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# Two dimensional attractors in the border collision normal form

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**Abstract.** New techniques are developed to show that the two-dimensional normal form for codimension one border collision bifurcations of fixed points of discrete time piecewise smooth dynamical systems has attractors which are themselves two dimensional. This makes it possible to prove the existence of these attractors for a countable set of parameter values which cannot be treated using the essentially one-dimensional methods in the literature.

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## 1. Introduction

As more and more applications of digital control techniques are found in electronics and other areas, issues surrounding the dynamics of systems with discrete and continuous components or evolution has become correspondingly more important. There is a growing literature of both applications and theory [2, 3, 6, 23] which show that systems which have discontinuities in their derivatives across switching surfaces arise in many situations. Border collision bifurcations are the simplest bifurcations which can be observed in these discrete time hybrid systems and which have no analogue in standard bifurcation theory, and strange attractors are known to exist over regions of parameter values for some models. For the two-dimensional normal form developed by Nusse and Yorke [24] this is known as 'robust chaos' [4]. Here the attractor is the closure of the one-dimensional unstable manifold of a saddle fixed point or periodic orbit, with a fractal structure in directions orthogonal to the local unstable manifold. In this paper we show that attractors can exist for models which allow area expansion, and prove that there are some parameter values for which the strange attractor can be a fully two-dimensional object rather than the usual fractal attractors with dimension less than two. Note that such fully two-dimensional attractors have been observed numerically [11], and the existence of two-dimensional trapping regions has been established in many cases [21]. However, in the two situations we are aware of for which the existence of a two-dimensional strange attractor has been proved the attractor is either the closure of a one-dimensional unstable manifold [10] (which is not the case in the examples described here) or it can be treated by analyzing an appropriate one-dimensional map ([7, 11, 21], the 'Cournot' case, see below). We aim to develop techniques which do not rely on one-dimensional techniques, and hence have a broader application.

The systems considered in this paper are piecewise smooth and have discrete time and continuous variables. There is a switching surface $\Sigma$ dividing the regions in which the dynamics is determined by smooth maps, and the equations are continuous across $\Sigma$. Thus the left and right sides of $\Sigma$ could be labelled by $L$ and $R$ respectively, and a discrete variable defined to take values in $\{L, R\}$ according to which side of $\Sigma$ the continuous variables are at time $n$. This discrete variable then determines which dynamical system is applied at the next time step.

The normal form for the bifurcation which occurs if a fixed point of one of the maps strikes the boundary was developed by Nusse and Yorke [24]. If the switching surface is transformed to be the $y-$axis ($x = 0$) then the local evolution with $\mathbf{x} = (x, y)^T$ is

$$\mathbf{x}_{n+1} = \begin{cases} A_L\mathbf{x} + \mathbf{m} & \text{if } x \leq 0 \\ A_R\mathbf{x} + \mathbf{m} & \text{if } x \geq 0 \end{cases} \tag{1}$$

where the matrices $A_L$ and $A_R$, and the vector $\mathbf{m}$ are defined as

$$A_\alpha = \begin{pmatrix} T_\alpha & 1 \\ -D_\alpha & 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{m} = \begin{pmatrix} \mu \\ 0 \end{pmatrix} \tag{2}$$

for $\alpha = L, R$. The constants $T_\alpha$ and $D_\alpha$ are the trace and determinant of the Jacobian of the defining equations evaluated at the bifurcation point on the left and right of $\Sigma$, whilst $\mu$ is the bifurcation parameter. If $\mu = 0$ then the origin is a fixed point, and this is clearly in $\Sigma$, whilst the challenge for bifurcation theory is to describe what can happen if $\mu > 0$ or $\mu < 0$. Note that by a change of scale only the sign of $\mu$ matters if $\mu \neq 0$.

If $|D_R|$ and $|D_L|$ are less than one then the local dynamics which can occur has been discussed in a number of papers [2, 4, 5, 6, 18]. In this case bifurcations analogous to the standard saddlenode are possible, as is a border crossing in which the fixed point simply moves across the boundary. Depending on the values of the other constants, more complicated possibilities occur, with the creation of other periodic orbits and even robust (attracting) chaos. The robust chaotic sets created in the normal form are associated with the unstable manifold of a saddle, and so the attractor can be thought of as one-dimensional (more accurately, one dimensional in one direction and a fractal with dimension less than one in another): there is only one positive Lyapunov exponent. Since the determinant of the Jacobian matrix of a map shows how areas are increased or decreased by iteration, attractors are easily observed in the determinant less than one cases, and for this reason these results have found a number of applications.

The piecewise linear maps (1) are natural objects of study, and a series of papers dating from the mid-1980s has investigated two-dimensional chaotic regions which can arise if at least one of the determinants has modulus greater than one. As well as extensive numerical investigations, bifurcation phenomena have been identified and the existence of absorbing regions has been proved in a variety of cases [7, 11, 21, 22]. However, there are very few proofs of the existence of two-dimensional attractors for these cases, i.e. two-dimensional regions with infinitely many unstable periodic orbits and dense orbits. We know of two exceptions to this statement, the Cournot map cases ($T_L = T_R = 0$) covered in [7, 21] and the special case considered by Dobryiskiy [10]. The former case reduces to analysis based on the standard theory of one-dimensional maps, and the latter is based on a proof that the unstable manifold of a saddle is dense. The aim of this paper is to extend the range of examples for which two-dimensional attractors can be proved to exist using two-dimensional Markov partitions. Unlike the example of Dobryiskiy [10] the attractor contains a repeller.

