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Determining the absorption in anisotropic media

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Abstract. The problem in Optical Tomography of determining the spatially dependent absorption coefficient in an anisotropic medium with a-priori known strong scattering is considered. The problem is modelled by the diffusion approximation of the Radiative Transfer Equation and the time-harmonic case is studied. In this particular situation the diffusion approximation leads to an elliptic second order partial differential equation with complex variable coefficients which allows to treat the problem equivalently to the inverse conductivity problem in Electrical Impedance Tomography (EIT). Results of uniqueness and stability for the absorption coefficient are proven by using the approach of the work in *SIAM J. Math. Anal.* 33 (2001), no. 1, 153–171 for the inverse conductivity problem in EIT.

1 Introduction.

The classical Calderón inverse conductivity problem is to recover an unknown coefficient in a elliptic partial differential equation from the Dirichlet-to-Neumann map at the boundary. This problem arises in electrical resistivity tomography (or more generally electrical impedance tomography EIT), a method used for subsurface geophysical imaging, industrial process monitoring and as an experimental medical imaging technique. Optical tomography (OT) of a highly scattering medium using near infra-red light [5] is another medical imaging technique that is closely related. As we shall see the usual mathematical model for this also results in an elliptic PDE with unknown coefficients to be determined from boundary data. In both EIT and OT anisotropic materials are common. Biological tissue and rocks commonly have a layered or fibrous structure on a small scale, which appears on a macroscopic scale as an anisotropic material property. While the problem of non-uniqueness of solution, and uniqueness with sufficient *a priori* information has been

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fairly well studied in EIT there has been little work on this in OT. In this paper we extend the methods of [3] to the case of OT. More precisely we recall that the P_1 approximation of the commonly used *Radiative Transfer Equation* in OT leads in the static and time-harmonic cases to an equation [13] of type

$$\nabla \cdot (K \nabla \varphi) - (\mu_a - ik)\varphi = 0, \quad (1.1)$$

where $k = \frac{\omega}{c}$ is the wave number, ω is the fixed harmonic frequency with which the input field is modulated and K is the complex valued matrix

$$K = \frac{1}{n} \left((\mu_a - ik)I + (I - B)\mu_s \right)^{-1}. \quad (1.2)$$

Here μ_a, μ_s are the absorption and scattering coefficient respectively i.e. the optical properties of the medium. One formulation of the OT inverse problem is to recover μ_a and μ_s from the knowledge of the so-called Robin-to-Robin map

$$\Upsilon_K : \Phi_- \in H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow \left(\gamma(n)\varphi + \frac{1}{2}K \nabla \varphi \cdot \nu \right) \Big|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega),$$

where $\varphi \in H^1(\Omega)$ is the unique solution to (1.1) corresponding to the input flux Φ_- . Here ν is the unit outer normal to $\partial\Omega$. The static Neumann-to-Dirichlet data is insufficient to recover both coefficients uniquely [7] unless *a priori* smoothness assumptions are employed [11]. We also refer to [22, section 6]. In the present paper we study the inverse problem of determining the absorption μ_a from the knowledge of the Dirichlet-to-Neumann map, in the case when the scattering μ_s and the matrix B appearing in (1.2) are given.

In medical applications OT has been proposed [6] as a method for *functional* rather than structural imaging. While the scattering coefficient μ_s varies from tissue to tissue, it is the absorption coefficient μ_a that carries the more interesting physiological information as it is related to the global concentrations of certain metabolites in their oxygenated and deoxygenated states. Many tissues including parts of the brain, muscle and breast tissue have a fibrous structure on a microscopic scale which results in anisotropic physical properties on a larger scale. Diffusion Tensor Magnetic Resonance Imaging (DT-MRI) measures a spacially varying rank two symmetric tensor field related to the diffusion coefficient for water in the tissue. This has been proposed as a method of inferring the anisotropic electrical conductivity as well as other properties including

anisotropic light scattering [21]. If we assume therefore that prior to the application of OT the structural information μ_s and B have been determined from conventional Magnetic Resonance Imaging and DT-MRI, we are interested in determination changes in μ_a and this presents the motivation for the present work.

More precisely we concentrate on the issue of determining boundary values of μ_a by pursuing the same line of investigation of [3] and considering anisotropic diffusion tensors that arise in the static and time-harmonic cases. If we assume that μ_s and B are known, then these tensors become of type $K(x) = K(x, \mu_a)$, where $K(x, t)$ is a known, complex matrix-valued function given by

$$K = \frac{1}{n} \left((t - ik)I + (I - B)\mu_s \right)^{-1} \quad (1.3)$$

and μ_a is the unknown scalar function we want to recover. The precise assumptions shall be illustrated in Section 3. We improve upon the results obtained in [3] by adapting the uniqueness and stability results at the boundary to the case in which the governing equation (1.1) has complex coefficients and an extra lower order term $\mu_a - ik$ which does not appear in the conductivity equation of [3]. The case in which μ_a and B are known and the scattering coefficient μ_s is to be determined can be treated in a similar manner to the one considered in this work. Moreover the results obtained in the present paper show that the so-called *monotonicity assumption* of [3] is a realistic hypothesis for example in the OT problem considered here.

The paper is organized as follows. Section 1 contains the formulation of the problem, starting with the mathematical model in OT of the *Radiative Transfer Equation* and ending with the recovery of the so-called P_1 approximation. Section 3 is devoted to the static and time-harmonic cases of the problem and the statement of the main assumptions on the given scattering coefficient μ_s and the matrix B . The boundary measurements of the problem are also introduced in this section. The main results are contained in Section 4, while Section 5 is devoted to the construction of singular solutions of equation (1.1) having the same type of singularity as those in [3]. Proofs of the main results are given in Section 6.

2 Formulation of the problem

2.1 The Radiative Transfer Equation

Although Maxwell's equations provide a complete model for the light propagation in a scattering medium on a micro scale, on the scale suitable for medical OT an appropriate model is given by the *Radiative Transfer Equation* (or *Boltzmann equation*)[8].

Let Ω be a domain in \mathbb{R}^n , with $n = 2$ or $n = 3$ with smooth boundary $\partial\Omega$. We will denote by ν the outer unit vector on $\partial\Omega$. If radiation is considered in the body Ω , then the radiation flux density at the point $x \in \Omega$, at the time t to the infinitesimal solid angle ds , in the direction $\hat{\theta}$ is given by

$$d\vec{J}(x, t, \hat{\theta}) = I(x, t, \hat{\theta})\hat{\theta}ds(\hat{\theta}), \quad (2.1)$$

where $I(x, t, \hat{\theta})$ is called the *radiance* and it satisfies the *Radiative Transfer Equation* (RTE)

$$\begin{aligned} \frac{1}{c}I_t(x, t, \hat{\theta}) + \hat{\theta} \cdot \nabla I(x, t, \hat{\theta}) + (\mu_a(x) + \mu_s(x))I(x, t, \hat{\theta}) \\ - \mu_s(x) \int_{S^{n-1}} f(x, \hat{\theta}, \hat{\omega})ds(\hat{\omega}) = 0, \end{aligned} \quad (2.2)$$

where c is the speed of light (assumed to be constant) and μ_a and μ_s are the absorption and the scattering coefficient respectively (the optical properties of the medium). The kernel f is the *scattering phase function* (see [12] for more details on this topic). The RTE is an integro-differential equation in $2n - 1$ variables, therefore it leads to numerical problems of very large size. A common simplification of it is the so-called *Diffusion Approximation* (see [5]).

2.2 The Diffusion Approximation

Let S^{n-1} be the unit sphere in \mathbb{R}^n and $\hat{\theta} = (\theta_1, \dots, \theta_n) \in S^{n-1}$. Note that $span\{1, \theta_1, \dots, \theta_n\} \subset L^2(S^{n-1})$ is a closed subspace, therefore we can consider the orthogonal projection

$$P : L^2(S^{n-1}) \longrightarrow span\{1, \theta_1, \dots, \theta_n\},$$

defined for any $g \in L^2(S^{n-1})$ by $Pg = \alpha + \vec{a} \cdot \hat{\theta}$, where

$$\alpha = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} g(\hat{\theta})ds \quad \text{and} \quad \vec{a} = \frac{n}{|S^{n-1}|} \int_{S^{n-1}} \hat{\theta}g(\hat{\theta})ds.$$

If we denote by \mathcal{B} the integro-differential operator on the left hand side of (2.2), then the diffusion approximation of the radiative transfer equation is defined by

$$P\mathcal{B}PI = 0. \quad (2.3)$$

If we define the energy flux and the energy current density corresponding to the radiance $I(x, t, \hat{\theta})$ by

$$\varphi(x, t) = \int_{S^{n-1}} I(x, t, \hat{\theta}) ds(\hat{\theta}) \quad \text{and} \quad \vec{J}(x, t) = \int_{S^{n-1}} I(x, t, \hat{\theta}) \hat{\theta} ds(\hat{\theta})$$

respectively, it turns out that (2.3) is equivalent to the coupled system in φ and \vec{J}

$$\frac{1}{c} \varphi_t = -\nabla \cdot \vec{J} - \mu_a \varphi \quad (2.4)$$

$$\frac{1}{c} \vec{J}_t = -\frac{1}{n} \nabla \varphi - (\mu_a + (I - B)\mu_s) \vec{J}, \quad (2.5)$$

where

$$B_{ij}(x) = B_{ji}(x) = \frac{n}{|S^{n-1}|} \int_{S^{n-1}} \int_{S^{n-1}} \theta_i \omega_j f(x, \hat{\theta}, \hat{\omega}) ds(\hat{\theta}) ds(\hat{\omega}) \in \mathbb{R}.$$

Remark 2.1. $I - B$ is a positive definite matrix (see [5], [12] and [14]).

