Geometric structure in the tempered dual of SL(N)

Chao, Kuok Fai

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GEOMETRIC STRUCTURE IN THE TEMPERED DUAL OF SL(N)

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Kuok Fai Chao
School of Mathematics

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Kuok Fai Chao
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Let F be a nonarchimedean local field with $char(F) = 0$. We prove the ABP-conjecture for $\mathrm{SL}_2(F)$ and part(3) of the conjecture for $\mathrm{SL}_3(F)$ and the toral part of $\mathrm{SL}_4(F)$ with $F = \mathbb{Q}_p$, $p \geq 3$.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

Let $\mathbf{G}$ be a connected reductive group over a nonarchimedean field F of $char(F) = 0$. We denote G the group of F-points of $\mathbf{G}$.

In [1], [2], Aubert, Baum and Plymen propose a geometric conjecture for the smooth dual of G. Part (3) of the conjecture concerns the tempered dual of G.

This conjecture can be divided into two parts: extended quotient and smooth dual (the set of equivalent classes of smooth representations). This means we have to figure out the problem of reducibility of induced representations. Hence, we will apply the important result in [14] to simplify the computation of R-group. We emphasis that such result we apply in the thesis is only valid over a nonarchimedean field F with $char = 0$.

In this thesis, we will give a detailed proof for $\mathrm{SL}_2(F)$ with $F = \mathbb{Q}_p$. We prove part (3) of the conjecture for $\mathrm{SL}_3(F)$; We prove part (3) of the conjecture for the toral part in the tempered dual of $\mathrm{SL}_4(F)$ with $F = \mathbb{Q}_p$, $p \neq 2$.

In chapter two, we discuss the basics of p-adic fields, linear algebraic groups and the representation theory of p-adic groups.

In chapter three, we will introduce the geometric conjecture for reductive p-adic

groups. In fact, we focus on part (3) of the conjecture for the case $\mathrm{SL}_3(F)$ and $\mathrm{SL}_4(F)$.

In chapter four, we give a detailed proof for $\mathrm{SL}_2(F)$ based on the result in [21] by Plymen.

In chapter five, we will figure out the representations of $\mathrm{SL}_3(F)$ and explain the relation to the conjecture.

In chapter six, we study the toral part of the tempered dual of $\mathrm{SL}_4(\mathbb{Q}_p)$, $p \neq 2$ and give a proof of part (3) of the conjecture. We also discuss the R-group in some special cases.

Aubert-Baum-Plymen have conjectured (unpublished) that K-theory of the reduced C^*-algebra of reductive p-adic group G can be computed from the topological K-theory of the compact extended quotients which features in their conjecture [1], [2]. In that case, the toral part of the K-theory of $C_r^*\mathrm{SL}_4(\mathbb{Q}_p)$ for $p \neq 2$, the reduced C^*-algebra of $\mathrm{SL}_4(\mathbb{Q}_p)$, can be computed from the main result in this thesis. This K-theory would appear to be otherwise intractable.

Chapter 2

Preliminaries

In this chapter we will give the background and the basic concepts which will be used throughout this thesis.

2.1 Local fields

In this section, we will give an account of results on the structure of local fields. A locally compact non-discrete field is called **local field**.

First of all, we introduce the concept of valuation. In fact, the valuation on a field is the generalization of the absolute value on the real number field $\mathbb{R}$ or the complex number filed $\mathbb{C}$.

Definition 2.1.1. *Let F be a field. A **valuation** on F is a map $|\ |{:}F \to \mathbb{R}$ satisfying*

1. $|x| \geq 0$, $|x| = 0$ *if and only if* $x = 0$;
2. $|x + y| \leq |x| + |y|$ *(the triangle inequality);*
3. $|xy| = |x||y|$.

for all $x, y \in F$.

If the valuation $|\ |$ on field F satisfies the strong triangle inequality

$$|x+y| \leq \max(|x|, |y|)$$

instead of the triangle inequality, we call such a valuation **nonarchimedean** and field F **nonarchimedean field**.

Let p be a prime number. Write $x \in \mathbb{Q}^\times$ as $x = p^\alpha \frac{u}{v}$, where $u, v, \alpha \in \mathbb{Z}$ and where $p \nmid uv$. Then we set

$$|x|_p = p^{-\alpha}$$

We also put $|0|_P = 0$. This absolute value satisfies the properties: the multiplicativity, $|x|_p|y_p| = |xy|_p$ and the strong triangle inequality:

$$|x+y|_p \leq \max\{|x|_p, |y|_p\}$$

It follows that $d(x, y) = |x - y|_p$ defines a metric on $\mathbb{Z}$. The ring of p-adic integers $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ with respect to this metric.

The ring of $\mathbb{Z}_p$ has (p) as a unique maximal ideal. The field of fractions $\mathbb{Q}_p$ is defined by adjoining $1/p$, i.e

$$\mathbb{Q}_p = \cup_{-\infty<i<\infty} p^i \mathbb{Z}_p$$

Denote by $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ with respect to $|\ |_p$. Then we call $\mathbb{Q}_p$ the **field of p-adic numbers**.

A character of $\mathbb{Q}_p^\times$ is called **smooth** if it is trivial on some $p^i\mathbb{Z}_p$. Moreover, it is called **unramified** if it is trivial on $\mathbb{Z}_p^\times$.

Let $\mathbb{Z}_p$ be the ring of integers of $\mathbb{Q}_p$. Then the p-adic expansion of a is of the form

$$a = a_0 + a_1 p + a_2 p^2 + \cdots$$

The prime ideal $\mathfrak{p}$ of $\mathbb{Z}_p$ is

$$\mathfrak{p} = p\mathbb{Z}_p = \{a \in \mathbb{Q}_p | a = a_1 p + a_2 p + \cdots\}$$

and the residue field $\mathbb{Z}_p/\mathfrak{p}$. We denote q the order of the residue field. At the same time, we normalize the valuation of F and set

$$|x|_F = (q^{-1})^{\mathrm{val}(x)}$$

for $x \in F$.

The ideals $\mathfrak{p}^n = p^n\mathbb{Z}_p$, $n \in \mathbb{N}$ form a neighborhood basis of 0 consisting of compact open sets. For $a \in \mathbb{Q}_p$, the set $\{a+p^n\mathbb{Z}_p | n \in \mathbb{N}\}$ is a neighborhood basis of a consisting of compact open sets.

We define

$$\mathcal{O} = \mathcal{O}_p = \{x \in F : |x|_F \leq 1\}$$

It is easy to see that $\mathcal{O}$ is a ring. We call it the **ring of integer** of F. Then the set

$$\mathfrak{p} = \{x \in F : |x|_F < 1\}$$

is the unique maximal ideal of $\mathcal{O}$. As above, we know this is a prime ideal. Then the quotient $\mathcal{O}/\mathfrak{p}$ is a field, called the **residue field** of F. The set

$$\mathfrak{U} = \{x \in F : |x|_F = 1\}$$

is called the ring of units.

There are two kinds of local field: archimedean and nonarchimedean. Indeed, there are two archimedean local fields: $\mathbb{R}$ and $\mathbb{C}$. A nonarchimedean local field of characteristic 0 is a finite algebraic extensions of $\mathbb{Q}_p$ for some prime number p.

2.2 Field Extensions

In this section, we will recall some facts about field extension. Let E be an algebraic extension of a field F. Then we can take E as a vector space over F. The **degree** of a field extension E/F, denoted by $[E : F]$, is the dimension of E as a vector space over F, $[E : F] = dim_F E$. For example, the complex number field $\mathbb{C}$ can be considered as a field which element is of the form $a + bi$ where $a, b \in \mathbb{R}$. Then we can see that the degree of the field extension $\mathbb{C}/\mathbb{R}$ is 2.

We denote by $Aut(E/F)$ the group of automorphisms of E which fix F. If H is a subgroup of $Aut(E/F)$, then the set

$$E^H = \{\alpha \in E : \sigma(\alpha) = \alpha, \forall \sigma \in H\}$$

is called the **fixed field** of H. This is a subfield of E. A extension E/F is called **normal** if E is a splitting field over F for a collection of polynomials $f(x) \in F[x]$.

A polynomial $f(x) \in F[x]$ is called **separable** if it have no multiple roots. The field E is called separable over F if every element in F is the root of a separable polynomial over F. In fact, every finite extension of $\mathbb{Q}$, $\mathbb{Q}_p$ and finite field is separable.

If a field extension E/F is normal and separable, we call such field extension is **Galois**. If E/F is Galois, the group $Aut(E/F)$ is called the Galois group of E/F, denoted by $Gal(E/F)$.

Now we define the norm of α from E to F to be

$$N_{E/F}(\alpha) = \prod_{\sigma} \sigma(\alpha)$$

where the products run through all the embeddings of E into an algebraic closure of F.

2.3 Classical Groups

In this section, we will introduce the general theory of classical groups. In particular, we will focus on the structure of the following groups, the general linear group $\mathrm{GL}_n(F)$ and the special linear group $\mathrm{SL}_n(F)$.

2.3.1 Algebraic varieties

A commutative ring R is said to be **Noetherian** if there is no infinite increasing chain of ideals in R, i.e., whenever $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ is can increasing chain of ideals of R, then there us a positive integer m such that $I_k = I_m$ for all $k \geq m$.

Theorem 2.3.1. *The following are equivalent:*

(1)R is a Noetherian ring.

(2)Every nonempty set of ideals of R contains a maximal element under inclusion.

(3)Every ideal of R is finitely generated.

Theorem 2.3.2. *(**Hilbert's Nullstellensatz**) If I is an ideal in $K[x_1, \cdots, x_n]$, then*

$$\mathcal{I}(\mathcal{V}(I)) = \mathrm{rad} I.$$

Theorem 2.3.3. *(**Hilbert's Basis Theorem**) If R is a Noetherian ring , then so is the polynomials ring $R[x]$.*

Let K be an algebraically closed field, such as $\mathbb{C}$ or $\overline{\mathbb{Q}}_p$. The set $L^n = K \times \cdots \times K$ will be called affine n-space and denoted $\mathbb{A}^n$. A subset $\mathcal{V}$ of $\mathbb{A}^n$ is called affine variety if $\mathcal{V}$ is the set of common zeros of a finite collection $s = \{f_\alpha\}$ of polynomials, $S \subset K[X] = K[x_1, \cdots, x_n]$. We write

$$\mathcal{V} = \mathcal{V}(S)$$

If I is the ideal generated by $\{f_\alpha\}$, then it is easy to see that I has the same set of common zeros as $\{f_\alpha\}$.

If I is an ideal of $K[X]$, we denote by $\mathcal{V}(I)$ the set of its common zeros in $\mathbb{A}^n$. $\mathcal{V}(I)$ is an affine variety. Let $U \subset \mathbb{A}^n$. We denote by $\mathcal{I}(U)$ the collection of polynomials vanishing on U. Then $\mathcal{I}(U)$ is an ideal. In fact, it is not hard to see that there are inclusions

$$U \subset \mathcal{V}(\mathcal{I}(\mathcal{U})),\ I \subset \mathcal{I}(\mathcal{V}(\mathcal{I}))$$

The **radical** of the ideal I is defined as

$$\mathrm{rad} I = \{f \in K[X] : f^r \in I \text{ for some } r \geq 0\}$$

2.3.2 Zariski topology

Definition 2.3.4. *The **Zariski topology** on affine n-space is the topology in which the closed sets are the affine varieties in $\mathbb{A}^n$.*

We see that every point in $\mathbb{A}^n$ is a closed set. But the Hausdorff separation axiom fails.

