,t}{\log^4 t}\Big|_2^x + \int_2^{e^8} \frac{48}{\log^5 t}\,dt \\ &\leq \frac{12\,x}{\log^4 x} - \frac{24}{\log^4 2} + \int_2^{e^8} \frac{48}{\log^5 t}\,dt \\ &= \frac{12\,x}{\log^4 x} + C_2\,.\end{aligned}$$

Combining, we get

$$\begin{aligned}0 &\leq li(x) - \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) - C_1 \\ &= \int_2^x \frac{6}{\log^4 t}\,dt \\ &\leq \frac{12\,x}{\log^4 x} + C_2\,.\end{aligned}$$

□

Using above theorem with Dusart's Theorem from chapter 3, we claim

Theorem 5.0.15 ([6, Theorem 5.2]) *If* $x \geq 355991$, *we have*
$-\frac{0.2x}{\log^3 x} - \frac{12x}{\log^4 x} - \big(C_1 + C_2\big) \leq \pi(x) - li(x) \leq \frac{0.51x}{\log^3 x} - C_1$.
Moreover, if $x \geq e^{40}$, *then* $|\pi(x) - li(x)| \leq \frac{0.51x}{\log^3 x} - C_1$.

We will split the proof of above theorem into three steps.

In **Step 1** (page 51), we use Theorem 5.0.14 and Dusart's Theorem to show that

$$\pi(x) - li(x) \geq -\frac{0.2\,x}{\log^3 x} - \frac{12\,x}{\log^4 x} - (C_1 + C_2)\,.$$

Step 2 (page 52) proves that

$$\pi(x) - li(x) \leq \frac{0.51\,x}{\log^3 x} - C_1$$

This gives us the first conclusion of the theorem.

In **Step 3** (page 52), we show that

$$\begin{aligned} -\frac{0.51\,x}{\log^3 x} + C_1 &\leq -\frac{0.2\,x}{\log^3 x} - \frac{12\,x}{\log^4 x} - (C_1 + C_2) \\ &\leq \pi(x) - li(x)\,. \end{aligned}$$

Combining this result with Step 2, we get the later conclusion of the theorem.

Proof For this proof, we closely follow Saouter-Demichel's proof of above theorem.

Step 1

From Theorem 5.0.14, we have

$$li(x) - \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) - C_1 \leq \frac{12x}{\log^4 x} + C_2\,.$$

Then

$$li(x) \leq \frac{12x}{\log^4 x} + \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) + C_1 + C_2\,.$$

By Theorem 3.0.9, we have

$$\frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}\right) \leq \pi(x)\,.$$

Then

$$\frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}\right) - li(x) \leq \pi(x) - li(x)\,.$$

Then

$$\begin{aligned} &\frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}\right) - \left[\frac{12x}{\log^4 x} + \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) + C_1 + C_2\right] \\ &\quad \leq \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}\right) - li(x) \\ &\quad \leq \pi(x) - li(x)\,. \end{aligned}$$

Hence

$$\begin{aligned}\pi(x) - li(x) &\geq \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}\right) \\ &\quad - \left[\frac{12\,x}{\log^4 x} + \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) + C_1 + C_2\right] \\ &= \frac{x}{\log x}\left(\frac{-0.2}{\log^2 x}\right) - \frac{12\,x}{\log^4 x} - (C_1 + C_2) \\ &= -\frac{0.2\,x}{\log^3 x} - \frac{12\,x}{\log^4 x} - (C_1 + C_2)\,.\end{aligned}$$

Step 2

From Theorem 5.0.14, we have

$$li(x) - \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) - C_1 \geq 0\,.$$

Further from Theorem 3.0.9, we have

$$\pi(x) \leq \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}\right).$$

Then

$$\begin{aligned}\pi(x) &\leq \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}\right) + \left[li(x) - \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) - C_1\right] \\ &= \frac{x}{\log x}\left(\frac{0.51}{\log^2 x}\right) + li(x) - C_1 \\ &= \frac{0.51\,x}{\log^3 x} + li(x) - C_1\,.\end{aligned}$$

Hence

$$\pi(x) - li(x) \leq \frac{0.51x}{\log^3 x} - C_1\,.$$

Step 3

Note that $\frac{0.4\,e^{40}}{40^4} \approx 3.7 \times 10^{10}$ and $\frac{48\,e^8}{40^5} \approx 0.1$.