We refer to [5], [12] and [14] for a full understanding of the diffusion approximation, being the purpose of this section here only a brief resume on the mathematical model used in OT.

3 The static and time-harmonic cases.

If the input field is modulated with a fixed harmonic frequency ω , i.e.

$$I(x, t, \hat{\theta}) = \text{Re} \left(e^{-i\omega t} I(x, \hat{\theta}) \right),$$

with ϕ and \vec{J} similarly complexified so that

$$\varphi_t = -i\omega \varphi; \quad \vec{J}_t = -i\omega \vec{J},$$

which reduces the system given by equations (2.4), (2.5) to the elliptic equation

$$\nabla \cdot (K \nabla \varphi) - (\mu_a - ik) \varphi = 0, \quad (3.1)$$

where $k = \frac{\omega}{c}$ is the wave number and K is the complex valued matrix

$$K = \frac{1}{n} \left((\mu_a - ik)I + (I - B)\mu_s \right)^{-1}. \quad (3.2)$$

Physically the infra-red light is assumed sinusoidally modulated, or a single Fourier component has been measured and k is the angular frequency of this component which is of course much lower than the frequency of the light.

Notice that (3.2) reduces in the static case to the real matrix

$$\frac{1}{n} \left(\mu_a + (I - B)\mu_s \right)^{-1}$$

when $k = 0$. In this paper we will treat the time harmonic case but results include the static case simply by setting $k = 0$.

In EIT it is usual to consider as data the Neumann-to-Dirichlet map. For a more general equation like

$$\nabla \cdot (K\nabla\varphi) - q\varphi = 0 \quad \text{in } \Omega,$$

where Ω is a domain with normal ν , this is $N_{K,q}g = u|_{\partial\Omega}$ where $\nu \cdot K\nabla\varphi|_{\partial\Omega} = g$. Here we will denote $N_{K,q}$ simply by N_K , q being always $\mu_a - ik$ in this context. A precise definition of N_K and the boundary measurements is given in Section 3.2 after assumptions are stated on the domain Ω in Section 3.1. One formulation of the OT inverse problem in the isotropic case ($B = 0$) is to determine μ_s and μ_a from $N_{K,q}$ for one or more known values of k . The static Neumann-to-Dirichlet data is insufficient to recover both coefficients uniquely [7] unless *a priori* smoothness assumptions are employed [11]. The key observation here is that for any function $\gamma \in C^2(\Omega) \cap C^1(\bar{\Omega})$, $\inf_{\bar{\Omega}} \gamma > 0$, $\nu \cdot K\nabla\gamma = 0$ on $\partial\Omega$

$$N_{K,\tilde{q}}^{-1} = \gamma^{-1} N_{\gamma^2 K, q}^{-1} \gamma^{-1} + \nu \cdot K\nabla\gamma. \quad (3.3)$$

with

$$\tilde{q} = \frac{q}{\gamma^2} + \frac{\nabla \cdot K\nabla\gamma}{\gamma}.$$

In this paper we will concentrate on the unique determination of μ_a with μ_s and B assumed known.

3.1 Main assumptions

Let Ω be a domain in \mathbb{R}^n ($n \geq 2$), with Lipschitz boundary $\partial\Omega$. We recall, for sake of completeness, the definition of Lipschitz regularity of the boundary. We stick to the notation already used in [3].

DEFINITION 3.1. *Given positive numbers L, r, h satisfying $h \geq Lr$, we say that a bounded domain $\Omega \in \mathbb{R}^n$ has Lipschitz boundary if, for every $x^0 \in \partial\Omega$, there exists a rigid transformation of coordinates which maps x^0 into the origin, such that, setting $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$, we have*

$$\begin{aligned} \Omega \cap \{x = (x', x_n) \mid |x'| < r, |x_n| < h\} &= \\ &= \{x = (x', x_n) \mid |x'| < r, |x_n| < h, x_n \geq f(x')\}, \end{aligned}$$

where $f = f(x')$ is a Lipschitz function defined for $|x'| < r$, which satisfies

$$\begin{aligned} f(0) &= 0 \\ |f(x') - f(y')| &\leq L|x' - y'|, \end{aligned}$$

for every $x', y' \in \mathbb{R}^{n-1}$, with $|x'|, |y'| < r$.

ASSUMPTION 3.1. (Assumption on the known parameters μ_s and B)
Given the positive constant E , we assume that μ_s and B satisfy

$$\|\mu_s\|_{W^{1,\infty}(\Omega)} \leq E \quad (3.4)$$

$$\|B\|_{W^{1,\infty}(\Omega)} \leq E. \quad (3.5)$$

Let us introduce a class of matrix valued functions $K(x, t)$ on $\Omega \times [\lambda^{-1}, \lambda]$ that we will denote by \mathcal{H}' .

DEFINITION 3.2. *Given $p > n$, the positive constants $\lambda, \mathcal{E}, \mathcal{F} > 0$, and denoting by Sym_n the class of $n \times n$ complex symmetric matrices, we say that $K(\cdot, \cdot) \in \mathcal{H}'$ if the following conditions hold*

$$K \in W^{1,p}(\Omega \times [\lambda^{-1}, \lambda], \text{Sym}_n), \quad (3.6)$$

$$D_t K \in W^{1,p}(\Omega \times [\lambda^{-1}, \lambda], \text{Sym}_n), \quad (3.7)$$

$$\begin{aligned} \text{ess sup}_{t \in [\lambda^{-1}, \lambda]} \left(\|K(\cdot, t)\|_{L^p(\Omega)} + \|D_x K(\cdot, t)\|_{L^p(\Omega)} \right. \\ \left. + \|D_t K(\cdot, t)\|_{L^p(\Omega)} + \|D_t D_x K(\cdot, t)\|_{L^p(\Omega)} \right) \leq \mathcal{E}, \quad (3.8) \end{aligned}$$

$$\begin{aligned} \lambda^{-1}|\xi|^2 \leq |K(x, t)\xi \cdot \xi| \leq \lambda|\xi|^2, \quad \text{for almost every } x \in \Omega, \\ \text{for every } t \in [\lambda^{-1}, \lambda], \xi \in \mathbb{R}^n \end{aligned} \quad (3.9)$$

$$\begin{aligned} \text{Re}\{D_t K(x, t)\} \xi \cdot \xi \leq -\mathcal{F}|\xi|^2, \quad \text{for almost every } x \in \Omega, \\ \text{for every } t \in [\lambda^{-1}, \lambda], \xi \in \mathbb{R}^n \end{aligned} \quad (3.10)$$

We observe that (3.10) is a condition of monotonicity with respect to the last variable t and that it replaces the monotonicity condition

$$\begin{aligned} D_t A(x, t) \xi \cdot \xi &\geq \mathcal{F}|\xi|^2, && \text{for almost every } x \in \Omega, \\ &&& \text{for every } t \in [\lambda^{-1}, \lambda], \xi \in \mathbb{R}^n \end{aligned} \quad (3.11)$$

used in [3] to define the class \mathcal{H} .