If $T_L = T_R = 0$ then (1) becomes

$$\begin{aligned} x_{n+1} &= y_n + \mu \\ y_{n+1} &= \begin{cases} -D_L x_n & \text{if } x_n \leq 0 \\ -D_R x_n & \text{if } x_n \geq 0 \end{cases} \end{aligned} \tag{3}$$

This is a *Cournot map* [7, 21], i.e. a map of the form $x_{n+1} = g(y_n)$, $y_{n+1} = h(x_n)$, which implies that $x_{n+2} = g \circ h(x_n)$ and $y_{n+2} = h \circ g(y_n)$. These maps for the second iterate are one-dimensional maps, and so they can be analyzed using standard one-dimensional techniques, which can then be re-interpreted for the original two-dimensional maps. In our case (3) gives

$$x_{n+2} = \begin{cases} \mu - D_L x_n & \text{if } x_n \leq 0 \\ \mu - D_R x_n & \text{if } x_n \geq 0 \end{cases} \tag{4}$$

If $D_R D_L < 0$ then this map can have chaotic dynamics with motion dense on an interval, and this translates to two-dimensional regions in the full two dimensional map. More interestingly, there are choices of the parameters at which there is motion dense on a union of disjoint intervals which are permuted by the one dimensional map [20]. This can correspond to multiple attractors in the two-dimensional Cournot map [7, 8], though the projection of the dynamics onto either co-ordinate axis is equivalent for each separate attractor.

More recently the cases which arise if at least one of the systems has determinant with modulus greater than one, so it is area expanding, have been considered. Simpson and Meiss [26, 27] describe complicated regions of mode-locking if $D_L D_R > 0$, whilst the existence of unstable chaotic sets via snap-back repellers has been described if $D_L D_R < 0$ [12, 15, 16, 21]. The chaos due to snap-back repellers is unstable because of the area expansion near the repeller, but here we describe how it is possible that this chaos can be part of a strange attractor. Specifically, we show that for appropriate parameter values there is an attracting two-dimensional region in phase space in which periodic orbits are dense in this region and there is a dense orbit. In this sense the attractor appears to have a very similar structure to that of a piecewise linear coupled map initially studied in [25] and which was proved to have dense periodic orbits and a dense orbit in [9, 14]. The attractor also contains a snap-back repeller with all the dynamics associated with this [16]. Moreover, there can be, on average, area expansion along orbits, and two positive Lyapunov exponents – the contraction required to keep orbits bounded and create an attractor is provided by the folding action across the switching boundary.

The existence of polygonal absorbing regions and some of the changes which can occur as parameters has been studied for many years (see [21] and references therein), where the polygonal construction is used to prove the existence of two-dimensional absorbing regions in a variety of cases. We are not aware of a rigorous proof of the existence of a two-dimensional transitive attractor except in the Cournot case described above.

The remainder of the paper is organized as follows. In section two we introduce the results from standard Markov partition theory of dynamical systems which will be used to prove the results. We also introduce a generalization of the affine locally eventually onto (ALEO) property developed in [14] which can be used to show that periodic orbits are dense in a two-dimensional region, and the map is topologically transitive on this region – these are the two properties often used to define chaos. In section three we give an example where expansion of appropriate iterates of the map can be used to prove the existence of attractors. Section four uses the ALEO property to establish the existence of two-dimensional attractors for countably many parameter values. At the end of this analysis we find ourselves in a similar situation to the analysis of the logistic map at a time when the existence of strange attractors could be proved at Misieurewicz points (parameter values where the orbit of the critical point has particularly simple properties) but not more generally. We believe that the techniques introduced here can be generalized to prove the existence of two-dimensional attractors for open sets of parameters.

## 2. Markov partitions, expansion and the ALEO Property

In this section we will develop some general theory for piecewise affine maps. We begin with a definition.

**Definition 1** *[19] Given $D \subseteq \mathbb{R}^2$ and $F : D \to D$, an affine subdivision is a finite collection $\mathcal{M} = \{M_1, \ldots, M_N\}$ of pairwise disjoint open sets in $D$ whose boundary $\cup \partial M_i$ is a finite union of close line segments (possibly infinite or semi-infinite) such that their $\cup M_i$ is dense in $D$ and $F|_{M_i}$ is an affine map, $i = 1, \ldots, N$.*

If such a subdivision exists we say $F$ is a piecewise affine map, the subdivision is minimal if the domains $M_i$ are the largest domains on which $F_{M_i}$ is affine. Note that this definition does not assume continuity on the boundary, and this means that piecewise affine maps may be multi-valued, on the other hand the border-collision normal form is a continuous piecewise affine map, so ultimately any multi-valued features of induced maps will be resolved.

Many different properties could be used to characterize the dynamics of affine maps; here we use the locally eventually onto property introduced by Guckenheimer and Williams [17, 30] in the context of expanding maps of the interval. This was used by Glendinning [14] to prove strong expansion properties of a piecewise affine map originally introduced by Pikovsky and Grassberger [25]. The definition below is slightly weaker than the version introduced in [14], but it is enough to imply standard chaotic properties, see Proposition 2 below.

**Definition 2** ***(ALEO)*** *A piecewise affine map $F : D \to D$ has the ALEO (affine locally eventually onto) property on the subdivision $(M_i)_{i=1}^N$ of $D$ if for every open set $U \subseteq D$ and $i \in \{1, \dots, N\}$ there exists $V \subseteq U$ and $n > 0$ such that $F^n(V) = M_i$ and $F^n|_V$ is affine.*

The main tool we use to prove this property in this paper is the existence of finite Markov partitions, although we believe that the ALEO property holds in many examples which do not have finite Markov partitions (cf. [14]). Throughout this paper $c\ell(U)$ denotes the closure of $U$ and $int(U)$ denotes the interior of $U$. The definitions below follow [1, 29, 28].

**Definition 3** *Let $F$ be a piecewise affine map. A finite Markov partition of an $F-$invariant set $D$ is a finite subdivision*

$$\mathcal{M} = \{M_1, M_2, \dots M_N\}$$

*such that $F(M_i)$ is a union of elements of $\mathcal{M}$, $i = 1, \dots, N$. If every set in $\mathcal{M}$ is convex then we say $\mathcal{M}$ is a convex Markov partition.*

The existence of a Markov partition makes it possible to set up a symbolic dynamics which describes the possible behaviour of orbits under $F$ in terms of passages through the different elements of $\mathcal{M}$. This labelling may not be unique (points may have non-trivial stable manifolds, and points on the boundary lie in two sets). This will cause us some technical difficulties below when proving the existence of chaotic properties for the maps. These difficulties will be resolved in one of two ways – either by proving an expansion result which ensures that there is a unique correspondence between trajectories and allowed symbol sequences, or by making an additional (weak) assumption on the Markov partition.