Then for $x = e^{40}$, we have

$$\begin{aligned}
-\frac{0.51x}{\log^3 x} &= -\frac{0.51\,e^{40}}{(\log e^{40})^3} \\
&= -\frac{0.51\,e^{40}}{40^3} \\
&= -\frac{20.4\,e^{40}}{40^4} \\
&= -\frac{20\,e^{40}}{40^4} - \frac{0.4\,e^{40}}{40^4} \\
&\leq -\frac{20\,e^{40}}{40^4} - \frac{48\,e^{8}}{40^5} + \frac{96}{40^5} - \frac{1}{10} \\
&= -\frac{20\,e^{40}}{40^4} - \left[\frac{48(e^8)}{40^5} - \frac{48(2)}{40^5}\right] - \left[\frac{2(2)}{40} - \frac{2(0)}{40}\right] \\
&= -\frac{20\,e^{40}}{40^4} - \frac{48t}{40^5}\bigg|_2^{e^8} - \frac{2\,t}{40}\bigg|_0^2 \\
&= -\frac{20\,e^{40}}{40^4} - \int_2^{e^8} \frac{48}{40^5}\,dt - \int_0^2 \frac{2}{40}\,dt \\
&= -\frac{12\,e^{40}}{40^4} - \frac{8\,e^{40}}{40^4} - \int_2^{e^8} \frac{48}{40^5}\,dt - \int_0^2 \frac{2}{40}\,dt \\
&= -\frac{12\,e^{40}}{(\log e^{40})^4} - \frac{0.2\,e^{40}}{(\log e^{40})^3} - \int_2^{e^8} \frac{48}{(\log e^{40})^5}\,dt - \int_0^2 \frac{2}{\log e^{40}}\,dt \\
&= -\frac{12\,x}{\log^4 x} - \frac{0.2\,x}{\log^3 x} - \int_2^{e^8} \frac{48}{\log^5 t}\,dt - 2\int_0^2 \frac{1}{\log t}\,dt \\
&\leq -\frac{0.2\,x}{\log^3 x} - \frac{12\,x}{\log^4 x} - \int_2^{e^8} \frac{48}{\log^5 t}\,dt - 2\,li(2) \\
&\quad + 2\left[\frac{2}{\log 2}\left(1 + \frac{1}{\log 2} + \frac{2}{\log^2 2}\right)\right] + \frac{24}{\log^4 2} \\
&= -\frac{0.2\,x}{\log^3 x} - \frac{12\,x}{\log^4 x} - 2\left[li(2) - \frac{2}{\log 2}\left(1 + \frac{1}{\log 2} + \frac{2}{\log^2 2}\right)\right] \\
&\quad - \left[\int_2^{e^8} \frac{48}{\log^5 t}\,dt - \frac{24}{\log^4 2}\right] \\
&= -\frac{0.2\,x}{\log^3 x} - \frac{12\,x}{\log^4 x} - 2\,C_1 - C_2\,.
\end{aligned}$$

Hence for $x = e^{40}$, we have

$$\begin{aligned}
-\frac{0.51\,x}{\log^3 x} + C_1 &\leq -\frac{0.2\,x}{\log^3 x} - \frac{12\,x}{\log^4 x} - \big(C_1 + C_2\big) \\
&\leq \pi(x) - li(x)\,.
\end{aligned}$$

Then it follows that this inequality holds for $x \geq e^{40}$. Hence for $x \geq e^{40}$, we have

$$-\left(\frac{0.51\,x}{\log^3 x} - C_1\right) \leq \pi(x) - li(x) \leq \frac{0.51\,x}{\log^3 x} - C_1\,.$$

Thus

$$|\pi(x) - li(x)| \leq \frac{0.51\,x}{\log^3 x} - C_1\,.$$

□

Using above theorem, we will now look at the tail parts of the integral

$$I(\omega,\eta) = \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\,u\,e^{-u/2}[\pi(e^u) - li(e^u)]\,du\,.$$