A nonempty affine variety V is called irreducible if it cannot be written as $V = V_1 \cup V_2$ where V_1 and V_2 are proper affine varieties.

Proposition 2.3.5. *1. The affine variety V is irreducible if and only if $\mathcal{I}(V)$ is a prime ideal.*

2. Every nonempty affine variety V may be written in the form

$$V = V_1 \cup V_2 \cup \cdots V_q$$

where each V_i is irreducible and $V_i \nsubseteq V_j$ for $i \neq j$.

Suppose $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ are two affine algebraic varieties. A map $\varphi : V \to W$ is called a **morphism of affine varieties** if there are polynomials $\varphi_1, \cdots, \varphi_m \in K[x_1, \cdots, x_n]$ such that

$$\varphi((a_1, \cdots, a_n)) = (\varphi_1(a_1, \cdots, a_n), \cdots, \varphi_m(a_1, \cdots, a_n))$$

for all $(a_1, \cdots, a_n) \in V$. The map $\varphi : V \to W$ is an **isomorphism of affine varieties** if there is a morphism $\psi : W \to V$ such that $\varphi \circ \psi = 1_W$ and $\psi \circ \varphi = 1_V$.

Projective n-space $\mathbb{P}^n$ is the set of equivalence classes of $K^{n+1} - (0, \cdots, 0)$ related to the equivalence relation

$$(c_0, c_1, \cdots, c_n) \sim (d_0, d_1, \cdots, d_n)$$

if and only if

$$(c_0, c_1, \cdots, c_n) = \alpha(d_0, d_1, \cdots, d_n) = (\alpha d_0, \alpha d_1, \cdots, \alpha d_n)$$

for some $\alpha \in K^\times$. Each point in $\mathbb{P}^n$ can be described by **homogeneous coordinates** $c_0, c_1, \cdots, c_n$ which are not unique but may be multiplied by any nonzero scalar in $K^\times$.

A polynomial $f \in K[X] = K[x_0, \cdots, x_n]$ is called **homogeneous** of degree d if it is a linear combination of monomials of degree F. In fact, it is equivalent to

$$f(\alpha x_0, \cdots, \alpha x_n) = \alpha^d f(x_0, \cdots, x_n)$$

where $\alpha \in K^\times$.

Now we can define projective variety by homogenous polynomials. It is similar to affine variety. A subset $\mathcal{V}$ of $\mathbb{P}^n$ ic called a **projective variety** if $\mathcal{V}$ is the set of common zeros of a finite collections $s = \{f_i\}$ of homogeneous polynomials in $K[X]$.

We write $\mathcal{V} = \mathcal{V}(S)$. let $U \in \mathbb{P}^n$. We denote by $\mathcal{I}(U)$ the collection of all polynomials vanishing on U. Then it is easy to see that $\mathcal{I}(U)$ is an ideal.

An arbitrary polynomial $f \in K[X]$ can be written in the form of

$$f = \sum f^d$$

where $f^d \in K[X]$ is a homogeneous polynomial of degree d. Then we call f^d the **homogeneous part** of f of degree d. An ideal I of $K[X]$ is called homogeneous if whenever $f \in I$ then each homogenous part f^d also lies in I.

2.3.3 Algebraic Groups

In this section, we will introduce algebraic groups. Such an algebraic group is defined by a variety (i.e. algebraic variety). Furthermore, we will focus on the structure of algebraic groups, particularly the root system. It will play an important role in the representation theory of algebraic groups(over p-adic field).

Let G be a variety endowed with the structure of a group. We define

$$\mu : G \times G \to G, \mu(ab) = \mu(a)\mu(b)$$

$$\iota : G \to G, \iota(a) = a^{-1}$$

If μ and ι are morphism of varieties, we will call G an **algebraic group**. A morphism of algebraic groups is a group homomorphism $\varphi : G \to G'$ which is a morphism of varieties. So we call that algebraic groups G and G' are **isomorphic** if there exists an isomorphism of varieties $\varphi : G \to G'$.

The following we give some examples of algebraic groups.

1. The additive group $\mathbb{G}_a$ is the affine line $\mathbb{A}^1$ with

$$\mu(a, b) = a + b.$$

2. The multiplicative group $\mathbb{G}_m$ is $F^\times$ with

$$\mu(a, b) = ab.$$

3. The general linear group $\mathrm{GL}_n(F)$ is the group of all $n \times n$ matrices with coefficients in F and non-zero determinant.

4. The group $D_n(F)$ of diagonal invertible $n \times n$ matrices

$$D_n(F) \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m$$

5. The special linear group $\mathrm{SL}_n(F)$ is the subgroup of $\mathrm{GL}_n(F)$ with determinant one.

2.3.4 Identity Component

Let G be an algebraic group . As an affine variety, we have

$$G = v_0 \cup v_1 \cup \cdots \cup v_m$$

Hence, we know that G is a disjoint union of irreducible variety. Here we denote the unique irreducible component containing e by G^0. We cell such G^0 the identity component of G.

Proposition 2.3.6. *Let G be an algebraic group. Then the identity component G^0 is a normal subgroup of finite index in G. The cosets of G^0 are irreducible component of G.*

An algebraic group G is called **connected** if $G = G^0$.

2.3.5 Lie algebra of Algebraic group

A Lie algebra $\mathfrak{g}$ is a vector space with a bilinear multiplication $[x, y]$ such that $[x, x] = 0$ and such that the Jacobi identity holds

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

For example, the Lie algebra $\mathfrak{gl}_n(F)$ of general linear group $\mathrm{GL}_n(F)$ which is the set $M_n(F)$ of all $n \times n$ matrices with the bracket operation

$$[g, h] = gh - hg$$

Let F be the complex field $\mathbb{C}$.

Example:

1. For $\mathrm{GL}_n(F)$, the Lie algebra is

$$\mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C})$$

2. For $\mathrm{SL}_n(F)$, the Lie algebra is given by

$$\mathfrak{sl}_n(\mathbb{C}) = \{x \in \mathfrak{gl}_n(\mathbb{C}) : \mathrm{tr}(x) = 0\}$$

A derivation D of ring R is a mapping $D : R \to R$ which is linear and satisfies the ordinary rule for derivatives, ie.e,

$$D(x + y) = Dx + Dy \text{ and } D(xy) = xDy + yDx$$

Let G be an algebraic group and $A = K[G]$. Let $DerA$ denote the set of all derivations of A. The $DerA$ is a Lie algebra, with respect to the $[x, y] = xy - yx$.

For $x \in G$, we define the left translation λ_x and the right translation ρ_x by

$$(\lambda_x f)(y) = f(x^{-1}y)$$

$$(\rho_x f)(y) = f(yx)$$

where f is a function on G and $y \in G$. Define

$$\mathcal{L}(G) = \{\xi \in DerA : \xi\lambda_x = \lambda_x\xi, \forall xi \in G\}$$

Then $\mathcal{L}(G)$ is a Lie algebra. We call $\mathcal{L}(G)$ the Lie algebra of the algebraic group G.

Let $g \in G$. We define an action $\mathrm{Ad}g$ on $\mathfrak{g}$ by

$$\mathrm{Ad}g(\xi) = \rho_x\xi\rho_x^{-1}$$

Then $\mathrm{Ad}g \in Aut(\mathfrak{g})$. We define

$$\mathrm{Ad} : G \to Aut(\mathfrak{g}) \text{ by } \mathrm{Ad}(g) = \mathrm{Ad}g$$

So $\mathrm{Ad} : G \to Aut(\mathfrak{g})$ is amorphism of algebraic groups and we call the adjoint representation.

Example: For $G = \mathrm{GL}_n(F)$, the Lie algebra is $\mathfrak{g} = \mathfrak{gl}_n(F)$. If $g \in G$, the adjoint representation is given by

$$\mathrm{Ad}_g(X) = gXg^{-1}$$

for all $X \in \mathfrak{g}$.

2.3.6 Root system

Let A and B be the subgroup of G, we denote (A, B) the subgroup generated by the set of commutators, i.e.

$$(A, B) = \{aba^{-1}b^{-1} : a \in A, b \in B\}$$

Definition 2.3.7. *We set $D^0 = G$, $D^n = (D^{n-1}, D^{n-1})$, then we get a series*

$$D^0 > D^1 > \cdots D^n \cdots$$

it is called derived series.

Definition 2.3.8. *The group G is called **solvable** if there exists an n such that $D^n = D^{n+1} = \cdots = \{e\}$.*

Suppose G is algebraic group. For any subset $S \subseteq G$, we set

$$Z_G(S) = \{g \in G : gsg^{-1} = s, \forall s \in S\}$$

$$N_G(S) = \{g \in G : gSg^{-1} = S\}$$

We call $Z_G(S)$, $N_G(S)$ the **centralizer** and **normalizer** of S in group G respectively.

Let G be an algebraic group. The **radical** of G, denoted by $R(G)$, is the maximal connected normal solvable subgroup of G. The subgroup of $R(G)$ consisting of all its unipotent elements is normal in G; we call it the **unipotent radical** and denote it by $R_u(G)$. Then we know $R_u(G)$ is the largest connected normal unipotent subgroup of G.

An algebraic group is called a **torus** if it is isomorphic to some $D_n(F)$, Let T be a torus, we have

$$T \cong D_n(F) \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m$$

Any morphism of algebraic groups $\lambda : \mathbb{G}_m \to T$ is called a **cocharacter** of T. We denote the set of cocharacter of T by $X_*(T)$

A connected algebraic group $G \neq \{e\}$ is called **semisimple** if $R(G)$ is trivial, i.e, $R(G) = \{e\}$. Furthermore, the group G is called **reductive** if $R_u(G) = \{e\}$. It is obvious to see that each semisimple group is reductive. For example, $\mathrm{SL}_n(F)$ is semisimple group. $\mathrm{GL}_n(F)$ and any torus are reductive group.

Definition 2.3.9. *Suppose G is an algebraic group. Then we call a homomorphism of algebraic group $\chi : G \to \mathrm{GL}_1(F)$ the **character** of G.*

We denote the set of all character of G by $X(G)$.