**Definition 4** *Let $F$ be a piecewise affine map and let $\mathcal{M}$ be a finite Markov partition with $N$ elements. Then the associated graph $\mathcal{G}$ is the directed graph with vertices labelled*

*$\{1, \dots N\}$ and edges from $i$ to $j$ iff $M_j \subseteq F(M_i)$. The transition matrix for this graph is $H_{\mathcal{G}} = (h_{ij})$ where $h_{ij} = 1$ if there is an edge from $i$ to $j$ and $h_{ij} = 0$ otherwise. The graph is strongly connected if there is a path from each vertex to every other vertex, so for each $i$ and $j$ there exists $n$ (depending on $i$ and $j$) such that $(H^n)_{ij} > 0$; such a transition matrix is called irreducible.*

The graph of a Markov partition defines a symbolic dynamics in the following standard way. Let $\Sigma(n)$ denote the set of all words $b_0 \dots b_n \in \{1, \dots, N\}^n$ such that

$$h_{b_i b_{i+1}} = 1, \quad i = 0, \dots, n-1$$

Then for each $\omega \in \Sigma(n)$ the set

$$R_\omega = M_{\omega_0} \cap F^{-1}(M_{\omega_1}) \cap \dots \cap F^{-n}(M_{\omega_n})$$

is closed and non-empty, where the inverse maps are chosen such that if $M_{\omega_{k+1}} \subseteq F_i(M_{\omega_k})$ then $F^{-1}(M_{\omega_{k+1}})$ is defined using the inverse of the map $F_i$, proceeding inductively along the word. Taking the limit as $n \to \infty$ we obtain one-sided infinite sequences of symbols and as a countable intersection of closed nested sets is non-empty, if $\omega \in \Sigma(\infty)$ then

$$R_\omega = \cap_{k=0}^{\infty} F^{-k}(M_{\omega_k})$$

(with the convention on the definition of the inverse described above) is non-empty and if $x \in R_\omega$ then $F(x) \in R_{\sigma(\omega)}$ where $\sigma$ is the shift map (just delete the first term in the sequence and relabel the resulting sequence).

Most of the following lemma is again standard for continuous maps and requires no modification when applied to piecewise affine maps. The final statement about convexity follows as the image or preimage of a convex set under a non-singular affine map is convex, and a non-empty intersection of convex sets is convex.

**Lemma 1** *Let $F : D \to D$ be a piecewise affine map with a finite Markov partition $\mathcal{M}$. Then for $2 \leq n \leq \infty$*
*(i) if $\omega \in \Sigma(n)$ then $R_\omega$ is closed and non-empty;*
*(ii) $\cup_{\omega \in \Sigma(n)} R_\omega = D$;*
*(iii) if $n < \infty$ then $F^n$ restricted to $R_\omega$ is affine;*
*(iv) if $n = \infty$ then $F^j$ restricted to $R_\omega$ is affine for all $j \in \mathbb{N}$;*
*(v) $F(R_\omega) = R_{\sigma(\omega)}$.*
*In addition, if $\mathcal{M}$ is convex then for all $\omega \in \Sigma(n)$ then $R_\omega$ is convex.*

Much of the technical effort surrounding the relationship between the symbolic dynamics and the map itself involves describing the sets $R_\omega$ for $\omega \in \Sigma(\infty)$. In the ideal case this is a point, for then the map from $\Sigma(\infty)$ to $D$ is surjective. This is usually proved using some expansive property (or conversely, a contraction property on the inverse), but as discussed at the end of this section, the border collision normal form is not expanding. It may well be that a better theory from the one developed below is possible, but the results here do allow us to prove the ALEO property in the examples considered here.

**Definition 5** *A Markov partition (or its transition graph $H$) is said to be contracting if $R_\omega$ is a point for all $\omega \in \Sigma(\infty)$.*

Thus for a contracting Markov partition there can be no concerns about whether points with nearby symbol sequences are close in $D$.

**Proposition 1** *Let $F : D \to D$ be piecewise affine and suppose that $F$ has a finite Markov partition $\mathcal{M} = \{M_1, \dots, M_N\}$ with irreducible transition matrix $H$. If for all open sets $U \subset D$ there exists $i \in \{1, \dots, N\}$, $k \geq 0$ and $V \subseteq U$ such that $F^k(V) = M_i$ and $F^k|_V$ is affine, then $F$ is ALEO on $\mathcal{M}$.*

*Proof:* $H$ is irreducible, so there exists $n > 0$ such that for every $M_i$ and $M_j$ in the partition $\mathcal{M}$ there exists $V_{ij} \subset M_i$ such that $F^n(V_{ij}) = M_j$ and $F^n|_{V_{ij}}$ is affine ($H$ is irreducible so there is an allowed path from $i$ to $j$ and by taking preimages backwards along this path we obtain $V_{ij}$).

By assumption, for all open $U$ there is $V \subseteq U$ and $i$ and $k$ such that $F^k(V) = M_i$ and $F^k|_V$, so for any $M_j$ let $\hat{V} \subseteq V$ be such that $F^k(\hat{V}) = V_{ij}$ and note that $F^{k+n}(\hat{V}) = F^n(V_{ij}) = M_j$ and the map is affine by construction. Hence $F$ is ALEO on $\mathcal{M}$.

□

The importance of this definition for the dynamics is given by the following result about chaotic properties. Recall that a map $F$ is topologically transitive on $D$ if for all open sets $U$ and $V$ in $D$ there exists $n$ such that $F^n(U) \cap V \neq \emptyset$ and $F$ has sensitive dependence on initial conditions (sdic) on $D$ if there exists $\delta > 0$ such that for all $x \in D$ and $\epsilon > 0$ there exists $n \geq 0$ and $y \in D$ with $|x - y| < \epsilon$ such that $|F^n(x) - F^n(y)| > \delta$. If a continuous map on a metric space has an uncountable invariant set on which it is topologically transitive and for which periodic orbits are dense, then it also has sensitive dependence on initial conditions [13].