LEMMA 3.2. *If μ_s, B satisfy conditions (3.4), (3.5) respectively, then the matrix $K(x, t)$ given by (3.2) belongs to the class \mathcal{H}' with \mathcal{E} being a positive constant depending on n, λ and E .*

Proof of Lemma 3.2. Notice that if μ_s and B satisfy (3.4), (3.5) respectively, then

$$K(x, t) \in L^\infty(\Omega). \quad (3.12)$$

We also have

$$D_t K(x, t) = -nK^2(x, t) \quad (3.13)$$

$$D_x K(x, t) = nK(x, t) [(D_x B)\mu_s - (I - B)D_x \mu_s] K(x, t) \quad (3.14)$$

$$D_x D_t K(x, t) = -2n^2 K^2(x, t) [(D_x B)\mu_s - (I - B)D_x \mu_s] K(x, t) \quad (3.15)$$

and by combining (3.12) together with (3.13)-(3.15) we obtain that $K(x, t)$ satisfies conditions (3.6)- (3.8) with $p = \infty$. Note that

$$\begin{aligned} \operatorname{Re}\{K(x, t)\} &= \frac{1}{n} \left[(tI + (I - B)\mu_s)^2 + k^2 I \right]^{-1} (tI + (I - B)\mu_s) \\ \operatorname{Im}\{K(x, t)\} &= \frac{k}{n} \left[(tI + (I - B)\mu_s)^2 + k^2 I \right]^{-1} \end{aligned}$$

i.e. $\operatorname{Re}\{K(x, t)\}$ and $\operatorname{Im}\{K(x, t)\}$ are both bounded and positive definite matrices which proves (3.9). (3.10) is a straightforward consequence of (3.13) and the fact that $K(x, t)$ is positive definite. \blacksquare

3.2 Boundary measurements

It is common practise in optical tomography to set the boundary measurements to be given by the so-called *Robin-to-Robin* map which provides the relation between the input and the output fluxes of the radiation through the object. We recall its definition below.

DEFINITION 3.3. *The Robin-to-Robin map (associated to (3.1)) is the map*

$$\begin{aligned} \Upsilon_K &: H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^{-\frac{1}{2}}(\partial\Omega) \\ \Phi_- &\mapsto \left(\gamma(n)\varphi + \frac{1}{2}K\nabla\varphi \cdot \nu \right) \Big|_{\partial\Omega} \end{aligned} \quad (3.16)$$

where $\varphi \in H^1(\Omega)$ is the unique solution to

$$\begin{cases} \nabla \cdot \left(K(\mu_a, \mu_s)\nabla\varphi \right) - (\mu_a - ik)\varphi = 0 & \text{in } \Omega \\ (\gamma(n)\varphi - \frac{1}{2}K\nabla\varphi \cdot \nu) \Big|_{\partial\Omega} = \Phi_- \end{cases}$$

Here ν is the unit outer normal on $\partial\Omega$ and K is given by

$$K = \frac{1}{n} \left((\mu_a - ik)I + (I - B)\mu_s \right)^{-1}.$$

Prescribing the Robin-to-Robin map (3.16) is mathematically equivalent to prescribing the so-called Neumann-to-Dirichlet map (associated to (3.1)). The traditionally used Dirichlet-to-Neumann map, Λ_K , inverse of N_K , will be considered instead for sake of simplicity. Denoting by $\langle \cdot, \cdot \rangle$ the $L^2(\partial\Omega)$ -pairing between $H^{\frac{1}{2}}(\partial\Omega)$ and its dual $H^{-\frac{1}{2}}(\partial\Omega)$, we recall that Λ_K is the map

$$\Lambda_K : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega) \quad (3.17)$$

defined by

$$\langle \Lambda_K\phi, \psi \rangle = \int_{\Omega} \left[K(x)\nabla u(x) \cdot \nabla\phi(x) - (\mu_a - ik)u\phi \right] dx \quad (3.18)$$

for any $\phi, \psi \in H^1(\Omega)$, where $u \in H^1(\Omega)$ is the solution to

$$\begin{cases} \nabla \cdot \left(K(\mu_a, \mu_s)\nabla\varphi \right) - (\mu_a - ik)\varphi = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = \phi. \end{cases}$$

We shall denote by $\| \cdot \|_{\star}$ the norm of the Banach space of bounded linear operators between $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$.

4 Main results

THEOREM 4.1. (Lipschitz stability of boundary values). *Given $p > n$, let Ω be a bounded domain with Lipschitz boundary with constants L, r, h . Let μ_{s_1}, μ_{s_2} satisfy (3.4), B satisfy (3.5) and moreover μ_{a_1}, μ_{a_2} be such that*

$$\lambda^{-1} \leq \mu_{a_1}(x), \mu_{a_2}(x) \leq \lambda, \quad \text{for every } x \in \Omega, \quad (4.1)$$

$$\| \mu_{a_1} \|_{W^{1,p}(\Omega)}, \| \mu_{a_2} \|_{W^{1,p}(\Omega)} \leq F, \quad (4.2)$$

then we have

$$\| \mu_{a_1}(x) - \mu_{a_2}(x) \|_{L^\infty(\partial\Omega)} \leq C \| \Lambda_{K_1} - \Lambda_{K_2} \|_* . \quad (4.3)$$

Here $C > 0$ is a constant depending on $n, p, L, r, h, \text{diam}(\Omega), \lambda, \mathcal{E}, \mathcal{F}, E$ and F .

Next we state our stability results for boundary values of the derivatives of the absorption coefficient.

THEOREM 4.2. (Hölder stability of derivatives at the boundary). *Let $p, \Omega, \mu_{a_i}, \mu_{s_i}, i = 1, 2$ and B be as in Theorem 4.1. Given $y \in \partial\Omega$ and a neighborhood U of y in $\bar{\Omega}$, assume that for some positive integer l and some $\alpha, 0 < \alpha < 1$ we have*

$$\| \mu_{s_i} \|_{C^{l,\alpha}(\bar{U})} \leq E_l \quad \text{for } i = 1, 2; \quad (4.4)$$

$$\| B \|_{C^{l,\alpha}(\bar{U})} \leq E_l \quad (4.5)$$

$$\| \mu_{a_1} - \mu_{a_2} \|_{C^{l,\alpha}(\bar{U})} \leq E_l. \quad (4.6)$$

Then, for every neighborhood W of y in $\bar{\Omega}$ such that $\bar{W} \subset U$,

$$\| D^l(\mu_{a_1} - \mu_{a_2}) \|_{L^\infty(\partial\Omega \cap \bar{W})} \leq C \| \Lambda_{K_1} - \Lambda_{K_2} \|_*^{\delta_l \alpha}, \quad (4.7)$$

where

$$\delta_l = \prod_{j=0}^l \frac{\alpha}{\alpha + j}. \quad (4.8)$$

Here $C > 0$ is a constant which depends only on $n, p, L, r, h, \text{diam}(\Omega), \rho_0, \rho, \lambda, E, F, \mathcal{E}, \alpha, l$, and E_l .

Under a slightly weaker assumption, we can also obtain the following uniqueness result.

THEOREM 4.3. (Uniqueness at the boundary). *Let $p, \Omega, \mu_{a_i}, \mu_{s_i}, i=1,2$ and B be as in Theorem 4.1. Given $y \in \partial\Omega$ and a neighborhood U of y in $\bar{\Omega}$, assume that, for some positive integer l we have*

$$\mu_{a_1} - \mu_{a_2} \in C^l(\bar{U}). \quad (4.9)$$

If

$$\Lambda_{K_1} = \Lambda_{K_2},$$

then

$$D^j(\mu_{a_1} - \mu_{a_2}) = 0 \quad \text{on } \partial\Omega \cap \bar{U}, \quad \text{for all } j \leq l. \quad (4.10)$$

What follows is a well-known consequence of the previous theorem (see [2] and [15] for related arguments).