For connected algebraic group G, if $\mathfrak{g}$ is the Lie algebra of G and let T be the maximal torus of G, then $\mathrm{Ad} : T \to \mathrm{GL}(\mathfrak{g})$ is a representation of T on the space $\mathfrak{g}$

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \mathrm{Ad}t \cdot X = \alpha(t)X\}$$

In particular, for $0 \in X(T)$, we set $0(t) = 1$ for any $t \in T$. Thus, we have

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} : \mathrm{Ad}t \cdot X = X\}$$

Definition 2.3.10. *We denote the set of $\alpha \in X(T)$ such that $\mathfrak{g}_\alpha \neq 0$ by $\Phi(G,T)$ and call such character α the **root** of $\Phi(G,T)$.*

Then we have the decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \bigoplus_{\alpha \in \Phi(G,T)} \mathfrak{g}_\alpha$$

Definition 2.3.11. *If the subset $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ of the root system Φ satisfies:*
(1)$E = \mathbb{R}\alpha_1 \oplus \mathbb{R}\alpha_2 \oplus \cdots \oplus \mathbb{R}\alpha_l$.
(2)For $\alpha \in \Phi$, there exists $c_1, c_2, \cdot, c_l \in \mathbb{Z}^+$ such that

$$\alpha = \sum_{i=1}^{l} c_i\alpha_i \text{ or } \alpha = -\sum_{i=1}^{l} c_i\alpha_i$$

*then Δ is called the **base** of root system Ψ. We call $\alpha_i \in \Delta$ the simple root of Φ. In the decomposition $\alpha = \sum_{i=1}^{l} c_i\alpha_i$, α is called **positive root(negative root)** if all $c_i \geq 0$ ($c_i \leq 0$).*

2.3.7 Abstract root systems

Let E be a Euclidean space: a finite dimensional real vector space with an inner product $(,)$. Let $\alpha \in E$. Then we define the **reflection** s_α with respect to the hyperplane orthogonal to α is given by

$$s_\alpha = x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha$$

We define a paring $<,>$ on E by

$$< x, y >= \frac{2(x,y)}{(y,y)}$$

A abstract root system is subset Φ of E such that

1. Φ is finite, spans E and doest not contain 0.
2. If $\alpha \in \Phi$, then the only multiples of α are $\pm\alpha$.
3. If $\alpha \in \Phi$, then $s_\alpha(\Phi) \subset \Phi$.
4. If $\alpha, \beta \in \Phi$, then $< \alpha, \beta > \in \mathbb{Z}$.

We say the **rank** of Φ is the dimension of E.

Let Φ be a root system in E. Denote by W the subgroup of $\mathrm{GL}(E)$ generated by the reflections $s_\alpha, \alpha \in \Phi$. By $(R3)$, we can see that W permutes the set Φ. In particular, W is finite. We call W the Weyl group of Φ.

For $\alpha \in \Phi$, we define

$$\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$$

Then we call $\alpha^\vee$ the dual of α. The set

$$\Phi^\vee = \{\alpha^\vee | \alpha \in |\Phi\}$$

is a root system as well. We call it the dual of Φ.

Let Ψ be a quadruple $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ where X and $X^\vee$ are free abelian groups of finite type. in duality by a paring $X \times X^\vee \to \mathbb{Z}$ denoted by $<,>$, Φ and $\Phi^\vee$ are finite subset of X and $X^\vee$ and there is a bijection

$$\Phi \to \Phi^\vee,\ \alpha \to \alpha^\vee$$

2.3.8 Langlands duality

Begin with G a connected reductive algebraic group defined over a local field F. We assume that the group G is quasi-split over F. This means G has a Borel subgroup B defined over F. Now we let $T \subset B$ be a maximal torus defined over k which is contained in B and we let $X^*(T)$ denote the group of rational characters of T and $X_*(T)$ be the group of cocharacter, i.e.

$$X_*(T) = \mathrm{Hom}(F^\times, T)$$

We denote the $\Phi = \Phi(G, T)$. $\Phi^\vee$ is the corresponding coroot system to Φ. We denote by $\triangle$ the set of reduced roots corresponding B. At the same time, we denote by $\triangle^\vee$ the simple coroots of $\Phi^\vee$.

Then G determines a root datum $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ and a based root datum $(X^*(T), \triangle, X_*(T), \triangle^\vee)$

There is unique (up to isomorphism) complex reductive Lie group $G^\vee$ whose based root datum us is dual to that of F. Moreover, there is a maximal torus and Borel subgroup $T^\vee \subset B^\vee$ in $G^\vee$ such that

$$X^*(T) \cong X_*(T^\vee),\ X_*(T) \cong X^*(T^\vee)$$

For example $G = \mathrm{GL}_3(F)$, the group of characters of maximal torus T is isomorphic to the maximal torus of $T^\vee$.

In particular, the group of unramified character of T is isomorphic to the maximal torus $T^\vee$ of its Langlands dual group, i.e.

$$\Psi(T) \cong T^\vee$$

We will use this result heavily in the later chapter.

The Langlands dual group $G^\vee$ is the unique complex connected reductive group whose root datum is dual to that of G.

If G has trivial center, then $G^\vee$ is simply connected, and if G is simply connected then $G^\vee$ has trivial center.

The following table shows the examples for some split reductive groups G and their dual groups $G^\vee$.

G	$G^\vee$
$\mathrm{GL}_n(F)$	$\mathrm{GL}_n(\mathbb{C})$
$\mathrm{SL}_n(F)$	$\mathrm{PGL}_n(\mathbb{C})$
adjoint type	simply connnected
simply connnected	adjoint type

2.3.9 Borel subgroups and parabolic subgroups

Let G be a connected reductive algebraic group. A Borel subgroup of G is a maximal connected solvable subgroup. Any two Borel subgroup of G are conjugate.

Example: Let $G = \mathrm{GL}_n(F)$ and $B = T_n(F)$, the group of upper triangular matrices in G. Then B is a Borel subgroup. We call it the standard Borel subgroup.

A closed subgroup P of G is called **parabolic** if G/P a projective variety. A closed subgroup of G is parabolic if and only if it contain a Borel subgroup. In fact, a Borel subgroup is minimal parabolic subgroup. If we fix a Borel subgroup B, then parabolic subgroup containing B is called a standard parabolic subgroup. Each parabolic subgroup is conjugate to a standard parabolic subgroup.

Now we fix a maximal torus T, a Borel subgroup B of G containing T. Let $\Psi = \Psi(G, T)$ be the root system and Δ a base of Ψ. Denote by W the **Weyl group** of Ψ.

$$W \cong N_G(T)/Z_G(T) = N_G(T)/T$$

where $N_G(T)$ is the normalizer of T and $Z_G(T)$ is the centralizer of T in G.

In fact, there is an group action by Weyl group W on root system $\Phi(G,T)$. In other word, for $\alpha \in \Phi(G,T)$ and any $w \in W$, $w \cdot \alpha$ is still in $\Phi(G,T)$.

In the following we will introduce some important decompositions of reductive group.

For $w \in W$, we denote by $\tilde{w}$ be a representative of w in $N_G(T)$. Then, G is a disjoint union

$$G = \bigcup_{w \in W} B\tilde{w}B,$$

with $B\tilde{w}B = B\tilde{w'}B$ if and only if $w = w'$. We call this decomposition the **Bruhat decomposition** of G.

If P is a parabolic subgroup of G, then P has a following decomposition

$$P = MN$$

where M is a reductive group and N is the unipotent radical of P. Such decomposition is called **Levi decomposition**. We call M a Levi subgroup(factor). Assume we have a fixed the maximal torus T and the Borel subgroup B constraining T. If P is a standard parabolic subgroup, then there is a unique Levi subgroup M of P which contains T and we call it the standard Levi subgroup.

2.3.10 Haar measure

For a general locally compact group G, there exists a positive measure on G which is invariant under right translations. Such a measure will be called a **right Haar measure** on G. Similarly, if there exists a positive measure on G which is invariant under left translation, we will call such measure a **left Haar measure**. The integral of a continuous compactly supported functions F with respect to a fixed right Haar measure will be denoted by

$$\int_G f(f)dg$$

There exists a continues positive-valued character Δ_G of F such that

$$\int_G f(sg)dg = \Delta_G^{-1} \int_G g(g)dg$$

for each continues compactly supported function F on G and each $x \in G$. Here the character Δ_G is called the modular character of G. If $\Delta_G \equiv 1_G$, then we say that G is a unimodular group. For example, the discrete groups and the abelian groups are unimodular. In particular, the special linear group $\mathrm{SL}_n(F)$ is also unimodular.

2.4 Representations of p-adic groups

In this section, we will introduce the representation theory of (p-adic) groups and give a large amount to the results on the general linear group $\mathrm{GL}_n(F)$ and special linear group $\mathrm{SL}_n(F)$.

Definition 2.4.1. *Let V be a complex vector space and $GL(V)$ denote the group of invertible linear transformation of V. We say a* ***representation*** *(π, V) of group G if π is a homomorphism $\pi : G \to GL(V)$.*

Let (π, V) be a representation of G. If U is a subspace of V which is invariant for all $\pi(g)$, $g \in G$, i.e: $\pi(G)U \subseteq U$, then we call U a **subrepresentation** of V. If V does not contain such subrepresentations except $\{0\}$ or V, then one says that representation (π, V) is **irreducible**.
For two representations (π_1, V_1) and (π_2, V_2), a linear map $\varphi : V_1 \longrightarrow V_2$ is called **G-intertwining** or **morphism of G-modules** if

$$\varphi\pi_1(g) = \pi_2(g)\varphi$$

for all $g \in G$. Representations π_1 and π_2 are called **isomorphic** or **equivalent** if there exists a G-intertwining φ which is a 1-1 mapping onto. Sometimes, we call φ **intertwining operator**.

Let (π, V) be a representation of G. We say that π is a representation of finite length if there exists a finite sequence of subrepresentations

$$0 = V_0 \subset V_1 \subset \cdots \subset V_k = V$$

such that V_i/V_{i-1} is irreducible. The representation $\pi_i = V_i/V_{i-1}$ are called **irreducible subquotients** or **components** of π. The series $\pi_1, \cdots, \pi_k$ is called the **Jordan-Hölder series** of π.

Let (π, V) be a representation of G. A vector $v \in V$ is called **smooth** if there exists an open subgroups K of G such that $\pi(k)v = v$ for all $k \in K$. The vector subspace of all smooth vectors in V is denoted by V^∞. It is obvious to see that V^∞ is invariant under the action of G. The representation of G on V^∞ is denoted by π^∞. Then we will call (π^∞, V^∞) the **smooth part** of (π, V) (some reference call it **algebraic part**).

Definition 2.4.2. *A representation is called a* ***smooth representation*** *if* $V = V^\infty$.

Let K be an open compact subgroup of G, then we denote K-invariants by

$$V^K = \{v \in V | \pi(k)v = v \text{ for any } k \in K\}$$

Definition 2.4.3. *A smooth representation is called a* ***admissible representation*** *if* $\dim_{\mathbb{C}} V^K < \infty$ *for any open compact subgroup* K *of* G.

An admissible representation (π, V) of G is called **unitarizable** if there exists an inner product $(,)$ in V such that

$$(\pi(g)v, \pi(g)w) = (v, w)$$

for all $v, w \in V$ and $g \in G$.

Definition 2.4.4. *Let* $\mathrm{Irr}(G)$ *be the set of all equivalence classes of irreducible smooth representation. Then we will call* $\mathrm{Irr}(G)$ *the* ***smooth dual*** *of* G. *Similarly, let* $\tilde{G}$ *be the set of all equivalence classes of non-zero irreducible admissible representations of* G. *We call the set* $\tilde{G}$ *the* ***admissible dual*** *of* G.

Denote by V^* the dual space of V. Define a representation π^* on V^* by

$$[\pi^*(g)(v^*)](v) = v^*(\pi(g^{-1})v)$$

Let $(\tilde{\pi}, \tilde{V})$ be the smooth part of (π^*, V^*). Then $(\tilde{\pi}, \tilde{V})$ is called the **contragredient representation** of (π, V).

Let (π, V) be a smooth representation of G. Take $v \in V$ and $\tilde{v} \in \tilde{V}$. The function

$$c_{v,\tilde{v}} : g \longmapsto \tilde{v}(\pi(g)v)$$

is called a **matrix coefficient** of G.