**Proposition 2** *Let $F : D \to D$ be piecewise affine and suppose that $F$ has a finite Markov partition $\mathcal{M} = \{M_1, \dots, M_N\}$ with irreducible transition matrix $H$. If $F$ is ALEO on $\mathcal{M}$ then periodic orbits are dense in $D$ and $F$ is topologically transitive on $D$. Thus $F$ also has sdic on $D$.*

*Proof:* We start with dense periodic orbits. Let $U$ be an open neighbourhood of $x \in D$, and as $U$ is open $U \cap int(M_i) \neq \emptyset$ for some $i$; denote one of these non-empty intersections as $W$. Then the ALEO property implies that there exists $V \subseteq W$ and $n \geq 1$ such that $V \subseteq M_i = F^n(V)$. Hence $V$ contains a periodic point.

Topological transitivity is just as easy: for any open $V$ then $V \cap int(M_r) \neq \emptyset$ for an appropriate choice of $r$, and any open $U$ contains a subset $V_1$ such that $F^k(V_1) = M_r$ for some $k$ by the ALEO property.

□

It is easy to see that a contracting Markov partition with positive topological entropy (i.e. there exists $n > 0$ such that $(H^n)_{ij} > 0$ for all $i, j$) implies the ALEO property. Unfortunately, in many cases this is not enough as neither $A_R$ nor $A_L$ can be expanding (in the sense that $|F(x) - F(y)| > k|x - y|$ for some $k > 1$) unless $T_R = 0$ or $T_L = 0$ (respectively), which brings us back to the Cournot cases.

The issue of expansion requires a little more discussion. Note that $A$ is expanding if $|x^T A^T Ax| > k|x^T x|$ for some $k > 1$, in other words the eigenvalues of the symmetric matrix $A^T A$ must lie outside the unit circle. For the case of our matrices,

$$A^T A = \begin{pmatrix} T_k^2 + D_k^2 & T_k \\ T_k & 1 \end{pmatrix} \tag{5}$$

with trace $T_k^2 + D_k^2 + 1$ and determinant $D_k^2$ $(k = R, L)$. Expansion implies that both eigenvalues of $A^T A$ lie outside the unit circle, a necessary condition for this is $Tr(A^T A) \geq 2$ and $det(A^T A) - Tr(A^T A) + 1 \geq 0$. Since the last inequality is $T_k^2 \leq 0$ the only possible expanding matrices of the normal form have trace zero.

We have developed two different techniques to get around this problem. In the next section we show that although $F$ itself is not expanding, there are some situations for which an appropriate iterate of $F$ *is* expanding, and that this is enough to prove that the Markov partition is contracting. This is the approach used in the next section. In the following section we use a different strategy, showing that any open interval eventually maps over an element of the Markov partition and thus proving the ALEO property directly.

In the next four sections we describe the attractor of the border collision normal form (1) at two carefully chosen sets of parameters. For these parameters there is a simple Markov partition which allows us to prove the ALEO property on two-dimensional attractors in these cases. We believe that many other attractors of these systems (attractors for an open set of parameter values) are transitive on a two-dimensional region, though we have not yet developed the techniques to prove this; the ALEO property has been shown to work in other cases, hence our insistence on using it!

## 3. A finite Markov partition with local expansion

Suppose that the parameters of the border collision bifurcation can be chose as shown in Fig. 1, where $P_1 = F(O)$ and $P_2 = F(P_1)$ are in $x > 0$ and $P_3 = F(P_2)$ has $x = 0$. Moreover, $P_4 = F(P_3)$ is in $x < 0$, $F(P_4) = P_2$ and the line $P_4P_2$ intersects the $y-$axis at $W = (0, -1)^T$, which is the preimage of $O$. Let $V$ be the point of intersection of $P_2P_4$ and $P_1P_3$ and $U$ the intersection of $P_1P_3$ with $OP_2$. Then by definition (points of intersection of lines map to the points of intersection of the images of the lines) $F(U) = V$ and $F(V) = W$. Consider the Markov partition involving the sets

$$\begin{array}{llll} M_1 = OUVW, & M_2 = OP_1U, & M_3 = P_1P_2U, & M_4 = P_2UV \\ M_5 = P_2P_3V, & M_6 = P_3VW, & M_7 = P_3P_4W, & M_8 = P_4OW \end{array} \tag{6}$$

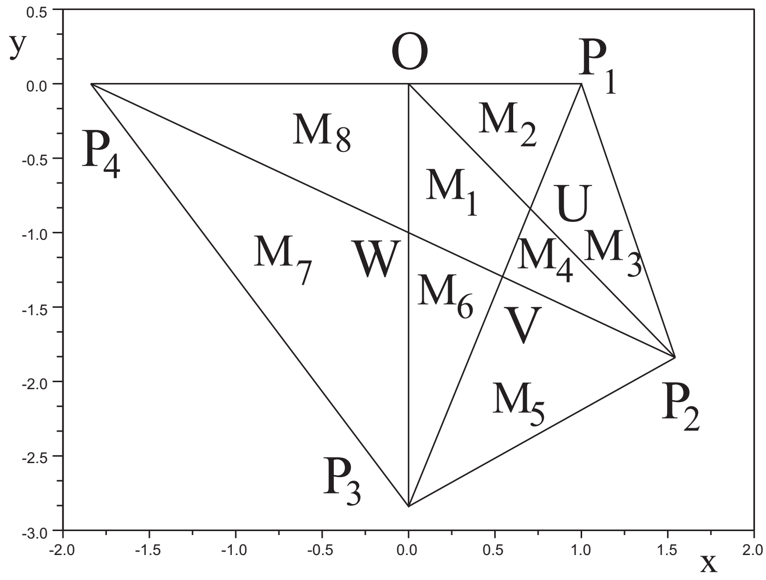


**Figure 1.** Attracting region for parameter values (11) with $F(O) = P_1$, $F(P_k) = P_{k+1}$, $k = 1, 2, 3$, $F(P_4) = P_2$, $F(W) = O$, $F(V) = W$ and $F(U) = V$; the Markov partition used to prove the region is transitive and has dense periodic orbits is labelled $M_i$.