Let η_0 be a real positive number such that $\eta_0 < \eta$. Then we can use case 1 for the estimate of area to get

$$\begin{aligned}
&\left|\int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)\,u\,e^{-u/2}[\pi(e^u) - li(e^u)]\,du\right| \\
&\leq \int_{\omega+\eta_0}^{\omega+\eta} \left|K(u-\omega)\,u\,e^{-u/2}[\pi(e^u) - li(e^u)]\right|\,du \\
&\leq \int_{\omega+\eta_0}^{\omega+\eta} \left|K(u-\omega)\right|\left|u\,e^{-u/2}\right|\left|\pi(e^u) - li(e^u)\right|\,du \\
&= \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)\,u\,e^{-u/2}\left|\pi(e^u) - li(e^u)\right|\,du\,.
\end{aligned}$$

Note that in Theorem 5.0.15 C_1 is negative and C_2 is positive. Using this theorem, we then have

$$\begin{aligned}
&\int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)\,u\,e^{-u/2}\left|\pi(e^u) - li(e^u)\right|\,du \\
&\leq \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)\,u\,e^{-u/2}\left[\frac{0.51(e^u)}{(\log e^u)^3} - C_1\right]du \\
&= \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)\,u\,e^{-u/2}\left[\frac{0.51\,e^u}{u^3} - C_1\right]du \\
&= \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)\left[\frac{0.51\,e^u\,u\,e^{-u/2}}{u^3} - C_1\,u\,e^{-u/2}\right]du \\
&= \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)\left[\frac{0.51\,e^{u/2}}{u^2} - C_1\,u\,e^{-u/2}\right]du \\
&\leq (\eta-\eta_0)\,K(\eta_0)\left[\frac{0.51\,e^{(\omega+\eta)/2}}{(\omega+\eta_0)^2} - C_1\,(\omega+\eta)\,e^{-(\omega+\eta_0)/2}\right].
\end{aligned}$$

Let the right-hand side of the above inequality be equal to T_1. Further

$$\begin{aligned}
&\left|\int_{\omega-\eta}^{\omega-\eta_0} K(u-\omega)\, u\, e^{-u/2}[\pi(e^u) - li(e^u)]\, du\right| \\
&\leq \int_{\omega-\eta}^{\omega-\eta_0} K(u-\omega)\left[\frac{0.51\, e^{u/2}}{u^2} - C_1\, u\, e^{-u/2}\right] du \\
&\leq (\eta-\eta_0)\, K(-\eta_0)\left[\frac{0.51\, e^{(\omega-\eta_0)/2}}{(\omega-\eta)^2} - C_1\,(\omega-\eta_0)\, e^{-(\omega-\eta)/2}\right].
\end{aligned}$$

Let the right-hand side of this inequality be equal to T_2. Then the sum of the two tail integrals is bounded above by $T_1 + T_2$. Using the previous numerical values, Saouter and Demichel let $\eta_0 = \frac{\eta}{2.074}$ which gives us the optimal value for T_1 and T_2 such that $I(\omega, \eta_0)$ is positive. Then we obtain

$$T_1 = 0.00001594194397$$

and

$$T_2 = 0.00001594167602\,.$$

Using the numerical results for T_1 and T_2 together with the estimate from chapter 4,

$$I(\omega, \eta) \geq 0.000037548111\,,$$

we get

$$I(\omega, \eta) \geq 0.000005664491481\,.$$

Further, $[\omega-\eta, \omega+\eta] = [\exp(727.951324783), \exp(727.951346801)]$. Then we have

$$\pi(e^u) - li(e^u) > 9.197773166 \times 10^{149}\,.$$

Hence we can state a new theorem:

Theorem 5.0.16 *There exists at least one value x in the interval* $[\exp(727.951324783), \exp(727.951346801)]$ *for which $\pi(x) - li(x) > 0$. Further, there are more than* $9.197773166 \times 10^{149}$ *successive integers in the vicinity of* $\exp(727.951335792)$ *where the inequality holds.*

The above theorem refines Saouter and Demichel's original theorem:

Theorem 5.0.17 ([6, Theorem 5.3]) *There exists one value x in the interval* $[\exp(727.95132478), \exp(727.95134681)]$ *such that* $\pi(x) - li(x) > 9.1472 \times 10^{149}$.

The improvement lies with the number of successive integers due to the improvement of term S_6 in Lehman's Theorem and a rounding error in the interval of Saouter-Demichel's theorem.

Next, we consider the case where the Riemann Hypothesis holds. For this we use a result by Schoenfeld:

Theorem 5.0.18 ([7, page 339]) *If the Riemann Hypothesis holds, then for* $x \geq 2657$ *we have* $|\pi(x) - li(x)| < \frac{1}{8\pi}\sqrt{x}\log x$.