In the following, we give a fundamental lemma in representation theory. It is called Schur's Lemma.

Theorem 2.4.5. *(**Schur's Lemma**) 1. Let (π_1, V_1) and (π_2, V_2) be irreducible representations of G and let $T : V_1 \longrightarrow V_2$ be the intertwining operator. Then either T is zero or an isomorphism.*
2. Let V be an irreducible smooth representation of G and $A : V \to V$ be a non-trivial homomorphism . Then $A = \mu I$ for some μ in $\mathbb{C}$, here I is the identity operator on V.

2.4.1 Parabolic induction

Let $P = MN$ be a parabolic subgroup of G. Take a smooth representation (σ, U) of M. Let $\mathrm{Ind}_P^G(\sigma)$ be the space of functions

$$f : G \longrightarrow U$$

such that

$$f(nmg) = \Delta_P(m)^{1/2}\sigma(m)f(g)$$

for all $m \in M$, $n \in N$, $g \in G$. Then we define the action R of G on $\mathrm{Ind}_P^G(\sigma)$ by the right translations

$$R_g f(x) = f(xg)$$

for $g, x \in G$, $f \in \mathrm{Ind}_P^G(\sigma)$. The smooth part of this representation is denoted by

$$(R, \mathrm{Ind}_P^G(\sigma))$$

The representation $\mathrm{Ind}_P^G(\sigma)$ is called **parabolically induced representation** of G from P by σ.

2.4.2 Jacquet modules and cuspidal representations

Let G be an reductive group and $P = MN$ be a standard parabolic subgroup of G. Let (π, V) be a representation of G. We set

$$V(N) = \mathrm{span}_{\mathbb{C}}\{\pi(n)v - v : n \in N, v \in V\}$$

Since M normalizes N, $V(N)$ is invariant for the action of M.

$$V_N = V/V(N)$$

We consider the natural quotient action of M on V_N

$$\pi_N(m)(v + V(N)) = \pi(m)v + V(N)$$

The M-representation (π_N, V_N) is called the **Jacquet module** of (π, V) with respect to $P = MN$. It is easy to see that

$$(\pi, V) \longmapsto (\pi_N, V_N)$$

is a functor from the category of smooth representations of G to the category of smooth representations of M.

We define $r_{M,G}(\pi) = \delta^{-1/2}\pi_U$. Then the functor

$$r_{M,G} : AlgG \to AlgM$$

is called **Jacquet functor**.

An admissible representation (π, V) of G is called **cuspidal** (or **supercuspidal**) if for any proper parabolic subgroup $P = MN$ of G and for any smooth representation σ of M, we have

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\sigma)) = 0$$

So from the definition and by the Frobenius reciprocity, (π, V) is cuspidal if and only if

$$V_N = 0$$

for any proper parabolic subgroup $P = MN$ of G. It is equivalent to $r_{M,G}(\pi) = 0$.

Proposition 2.4.6. *Let $P = MN$ be a standard parabolic subgroup of G. Then we have*

(1) The functors $i_{G,M}$ and $r_{M,G}$ are exact.

(2) The functor $r_{M,G}$ is left adjoint to $i_{G,M}$. In particular,

$$\mathrm{Hom}_G(\pi, i_{G,M}(\sigma)) = \mathrm{Hom}_M(r_{M,G}(\pi), \sigma)$$

where σ is a representation of M. (Frobenius reciprocity)

(3) If $P' = M'N'$ is a standard parabolic subgroup with $M' < M$, then

$$i_{G,M} \circ i_{M,M'} = i_{G,M'}$$

(induction in stages) and

$$r_{M',M} \circ r_{M,G} = r_{M',M}$$

(4) $\widetilde{i_{G,M}(\sigma)} = i_{G,M}(\widetilde{\sigma})$.

2.4.3 Bernstein variety

In this part, we introduce an important object in the representation theory of p-adic groups. Later, we will see that how important it will be. We denote $\mathrm{Cusp}(G)$ the union of cuspidal representations of G.

Definition 2.4.7. *A **cuspidal pair** is a pair (M, ρ) where $M \subset G$ is a Levi subgroup and $\rho \in \mathrm{Irr}_c(M)$. Two cuspidal pairs, (M, ρ),(M', ρ') are conjugate if there is a $g \in G$ so that*

$$\mathrm{Ad}_g : M \longrightarrow M' \textit{and } \mathrm{Ad}_g : \rho \longrightarrow \rho'$$

The following results by Bernstein show that how important relation between the cuspidal pair (M, ρ) and irreducible representation π of G.

Lemma 2.4.8. *Let π be an irreducible representation of G. Then there is a standard Levi subgroup M, an irreducible cuspidal representation ρ and a surjection $r_{M,G}(\pi) \twoheadrightarrow \rho$.*

Corollary 2.4.9. *If π is an irreducible representation of G, then there is a cuspidal data (M, ρ) such that*

$$\pi \hookrightarrow i_{G,M}(\rho).$$

Theorem 2.4.10. *Let k be an irreducible representation of G. Then all cuspidal pair (M, ρ), such that JH$(\rho \in r_{M,G}(k))$ are conjugate.*

Then we can have

Corollary 2.4.11. *Up to conjugacy, there exists a unique cuspidal pair with $k \hookrightarrow i_{G,M}(\rho)$.*

That means for any irreducible representation k, we can find an embedding from an irreducible representation k to a representation induced by a cuspidal pair (M, ρ).

From above, we can realize that there exists an embedding from any irreducible representation π to a subrepresentation of induced representation of G by an irreducible cuspidal representation ρ of M for some parabolic subgroup P of G.

Definition 2.4.12. *We let $\mathfrak{B}(G)$ be the set of conjugacy classes of cuspidal pair (M, ρ).*

By Bernstein[6], he proves that $\mathfrak{B}(G)$ is an algebraic variety. Recall that $\Psi(G)$ is the group of unramified characters of G. We consider the action of $\Psi(G)$ on $\mathrm{Irr}(G)$, namely $\psi : \pi \to \psi\pi$ where $\psi \in \Psi(G)$. By Langlands duality, we know the structure of $\Psi(G)$. Indeed, it is an algebraic variety.

From now, we will use $[M, \rho]_G$ to be the conjugacy class containing cuspidal pairs (M, ρ).

By the theorem, now we define **infinitesimal character** $(inf.ch)$

Definition 2.4.13. *The infinitesimal character $inf.ch$ is given by*

$$inf.ch : \mathrm{Irr}(G) \longmapsto \mathfrak{B}(G) \text{ by } \pi \longmapsto \{(M, \rho)\} | \rho \in \mathrm{JH}(r_{M,G}(\pi))$$

In fact, it is well defined by 2.4.10.

If $\mathfrak{s}$ is component of variety $\mathfrak{B}(G)$, then will denote by $\mathrm{Irr}(G)^{\mathfrak{s}}$ its pre-image under the surjection $\mathrm{Irr}(G) \to \mathfrak{B}(G)$. We can also say that $\mathrm{Irr}(G)^{\mathfrak{s}}$ is a component of $\mathrm{Irr}(G)$. From above we have

$$\mathrm{Irr}(G) = \bigsqcup_{\mathfrak{s}} \mathrm{Irr}_(G)^{\mathfrak{s}}$$

2.4.4 Square-integrable and tempered representations

In this part, we will introduce two important classes of representations: square-integrable representations and tempered representations. These are fundamental objects in representation theory.

Definition 2.4.14. *Suppose that (π, V) is an admissible representation of G which has a unitary central character. Then absolute value of each matrix coefficient*

$$|c_{v,\tilde{v}}| : g \longmapsto |c_{v,\tilde{v}}(g)|$$

is a function on $G/Z(G)$. Then the representation π is called ***square-integrable representation*** *if all functions $|c_{v,\tilde{v}}|$ are integrable functions on $G/Z(G)$, i.e*

$$\int_{G/Z(G)} |c_{v,\tilde{v}}(g)|^2 dg < \infty$$

If for an admissible representation τ of G, there exists a character χ of G such that the representation

$$\chi\tau : g \longmapsto \chi(g)\tau(g)$$

is square-integrable, then τ will be called an **essentially square-integrable representation**. Here, it is easy to see that every square-integrable representation is an essentially square-integrable representation.

The classification of discrete series representations is due to Bernstein and Zelevinsky [8],[24]. We will summarize the results below. In fact, we will apply these facts to generate special representation for proving the geometric conjecture.

Theorem 2.4.15. *Let $P = P(n_1, \cdots, n_k) = MN$ be a standard parabolic subgroup of F. Let $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ be an irreducible representation of M with every σ_i an irreducible supercuspidal representation of $\mathrm{GL}_{n_i}(F)$. The parabolically induced representation $\mathrm{Ind}_M^G(\sigma)$ is reducible if and only if there exists $1 \leq i, j \leq k$ with $i \neq j$, $n_i = n_j$ and $\sigma_i \cong \sigma_j | \cdot |_F$.*

Now we consider some specific reducible parabolically induced representations. Let $n = mk$ and σ be an irreducible cuspidal representation of $\mathrm{GL}_m(F)$. Let $P = MN$ be the standard parabolic subgroup corresponding to the partition $n = m + \cdots + m$, so $M = \mathrm{GL}_m(F) \times \cdots \times \mathrm{GL}_m(F)$ the product taken k times. From now, we denote $\nu := | \, |_F$. Let Δ denote the **segment**

$$\Delta = (\sigma, \sigma\nu, \cdots, \sigma\nu^{k-1})$$

as the representation $\sigma \otimes \sigma\nu \otimes \cdots \otimes \sigma\nu^{k-1}$ of M. Let $\mathrm{Ind}_M^G(\Delta)$ denote the corresponding parabolically induced representation of G. By 2.4.15, we know this representation is reducible. In the following, we state a theorem by Zelevinsky.

Theorem 2.4.16.

1. *For any segment Δ, the induced representation $\mathrm{Ind}_M^G(\Delta)$ has an unique irreducible quotient. This irreducible quotient will be denote $Q(\Delta)$.*

2. *For any segment Δ, the representation $Q(\Delta)$ is an essentially square integrable representation. Every essentially square-integrable representation of G is equivalent to some $Q(\Delta)$ for an uniquely determined Δ, i.e. for an uniquely determined m, k and σ.*

3. *The representation $Q(\Delta)$ is square-integrable if and only if it is unitary and if and only if $\sigma\nu^{\frac{k-1}{2}}$ is unitary.*

In the following, we take $m = 1$ and $k = n$ with the notations above. Indeed, the parabolic subgroup is Borel subgroup B. Let $\sigma = \nu^{\frac{1-k}{2}}$ as a representation of $GL_1(F)$. The corresponding $Q(\Delta)$ is called the **Steinberg representation** of $\mathrm{GL}_n(F)$. We denote the Steinberg representation for $\mathrm{GL}_n(F)$ by St_n. It is a square-integrable representation with trivial central character. The essentially square-integrable representations $Q(\Delta)$ are also called generalized Steinberg representations.

Definition 2.4.17. *An irreducible admissible representation π of G is called an irreducible* ***tempered representation*** *of G if there exist a parabolic subgroup $P = MN$ and a square-integrable representation σ of M such that π is equivalent to a subrepresentation of* $\mathrm{Ind}_P^G(\sigma)$.