Then

$$\begin{array}{llll} F(M_1) & = M_1 \cup M_2, & F(M_2) & = M_3 \cup M_4, \\ F(M_3) & = M_5, & F(M_4) & = M_6 \\ F(M_5) & = M_7, & F(M_6) & = M_8, \\ F(M_7) & = M_1 \cup M_4 \cup M_8, & F(M_8) & = M_2 \cup M_3 \end{array} \quad (7)$$

which shows that $P_1P_2P_3P_4$ is an absorbing region.

**Theorem 1** *Suppose* $\mu > 0$ *in the border collision normal form (1). If* $T_R = t$, *where* $t$ *is the solution of*

$$t^3 + t^2 + t - 1 = 0 \quad (8)$$

*in* $[0, 1]$, *and*

$$D_R = t^2 + t + 1 = 1/t \quad (9)$$

*and*

$$T_L = t^2, \quad D_L = -1 \quad (10)$$

*then the sets* $M_1$ *to* $M_8$ *defined in (6) form a Markov partition with covering (7) and* $F$ *is ALEO on* $P_1P_2P_3P_4$.

The implicit equations for the parameters above translate to the approximate values

$$T_R \approx 0.543689, \quad D_R \approx 1.839287, \quad T_L \approx 0.295598, \quad D_L = -1. \quad (11)$$

*Proof:* Without loss of generality choose $\mu = 1$. Direct calculation shows that

$$P_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} T_R + 1 \\ -D_R \end{pmatrix} \quad (12)$$

and

$$P_3 = \begin{pmatrix} T_R^2 + T_R + 1 - D_R \\ -D_R(T_R+1) \end{pmatrix}, \tag{13}$$

so the $x-$component of $P_3$ is zero if

$$D_R = T_R^2 + T_R + 1 \tag{14}$$

in which case $P_4 = (-D_R(T_R+1)+1, 0)^T$ and the line $P_4P_2$ intersects the $y-$axis at $(-1,0)^T$ if

$$\frac{1}{D_R(T_R+1)-1} = \frac{D_R}{T_R+1+D_R(T_R+1)-1} \tag{15}$$

using similar triangles. Rewriting this as $(D_R+1)(T_R+1)-1 = D_R^2(T_R+1)-D_R$, and using (14) gives

$$T_R^5 + 3T_R^4 + 4T_R^3 + 2T_R^2 - T_R - 1 = (T_R^3 + T_R^2 + T_R - 1)(T_R^2 + 2T_R + 1) = 0$$

and hence (8) holds. By direct evaluation this has a solution $T_R \in [0,1]$ and then $D_R$ given by (9) is greater than one, and $P_2$ is in $x > 0$ as required.

The conditions (10) on $D_L$ and $T_L$ simply ensure that $F(P_4) = P_2$; numerical calculations gives the approximations $t \approx 0.5437$ (or $D_R = 1/t \approx 1.8393$).

To prove the ALEO property we want to show that local distances eventually expand under iteration if points are always in the same element of the Markov partition, hence getting a contradiction and showing that any allowed itinerary corresponds to a unique point. It is easy to see that the transition matrix associated with the coverings (7) has positive entropy.

If the orbit of $x$ lies entirely in $x > 0$ then the point is the fixed point of the affine map in $x > 0$ and is unique – it corresponds to the allowed path $M_1M_1M_1\ldots$.

Now suppose that $x < 0$. If $x \in M_8$ then the allowed path starts either

$$M_8M_2M_4M_6M_8\ldots, \quad M_8M_2M_3M_5M_7\ldots, \quad \text{or} \quad M_8M_3M_5M_7\ldots.$$

In each case there is one iterate in $x < 0$ followed by either two or three iterates in $x \geq 0$ before returning to $x \leq 0$. Similarly if $x \in M_7$ the allowed paths are either

$$M_7M_1^pM_2M_4M_6M_8\ldots, \quad M_7M_1^pM_2M_3M_5M_7\ldots, \quad \text{or} \quad M_7M_4M_6M_8\ldots$$

with $p \geq 1$ ($p = \infty$ being allowed), or $M_7$ is followed by $M_8$ and we have one of the cases for $M_8$ with an extra iteration in $x < 0$ at the beginning. In all cases it is easy to see that (labelling with $L$ and $R$ for left and right of the $y-$axis respectively) the sequences of visits to the left and right can be obtained by concatenating the symbols

$$LLRR, \quad LLRRR, \quad LRR, \quad LRRR, \quad \text{and} \quad RR.$$

For example,

$$M_7M_1^4M_2M_4M_6M_8M_3M_5M_7M_8M_3M_5M_7\ldots$$

corresponds to $(LRRR)(RR)^2(LRR)(LLRR)L\ldots$. Hence if each of the five combinations of $L$s and $R$s corresponds to an effective expansion by a factor greater than one, then an infinite allowed path corresponds to a unique point.

| operator $B$ | $\delta = \det(B^T B)$ | $\tau = \mathrm{Trace}(B^T B)$ | $\tau - 2$ | $\delta - \tau + 1$ |
|---|---|---|---|---|
| $A_R^2 A_L^2$ | 11.44 | 7.06 | 5.06 | 5.38 |
| $A_R^3 A_L^2$ | 38.71 | 14.21 | 12.21 | 25.51 |
| $A_R^2 A_L$ | 11.44 | 6.77 | 4.77 | 5.68 |
| $A_R^3 A_L$ | 38.71 | 15.83 | 13.82 | 23.89 |
| $A_R^2$ | 11.44 | 7.06 | 5.06 | 5.38 |

**Table 1.** Expansion properties for higher iterates of the map.

As shown earlier, a $2 \times 2$ matrix $B$ is expanding if the trace $\tau$ and determinant $\delta$ of the symmetric matrix $B^T B$ satisfy $\tau > 2$ and $\delta - \tau + 1 > 0$. We have checked numerically that each of the five combinations for $B$:

$$A_R^2 A_L^2, \quad A_R^3 A_L^2, \quad A_R^2 A_L, \quad A_R^3 A_L, \quad \text{and} \quad A_R^2$$

(note that the order of the operators is reversed here from the order of the symbol sequences) do indeed correspond to expanding matrices. We could have done the calculation explicitly and made more thoughtful approximations to prove these inequalities rigorously, but the numerical results are unambiguous so we felt this was unnecessary – numerical results are shown in the following table (if the final two columns are positive then the matrix is expanding).