COROLLARY 4.4. (Uniqueness in the interior). *Let $p, \Omega, \mu_{s_i}, i = 1, 2$ and B be as in Theorem 4.1 and suppose $\mu_{a_i}, i = 1, 2$ satisfy (4.1), (4.2) with $p = \infty$. Suppose that Ω can be partitioned into a finite number of Lipschitz domains, $\{A_j\}_{j=1,\dots,N}$, such that $\mu_{a_1} - \mu_{a_2}$ is analytic on each \bar{A}_j .*

If

$$\Lambda_{K_1} = \Lambda_{K_2},$$

then we have

$$\mu_{a_1} = \mu_{a_2} \quad \text{in } \Omega. \quad (4.11)$$

5 Singular solutions for the operator $L = \nabla \cdot (K \nabla \cdot) - q$.

This section is devoted to the construction of singular solutions of an elliptic equation in divergence form with a lower extra term of order zero. The coefficients K_{ij} and q will be complex valued functions unless stated otherwise. More precisely let us consider the operator

$$L = \frac{\partial}{\partial x_i} \left(K_{ij} \frac{\partial}{\partial x_j} \right) - q \quad \text{in } B_R = \{x \in \mathbb{R}^n \mid |x| < R\} \quad (5.1)$$

where $q(x)$ is a complex valued function and the coefficient matrix $(K_{ij}(x))$ is a complex symmetric valued function which satisfies

$$\lambda^{-1} |\xi|^2 \leq |K_{ij}(x) \xi_i \xi_j| \leq \lambda |\xi|^2, \quad \text{for every } x, \xi, x \in B_R, \xi \in \mathbb{R}^n, \quad (5.2)$$

and

$$\|K_{ij}\|_{W^{1,p}(B_R)} \leq \mathcal{E}, \quad i, j = 1, \dots, n, \quad (5.3)$$

where $p > n$. We also assume that q satisfies

$$\lambda^{-1} \leq |q(x)| \leq \lambda, \quad \text{for any } x, x \in B_R. \quad (5.4)$$

THEOREM 5.1. (Complex Singular solutions). *Let L satisfy (5.1)-(5.4). For any spherical harmonic S_m of degree $m = 0, 1, 2, \dots$, there exists $u \in W_{loc}^{2,p}(B_R \setminus \{0\})$ such that*

$$Lu = 0, \text{ in } B_R \setminus \{0\}, \quad (5.5)$$

and furthermore

$$u(x) = \log |Jx| S_0 \left(\frac{Jx}{|Jx|} \right) + w(x), \quad \text{when } n = 2 \text{ and } m = 0 \quad (5.6)$$

$$u(x) = |Jx|^{2-n-m} S_m \left(\frac{Jx}{|Jx|} \right) + w(x) \quad \text{otherwise}, \quad (5.7)$$

where J is a positive definite complex symmetric matrix such that $J = \sqrt{(K_{ij}(0))^{-1}}$ and w satisfies

$$|w(x)| + |x| |Dw(x)| \leq C |x|^{2-n-m+\alpha}, \quad \text{in } B_r \setminus \{0\}, \quad (5.8)$$

$$\left(\int_{r < |x| < 2r} |D^2 w|^p \right)^{\frac{1}{p}} \leq C s^{-n-m+\alpha+\frac{n}{p}}, \quad \text{for every } r, 0 < r < R/2. \quad (5.9)$$

Here α is any number such that $0 < \alpha < 1 - \frac{n}{p}$, and C is a constant depending only on α, n, p, r, λ , and \mathcal{E} .

Before proceeding with the proof of Theorem 5.1 we recall three technical lemmas (lemma 5.2, 5.3, 5.4) we shall need later on. The proofs of these results are treated in details for example in [2] or in [9] for the case in which the operator L is simply

$$L = \nabla \cdot K \nabla, \quad (5.10)$$

with K real valued matrix function. In the present work the authors will consider the more general case

$$L = \nabla \cdot K \nabla - q, \quad (5.11)$$

with K complex valued matrix function and q complex valued functions. The extension of lemma 5.2, 5.3, 5.4 to the case (5.11) is straight forward, therefore only the key points for the transition from case (5.10) to case (5.11) in these proofs will be stressed out in what follows.

LEMMA 5.2. *Let $p > n$ and $u \in W_{loc}^{2,p}(B_R \setminus \{0\})$ be a complex valued function, such that, for some positive s ,*

$$|u(x)| \leq |x|^{2-s}, \quad \text{for any } x \in B_R \setminus \{0\} \quad (5.12)$$

$$\left(\int_{r < |x| < 2r} |Lu|^p \right)^{\frac{1}{p}} \leq Ar^{\frac{n}{p}-s}, \quad \text{for any } r, 0 < r < \frac{R}{2}. \quad (5.13)$$

Then we have

$$|Du(x)| \leq C|x|^{1-s}, \quad \text{for any } x \in B_R \setminus \{0\} \quad (5.14)$$

$$\left(\int_{r < |x| < 2r} |D^2u|^p \right)^{\frac{1}{p}} \leq Cr^{\frac{n}{p}-s} \quad \text{for any } r, \quad 0 < r < \frac{R}{4}, \quad (5.15)$$

where C is a positive constant depending only on A , n , p , λ and E .

Proof of Lemma 5.2. The proof is a consequence of the L^p interior Schauder estimate

$$\|D^2u\|_{L^p(B_{\sigma\rho})} \leq \frac{C}{(1-\sigma^2)\rho^2} (\rho^2\|Lu\|_{L^p(B_\rho)} + \|u\|_{L^p(B_\rho)}), \quad (5.16)$$

where $C = C(n, p, \lambda, E)$ is a positive constant, $0 < \sigma < 1$ and $B_\rho, B_{\sigma\rho}$ are two concentric balls such that $u \in W^{2,p}(B_\rho)$ (see [19, Lemma 5.6.1]). See [2, Proof of Lemma 2.1] or [9, Proof of Lemma 1.4] for a detailed proof of this lemma. \blacksquare

LEMMA 5.3. *Let $f \in L_{loc}^p(B_R \setminus \{0\})$ be a complex valued function satisfying*

$$\left(\int_{r < |x| < 2r} |f|^p \right)^{\frac{1}{p}} \leq Ar^{\frac{n}{p}-s}, \quad \text{for any } r, 0 < r < \frac{R}{2}, \quad (5.17)$$

with $2 < s < n < p$. Then there exists $u \in W_{loc}^{2,p}(B_R \setminus \{0\})$ satisfying

$$Lu = f, \quad \text{in } B_R \setminus \{0\} \quad (5.18)$$

and

$$|u(x)| \leq C|x|^{2-s}, \quad \text{for any } x \in B_R \setminus \{0\}, \quad (5.19)$$

where C is a positive constant depending only on A , s , n , p , R , λ and E .

DEFINITION 5.1. We shall denote solution u of (5.18) by

$$u = T_L u.$$

Proof of Lemma 5.3. The proof is based on the construction of a fundamental solution Γ of the equation $Lu = 0$ so that

$$|\Gamma(x, y)| \leq C(n, \lambda)|x - y|^{2-n}, \quad \text{for any } x \neq y. \quad (5.20)$$

Such a Γ is constructed in the real case (5.10) in [20]. A similar construction has been done for the complex constant coefficients case by F. John in his book [17, pp.69-70]. See also [1, section 4] for a brief description of this construction. See [2, Proof of Lemma 2.2] or [9, Proof of Lemma 1.6] for a complete proof of this lemma. ■

LEMMA 5.4. *Let $s > n$ be a non-integer real number. Let f be as in lemma 5.3 and satisfying (5.17) with $p > n$. Then there exists $u \in W_{loc}^{2,p}(B_R \setminus \{0\})$ satisfying*

$$\Delta u = f, \text{ in } B_R \setminus \{0\} \quad (5.21)$$

and (5.19) holds with C depending only on A, s, n, p and R .

DEFINITION 5.2. *We shall denote solution u of (5.21) by*

$$u = T_S u.$$

Proof of Lemma 5.4. See [2, Proof of Lemma 2.3] or [9, Proof of Lemma 1.7]. ■

Proof of Theorem 5.1 The proof follows the same line of [2, Proof of Theorem 1.1]. We will therefore only rephrase the key points of this proof showing how it can be adapted to the more general case treated here. We consider in $B_R \setminus \{0\}$ the harmonic

$$H(x) = |x|^{2-n-m} S_m \left(\frac{x}{|x|} \right).$$

As in [2, Proof of Theorem 1.1] the idea is to find w satisfying (5.6)-(5.9) and such that

$$Lw = -LH, \text{ in } B_R \setminus \{0\}.$$

We have

$$-LH = (\Delta - L)H = (\delta_{ij} - a_{ij}) \frac{\partial^2 H}{\partial x_i \partial x_j} - \frac{\partial a_{ij}}{\partial x_i} \frac{\partial H}{\partial x_j} - qH. \quad (5.22)$$

From [2, Proof of Theorem 1.1] we have

$$\left(\int_{r < |x| < 2r} |\delta_{ij} - a_{ij}|^p \left| \frac{\partial^2 H}{\partial x_i \partial x_j} \right|^p \right)^{\frac{1}{p}} \leq Cr^{\frac{n}{p} - n - m + \beta} \quad (5.23)$$

$$\left(\int_{r < |x| < 2r} \left| \frac{\partial a_{ij}}{\partial x_i} \right|^p \left| \frac{\partial H}{\partial x_j} \right|^p \right)^{\frac{1}{p}} \leq Cr^{\frac{n}{p} - n - m + \beta}, \quad (5.24)$$

where $\beta = 1 - \frac{n}{p}$. Here the extra lower order term $-cH$ needs to be estimated

$$\begin{aligned} \left(\int_{r < |x| < 2r} |qH|^p \right)^{\frac{1}{p}} &\leq C(\lambda, R) \left(\int_{r < |x| < 2r} |x|^{(2-n-m)p} \right)^{\frac{1}{p}} \\ &\leq C(\lambda, R) \left(\int_r^{2r} \rho^{(2-n-m)p+n-1} \right)^{\frac{1}{p}} \\ &\leq Cr^{\frac{n}{p} - n - m + \beta} \end{aligned} \quad (5.25)$$

and by combining (5.23)-(5.25) together we obtain

$$\left(\int_{r < |x| < 2r} |LH|^p \right)^{\frac{1}{p}} \leq Cr^{\frac{n}{p} - n - m + \beta}. \quad (5.26)$$