In fact, every irreducible tempered representation is unitary. A discrete series representation is an irreducible unitary representation of a topological group G that is a subrepresentation of the left (or equivalently right) regular representation of G on $L^2(G)$.

There is a particular representation which is called elliptic representation. A tempered representation π is called an **elliptic representation**, if its Harish-Chandra character θ_π, viewed as a locally integrable function, does not vanish on the set of regular elliptic elements of G. The set of elliptic representations includes the discrete series.

If G is unimodular, an irreducible unitary representation ρ of G is in the discrete series if and only if one matrix coefficient $< \rho(g)v, w >$ with v, w non-zero vectors is square-integrable on G, with respect to Haar measure.

2.5 Reducibility of induced representations and R-groups

In this section, we will focus on the problem of reducibility of induced representation. In fact, this is a central problem in representation theory. If we want to exhaust all irreducible representations, we have to understand the reducibility of induced representations. We will concentrate on the group $\mathrm{SL}_n(F)$. The discovery of the R-group gives a new route to probe the reducibility of induced representation. The R-group is not easy to compute. But, luckily, the formula

$$R(\sigma) \cong \overline{L}(\pi_\sigma)/X(\pi_\sigma)$$

given by Goldberg simplifies the computation of R-groups and gives a direct way to compute. The definitions of π_σ, $\overline{L}(\pi_\sigma)$ and $X(\pi_\sigma)$ refer to the section 2.5.1. In fact, we have applied this formula a lot to figure out the reducibility of induced representation of $\mathrm{SL}_n(F)$. Remark: This formula is only valid for $\mathrm{SL}_n(F)$.

From Bruhat theory, one knows that $\mathrm{Ind}_P^G(\sigma \otimes 1)$ and $\mathrm{Ind}_{P_1}^G(\sigma \otimes 1)$ have no composition factors in common if P and P_1 are not conjugate in G.

Let σ be a tempered irreducible representation of the Levi subgroup M of P. Let $W(M) := N_G(M)/M$ denote the Weyl group of M. Let $\tilde{w} \in W(M)$. Then w is the representative for $\tilde{w}$ in $N_G(M)$. The unitary operator $\mathcal{A}(w, \sigma)$ intertwines the representations $\pi(\sigma) = \mathrm{Ind}_P^G(\sigma \otimes 1)$ and $\pi(\tilde{w}\sigma) = \mathrm{Ind}_P^G(\tilde{w}\sigma \otimes 1)$:

$$\pi(\tilde{w}\sigma)(g) \cdot \mathcal{A}(w, \sigma) = \mathcal{A}(w, \sigma) \cdot \pi(\sigma)(g)$$

We suppose $\tilde{w}\sigma \cong \sigma$. Choose an intertwining operator $T(w)$ with $T(w)(\tilde{w}\sigma) = \sigma T(w)$. Then $\mathcal{A}'(w, \sigma) = T(w)\mathcal{A}(w, \sigma)$ is a self-intertwining operator for $\mathrm{Ind}_P^G(\sigma)$. Let $W(\sigma) = \{\tilde{w} \in W(A) : \tilde{w}\sigma \cong \sigma\}$. Then $\mathcal{A}'(w, \sigma)$ will lie in the commuting algebra $\mathcal{C}(\sigma)$ where $\mathcal{C}(\sigma)$ is the set of all intertwining operators for $\mathrm{Ind}_P^G(\sigma \otimes 1)$.

Theorem 2.5.1. *Suppose (π, V) is a unitary representation of G. Then π is irreducible if and only if its commuting algebra is one-dimensional.*

We can see how important the commuting algebra is. In other word, it controls the reducibility of induced representations. In fact, the commuting algebra is spanned by $\{\mathcal{A}(w, \sigma) : \tilde{w}\sigma \cong \sigma\}$. By Harish-Chandra, we have

Theorem 2.5.2. *The collection $\{\mathcal{A}'(w, \sigma) : \tilde{w} \in W(\sigma)\}$ spans the commuting algebra.*

The representation $\pi(\sigma)$ is irreducible if and only if each intertwining operator $\mathcal{A}'(w, \sigma)$ is just a scalar whenever $\tilde{w}\sigma \cong \sigma$. This implies that if $\mathcal{A}'(w, \sigma)$ is non-scalar, then we know that $\pi(\sigma)$ is reducible. From above, we know that the reducibility of representation is determined by its commuting algebra.

The theory of the Knapp-Stein R-group tells us how to determine a basis for $\mathcal{C}(\sigma)$ from among the $\mathcal{A}'(\sigma, w)$. Let $\Phi(P, A)$ be the reduced roots of P with respect to A and let $\beta \in \Phi(P, A)$. Let A_β be the torus $\ker \beta \cap A)^0$. Let M_β denote the centralizer of A_β in G. Then M is a maximal proper Levi subgroup of M_β. Let μ_β be the Placnherel measure attached to $\mathrm{Ind}_M^{M_\beta}(\sigma)$. Since M is maxiaml proper Levi subgroup of M_β, we know that $\mu_\beta(\sigma) = 0$ if and only if $\tilde{w}\sigma \cong \sigma$, for some $\tilde{w} \neq 1$ in $W(M_\beta/A)$, and $\mathrm{Ind}_M^{M_\beta}(\sigma)$ is irreducible. We denote by $\triangle'$ the collection of $\beta \in \Phi(P, A)$ such that $\mu_\beta(\sigma) = 0$. We let

$$R = R(\sigma) = \{\tilde{w} \in W(\sigma) | \bar{w}\beta > 0, \forall \beta \in \triangle'\}$$

Let W' be the subgroup of $W(\sigma)$ generated by the reflections in roots of $\triangle'$.

Theorem 2.5.3. *For any $\sigma \in \mathcal{E}_2(M)$, we have $W(\sigma) = R \ltimes W'$. Furthermore, $W' = \{\tilde{w} \in W(\sigma | \mathcal{A}'(\sigma, w)$ is scalar$)\}$.*

Thus, we know that $\{\mathcal{A}'(\sigma, w)\}$ is a basis for $\mathcal{C}(\sigma)$. The number of irreducible constituents of $\mathrm{Ind}_M^G(\sigma)$ is the number of irreducible representations of R. It is known that $\mathcal{C}(\sigma) \cong \mathbb{C}[R]$.

2.5.1 A method

From now, we will introduce the method which we will use in the following chapter for proving the main result of this thesis. This method is based on Goldberg [14] when he computes R-group for the induced representations of $\mathrm{SL}_n(F)$.

Let F be a nonarchimedean local field of characteristic zero. We say that an element x of G is **elliptic** if its centralizer is compact, modulo the center of G. It is called **regular** if the centralizer of x in G has dimension equal to $rank(G)$. Let $\mathcal{E}_2(G)$ be the set of equivalence classes of irreducible square integrable representations of G. Hence, these representations are *discrete series* representations and have matrix coefficients that are square integrable modulo center. Let $\mathcal{E}_t(G)$ be the set of equivalence classes of irreducible *tempered representations*. We have $\mathcal{E}_2(G) \subset \mathcal{E}_t(G)$.

If $P = MN$ is a standard parabolic subgroup of G then $\mathrm{Ind}_M^G(\sigma)$ will denote the induced representation $\mathrm{Ind}_{MN}^G(\sigma \otimes 1)$ (normalized induction). The R-group attached to σ will be denoted $R(\sigma)$.

We will use the framework, notation and main result in [14]. Let $\sigma \in \mathcal{E}_2(M)$ and let $\pi_\sigma \in \mathcal{E}_2(\widetilde{M})$ with $\pi_\sigma|M \supset \sigma$. We denote $\pi_\sigma \otimes \eta \circ \det$ by $\pi_\sigma \otimes \eta$ with η a smooth (unitary) character of $F^\times$. Let $W(\sigma)$ be the isotropy subgroup of Levi subgroup $W(M)$ which fixes σ. According to [14, Lemma 2.3] we have

$$W(\sigma) = \{w \in W : w\pi_\sigma \simeq \pi_\sigma \otimes \eta \quad \text{for some} \quad \eta \in \widehat{F^\times}\}.$$

Let

$$\bar{L}(\pi_\sigma) := \{\eta \in \widehat{F^\times} | \pi_\sigma \otimes \eta \simeq w\pi_\sigma \quad \text{for some} \quad w \in W\}$$

and

$$X(\pi_\sigma) := \{\eta \in \widehat{F^\times} | \pi_\sigma \otimes \eta \simeq \pi_\sigma\}$$

By [14, Theorem 2.4], the R-group of σ is given by

$$R(\sigma) \simeq \bar{L}(\pi_\sigma)/X(\pi_\sigma). \tag{2.1}$$

We note that this formula will play the most important role in this thesis. So the reason is that this formula simplifies the computation of R-group. In general, R-group is not easy to compute. From the right hand side of (2.1), we just need to compute the groups $\bar{L}(\pi_\sigma)$ and $X(\pi_\sigma)$. It implies we can transfers the problem to π_σ directly and offers a hints to see the structure of the R-group.

2.6 The special representations of $\mathrm{SL}_n(F)$

In this section, we discuss the special representation of $\mathrm{SL}_n(F)$. Let St_n be the Steinberg representation of $\mathrm{GL}_n(F)$ and ϕ be a character of $F^\times$. Then we call representation of the form $\phi \cdot St_n$ **special representation**. In fact, the Steinberg representation of $\mathrm{SL}_n(F)$ comes from the representation of $\mathrm{GL}_n(F)$. Here, we thank Anne-Marie

Aubert for the valuable discussion about the Steinberg representation.

It is known in [13] that the character of the Steinberg representation of an arbitrary p-adic group G is the class function

$$\sum_I (-1)^{|I|} ch(\mathrm{Ind}_{P_I}^G \delta_I^{-1/2}).$$

(Here I runs over the subsets the set of positive simple roots, P_I is the corresponding standard parabolic subgroup, δ_I is the modulus character of P_I, and $ch\pi$ means the character of the representation π.)

Now we use the framework as above to describe the relation of special representations between $\mathrm{GL}_n(F)$ and $\mathrm{SL}_n(F)$. Let $\tilde{M}$ and $\tilde{P}$ be a Levi subgroup and parabolic subgroup of $\mathrm{GL}_n(F)$ respectively. We set $M := \tilde{M} \cap \mathrm{SL}_n(F)$ and $P := \tilde{P} \cap \mathrm{SL}_n(F)$. Let $\tilde{\sigma}$ be a smooth representation of $\tilde{M}$. We know that $(r_{\mathrm{SL}_n(F)}^{\mathrm{GL}_n(F)} \circ \mathrm{Ind}_{\tilde{P}}^{GL_n(F)})(\tilde{\sigma})$ is isomorphic to $\mathrm{Ind}_P^{\mathrm{SL}_n(F)} \circ r_M^{\tilde{M}}(\tilde{\sigma})$.

Then we can apply this "commutation" to the alternated sum above, and one get the restriction of the character of the Steinberg representation of $\mathrm{GL}_n(F)$ (which is also the character of the restriction) is the character of the Steinberg representation of $\mathrm{SL}_n(F)$.

Now we have a different way to look at the question:

1. The first step is to check that the Steinberg representation of $\mathrm{SL}_n(F)$ is contained in the restriction to $\mathrm{SL}_n(F)$ of Steinberg representation of $\mathrm{GL}_n(F)$.