Thus since the transition graph has positive topological entropy, the map is ALEO. □

**Corollary 1** *For parameters defined in Theorem 1 the dynamics restricted to the quadrilateral $P_1P_2P_3P_4$ is transitive and periodic orbits are dense in the quadrilateral.*

## 4. A countable set of examples

The examples of previous sections demonstrate that the Markov partition technique can be applied effectively to study attractors of the border-collision normal form, and we give a last set of examples which show how to generate a countable set of parameters for which a Markov partition exists. An example is illustrated in Fig. 2.

Suppose that $\mu > 0$, so without loss of generality we can take $\mu = 1$ in the calculations below. The fixed point $P^*$ in $x > 0$ is

$$P^* = \tfrac{1}{1+D_R-T_R} \begin{pmatrix} 1 \\ -D_R \end{pmatrix} \tag{16}$$

and $A_R$ has complex conjugate eigenvalues $re^{\pm i\theta}$, if

$$T_R = 2r\cos\theta, \quad D_R = r^2. \tag{17}$$

We will choose $r$ and $\theta$ so that there exists $n > 1$ such that the $x-$coordinate of $P_n = F^n(O)$ has $x = 0$, with $P_k = F^k(O)$ in $x > 0$ for $k = 1, \ldots, n-1$ (this gives one condition which $T_R$ and $D_R$ must satisfy). Then $F(P_n) = P_{n+1}$ will lie on the $x-$axis

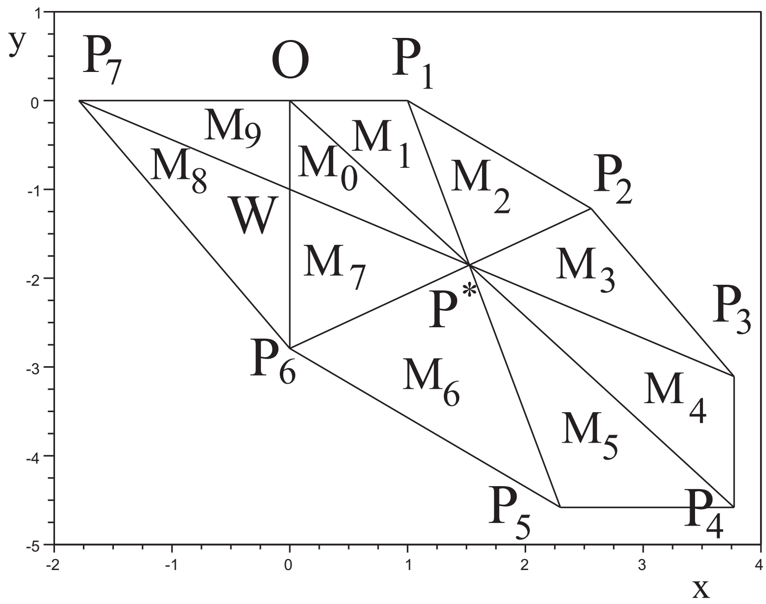


**Figure 2.** Attracting region showing the construction of the Markov partition used to prove the region is transitive and has dense periodic orbits. The case $n = 6$ is illustrated with $T_R \approx 1.55842898$, $D_R \approx 1.21435044$ and $T_L$ and $D_L$ given by (28). $F(O) = P_1$; $F(P_i) = P_{i+1}$, $i = 1, \ldots, 6$; $F(P^*) = F(P_7) = P^*$; and $F(W) = O$.

in $x < 0$. A second condition on $T_R$ and $D_R$ is obtained by imposing that the line from $P_{n+1}$ to $P^*$ intersects the $y-$axis at $W = (0, -1)^T$, which (by definition) maps to $O$ under $F$. Finally we choose $T_L$ and $D_L$ such that $F(P_{n+1}) = P^*$. This yields the geometry shown in Fig. 2, creating regions $M_0, \ldots, M_{n+4}$ defined below and which form a Markov partition. We begin by proving that parameters at which these conditions hold do exist for each $n > 2$ (recall that $\mu = 1$ in the calculations throughout this section).

By definition

$$F^k(O) = (A_R^{k-1} + \ldots + A_R + I) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (A_R - I)^{-1}(A_R^k - I) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

provided $r \neq 1$. A tedious calculation (working in the basis of eigenvectors of $A_R$ or otherwise) yields

$$\begin{aligned} P_n &= F^n(O) \\ &= \tfrac{r^n}{(1+D_R-T_R)\sin\theta} \begin{pmatrix} r\sin(n\theta) - \sin(n+1)\theta \\ D_R \sin(n+1)\theta - (T_R - 1) r \sin(n\theta) \end{pmatrix} \\ &\quad + \tfrac{1}{1+D_R-T_R} \begin{pmatrix} 1 \\ -D_R \end{pmatrix} \end{aligned} \qquad (18)$$

where, of course, $T_R$ and $D_R$ are given by (17) and the last term is $P^*$. The condition that $P_n$ has $x = 0$ is therefore

$$r^{n+1}\sin(n\theta) - r^n \sin(n+1)\theta + \sin\theta = 0. \qquad (19)$$

This is the first condition for $T_R$ and $D_R$ referred to above.