Let α be an irrational number such that $0 < \alpha < \beta$ and define

$$K = \left[\frac{m}{\alpha} \right].$$

If $w_0 = T_S(-LH)$, then we have

$$|w_0(x)| \leq C|x|^{2-n-m+\beta}, \quad \text{for any } x, x \in B_R \setminus \{0\}.$$

We define

$$w_j = \begin{cases} w_0, & j = 0 \\ T_S f, & f = (\Delta - L)w_{j-1}, \quad j = 1, \dots, K-1 \end{cases} \quad (5.27)$$

LEMMA 5.5. *For any $j = 0, \dots, K-1$ we have*

$$|w_j(x)| \leq C|x|^{2-n-m+(j+1)\alpha} \quad (5.28)$$

$$\left(\int_{r < |x| < 2r} |(\Delta - L)w_j|^p \right)^{\frac{1}{p}} \leq Cr^{\frac{n}{p} - n - m + (j+2)\alpha}. \quad (5.29)$$

Proof of Lemma 5.5. We prove (5.30) and (5.30) by induction on j . For $j = 0$ we have

$$|w_0(x)| \leq C|x|^{2-n-m+\beta} \leq C|x|^{2-n-m+\alpha}$$

and

$$\begin{aligned} \left(\int_{r<|x|<2r} |(\Delta - L)w_j|^p \right)^{\frac{1}{p}} &\leq Cr^{\frac{n}{p}-n-m+2\alpha} + C \left(\int_{r<|x|<2r} |cw_0|^p \right)^{\frac{1}{p}} \\ &\leq Cr^{\frac{n}{p}-n-m+2\alpha} + C \left(\int_{r<|x|<2r} |x|^{(2-n-m+\alpha)p} \right)^{\frac{1}{p}} \\ &\leq Cr^{\frac{n}{p}-n-m+2\alpha} + Cr^{\frac{n}{p}-n-m+\alpha} \\ &\leq Cr^{\frac{n}{p}-n-m+\alpha}. \end{aligned}$$

Suppose now that (5.30), (5.30) are true for j i.e.

$$\begin{aligned} |w_j(x)| &\leq C|x|^{2-n-m+(j+1)\alpha} \\ \left(\int_{r<|x|<2r} |(\Delta - L)w_j|^p \right)^{\frac{1}{p}} &\leq Cr^{\frac{n}{p}-n-m+(j+2)\alpha}, \end{aligned}$$

then if we define $s = n + m - (j + 2)\alpha$, we have that $s > n$ and if we take

$$w_{j+1} = T_S f, \quad \text{with } f = (\Delta - L)w_j,$$

then

$$|w_{j+1}(x)| \leq C|x|^{2-n-m+(j+2)\alpha} \quad (5.30)$$

and

$$\begin{aligned} \left(\int_{r<|x|<2r} |(\Delta - L)w_{j+1}|^p \right)^{\frac{1}{p}} &\leq Cr^{\frac{n}{p}-n-m+(j+3)\alpha} + C \left(\int_{r<|x|<2r} |cw_{j+1}|^p \right)^{\frac{1}{p}} \\ &\leq Cr^{\frac{n}{p}-n-m+(j+3)\alpha} \\ &\quad + C \left(\int_{r<|x|<2r} |x|^{(2-n-m+(j+2)\alpha)p} \right)^{\frac{1}{p}} \\ &\leq Cr^{\frac{n}{p}-n-m+(j+3)\alpha} + Cr^{\frac{n}{p}-n-m+(j+2)\alpha} \\ &\leq Cr^{\frac{n}{p}-n-m+(j+3)\alpha}, \end{aligned} \quad (5.31)$$

which conclude the proof. \blacksquare

(5.30) with $j = K - 1$ gives

$$\left(\int_{r < |x| < 2r} |(\Delta - L)w_{K-1}|^p \right)^{\frac{1}{p}} \leq Cr^{\frac{n}{p} - n - m + (K+1)\alpha}$$

and if we define $s = n + m - (K + 1)\alpha$, we have $s < n$. If we define

$$W_K = T_L f, \quad \text{with } f = (\Delta - L)w_{K-1},$$

we have

$$|W_K(x)| \leq C|x|^{2-n-m+(K+1)\alpha}, \quad \text{for any } x \in B_R \setminus \{0\}. \quad (5.32)$$

We define now like in [2, Proof of Theorem 1.1] the function w

$$w = \sum_{j=0}^{K-1} w_j + W_K. \quad (5.33)$$

$w \in W_{loc}^{2,p}(B_R \setminus \{0\})$ and satisfies

$$|w(x)| \leq C|x|^{2-n-m+\alpha} \quad \text{for any } x \in B_R \setminus \{0\},$$

moreover

$$\begin{aligned} \left(\int_{r < |x| < 2r} |Lw|^p \right)^{\frac{1}{p}} &\leq Cr^{\frac{n}{p} - n - m + \alpha} + \left(\int_{r < |x| < 2r} |qw|^p \right)^{\frac{1}{p}} \\ &\leq Cr^{\frac{n}{p} - n - m + \alpha} + C \left(\int_{r < |x| < 2r} |x|^{(2-n-m+\alpha)p} \right)^{\frac{1}{p}} \\ &\leq Cr^{\frac{n}{p} - n - m + \alpha} + Cr^{\frac{n}{p} + 2 - n - m + \alpha} \\ &\leq Cr^{\frac{n}{p} - n - m + \alpha}. \end{aligned} \quad (5.34)$$

Estimate (5.34) together with Lemma 5.2 lead to

$$|Dw(x)| \leq C|x|^{1-n-m+\alpha} \quad (5.35)$$

$$\left(\int_{r < |x| < 2r} |D^2w|^p \right)^{\frac{1}{p}} \leq Cr^{\frac{n}{p} - n - m + \alpha}, \quad (5.36)$$

which conclude the proof. \blacksquare

We shall also need the following lemma.

LEMMA 5.6. *Let the hypotheses of Theorem 5.1 be satisfied. For every $m = 1, 2, \dots$ there exists a spherical harmonic S_m of degree m such that the solution u given by Theorem 5.1 also satisfies*

$$|Du(x)| > |x|^{1-(n+m)}, \quad \text{for every } x, 0 < |x| < r_0, \quad (5.37)$$

where r_0 depends only on λ, E, p, m and R .

Proof. The proof of this lemma can be obtained along the same lines as of [2, Lemma 3.1] and [3, Section 3]. ■

6 Proofs of main results.

This section contains a detailed proof of Theorem 4.1 and the sketch of proofs of Theorems 4.2, 4.3. The proofs of the latest follow the idea of proof of Theorem 4.1 and [3, Proof of Theorem 2.2], [3, Proof of Theorem 2.3] respectively, therefore only the crucial points of the proofs will be highlighted here. The proof of Corollary 4.4 is left out since it follows the same line of [3, Proof of Theorem 2.4].

Since the boundary $\partial\Omega$ is Lipschitz, the normal unit vector field might not be defined on $\partial\Omega$. We shall therefore introduce a unitary vector field $\tilde{\nu}$ locally defined near $\partial\Omega$ such that: (i) $\tilde{\nu}$ is C^∞ smooth, (ii) $\tilde{\nu}$ is non-tangential to $\partial\Omega$ (see [3], [4]). With the same arguments used in [3, Section 3] we can state that the point $z_\sigma = x^0 + \sigma\tilde{\nu}$, where $x^0 \in \partial\Omega$, satisfies

$$C\sigma \leq d(z_\sigma, \partial M) \leq \sigma, \quad \text{for any } \sigma, \quad 0 \leq \sigma \leq \sigma^0, \quad (6.1)$$

where σ^0 and C depend only on L, r, h .