2. The Steinberg representation (of any p-adic group G) is in the discrete series of G, and hence it is a (very special case) of what is called an elliptic representation (that is, its Harish-Chandra character, viewed as a locally integrable function, does not vanish on the set of regular elliptic elements of G).

In [14, Theorem 3.4], one can consider the particular case where the Levi subgroup M there is equal to $\mathrm{SL}_n(F)$ itself and take the representation σ to be equal to the Steinberg representation of $\mathrm{SL}_n(F)$. Then the Levi M has only one block on the diagonal, that is, the integer r of the Theorem is equal to 1. Now condition (a) of the Theorem is satisfied since $\mathrm{Ind}_M^G(\sigma) = \sigma$. Then condition (c) says that the R-group of σ is the trivial group $\{1\}$. Since the elliptic representations occurring in $\mathrm{Ind}_M^G(\sigma)$ form an L-packet, it follows that the L-packet of the Steinberg representation of $\mathrm{SL}_n(F)$ is a singleton (that is, it contains only the Steinberg representation). On the other hand, each L-packet of representations of $\mathrm{SL}_n(F)$ consists in all the irreducible representations which occur in the restriction to $\mathrm{SL}_n(F)$ of a given irreducible representation of $\mathrm{GL}_n(F)$. This says that the restriction of the Steinberg representation of $\mathrm{GL}_n(F)$ is irreducible. Hence it must be equal to the Steinberg representation of $\mathrm{SL}_n(F)$. This will be used in Chapter four, five and six.

Hence we know that the Steinberg representation of $\mathrm{SL}_n(F)$ is the same as the restriction to $\mathrm{SL}_n(F)$ of Steinberg representation of $\mathrm{GL}_n(F)$. These representations will appear in the later chapter when we consider the tempered dual.

Chapter 3

Geometric conjecture in the representation theory of reductive p-adic groups

In this chapter, we will introduce a conjectural geometric structure of representations of p-adic groups. In [1], [2] and [3], Aubert, Baum and Plymen propose a geometric conjecture related to the reductive p-adic group which we will describe below. If the conjecture is true, it implies there will exists a very beautiful relation between the extended quotients and the tempered dual of p-adic groups.

3.1 ABP geometric conjecture

Now we will define a concept which is called **extended quotient**. It was introduced by Baum and Connes [5]. In fact, the extended quotient will play an important role in the conjecture. In particular, its structure will predict the reducibility of induced representations.

3.1.1 Extended quotient

Firstly, we have to introduce the concept: extended quotient. Indeed, it is a quotient space.

Definition 3.1.1. *(**Extended Quotient**) Let X be a Hausdorff topological space. Let Γ be a finite group acting on X by a left continuous action. Let*

$$\tilde{X} = \{(x, \gamma) \in X \times \Gamma : \gamma x = x\}$$

with group action on $\tilde{X}$ given by

$$g \cdot (x, \gamma) = (gx, gxg^{-1})$$

for $g \in \Gamma$. Then the extended quotient is given by

$$X//\Gamma := \tilde{X}/\Gamma = \bigsqcup_{\gamma \in \Gamma} X^{\gamma}/\Gamma$$

where γ runs through the representatives of conjugacy classes of Γ.

3.2 The extended variety $\mathfrak{A}(G)$

Let F be a local nonarchimedean field and let G be the group of F-rational points in a connected reductive algebraic group defined over F. Let $\mathrm{Irr}(G)$ be the set of equivalence classes of irreducible smooth representations of G. For M a Levi subgroup of G, we denote by $\mathrm{Cusp}(M)$ the set of equivalence classes of irreducible supercuspidal representations of M and by $W(M)$ the group $\mathrm{N}_G(M)/M$. By a *cuspidal triple* we shall mean a triple of the form (M, σ, w), where M is a Levi subgroup of G, $\sigma \in \mathrm{Cusp}(M)$, $w \in W(M)$, and $w\sigma = \sigma$. The group G acts on the set of all cuspidal triples:

$$g \cdot M = gMg^{-1}, \quad g \cdot \sigma = {}^{g}\sigma, \quad g \cdot w = {}^{g}w.$$

Denote by $\mathfrak{A}(G)$ the quotient by G of the set of all cuspidal triples:

$$\mathfrak{A}(G) := \{(M, \sigma, w) : w\sigma = \sigma\}/G.$$

We recall standard notation of Bernstein [6]: $\Psi(M)$ is the group of unramified quasicharacters of M,

$$D := \Psi(M) \otimes \sigma \subset \mathrm{Irr}(M).$$

We have the short exact sequence $1 \to \mathcal{G} \to \Psi(M) \to D \to 1$. We emphasize that D is the quotient of the complex torus $\Psi(M)$ by $\mathcal{G}$, where $\mathcal{G}$ is a finite subgroup of $\Psi(M)$, and so has the structure of a complex torus.

Denote by $[\Psi(M)/\mathcal{G}]^w$ the w-fixed set, $w \in W(M)$. This has the structure of complex affine algebraic variety. Now hold M and w fixed, and consider $\{(M, \psi \otimes \sigma, w) : \psi \in [\Psi(M)/\mathcal{G}]^w\}$. Note that

$$w \cdot (\psi \otimes \sigma) = w \cdot \psi \otimes w\sigma = \psi \otimes w\sigma = \psi \otimes \sigma$$

so that the new triples are cuspidal triples. The map $[\Psi(M)/\mathcal{G}]^w \to \{(M, \psi \otimes \sigma, w) : \psi \in [\Psi(M)/\mathcal{G}]^w\}$ is a bijection. This defines on $\mathfrak{A}(G)$ the structure of a disjoint union of countably many complex affine algebraic varieties. When $G = \mathrm{GL}(n)$, each of these varieties is smooth, of dimension d with $1 \leq d \leq n$. In general, the varieties may be singular.

We have a map from $\mathfrak{A}(G)$ to the Bernstein variety $\Omega(G)$, induced by the map $(M, \sigma, w) \mapsto (M, \sigma)$ which sends a cuspidal triple to the corresponding cuspidal pair. We denote this by

$$\pi \colon \mathfrak{A}(G) \to \Omega(G).$$

This determines the Bernstein decomposition of $\mathfrak{A}(G)$:

$$\mathfrak{A}(G) = \bigsqcup \mathfrak{A}(G)^{\mathfrak{s}}$$

the disjoint union taken over all the components $\mathfrak{s}$ of $\Omega(G)$. We set

$$W^{\mathfrak{s}} := \{w \in W(M) \,:\, w \cdot \sigma \in D\}\,, \tag{3.1}$$

where $D = D^{\mathfrak{s}}$ is the $\Psi(M)$-orbit of σ in $\mathrm{Irr}(M)$. The map $\mathfrak{A}(G) \to \Omega(G)$, restricted to $\mathfrak{A}(G)^{\mathfrak{s}}$ is given by the standard projection

$$\mathfrak{A}(G)^{\mathfrak{s}} = D^{\mathfrak{s}}//W^{\mathfrak{s}} \to D^{\mathfrak{s}}/W^{\mathfrak{s}}.$$

We will fix a component $\mathfrak{s}$ of $\Omega(G)$ and write $D = D^{\mathfrak{s}}$, $W = W^{\mathfrak{s}}$. Let $\mathrm{Irr}(G)^{\mathfrak{s}}$ denote the $\mathfrak{s}$-component of $\mathrm{Irr}(G)$ in the Bernstein decomposition of $\mathrm{Irr}(G)$. We will give the quotient variety D/W the Zariski topology, and $\mathrm{Irr}(G)^{\mathfrak{s}}$ the Jacobson topology. We

note that irreducibility is an *open* condition, and so the set $(D/W)_{\mathrm{red}}$ of reducible points in D/W, i.e. those $(M, \psi \otimes \sigma)$ such that when parabolically induced to G, $\psi \otimes \sigma$ becomes reducible, is a sub-variety of D/W. Let q denote the cardinality of the residue field of F. Let E be the maximal compact subgroup of D.

3.2.1 Geometric conjecture

Conjecture 3.2.1. *1. There is a flat family $\mathfrak{X}_t$ of subvarieties of D/W, with $t \in \mathbb{C}^\times$, such that*

$$\mathfrak{X}_1 = \pi(D//W - D/W), \qquad \mathfrak{X}_{\sqrt{q}} = (D/W)_{\mathrm{red}}.$$

2. For each irreducible component $\mathbf{c} \subset D//W$ there is a cocharacter (i.e. a homomorphism of algebraic groups) $h_{\mathbf{c}} \colon \mathbb{C}^\times \to D$ such that, if we set $\pi_t(x) = \pi(h_{\mathbf{c}}(t) \cdot x)$ for all $x \in \mathbf{c}$, then, for each $t \in \mathbb{C}^\times$, π_t is a finite morphism of algebraic varieties with $\mathfrak{X}_t = \pi_t(D//W - D/W)$. If $\mathbf{c} = D/W$ then $h_{\mathbf{c}} = 1$.

3. There exists a continuous bijection $\mu \colon D//W \to \mathrm{Irr}(G)^{\mathfrak{s}}$ with $(inf.ch.) \circ \mu = \pi_{\sqrt{q}}$ and with $\mu(E//W) = \mathrm{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}}$.

From the conjecture above, we can see that if is true, there are some information about the reducibility of induced representations with respect to $\mathfrak{s}$. It will appear in the structure of extended quotient. In the following chapter, we will give a detail proof for the case $\mathrm{SL}_2(F)$ and $SL_3(F)$ and a partial proof for the case $\mathrm{SL}_4(F)$.

3.3 Some well-known examples and conclusion

There are some examples when Aubert, Baum and Plymen propose this geometric conjecture in [2]. They have proved that the conjecture is true for $\mathrm{GL}_n(F)$ using the Langlands correspondence map. In [3], they also prove that it is also true for the principal series of G_2. In [19], Jawdat and Plymen have given some conclusions for the special linear group $\mathrm{SL}_n(F)$. In particular, they figure out the picture of elliptic representations. In the following, we will introduce these results. The following one

is about the decomposition of extended quotient for particular isotropy group which is cyclic group.

Theorem 3.3.1. *[19, Theorem 4.2] The extended quotient* $(\mathbb{T}^{\mathfrak{s}}/\mathbb{T})//(\mathbb{Z}/n\mathbb{Z})$ *is homeomorphic to a disjoint union of compact orbifolds:*

$$(\mathbb{T}^n/\mathbb{T})//(\mathbb{Z}/n\mathbb{Z}) \cong \bigsqcup_{d|n,\omega^{n/d}=1} (n \cdot \phi(n)/d)\mathfrak{X}_d(\omega)$$

where $\mathbb{T}$ *is the unit circle in complex plane and* $\mathfrak{X}_d(\omega) := (\mathbb{T}^d/\mathbb{T})/(\mathbb{Z}/n\mathbb{Z})$.

Remark: If ℓ is a prime number, then we have the such decomposition:

$$(\mathbb{T}^\ell/T)//(\mathbb{Z}/\ell\mathbb{Z}) \cong \mathfrak{X}_\ell \sqcup \ell(\ell-1)\mathfrak{X}_1$$

We still assume $E^{\mathfrak{s}} = \mathbb{T}^n/\mathbb{T}$ and $W^{\mathfrak{s}} = \mathbb{Z}/n\mathbb{Z}$. The n-tuple $t = (z_1, \cdots, z_n) \in \mathbb{T}^n$ determines an unramified unitary character χ_t

$$\chi_t := (z_1^{\mathrm{val} \circ det}, \cdots, z_n^{\mathrm{val} \circ det}).$$

The following theorem will play important role in later chapter.