Using (19), the expression for $P_n$, (18) simplifies to $P_n = (0, q)^T$ where

$$q = \frac{r^{n+1}\sin(n\theta)}{\sin\theta} \tag{20}$$

and so $P_{n+1} = (1+q, 0)^T$ and we will need to check that $q < -1$. The condition that $W$ is on the line from $P_{n+1}$ to $P^*$ is equivalent to the statement that the slope of the line $P_{n+1}W$ equals the slope of the line $WP^*$ which, after a little manipulation is

$$\frac{r^{n+1}\sin(n\theta) + \sin\theta}{\sin\theta} = \frac{1}{1 - T_R} \tag{21}$$

or

$$2r^{n+1}\cos\theta\sin(n\theta) - r^n\sin(n\theta) + 2\cos\theta\sin\theta = 0. \tag{22}$$

This is the second of the two conditions which define $T_R$ and $D_R$ via (17). Expressing $r^n\sin(n\theta)$ in terms of $r$ and $\theta$ using (22), and $r^n\sin(n+1)\theta$ similarly in terms of (19) and dividing gives

$$2\cos\theta\sin(n+1)\theta = \sin(n\theta). \tag{23}$$

and by using standard double angle formulae on the left hand side gives (at last)

$$\sin(n+2)\theta = 0, \tag{24}$$

and we choose the solution with

$$\theta_n = \frac{2\pi}{n+2}. \tag{25}$$

Substituting this value into (19) and simplifying we find that (19) is satisfied if and only if $g_n(r) = 0$ where

$$g_n(r) = 2r^{n+1}\cos\theta_n - r^n - 1. \tag{26}$$

For large $r$, $g_n(r) > 0$, and if $r = 1$

$$g_n(1) = -2(1 - \cos\theta_n) \le 0. \tag{27}$$

Hence, by the intermediate value theorem $g_n$ has a zero, $r_n > 1$.

This establishes the existence of a solution $(r_n, \theta_n)$ to the conditions determining the desired behaviour for $F^k(O)$.

For any given $n > 1$, $P_{n+1} = (1+q, 0)^T$ with $q$ given by (20) for parameters determined by $(r_n, \theta_n)$. The condition $F(P_{n+1}) = P^*$ is thus

$$1 + T_L(1+q) = 1/(1 + D_R - T_R), \quad D_L(1+q) = D_R/(1 + D_R - T_R) \tag{28}$$

which determines $T_L$ and $D_L$ (note that since $1 + q < 0$ both are negative).

**Theorem 2** *Suppose $\mu > 0$ in the border collision normal form (1). For each $n > 0$ sufficiently large let $(T_R, D_R, T_L, D_L)$ be determined from $(r_n, \theta_n)$ defined above via (17) and (28). Then the border collision normal form is ALEO on a two-dimensional region.*

*Remark:* Numerical calculations suggest that 'sufficiently large' means $n \geq 6$.

With the notation above (fixing $\mu = 1$ throughout this section) and illustrated in Fig. 2, the Markov partition will be constructed using the sets $M_0, \ldots, M_{n+3}$ defined by $M_0 = WOP^*$, $M_1 = OP_1P^*$ and $M_k = P_{k-1}P_kP^*$, $k = 2, \ldots, n$ (so $F(M_{k-1}) = M_k$, $k = 1, \ldots, n$), $M_{n+1} = WP_nP^*$, $M_{n+2} = WP_nP_{n+1}$ and $M_{n+3} = WP_{n+1}O$. Then by definition of the end points and the parameters

$$\begin{array}{ll} F(M_n) = M_{n+1} \cup M_{n+2}, & F(M_{n+1}) = F(M_{n+2}) = M_{n+3} \cup M_0, \\ F(M_{n+3}) = F(M_1) & \end{array} \tag{29}$$

which implies that the union of these regions, the polygon $P_1P_2 \ldots P_{n+1}$, is invariant and $M_0, \ldots, M_{n+3}$ form an irreducible convex Markov partition.

Let $a < b < c$. Then a skew tent map $S : [a, c] \to [a, c]$ is a continuous map such that $S([a, b]) = [a, c]$ and $S([b, c]) = [a, c]$, and such that $S$ is an affine map on both $[a, b]$ and $[b, c]$. The point $b$ is called the turning point of the skew tent map. Note that if $I$ is any open interval in $[a, b]$ then there exists $J \subseteq I$ and $n > 0$ such that $S^n|_J$ is affine and $S^n(J) = [a, b]$ (e.g. [14]). The proof of Theorem 2 relies on a simple lemma which connects the dynamics of the normal form restricted to a line to the skew tent map. This line segment will play the same role as the diagonal in the proof of the ALEO property for the example in [14]. The following lemma will be useful in the proof of Theorem 2.

**Lemma 2** *Consider the normal form for parameters defined in Theorem 2. Then*
*(i)* $F^{n+2}$ *restricted to the line segment* $P_{n+1}P^*$ *is a skew tent map with turning point at* $W$*; and*
*(ii) Let* $Q$ *be any region in* $M_k$, $k = 0, \ldots, n+1$, *which fills the angle at* $P^*$*. Then there exists* $Q_1 \subseteq Q$ *and* $m$ *such that* $F^m(Q_1) = M_0$ *and* $F^m|_{Q_1}$ *is affine.*

*Proof:* Part (i) follows easily from the observation that $F(P_{n+1}W) = F(WP^*) = OP^*$, and then that $F^{n+1}(OP^*) = P_{n+1}P^*$.

To prove (ii) consider preimages of $M_0 = WP^*O$ under the map in $R$. The preimage of $W$, $W_{-1}$ lies on the open line segment $P_nP^*$, so the preimage will be the set $WP^*W_{-1} \subset M_{n+1}$. Its preimage is $W{-}1P^*W_{-2} \subset M_n$ with $W_{-2}$ on the open segment $P_{n-1}P^*$. Continuing in this way we see that $F^{-(n+2)}(M_0) \subset M_0$ and contains the angle $\angle P_{n+1}P^*O$, and more generally the sets $F^{-m(n+2)}(M_0)$, $m > 0$, form a nested sequence of such regions tending to $P^*$. Similarly with $k \in \{1, \ldots, n+1\}$ fixed, $F^{-m(n+2)+k}(M_0)$ is in $M_k$ and tends to $P^*$ as $m \to \infty$.