Then looking again at the tail parts of the integral $I(\omega, \eta)$ we get

$$\begin{aligned}
&\left|\int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)\, u\, e^{-u/2}[\pi(e^u) - li(e^u)]\, du\right| \\
&\leq \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)\, u\, e^{-u/2}\big|\pi(e^u) - li(e^u)\big|\, du \\
&< \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)\, u\, e^{-u/2}\left[\frac{1}{8\pi}\sqrt{e^u}\,\log e^u\right] du \\
&= \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)\, u\, e^{-u/2}\left[\frac{1}{8\pi}\, e^{u/2}\, u\right] du \\
&= \frac{1}{8\pi}\int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)\, u^2\, du \\
&\leq (\eta - \eta_0)\,\frac{1}{8\pi}\, K(\eta_0)\,(\omega+\eta)^2\,.
\end{aligned}$$

Let the right-hand side of above inequality be equal to $T_1^{'}$. Further

$$\begin{aligned}
\left|\int_{\omega-\eta}^{\omega-\eta_0} K(u-\omega)\, u\, e^{-u/2}[\pi(e^u) - li(e^u)]\, du\right| &< \frac{1}{8\pi}\int_{\omega-\eta}^{\omega-\eta_0} K(u-\omega)\, u^2\, du \\
&\leq (\eta - \eta_0)\,\frac{1}{8\pi}\, K(-\eta_0)\,(\omega-\eta_0)^2\,.
\end{aligned}$$

Let the right-hand side of this inequality be equal to $T_2^{'}$. Then the sum of the two tail integrals is bounded above by $T_1^{'} + T_2^{'}$. Using the previous numerical values, we let $\eta_0 = \frac{\eta}{8.10}$ which gives us the optimal values for $T_1^{'}$ and $T_2^{'}$ such that $I(\omega, \eta_0)$ is positive. Hence we obtain

$$T_1^{'} = 0.00001828536920$$

and

$$T_2^{'} = 0.00001828536792\,.$$

Using the numerical results for $T_1^{'}$ and $T_2^{'}$ together with the estimate from chapter 4,

$$I(\omega, \eta) \geq 0.000037548111$$

we get

$$I(\omega, \eta_0) \geq 9.97737436 \times 10^{-7}\,.$$

Further $[\omega - \eta_0, \omega + \eta_0] = [\exp(727.951332973), \exp(727.951338611)]$ and $\pi(e^u) - li(e^u) > 1.58702111 \times 10^{149}$. Hence we can state a new theorem:

Theorem 5.0.19 *If the Riemann Hypothesis holds, then there exists at least one value x in the interval* $[\exp(727.951332973), \exp(727.951338611)]$ *for which* $\pi(x) - li(x) > 0$. *Further, there are more than* $1.58702111 \times 10^{149}$ *successive integers in the vicinity of* $\exp(727.951335792)$ *where the inequality holds.*

The above theorem improves Saouter-Demichel's original theorem:

Theorem 5.0.20 ([6, Theorem 5.5]) *If the Riemann Hypothesis holds, then there exists one value x in the interval* $[\exp(727.95133239), \exp(727.95133919)]$ *such that* $\pi(x) - li(x) > 1.7503 \times 10^{148}$.

The improvement is due to a new value for η_0 and the improvement of term S_6 in Lehman's Theorem. For Theorem 5.0.19, we use $\eta_0 = \frac{\eta}{8.10}$ rather than Saouter-Demichel's value $\eta_0 = \frac{\eta}{6.72}$. This is possible because we are using

$$(\eta - \eta_0)\,\frac{1}{8\pi}\,K(\eta_0)\,(\omega + \eta)^2$$

and

$$(\eta - \eta_0)\,\frac{1}{8\pi}\,K(-\eta_0)\,(\omega - \eta_0)^2$$

as and estimate for the upper bound instead of Saouter-Demichel's estimate

$$\frac{1}{8\pi}\,K(\eta_0)\,(\omega+\eta)^2$$

and

$$\frac{1}{8\pi}\,K(-\eta_0)\,(\omega-\eta_0)^2\,.$$

Chapter 6

Interval of Positivity

In this chapter, we will consider integers greater than x for which $\pi(x) - li(x)$ is positive.