LEMMA 6.1. *If K is an $n \times n$ positive definite complex symmetric matrix, then there exists a positive constant C such that*

$$\operatorname{Re}(K\xi \cdot \bar{\xi}) \geq C|\xi|^2, \quad \text{for any } \xi, \quad \xi \in \mathbb{C}^n \quad (6.2)$$

Proof. Let K_R and K_I denote the real and imaginary parts of K and ξ_R and ξ_I the real and the imaginary parts of a complex vector ξ

respectively. We have

$$\begin{aligned}
K\xi \cdot \bar{\xi} &= (K_R + iK_I)(\xi_R + i\xi_I) \cdot (\xi_R - i\xi_I) \\
&= (K_R\xi_R - K_I\xi_I + iK_I\xi_R + iK_R\xi_I) \cdot (\xi_R - i\xi_I) \\
&= K_R\xi_R \cdot \xi_R - iK_R\xi_R \cdot \xi_I \\
&\quad - K_I\xi_I \cdot \xi_R + iK_I\xi_I \cdot \xi_I \\
&\quad + K_I\xi_R \cdot \xi_I + iK_I\xi_R \cdot \xi_R \\
&\quad + K_R\xi_I \cdot \xi_I + iK_R\xi_I \cdot \xi_R.
\end{aligned}$$

Therefore

$$\begin{aligned}
\operatorname{Re}(K\xi \cdot \bar{\xi}) &= K_R\xi_R \cdot \xi_R + K_R\xi_I \cdot \xi_I \\
&\geq C(|\xi_R|^2 + |\xi_I|^2) = C|\xi|^2,
\end{aligned}$$

for some positive constant C . \blacksquare

LEMMA 6.2. *Let K be given by (3.2) and μ_s, B satisfy conditions (3.4), (3.5) respectively. If moreover μ_a satisfies (4.1), (4.2), then*

$$K(\cdot, \mu_a(\cdot)) \in W^{1,p}(\Omega, \operatorname{Her}_n), \quad (6.3)$$

and furthermore

$$\|K(\cdot, \mu_a(\cdot))\|_{W^{1,p}(\Omega)} \leq C\mathcal{E}(1 + \|\mu_a\|_{W^{1,p}(\Omega)}), \quad (6.4)$$

where C is a positive constant depending only on λ, Ω, n and p .

Proof of Lemma 3.2. We refer to [3, Lemma 3.6] for the proof of this lemma. \blacksquare

We can proceed with the proof of Theorem 4.1.

Proof of Theorem 4.1. We start with the identity (see [10, (6.35), p.99], [16, (5.0.4), p.129])

$$\begin{aligned}
\langle (\Lambda_{K_2} - \Lambda_{K_1})u, v \rangle &= \int_{\Omega} (K(x, \mu_{a_2}) - K(x, \mu_{a_1})) \nabla u \cdot \nabla v \\
&\quad - \int_{\Omega} (\mu_{a_2} - \mu_{a_1})uv,
\end{aligned} \quad (6.5)$$

which holds for any u, v solutions to

$$\begin{aligned}
\nabla \cdot K(x, \mu_{a_1})\nabla u - (\mu_{a_1} - ik)u &= 0 \text{ in } \Omega \\
\nabla \cdot K(x, \mu_{a_2})\nabla v - (\mu_{a_2} - ik)v &= 0 \text{ in } \Omega
\end{aligned} \quad (6.6)$$

respectively. Let $x^0 \in \partial\Omega$ such that

$$(\mu_{a_1} - \mu_{a_2})(x^0) = \|\mu_{a_1} - \mu_{a_2}\|_{L^\infty(\partial M)}$$

and $x_\sigma = x^0 + \sigma\tilde{\nu}$, with $0 < \sigma \leq \{\sigma_0, \frac{r_0}{4}\}$, where σ_0 is the number fixed in (6.1) and r_0 is the number appearing in (5.37) (see [3, Proof of Theorem 2.1, p. 160]). By following the same procedure of the proof of Theorem 2.1 of [3], we fix $m = 0$ and consider the two singular solutions $u, v \in W^{2,p}(\Omega)$ found in Theorem 5.1 having a Green function type of singularity at z_σ

$$u(x) = |J_{\mu_{a_1}}(x - z_\sigma)|^{2-n} + O(|x - z_\sigma|^{2-n+\alpha}) \quad (6.7)$$

$$v(x) = |J_{\mu_{a_2}}(x - z_\sigma)|^{2-n} + O(|x - z_\sigma|^{2-n+\alpha}). \quad (6.8)$$

If $B_\rho(z_\sigma)$ is the ball with center at z_σ and radius $\rho = r^0$, then (6.5) leads to

$$\begin{aligned} \|\Lambda_{K_2} - \Lambda_{K_1}\| \|u\| \|v\| &\geq \left| \operatorname{Re} \int_{\Omega \cap B_\rho(z_\sigma)} (K(x, \mu_{a_2}) - K(x, \mu_{a_1})) \nabla u \cdot \nabla \bar{v} \right| \\ &\quad - \int_{\Omega \cap B_\rho(z_\sigma)} |\mu_{a_2} - \mu_{a_1}| |u| |v| \\ &\quad - \left| \int_{\Omega \setminus B_\rho(z_\sigma)} (K(x, \mu_{a_2}) - K(x, \mu_{a_1})) \nabla u \cdot \nabla \bar{v} \right| \\ &\quad - \int_{\Omega \setminus B_\rho(z_\sigma)} |\mu_{a_2} - \mu_{a_1}| |u| |v|. \end{aligned} \quad (6.9)$$

By combining (6.7), (6.8) with (6.9) and recalling that $K(x, \mu_{a_i})$ is Hölder continuous with exponent $\beta = 1 - \frac{n}{p}$ and $J_{\mu_{a_i}}$ is complex symmetric for

$i = 1, 2$, we obtain

$$\begin{aligned}
& \operatorname{Re} \int_{\Omega \cap B_\eta(z_\sigma)} \frac{J_{\mu_{a_2}}^2 (K(x^0, \mu_{a_2}) - K(x^0, \mu_{a_1})) J_{\mu_{a_1}}^2 (x - z_\sigma) \cdot (x - z_\sigma)}{|J_{\mu_{a_1}}(x - z_\sigma)|^n |J_{\mu_{a_2}}(x - z_\sigma)|^n} \\
& \leq C \left\{ \int_{\Omega \cap B_\rho(z_\sigma)} |x - z_\sigma|^{2-2n+\alpha} \right. \\
& + \int_{\Omega \cap B_\rho(z_\sigma)} |x - z_\sigma|^{2-2n} |x - x^0|^\beta \\
& + \int_{\Omega \setminus B_\rho(z_\sigma)} |K(x, \mu_{a_2}) - K(x, \mu_{a_1})| |x - z_\sigma|^{2-2n} \\
& + \int_{\Omega \cap B_\rho(z_\sigma)} |x - z_\sigma|^{4-2n} \\
& \left. + \int_{\Omega \setminus B_\rho(z_\sigma)} |\mu_{a_2} - \mu_{a_1}| |x - z_\sigma|^{4-2n} \right\} \\
& + \|\Lambda_{K_1} - \Lambda_{K_2}\|_* \|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \|v\|_{H^{\frac{1}{2}}(\partial\Omega)}.
\end{aligned}$$

By recalling ([3, p.161]) that

$$\begin{aligned}
|J_{\mu_{a_1}} - K(x^0, \mu_{a_1})^{-1}| &\leq C |x - z_\sigma - x^0|^\beta \leq C\sigma^\beta \\
|J_{\mu_{a_2}} - K(x^0, \mu_{a_2})^{-1}| &\leq C |x - z_\sigma - x^0|^\beta \leq C\sigma^\beta,
\end{aligned}$$

we obtain

$$\begin{aligned}
& \operatorname{Re} \left[J_{\mu_{a_2}}^2 (K(x^0, \mu_{a_2}) - K(x^0, \mu_{a_1})) J_{\mu_{a_1}}^2 (x - z_\sigma) \cdot (x - z_\sigma) \right] \\
& \geq (K(x^0, \mu_{a_1})^{-1} - K(x^0, \mu_{a_2})^{-1})(x - z_\sigma) \cdot (x - z_\sigma) \\
& + C\sigma^\beta (\mu_{a_1} - \mu_{a_2})(x^0) |x - z_\sigma|^2. \tag{6.10}
\end{aligned}$$