Theorem 3.3.2. *[19, Theorem 5.3] The commutative diagram:*

$$\begin{tikzcd} E^{\mathfrak{s}}//W^{\mathfrak{s}} \arrow[r, "\mu^{\mathfrak{s}}"] \arrow[dr, "\pi_{\sqrt{q}}"'] & \mathrm{Irr}^{\mathfrak{t}}(G)^{\mathfrak{s}} \arrow[d, "inf.ch"] \\ & E^{\mathfrak{s}}/W^{\mathfrak{s}} \end{tikzcd}$$

is commutative where $\mu^{\mathfrak{s}}$ *is a continuous bijection. In particular, the part 3 of geometric conjecture is true for this case. Here, the bijection* $\mu^{\mathfrak{s}}$ *is defined as follows:*

$$\mu^{\mathfrak{s}} : (t, \gamma^{rd}) \mapsto \mathrm{Ind}_M^G(\chi_t \otimes \sigma : r)$$

We also mention that all the irreducible constituents are elliptic.

Chapter 4

Geometric structure: $\mathrm{SL}_2(F)$

In this chapter, we will consider the geometric conjecture for $\mathrm{SL}_2(F)$ with $char(F) = 0$. Here, we just focus on the case $\mathbb{Q}_p$. For the proof of the conjecture, we have to exhaust all the elements in the tempered dual $\mathrm{Irr}^t(\mathrm{SL}_2(\mathbb{Q}_p))$. Most of the content in this chapter in based on the work [20] by Plymen. Our discussion will be divided into two parts, i.e., $p \neq 2$ and $p = 2$.

Let G be the group of F-rational points in a connected reductive algebraic group defined over F and let $\mathrm{Irr}(G)$ be the set of equivalence classes of irreducible smooth representation of G. For M a Levi subgroup of G, we denote by $\mathrm{Cusp}(M)$ the set of equivalence classes of irreducible supercuspidal representations of M and by $W(M)$ the group $N_G(M)/M$. By a cuspidal triple we shall means a triple of the form (M, σ, w), where M is a Levi subgroup of G, $\sigma \in \mathrm{Cusp}(M)$, $w \in W(M)$ and $w\sigma = \sigma$. The group G acts on the set of all cuspidal triples:

Let E/F be a cyclic extension of order ℓ of F. The reciprocity law in local class field theory is an isomorphism

$$F^\times / N_{E/F}(E^\times) \cong \Gamma(E/F) = \mathbb{Z}/\ell\mathbb{Z}$$

where $\Gamma(E/F)$ is the Galois group of E over F. Let now $\mu_\ell(\mathbb{C})$ be the group of lth roots of unity in $\mathbb{C}$. Choose an isomorphism $\mathbb{Z}/\ell\mathbb{Z} \cong \mu_\ell(\mathbb{C})$. This produces a

character

$$\mathcal{K}_E : F^\times \to F^\times / N_{E/F}(E^\times) \cong \mathbb{Z}/\ell\mathbb{Z} \cong \mu_\ell(\mathbb{C})$$

It is obvious to see that $\mathcal{K}_E$ is a character of order l.

For $u \in \mathbb{Z}$, we define a symbol called Legendre symbol $(\frac{u}{p})$. This comes from the classic number theory.

Definition 4.0.3. *Let p be a prime number except 2 and let $u \in {\mathbb{F}_p}^\times$. The Legendre symbol of u, denoted by $(\frac{u}{p})$, is the integer satisfying $u^{(p-1)/2} = \pm 1$.*

It is convenient to extend $(\frac{u}{p})$ to all of F_q by putting $(\frac{0}{p}) = 0$. Moreover, if $x \in \mathbb{Z}$ has for image of $x' \in F_q$, we write $(\frac{x}{p}) = (\frac{x'}{p})$. It is easy to check: $(\frac{x}{p})(\frac{y}{p}) = (\frac{xy}{p})$. In fact, the Legendre symbol is a character of order 2 (quadratic character).

The above definition is defined by Legendre originally. In fact, the Legendre symbol can be explained by another way:

$$(\frac{u}{p}) = \begin{cases} 0, & \text{if } u \equiv 0 (\text{mod } p); \\ 1, & \text{if } u \neq 0 (\text{mod } p) \text{ and for some } x,\ x^2 \equiv u (\text{mod } p)\ ; \\ -1, & \text{if there is no such } x. \end{cases}$$

Theorem 4.0.4. *There are exactly 7 distinct quadratic extensions of $\mathbb{Q}_2$ and these may be represented by*

$$\mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{5}), \mathbb{Q}_2(\sqrt{-5}), \mathbb{Q}_2(\sqrt{10}), \mathbb{Q}_2(\sqrt{-10}))$$

If $p \geq 3$, then $\mathbb{Q}_p$ has exactly 3 distinct quadratic extensions and these may be represented by

$$\mathbb{Q}_p(\sqrt{u}), \mathbb{Q}_p(\sqrt{p}), \mathbb{Q}_p(\sqrt{up})$$

where u is the smallest non-quadratic residue of p, i.e., the Legendre symbol $(\frac{u}{p})$ is -1.

For example, if $p = 5$, then there are 3 quadratic extension of $\mathbb{Q}_5(\sqrt{2})$, $\mathbb{Q}_5(\sqrt{5})$ and $\mathbb{Q}_5(\sqrt{10})$.

Corollary 4.0.5. *There are 7 quadratic characters of $\mathbb{Q}_2$ and there are 3 quadratic characters of $\mathbb{Q}_p$ where $p \geq 3$.*

4.1 Case: $F = \mathbb{Q}_p$, $p \neq 2$

First of all, we investigate the case $p \neq 2$. Since we are working on $\mathbb{Q}_p$, then, from above, we know that there are three quadratic characters of $\mathbb{Q}_p^\times$, i.e. one unramified quadratic character $\epsilon : x \longmapsto (-1)^{val(x)}$ and two ramified quadratic characters.

For $G = \mathrm{SL}_2(F)$, there are two kinds of standard Levi subgroup: the maximal torus T and group G itself.

4.1.1 $M = T$

In this section, we consider when the Levi factor M is maximal torus T. Let $\mathfrak{s} = [T, \sigma]_G$ be the Bernstein variety. In this case, $D^{\mathfrak{s}} := \{\Psi(T) \otimes \sigma\}$ where $\Psi(T) \cong T^\vee \cong \mathbb{C}^\times$ and $E^{\mathfrak{s}} := \{\Psi^{\mathfrak{t}}(T) \otimes \sigma\} \cong \mathbb{T}$.

1. $\mathfrak{s} = [T, 1]_G$

In this case, We take $D = D^{\mathfrak{s}}$ and we know $W = W^{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z}$. So we can easily get $D//W = D/W \sqcup \{-1\} \sqcup \{1\}$. Thus, we have two reducible points and denote these two points by $\{-1\}$ and $\{1\}$. We denote the component D/W, $\{-1\}$ and $\{1\}$ by c_0,c_1,c_2 respectively.

Each unramified character χ_s of the maximal torus $T \subset SL(2)$ is given by $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \longmapsto s^{val_F(x)}$ with $s \in \mathbb{C}^\times$.

The unramified unitary principal series is defined as follows:

$$\omega(s) := \mathrm{Ind}(\chi_s \otimes 1)$$

By [15], we know that $\mathrm{Ind}_P^G(\chi \otimes 1)$ is reducible when the character χ is order of 2. Then we know the representation $\omega(-1)$ is reducible:

$$\omega(-1) = \omega^+(-1) \oplus \omega^-(-1)$$

On the other hand, there is a special representation admitted in $\omega(q^{-1})$. This is Steinberg representation St_2 and actually is a quotient representation of induced representation $\omega(q^{-1})$. Recall that the norm of F is given by $|x|_F = q^{-val_F(x)}$, where q is the cardinal of residue field.

Therefore, we exhaust the tempered dual with respect to $\mathfrak{s}$. They are $\mathrm{Ind}_P^G(\chi_s \otimes 1)$ where $s \neq -1$, St_2 and $\omega^+(-1)$, $\omega^-(-1)$.

Now we define a map π_t sending $D//W$ to D/W as the diagram as follow:

$$\begin{array}{ccccccc}
D//W & = & \mathbb{C}^\times/(\mathbb{Z}/2\mathbb{Z}) & \sqcup & \{-1\} & \sqcup & \{1\} \\
\downarrow{\scriptstyle \pi_t} & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow \\
D/W & & \mathbb{C}^\times/(\mathbb{Z}/2\mathbb{Z}) & & \{-1\} & & \{t^2\}
\end{array}$$

In particular, when we interpolate $t = \sqrt{q}$. we have such map as below:

$$\begin{array}{ccccccc}
D//W & = & \mathbb{C}^\times/(\mathbb{Z}/2\mathbb{Z}) & \sqcup & \{-1\} & \sqcup & \{1\} \\
\downarrow{\scriptstyle \pi_{\sqrt{q}}} & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow \\
D/W & & \mathbb{C}^\times/(\mathbb{Z}/2\mathbb{Z}) & & \{-1\} & & \{q\}
\end{array}$$

Then we check the diagram is commutative, where the diagram is the relation between the extended quotient and the smooth dual of G:

$$\begin{tikzcd}
D//W \arrow[r, "\mu"] \arrow[dr, "\pi_{\sqrt{q}}"'] & \mathrm{Irr}(G)^{\mathfrak{s}} \arrow[d, "inf.ch"] \\
& D/W
\end{tikzcd}$$

$$\mu : D/W \sqcup \{-1\} \sqcup \{1\} \longrightarrow \mathrm{Irr}^t(G)^{\mathfrak{s}}$$

The detail of this map is as follow:

$$\begin{array}{lccc}
 & \{1\} \longmapsto & St_2 \hookrightarrow & \mathrm{Ind}_M^G(\nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}) \\
\mu : & \{-1\} \longmapsto & \omega^+(-1) \hookrightarrow & \mathrm{Ind}_M^G((-1)^{val \circ det} \otimes 1) \\
 & z \longmapsto & \mathrm{Ind}_M^G(z^{val \circ det} \otimes 1)) &
\end{array}$$

Now we investigate the image of $inf.ch$ (infinitesimal character)

$$inf.ch(\mu(1)) = [T, \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}]_G \longmapsto \{q^{-1}, q\} \in D/W$$

$$inf.ch(\mu(-1)) = [T, (-1)^{val \circ det} \otimes 1]_G \longmapsto \{-1\} \in D/W$$

$$inf.ch(\mu(z)) = [T, z^{val \circ det} \otimes 1]_G \longmapsto \{z^{-1}, z\} \in D/W$$

We set $\pi_t(x) = \pi(h_c(t) \cdot x)$. Now, there exists the cocharacter : $h_{c_0}(t) = 1$, $h_{c_1}(t) = 1$ and $h_{c_2}(t) = t^2$. Such the cocharacters make the diagram commutative. For $t \in \mathbb{C}^\times$, the flat family $\mathfrak{X}_t$ is given by $(x+1)(x^{-1}+1)(x-t^2)(x^{-1}-t^2) = 0$.