Let b be a positive. From the definition of $li(x)$, it follows that

$$\begin{aligned} li(x-b) &= \int_0^{x-b} \frac{1}{\log t}\, dt \\ &\leq \int_0^{x} \frac{1}{\log t}\, dt \\ &= li(x)\,. \end{aligned}$$

Let n denote the number of primes up to and including x. We will consider two cases. For the first case, we let x be prime. Then

$$\begin{aligned} \pi(x-1) &= n-1 \\ &= \pi(x) - 1\,. \end{aligned}$$

For the second case, we assume that x is not prime. Then

$$\begin{aligned} \pi(x-1) &= n \\ &\geq n-1 \\ &= \pi(x) - 1\,. \end{aligned}$$

Hence in general,

$$\pi(x-1) \geq \pi(x) - 1\,.$$

From the above, it follows that

$$\begin{aligned}\pi(x-b) - li(x-b) &\geq \pi(x) - b - li(x-b)\\ &\geq \pi(x) - b - li(x)\\ &= \pi(x) - li(x) - b\,.\end{aligned}$$

In Theorem 4.0.12, the interval [exp(727.951312959, exp(727.951358625)] exhibits $6.096911165 \times 10^{150}$ consecutive integers where $\pi(x) - li(x)$ is positive. Hence we have

$$\begin{aligned}\pi(x-b) - li(x-b) &\geq \pi(x) - li(x) - b\\ &> 6.096911165 \times 10^{150}\,.\end{aligned}$$

Thus we can confirm that the successive integers preceding x belong to the interval of positivity. To obtain this result, we considered integers less than x. However, we can get a better result if we consider integers greater than x. For this we use the following theorem:

Theorem 6.0.21 ([6, Theorem 6.1]) *Let $x > 1$, and $y > 0$. Then we have* $li(x+y) - li(x) = \int_x^{x+y} \frac{1}{\log t}\,dt < \frac{y}{\log x}$.

Proof The graph below shows $y = \frac{1}{\log x}$.

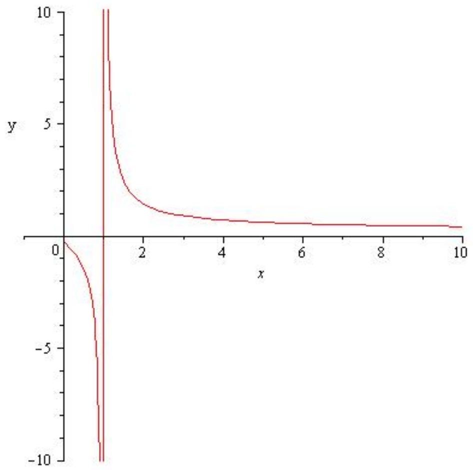

Figure 6.1: $y = \dfrac{1}{\log x}$

We can clearly see that since $x + y > x > 1$, we have

$$\begin{aligned}\int_x^{x+y} \frac{1}{\log t}\, dt &< [(x+y) - x]\left(\frac{1}{\log x}\right)\\ &= y\left(\frac{1}{\log x}\right)\\ &= \frac{y}{\log x}\,.\end{aligned}$$

□

Using above theorem, we then claim

Theorem 6.0.22 ([6, Theorem 6.2]) *Let x be a real number such that $\pi(x) - li(x) = A$ where $A > 0$. Then if y is a real number such that $0 \leq y < A \log x$, we have $\pi(x+y) - li(x+y) > 0$.*

Proof For this proof we closely follow Saouter-Demichel's proof of above theorem. Let $A > 0$ and $y > 0$ such that $0 < y < A \log x$ and $\pi(x) - li(x) = A$. Note that

$\pi(x+y) - \pi(x) \geq 0$. Then

$$\begin{aligned}
\pi(x+y) - li(x+y) &= \pi(x+y) - li(x+y) + [\pi(x) - \pi(x)] + [li(x) - li(x)] \\
&= [\pi(x+y) - \pi(x)] + [\pi(x) - li(x)] + [li(x) - li(x+y)] \\
&= [\pi(x+y) - \pi(x)] + A + [li(x) - li(x+y)] \\
&\geq A + [li(x) - li(x+y)] \\
&= A - [li(x+y) - li(x)] \\
&> A - \frac{y}{\log x} \\
&> 0\,.
\end{aligned}$$