□

*Remark:* In fact, this shows that preimages of $M_0$ exist in any region filling an angle $P_{k-1}P^*P_k$ in $M_k$, $k = 2, \ldots n$.

*Proof of Theorem 2:* To apply Proposition 1 we need to check that for all open sets $U$ in the absorbing region $\cup M_k$ there exists $i$, $n$ and $V \subseteq U$ such that $F^n(V) = M_i$ and $F|_V$ is affine. Since the transition matrix associated with the Markov partition is easily seen to be irreducible it is enough to show this for $i = 0$ (cf. Proposition 1).

First note that $r_n > 1$ and so $D_R = r_n^2 > 1$. Simplifying the expression for $D_L$ in (28) using $g_n(r_n) = 0$ in (26) gives

$$D_L = -r_n^{-(n-2)}\left(1 + r_n^2 - 2r_n\cos\theta_n\right)^{-1} \tag{30}$$

so $|D_L| > 1$ if $r_n^{n-2}\left(1 + r_n^2 - 2r_n\cos\theta_n\right) < 1$. Since $r_n \to 1$ and $\theta_n \to 0$ as $n \to \infty$, $D_L \to -\infty$ (with $D_L$ the function of $n$ in (28)) as $n \to \infty$ and in particular $|D_L| > 1$ for sufficiently large $n$. Therefore for large $n$ both $|D_R|$ and $|D_L|$ are greater than 1 and hence areas are increased under iteration. Numerical simulations show that $r_n^{n-2}\left(1 + r_n^2 - 2r_n\cos\theta_n\right) < 1$ provided $n \geq 6$.

Let $U$ be an open set in the attracting polygon and fix $n$ large enough so that $D_R$ and $|D_L|$ are greater than one. Let $m_1 \geq 0$ be the smallest positive integer such that $F^{m_1}(U)$ intersects the critical line $x = 0$ ($m_1$ exists because otherwise $F^m|_U$ would be affine for each $m$ and the area of $F^m(U)$ would increase unboundedly, but $F^m(U)$ is in the finite absorbing region).

Suppose that $F^{m_1}(U)$ intersects $WP_n$ and let $U_1 \subseteq M_{n+1}$ be the component of $F^{m_1}(U)$ in $R$, so that $F^{m_1}(U) \cap WP_n$ is on its boundary. Let $U_2 \subseteq U$ such that $F^{m_1}(U_2) = U_1$ and $F^{m_1}|_{U_2}$ is affine.

Since $F(M_{n+1}) = M_{n+3} \cup M_0$ and $WP_n$ maps to $OP_{n+1}$, $F(U_1) \cap M_{n+3} \neq \emptyset$ and by choosing $U_1$ and $U_2$ smaller if necessary we may assume that $F(U_1)$ is contained in $M_{n+3}$, so $F^{n+2}|_{U_1}$ is affine and $F^{n+2}(U_1) \subseteq M_{n+1} \cup M_{n+2}$ has a segment $I \subseteq P_{n+1}P^*$ on its boundary. By Lemma 2 (i) (using the expansion property of skew tent maps described above the statement of Lemma 2) there exists $J \subseteq I$ and $m_2$ such that $F^{m_2(n+2)}|_J$ is affine and $F^{m_2(n+2)}(J) = P_{n+1}P^*$, so there exists $U_3 \subseteq F^{n+2}(U_1)$ with $J$ on the boundary such that $F^{m_2(n+2)}|_{U_3}$ is affine and $P_{n+1}P^*$, and in particular $W$, is on the boundary of $F^{m_2(n+2)}(U_3)$. By (29), $F^{m_2(n+2)}(U_3) \subseteq M_{n+1} \cup M_{n+2}$, and since it contains $W$ on its boundary it contains the intersection of an open neighbourhood of $W$ with $M_{n+1}$, i.e. it fills the angle $P_nWP^*$ at $W$. Let $U_4 \subseteq F^{m_2(n+2)}(U_3)$ be the component of $F^{m_2(n+2)}(U_3)$ in $R$ with $F(U_4) \subseteq M_0$ and which fills the angle $\angle WOP^*$ at $O$, then $F^{n+3}(U_4) \subseteq M_0$ and fills $\angle WP^*O$ at $P^*$. Thus Lemma 2 (ii) ensures that there exists $U_5 \subseteq F^{n+3}(U_4)$ and $m_3$ such that $F^{m_3}(U_5) = M_0$ and $F^{m_3}|_{U_5}$ is affine. Therefore, given an open set $U$, there exists $V (\subseteq U_2) \subseteq U$ and $N = m_1 + (n+2) + m_2(n+2) + (n+3) + m_3$ such that $F^N|_V$ is affine and $F^N(V) = M_0$, and hence $F$ is ALEO on the polygon $P_1P_2\dots P_{n+1}$ by Proposition 1.

If $F^{m_1}(U)$ intersects $WO$, let $U_1$ be the component of $F^{m_1}(U)$ in $R$, then $F^{n+2}(U_1) \subseteq M_{n+3} \cup M_0$ and has a segment $P^*P_{n+1}$ on the boundary and above argument applies.

□

**Corollary 2** *For parameters such that the previous theorem holds, the closed polygon $P_1P_2\dots P_{n+1}$ is a transitive attractor and periodic orbits are dense in the polygon.*

## 5. Conclusion

We have shown that two-dimensional attractors with finite Markov partitions exist for the two dimensional normal form of border collisions and this provides a means of proving the existence of transitive two-dimensional attractors for the border collision normal form. Numerical evidence [11, 21] suggests that these two-dimensional attractors exist over much larger regions of parameter space than those amenable to either the Cournot map analysis of [7] or the two-dimensional Markov partitions described here. Our hope is that the use of the ALEO property will eventually allow us to provide a mathematical proof of the existence of these attractors.

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