It is only at this stage that the monotonicity property of $K(x, t)$ is needed. Note that $D_t K(x, t)$ is negative definite in this context, it was positive

definite in [3], therefore we have

$$\begin{aligned}
& \operatorname{Re} \left[(K(x^0, \mu_{a_1})^{-1} - K(x^0, \mu_{a_2})^{-1}) (x - z_\sigma) \cdot (x - z_\sigma) \right] \\
&= \int_{\mu_{a_2}(x^0)}^{\mu_{a_1}(x^0)} \operatorname{Re} \left[D_t (K(x^0, t))^{-1} \right] (x - z_\sigma) \cdot (x - z_\sigma) dt \\
&= \int_{\mu_{a_2}(x^0)}^{\mu_{a_1}(x^0)} \operatorname{Re} \left[-K^{-1}(x^0, t) D_t K(x^0, t) K^{-1}(x^0, t) \right] (x - z_\sigma) \cdot (x - z_\sigma) dt \\
&= \int_{\mu_{a_2}(x^0)}^{\mu_{a_1}(x^0)} \operatorname{Re} \left[-D_t K(x^0, t) K^{-1}(x^0, t) (x - z_\sigma) \cdot \overline{K^{-1}(x^0, t) (x - z_\sigma)} \right] dt \\
&\geq \mathcal{E}^{-1} \int_{\mu_{a_2}(x^0)}^{\mu_{a_1}(x^0)} |K^{-1}(x^0, t) (x - z_\sigma)|^2 dt \\
&\geq C(\mathcal{E}, \lambda) (\mu_{a_1} - \mu_{a_2})(x^0) |(x - z_\sigma)|^2. \tag{6.11}
\end{aligned}$$

By combining (6.10) together with (6.11) we obtain

$$\begin{aligned}
& \operatorname{Re} \left[J_{\mu_{a_2}}^2 (A(x^0, \mu_{a_2}) - A(x^0, \mu_{a_1})) J_{\mu_{a_1}}^2 (x - z_\sigma) \cdot (x - z_\sigma) \right] \\
&\geq (C(\mathcal{E}, \lambda) + C\sigma^\beta) (\mu_{a_1} - \mu_{a_2})(x^0) |x - z_\sigma|^2. \tag{6.12}
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& \| \mu_{a_1} - \mu_{a_2} \|_{L^\infty(\partial\Omega)} \int_{\Omega \cap B_\rho(z_\sigma)} |x - z_\sigma|^2 \\
&\leq C \left\{ \| \mu_{a_1} - \mu_{a_2} \|_{L^\infty(\partial\Omega)} \sigma^\beta \int_{\Omega \cap B_\rho(z_\sigma)} |x - z_\sigma|^{2-2n} \right. \\
&\quad + \int_{\Omega \cap B_\rho(z_\sigma)} |x - z_\sigma|^{2-2n+\alpha} \\
&\quad + \int_{\Omega \cap B_\rho(z_\sigma)} |x - z_\sigma|^{2-2n} |x - x^0|^\beta \\
&\quad + \int_{\Omega \setminus B_\rho(z_\sigma)} |A(x, \mu_{a_2}) - A(x, \mu_{a_1})| |x - z_\sigma|^{2-2n} \\
&\quad + \int_{\Omega \cap B_\rho(z_\sigma)} |x - z_\sigma|^{4-2n} \\
&\quad \left. + \int_{\Omega \setminus B_\rho(z_\sigma)} |\mu_{a_2} - \mu_{a_1}| |x - z_\sigma|^{4-2n} \right\} \\
&\quad + \| \Lambda_{K_1} - \Lambda_{K_2} \|_* \| u \|_{H^{\frac{1}{2}}(\partial\Omega)} \| v \|_{H^{\frac{1}{2}}(\partial\Omega)}
\end{aligned}$$

and by estimating the above integrals and the $H^{\frac{1}{2}}(\partial\Omega)$ norms of u and v (see [2], [3]) we finally obtain

$$\begin{aligned} \|\mu_{a_1} - \mu_{a_2}\|_{L^\infty(\partial\Omega)} \sigma^{2-n} \leq & C \left\{ \|\mu_{a_1} - \mu_{a_2}\|_{L^\infty(\partial\Omega)} \sigma^{2-n+\beta} \right. \\ & + \sigma^{2-n+\alpha} + \sigma^{2-n+\beta} + C + \sigma^{4-n} \\ & \left. + \|\Lambda_{K_1} - \Lambda_{K_2}\|_* \sigma^{n-2} \right\}, \end{aligned}$$

therefore

$$\|\mu_{a_1} - \mu_{a_2}\|_{L^\infty(\partial\Omega)} \leq C \{\omega(\sigma) + \|\Lambda_{K_1} - \Lambda_{K_2}\|_*\} \quad (6.13)$$

where $\omega(\sigma) \rightarrow 0$ as $\sigma \rightarrow 0$ and from (6.13) we obtained (4.3), which concludes the proof. \blacksquare

Proof of Theorem 4.2. The main points in which this proof differs from [3, Proof of Theorem 2.2] will be highlighted here. We shall prove the following inequality (see [3, Proof of Theorem 2.2] for more details)

$$\left\| \frac{\partial^j}{\partial \tilde{\nu}^j} (\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial\Omega \cap \bar{W})} \leq C \|\Lambda_1 - \Lambda_2\|_*^{\delta_j}, \quad \text{for every } j \leq k, \quad (6.14)$$

where δ_j is given by (4.8) and $\tilde{\nu}$ is the unit vector introduced in section 6. We proceed by induction on k . Using (4.1) we have that (6.14) is satisfied when $k = 0$. Let us assume that (6.14) is true for $j = k - 1$ and prove that it is true for $j = k$ too.

Let m be a positive integer and $x^0 \in \partial\Omega \cap W$ be such that

$$(-)^k \frac{\partial^k}{\partial \tilde{\nu}^k} (\mu_{a_1} - \mu_{a_2})(x^0) = \left\| \frac{\partial^k}{\partial \tilde{\nu}^k} (\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial\Omega \cap \bar{W})}. \quad (6.15)$$

Let $z_\sigma = x^0 + \sigma \tilde{\nu}$ and ρ be the point and the positive real number respectively chosen as in [3, Proof of Theorem 2.2] so that

$$B_\rho(z_\sigma) \cap \bar{\Omega} \subset U. \quad (6.16)$$

By following the same line of [3, Proof of Theorem 2.2] we obtain that for every $x \in \bar{U}$ there exists $t(x)$, $0 < t(x) < 1$, such that

$$K(x, \mu_{a_1}) - K(x, \mu_{a_2}) = (\mu_{a_2} - \mu_{a_1})(x) \left(-D_t K(x, t)|_{t=c(x)} \right), \quad (6.17)$$

where $c(x) = a(x) + t(x)(\mu_{a_2}(x) - \mu_{a_1}(x))$. Therefore estimate (3.35) of [3, Proof of Theorem 2.2] can be replaced by

$$\operatorname{Re} \left(-D_t K(x, t)|_{t=c(x)} Du \cdot D\bar{v} \right) \geq C|x - z_\sigma|^{2-2(n+m)}, \quad (6.18)$$

for almost every $x \in B_\rho(z_\sigma) \cap \Omega$. By combining Taylor's formula

$$\begin{aligned} \left\| \frac{\partial^k}{\partial \tilde{v}^k} (\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial\Omega \cap \bar{W})} &\leq k! (\mu_{a_1} - \mu_{a_2})(x) \\ &+ C \left\{ \sum_{j=0}^{k-1} \left\| \frac{\partial^j}{\partial \tilde{v}^j} (\mu_{a_1} - \mu_{a_2}) \right\| s^j \right. \\ &\left. + s^k |x - x^0|^\alpha \right\} \end{aligned} \quad (6.19)$$

with the inequality

$$\begin{aligned} & \| \Lambda_{K_1} - \Lambda_{K_2} \|_* \|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \|v\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ & \geq \operatorname{Re} \int_{\Omega \cap B_\eta(z_\tau)} (\mu_{a_2} - \mu_{a_1})(x) (D_t(x, t)|_{t=c(x)} Du \cdot D\bar{v}) \\ & - \int_{\Omega \cap B_\rho(z_\sigma)} |(\mu_{a_1} - \mu_{a_2})(x)| |u| |v| \\ & - \int_{\Omega \setminus B_\rho(z_\sigma)} |K(x, \mu_{a_1}) - K(x, \mu_{a_2})| Du \cdot D\bar{v} \\ & - \int_{\Omega \cap B_\rho(z_\sigma)} |(\mu_{a_1} - \mu_{a_2})(x)| |u| |v| \end{aligned}$$

we obtain

$$\begin{aligned}
& \|\Lambda_{K_1} - \Lambda_{K_2}\|_* \|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \|v\|_{H^{\frac{1}{2}}(\partial\Omega)} \\
& \geq \left\| \frac{\partial^k}{\partial \tilde{\nu}^k} (\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial\Omega \cap \bar{W})} \\
& \times \int_{\Omega \cap B_\eta(z_\tau)} (d(x, \partial\Omega))^k \operatorname{Re} (-D_t K(x, t)|_{t=c(x)} Du \cdot D\bar{v}) \\
& - \sum_{j=0}^{k-1} \left\| \frac{\partial^j}{\partial \tilde{\nu}^j} (\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial\Omega \cap \bar{W})} \\
& \times \int_{\Omega \cap B_\eta(z_\tau)} (d(x, \partial\Omega))^j \operatorname{Re} (-D_t K(x, t)|_{t=c(x)} Du \cdot D\bar{v}) \\
& - \int_{\Omega \cap B_\eta(z_\tau)} (d(x, \partial\Omega))^k |x - x^0|^\alpha \operatorname{Re} (-D_t K(x, t)|_{t=c(x)} Du \cdot D\bar{v}) \\
& - \int_{\Omega \cap B_\rho(z_\sigma)} |(\mu_{a_1} - \mu_{a_2})(x)| |u| |v| \\
& - \int_{\Omega \setminus B_\rho(z_\sigma)} |K(x, \mu_{a_1}) - K(x, \mu_{a_2})| |Du| |Dv| \\
& - \int_{\Omega \setminus B_\rho(z_\sigma)} |(\mu_{a_1} - \mu_{a_2})(x)| |u| |v|. \tag{6.20}
\end{aligned}$$