□

From Theorem 5.0.16, we have $A = 9.197773166 \times 10^{149}$. Then $A \log x = A \log e^u = A$ where $u = \omega$. Thus

$$A \log x = 6.695531258 \times 10^{152}\,.$$

Let $y = 0$. Then $A \log x > y \geq 0$, so the conditions of Theorem 6.0.22 are met. Hence we have $6.695531258 \times 10^{152}$ successive integers. However, we do not know where the first x lies in the interval [exp(727.951324783), exp(727.951346801)] of Theorem 5.0.16. We only know that the maximal value is at exp(727.951346801). But

$$\begin{aligned}
\exp(727.951346802) - \exp(727.951346801) &\approx 1.3972 \times 10^{307} \\
&> 6.695531258 \times 10^{152}\,.
\end{aligned}$$

Then the $6.695531258 \times 10^{152}$ successive integers following x belong to the interval [exp(727.951324783), exp(727.951346802)]. Hence we can claim:

Theorem 6.0.23 *There are at least* $6.695531258 \times 10^{152}$ *consecutive integers in the interval* [exp(727.951324783), exp(727.951346802)] *for which* $\pi(x) - li(x) > 0$.

The above theorem improves Saouter and Demichel's original theorem:

Theorem 6.0.24 ([6, Theorem 6.3]) *There are at least* 6.6587×10^{152} *consecutive integers* x *in the interval* $[\exp(727.95132478), \exp(727.95134682)]$ *such that* $\pi(x) - li(x) > 0$.

The improvements are due to the improvements of Theorem 5.0.16.

Further if the Riemann Hypothesis holds, we can use Theorem 6.0.22 and Theorem 5.0.19. Then $A = 1.58702111 \times 10^{149}$. Hence

$$A \log x = 1.15527413 \times 10^{152}$$

and

$$\begin{aligned}\exp(727.951338612) - \exp(727.951338611) &\approx 1.3972 \times 10^{307} \\ &> 1.15527413 \times 10^{152}.\end{aligned}$$

Hence we can claim:

Theorem 6.0.25 *If the Riemann Hypothesis holds, then there are at least* $1.15527413 \times 10^{152}$ *successive integers in the interval* $[\exp(727.951332973), \exp(727.951338612)]$ *for which* $\pi(x) - li(x) > 0$.

Due to the improvements of Theorem 5.0.19, the above theorem refines Saouter-Demichel's original theorem:

Theorem 6.0.26 ([6, Theorem 6.4]) *If the Riemann Hypothesis holds, then there are at least* 1.2741×10^{151} *consecutive integers in the interval* $[\exp(727.95133239), \exp(727.95133920)]$ *such that* $\pi(x) - li(x) > 0$.

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Appendix A

Maple

$\omega := 727.951335792;$

$$727.951335792 \quad (1)$$

$A := 6.85 \cdot 10^7;$

$$6.850000000\ 10^7 \quad (2)$$

$\alpha := 6 \cdot 10^{12};$

$$6000000000000 \quad (3)$$

$\eta := \frac{2 \cdot A}{\alpha};$

$$0.00002283333334 \quad (4)$$

$\eta[0] := \frac{\eta}{2.074};$

$$0.00001100932176 \quad (5)$$

$T := 10379599.727431060;$

$$1.0379599727431060\ 10^7 \quad (6)$$

$S1 := \frac{2}{\omega - \eta} + \frac{10.04}{(\omega - \eta)^2} + \ln(2) \cdot (\omega + \eta) \cdot \exp\left(-\frac{\omega - \eta}{2}\right) + \frac{2}{\ln(2)} \cdot (\omega + \eta) \cdot \exp\left(-\frac{\omega - \eta}{6}\right);$

$evalf(S1);$

$$0.002766382992 \quad (7)$$

$S2 := \frac{2 \cdot \exp\left(-\frac{\alpha \cdot \eta^2}{2}\right)}{\mathrm{sqrt}(2 \cdot \pi \cdot \alpha) \cdot \eta};$

$evalf(S2);$

$$7.612616054\ 10^{-682} \quad (8)$$

$S3 := 0.08 \cdot \mathrm{sqrt}(\alpha) \cdot \exp\left(-\frac{\alpha \cdot \eta^2}{2}\right);$