By applying now estimate (6.18) we finally obtain

$$\begin{aligned}
& \left\| \frac{\partial^k}{\partial \tilde{\nu}^k} (\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial\Omega \cap \bar{W})} \int_{\Omega \cap B_\eta(z_\tau)} (d(x, \partial\Omega))^k |x - z_\sigma|^{2-2(n+m)} \\
& \leq \sum_{j=0}^{k-1} \left\| \frac{\partial^j}{\partial \tilde{\nu}^j} (\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial\Omega \cap \bar{W})} \int_{\Omega \cap B_\eta(z_\tau)} (d(x, \partial\Omega))^j |x - z_\sigma|^{2-2(n+m)} \\
& + \int_{\Omega \cap B_\eta(z_\tau)} (d(x, \partial\Omega))^k |x - x^0|^\alpha |x - z_\sigma|^{2-2(n+m)} \\
& + \int_{\Omega \cap B_\rho(z_\sigma)} |(\mu_{a_1} - \mu_{a_2})(x)| |x - z_\sigma|^{4-2(n+m)} \\
& + C \|\Lambda_{K_1} - \Lambda_{K_2}\|_* \|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \|v\|_{H^{\frac{1}{2}}(\partial\Omega)}, \tag{6.21}
\end{aligned}$$

which leads to

$$\left\| \frac{\partial^k}{\partial \tilde{\nu}^k} (\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial\Omega \cap \bar{W})} \leq C \left\{ \|\Lambda_{K_1} - \Lambda_{K_2}\|_*^{\delta_{k-1}} \sigma^{-k} + \sigma^\alpha \right\}. \tag{6.22}$$

From here on the reader can follow [3, Proof of Theorem 2.2] to conclude the proof. ■

Proof of Theorem 4.3. This proof follows the same line of [3, proof of theorem 2.2] therefore the authors only wish to highlight the main points in which this proof differs from [3, proof of theorem 2.2]. Let us recall that we only need to prove

$$\frac{\partial^j}{\partial \tilde{\nu}^j}(\mu_{a_1} - \mu_{a_2}) = 0 \quad \text{on } \partial\Omega \cap \bar{W} \quad \text{for ever } j \leq k, \quad (6.23)$$

by induction on k . Here W is and arbitrary open subset of $\bar{\Omega}$ such that $\bar{W} \subset U$ and we can choose it as in proof of theorem 4.2. When $k = 0$, (6.23) is a consequence of Theorem 4.1. Let us assume that (6.23) is true for $j \leq k - 1$ and suppose by contradiction that there exists a point $x^0 \in \partial\Omega \cap \bar{W}$ so that

$$(-1)^k \frac{\partial^k}{\partial \tilde{\nu}^k}(\mu_{a_1} - \mu_{a_2})(x^0) > 0.$$

Let $z_\sigma = x^0 + \sigma \tilde{\nu}$, $\sigma > 0$ and $\rho > 0$ be chosen as we in Theorem 4.2. By Taylor's formula we have

$$(\mu_{a_1} - \mu_{a_2})(x) \geq \frac{1}{2} (-s)^k \frac{\partial^k}{\partial \tilde{\nu}^k} (\mu_{a_1} - \mu_{a_2})(x^0), \quad \text{for every } x \in U,$$

where the representation formula $x = x^0 - s\tilde{\nu}$ is the same of (see [3, proof of theorem 2.2]). We replace equality (4.2) of [proof of theorem 2.3]AG with

$$\text{Re} \left(K(x, \mu_{a_2}) - K(x, \mu_{a_1}) \right) = \int_{\mu_{a_2}}^{\mu_{a_1}} \text{Re} \left(-D_t K(x, t) \xi \cdot \bar{\xi} \right), \quad (6.24)$$

and by the monotonicity assumption (3.10) we obtain

$$\begin{aligned} \text{Re} \left(K(x, \mu_{a_2}) - K(x, \mu_{a_1}) \right) &= \int_{\mu_{a_2}}^{\mu_{a_1}} \text{Re} \left(-D_t K(x, t) \xi \cdot \bar{\xi} \right) \\ &= (\mu_{a_1} - \mu_{a_2}) M(x), \end{aligned} \quad (6.25)$$

where the matrix M satisfies

$$M(x) \xi \cdot \bar{\xi} \geq E^{-1} |\xi|^2, \quad \text{for almost every } x \in U, \quad \text{for every } \xi \in \mathbb{C}^n$$

and therefore (see [3, proof of theorem 2.3])

$$\begin{aligned} M(x) Du \cdot D\bar{v} &\geq C_1 |x - z_\sigma|^{2-2(n+m)}, \quad \text{for almost every } x \in U, \\ &\text{for every } \xi, \xi \in \mathbb{C}^n, \end{aligned} \quad (6.26)$$

where C_1 is a positive constant. By (6.5) and (6.25) we have

$$\begin{aligned}
0 &\geq \operatorname{Re} \int_{\Omega \cap B_\rho(z_\sigma)} K(x, \mu_{a_2} - K(x, \mu_{a_1})) Du \cdot D\bar{v} - \int_{\Omega \cap B_\rho(z_\sigma)} |\mu_{a_2} - \mu_{a_1}| |u| |v| - C \\
&\geq \int_{\Omega \cap B_\rho(z_\sigma)} (\mu_{a_1} - \mu_{a_2})(x) M(x) Du \cdot D\bar{v} - \int_{\Omega \cap B_\rho(z_\sigma)} |\mu_{a_2} - \mu_{a_1}| |u| |v| - C
\end{aligned} \tag{6.27}$$

and by combining (6.26) together with (6.27)

$$\begin{aligned}
0 &\geq C_1 \int_{\Omega \cap B_\rho(z_\sigma)} (\mu_{a_1} - \mu_{a_2})(x) |x - z_\sigma|^{2-2(n+m)} \\
&\quad - C_2 \int_{\Omega \cap B_\rho(z_\sigma)} |\mu_{a_2} - \mu_{a_1}| |x - z_\sigma|^{4-2(n+m)} - C \\
&= \int_{\Omega \cap B_\rho(z_\sigma)} (\mu_{a_1} - \mu_{a_2})(x) |x - z_\sigma|^{2-2(n+m)} \left(C_1 - C_2 |x - z_\sigma|^2 \right) \\
&\quad - C,
\end{aligned} \tag{6.28}$$

where C_2 and C are positive constants. By reducing ρ we can make

$$|x - z_\sigma| < \frac{1}{\sqrt{2}} \frac{C_2}{C_1}$$

and with the above choice, (6.28) leads to

$$0 \geq \frac{1}{2} C_2 \int_{\Omega \cap B_\rho(z_\sigma)} (\mu_{a_1} - \mu_{a_2})(x) |x - z_\sigma|^{2-2(n+m)} - C \tag{6.29}$$

and therefore

$$\frac{(-1)^k}{k!} \frac{\partial^k}{\partial \tilde{\nu}^k} (\mu_{a_1} - \mu_{a_2})(x^0) \leq C \sigma^{n+2m-2-k}, \tag{6.30}$$

which leads to a contradiction if we let $\sigma \rightarrow 0$. ■

Proof of Corollary 4.4. After noticing that $K \in W^{1, \infty}(\Omega \times [\lambda^{-1}, \lambda], \operatorname{Her}_n)$, the reader can refer to [3, Proof of Theorem 2.4] or [9, Proof of Theorem 2.4, p.47] for the proof of this result. ■

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