$evalf(S3);$

$$1.045693526\ 10^{-674} \quad (9)$$

$S4 := \exp\left(-\frac{T^2}{2 \cdot \alpha}\right) \cdot \left(\frac{\alpha}{\pi \cdot T^2} \cdot \ln\left(\frac{T}{2 \cdot \pi}\right) + \frac{8 \cdot \ln(T)}{T} + \frac{4 \cdot \alpha}{T^3}\right);$

$evalf(S4);$

$$0.00003202055302 \quad (10)$$

$S5 := \frac{0.05}{\omega - \eta};$

$$0.00006868591225 \quad (11)$$

$S6 := A \cdot \ln(A) \cdot \exp\left(-\frac{A^2}{2 \cdot \alpha} + \frac{\omega + \eta}{2}\right) \cdot \left(3.2 \cdot \alpha^{-\frac{1}{2}} + 14.4 \cdot \eta\right);$

$evalf(S6);$

$$7.329532854\ 10^{-7} \quad (12)$$

$J1 := 0.002906086981405 - 7.1645945511 \cdot 10^{-7} - S1 - S2 - S3 - S4 - S5 - S6;$

$evalf(J1);$

$$0.00003754811147 \quad \textbf{(13)}$$

$NoIntegers1 := \omega^{-1} \cdot \exp\left(\frac{\omega}{2}\right) \cdot J1;$
$evalf(NoIntegers1);$

$$6.096911165\ 10^{150} \quad \textbf{(14)}$$

$K := \mathrm{sqrt}\left(\frac{\alpha}{2 \cdot \pi}\right) \cdot \exp\left(-\frac{\alpha \cdot y^2}{2}\right);$

$T1 := (\eta - \eta[0]) \cdot subs(y = \eta[0], K) \cdot \left(\frac{0.51 \cdot \exp\left(\frac{\omega + \eta}{2}\right)}{(\omega + \eta[0])^2} + 1.80141 \cdot (\omega + \eta) \cdot \exp\left(-\frac{\omega + \eta[0]}{2}\right)\right);$
$evalf(T1);$

$$0.00001594194397 \quad \textbf{(15)}$$

$T2 := (\eta - \eta[0]) \cdot subs(y = -\eta[0], K) \cdot \left(\frac{0.51 \cdot \exp\left(\frac{\omega - \eta[0]}{2}\right)}{(\omega - \eta)^2} + 1.80141 \cdot (\omega - \eta[0]) \cdot \exp\left(-\frac{\omega - \eta}{2}\right)\right);$
$evalf(T2);$

$$0.00001594167602 \quad \textbf{(16)}$$

$J2 := J1 - T1 - T2;$
$evalf(J2);$

$$0.000005664491481 \quad \textbf{(17)}$$

$NoIntegers2 := \omega^{-1} \cdot \exp\left(\frac{\omega}{2}\right) \cdot J2;$
$evalf(NoIntegers2);$

$$9.197773166\ 10^{149} \quad \textbf{(18)}$$

$$T1\eta[00] := \frac{(\eta - \eta[00])}{8 \cdot \pi} \cdot subs(y = \eta[00], K) \cdot (\omega + \eta)^2;$$

$$T2\eta[00] := \frac{(\eta - \eta[00])}{8 \cdot \pi} \cdot subs(y = -\eta[00], K) \cdot (\omega - \eta[00])^2;$$

$$J\eta[00] := J1 - T1\eta[00] - T2\eta[00];$$

$$\eta[00] := \frac{\eta}{8.10};$$

$$0.000002818930041 \quad \textbf{(19)}$$

$evalf(J\eta[00]);$

$$9.7737436\ 10^{-7} \quad \textbf{(20)}$$

$evalf(T1\eta[00]);$

$$0.00001828536920 \quad \textbf{(21)}$$

$evalf(T2\eta[00]);$

$$0.00001828536792 \quad \textbf{(22)}$$

$$NoIntegers3 := \omega^{-1} \cdot \exp\left(\frac{\omega}{2}\right) \cdot J\eta[00];$$

$evalf(NoIntegers3);$

$$1.58702111\ 10^{149} \quad \textbf{(23)}$$

$NoIntegers4 := NoIntegers2 \cdot \omega;$
$evalf(NoIntegers4);$

$$6.695531258\ 10^{152} \quad \textbf{(24)}$$

$NoIntegers5 := NoIntegers3 \cdot \omega;$
$evalf(NoIntegers5);$

$$1.15527413\ 10^{152} \quad \textbf{(25)}$$