2. $\mathfrak{s} = [T, \lambda]_G$

Let λ be the Legendre Character. In this case, We take $D = D^{\mathfrak{s}}$ and we know $W = W^{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z}$. So we can easily get

$$D//W = D/W \sqcup \{-1\} \sqcup \{1\}$$

Thus, we have two reducible points in this case, due to $\Psi(T) \cong T^\vee \cong \mathbb{C}$, we denote these two points by $\{-1\}$ and $\{1\}$.

We denote the component D/W, $\{-1\}$ and $\{1\}$ by c_0,c_1,c_2 respectively.

The unramified unitary principal series is defined as follows:

$$\pi(s) := \mathrm{Ind}_M^G(\chi_s \lambda \otimes 1)$$

We note that $\pi(s)$ is reducible when the character is $\chi = 1$ or $\chi = (-1)^{val \circ det}$. Thus we also have two reducible points in this case, i.e, $\pi(-1)$ and $\pi(1)$

Then the representation $\pi(-1)$ and $\pi(1)$ are reducible and split into irreducible components:

$$\pi(-1) = \pi^+(-1) \oplus \pi^-(-1)$$

$$\pi(1) = \pi^+(1) \oplus \pi^-(1)$$

Now we exhaust the tempered dual with respect to $\mathfrak{s}$. They are $\pi(s)$ ($s \neq 1, -1$),$\pi^+(1)$, $\pi^-(1)$ and $\pi^+(-1)$, $\pi^-(-1)$.

Now we define a map π_t sending $D//W$ to D/W as the diagram as follow:

$$\begin{tikzcd}
D//W \arrow[d, "\pi_t"'] & = & \mathbb{C}^\times/(\mathbb{Z}/2\mathbb{Z}) \arrow[d, "id"'] & \sqcup & \{-1\} \arrow[d] & \sqcup & \{1\} \arrow[d] \\
D/W & & \mathbb{C}^\times/(\mathbb{Z}/2\mathbb{Z}) & & \{-1\} & & \{1\}
\end{tikzcd}$$

(where $1 \leq t \leq \sqrt{q}$)

Then we check the diagram is commutative, where the diagram is the relation between the extended quotient and the smooth dual of G:

$$\begin{tikzcd}
D//W \arrow[r, "\mu"] \arrow[dr, "\pi_{\sqrt{q}}"'] & \mathrm{Irr}(G)^{\mathfrak{s}} \arrow[d, "inf.ch"] \\
& D/W
\end{tikzcd}$$

$$\mu : D/W \sqcup \{-1\} \sqcup \{1\} \longrightarrow \mathrm{Irr}^t(G)^{\mathfrak{s}}$$

The detail of this map is as follow:

$$\{1\} \longmapsto \pi^+(1) \hookrightarrow \mathrm{Ind}_T^G(\lambda)$$

$$\mu : \qquad \{-1\} \longmapsto \pi^+(-1) \hookrightarrow \mathrm{Ind}_T^G((-1)^{val \circ det}\lambda)$$

$$z \longmapsto \mathrm{Ind}_T^G(z^{val \circ det}\lambda)$$

Now we investigate the image of $inf.ch$ (infinitesimal character)

$$inf.ch(\mu(1)) = [T, \lambda]_G \longmapsto \{1\} \in D/W$$

$$inf.ch(\mu(-1)) = [T, (-1)^{val_F}\lambda]_G \longmapsto \{-1\} \in D/W$$

$$inf.ch(\mu(z)) = [T, z^{val_F}\lambda]_G \longmapsto \{z\} \in D/W$$

We set the cocharacters: $h_c(t) = 1$ for all component. Then we set $\pi_t(x) = \pi(h_c(t) \cdot x)$ where π is the projection from $D//W$ to D/W and $x \in D//W$. It is easy to check these cocharacters make the above diagram commutative. For $t \in \mathbb{C}^\times$, the flat family $\mathfrak{X}_t$ is given by $(x+1)^2(x^{-1}+1)^2 = 0$.

3. $\mathfrak{s} = [T, \eta]_G$

Here η is not trivial character or Legendre symbol λ. It means that η is not quadractic. In this case, we take $D = D^{\mathfrak{s}}$ and we know $W^{\mathfrak{s}} = 1$. Hence, we have

$$D//W = D/W$$

On the other hand, we denote

$$\tau(s) := \mathrm{Ind}_M^G(\chi_s\eta \otimes 1)$$

Recall that $\tau(s)$ is reducible if and only if $\chi_s\eta$ is quadratic. By assumption, $\chi_s\eta$ is not of order 2. This means the induced representation $\tau(s)$ is irreducible. Hence the map μ is trivial.

Then we conclude the three cases above and get

Lemma 4.1.1. *The geometric conjecture is true for* $\mathfrak{s} = [T, \sigma]_G$.

4.1.2 $M = G$

In this section, we will consider the last case $M = G$. For $\mathfrak{s} = [G, \sigma]_G$ where σ is a discrete series of G, it is easy to see that each discrete series contributes a isolated point in tempered dual. So, this case is trivial.

Recall that the Weyl group $W(M)$ of M is given by $N_G(M)/Z_G(M)$. It is easy to get $W(M) = 1$. This implies the isotropy group $W^{\mathfrak{s}}$ is also trivial, i.e. $W^{\mathfrak{s}} = 1$.

Lemma 4.1.2. *The geometric conjecture is true for* $\mathfrak{s} = [G, \sigma]_G$.

Therefore, we conclude that

Theorem 4.1.3. *The geometric conjecture is true for* $\mathrm{SL}_2(\mathbb{Q}_p)$ *for* $p \neq 2$.

4.2 Case: $F = \mathbb{Q}_2$

From now, we consider the case when $p = 2$. In fact, by corollary 4.0.5, we know there exists 7 ramified quadratic characters of $\mathbb{Q}_2$. They come from the quadratic extension of $\mathbb{Q}_2$, i.e. $\mathbb{Q}_2(\sqrt{-1})$,$\mathbb{Q}_2(\sqrt{-2})$,$\mathbb{Q}_2(\sqrt{2})$,$\mathbb{Q}_2(\sqrt{-5})$,$\mathbb{Q}_2(\sqrt{-5})$,$\mathbb{Q}_2(\sqrt{-10})$,$\mathbb{Q}_2(\sqrt{10})$).

In the following, we denote these character:

$$\mathbb{Q}_2(\sqrt{-1}) \rightarrow \tau_1 \tag{4.1}$$

$$\mathbb{Q}_2(\sqrt{-2}) \rightarrow \tau_2 \tag{4.2}$$

$$\mathbb{Q}_2(\sqrt{2}) \rightarrow \tau_3 \tag{4.3}$$

$$\mathbb{Q}_2(\sqrt{-5}) \rightarrow \tau_4 \tag{4.4}$$

$$\mathbb{Q}_2(\sqrt{5}) \rightarrow \tau_5 \tag{4.5}$$

$$\mathbb{Q}_2(\sqrt{-10}) \rightarrow \tau_6 \tag{4.6}$$

$$\mathbb{Q}_2(\sqrt{10}) \rightarrow \tau_7 \tag{4.7}$$

In fact, they are similar. We denote τ be the quadratic character of $\mathbb{Q}_2$.

4.2.1 $M = T$

Then we consider the case $\mathfrak{s} = [T, \tau]_G$.

1. $\mathfrak{s} = [T, \tau]_G$

As above, τ is quadractic character. In this case, We have $D = D^{\mathfrak{s}}$ and we know $W = W^{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z}$. So we can easily get

$$D//W = D/W \sqcup \{-1\} \sqcup \{1\}$$

Thus, we have two reducible points in this case and denote these two points by $\{-1\}$ and $\{1\}$.

We denote the component D/W, $\{-1\}$ and $\{1\}$ by c_0,c_1,c_2 respectively.

The unramified unitary principal series is defined as follows:

$$\pi(s) := \mathrm{Ind}(\chi_s \tau \otimes 1)$$

We note that $\pi(s)$ is reducible when the character is $\chi_s = 1$ or $\chi_s = (-1)^{val \circ det}$. Thus we also have two reducible points in this case, i.e, $\pi(-1)$ and $\pi(1)$.

Then the representation $\lambda(-1)$ and $\lambda(1)$ are reducible and split into irreducible components:

$$\pi(-1) = \pi^+(-1) \oplus \pi^-(-1)$$

$$\pi(1) = \pi^+(1) \oplus \pi^-(1)$$

Now we exhaust the tempered dual with respect to $\mathfrak{s}$. They are $\pi(s)$ $(s \neq 1, -1)$,$\pi^+(1)$, $\pi^-(1)$ and $\pi^+(-1)$, $\pi^-(-1)$.

Now we define a map π_t sending $D//W$ to D/W as the diagram as follow:

$$\begin{tikzcd}
D//W \arrow[d, "\pi_t"'] & = & \mathbb{C}^\times/(\mathbb{Z}/2\mathbb{Z}) \arrow[d, "id"'] & \sqcup & \{-1\} \arrow[d] & \sqcup & \{1\} \arrow[d] \\
D/W & & \mathbb{C}^\times/(\mathbb{Z}/2\mathbb{Z}) & & \{-1\} & & \{1\}
\end{tikzcd}$$

(where $1 \leq t \leq \sqrt{q}$)

Then we check the diagram is commutative, where the diagram is the relation between the extended quotient and the smooth dual of G:

$$\begin{tikzcd}
D//W \arrow[r, "\mu"] \arrow[dr, "\pi_{\sqrt{q}}"'] & \mathrm{Irr}(G)^{\mathfrak{s}} \arrow[d, "inf.ch"] \\
& D/W
\end{tikzcd}$$

$$\mu : D/W \sqcup \{-1\} \sqcup \{1\} \longrightarrow \mathrm{Irr}^t(G)^{\mathfrak{s}}$$

The detail of this map is as follow:

$$\{1\} \longmapsto \pi^+(1) \hookrightarrow \mathrm{Ind}_T^G(\lambda)$$

$$\mu : \qquad \{-1\} \longmapsto \pi^+(-1) \hookrightarrow \mathrm{Ind}_T^G((-1)^{val \circ det}\lambda)$$

$$z \longmapsto \mathrm{Ind}_T^G(z^{val \circ det}\lambda)$$

Now we investigate the image of $inf.ch$ (infinitesimal character)

$$inf.ch(\mu(1)) = [T, \tau]_G \longmapsto \{1\} \in D/W$$

$$inf.ch(\mu(-1)) = [T, (-1)^{val \circ det}\tau]_G \longmapsto \{-1\} \in D/W$$

$$inf.ch(\mu(z)) = [T, z^{val \circ det}\tau]_G \longmapsto \{z\} \in D/W$$

We set the cocharacters $h_c(t) = 1$ and $\pi_t(x) = \pi(h_c(t) \cdot x)$ where π is the projection from $D//W$ to D/W and $x \in D//W$. It is easy to check these cocharacters make the above diagram commutative. For $t \in \mathbb{C}^\times$, the flat family $\mathfrak{X}_t$ is given by $(x+1)^2(x^{-1}+1)^2 = 0$.

The graph below shows the quadractic characters for $\mathbb{Q}_2$:

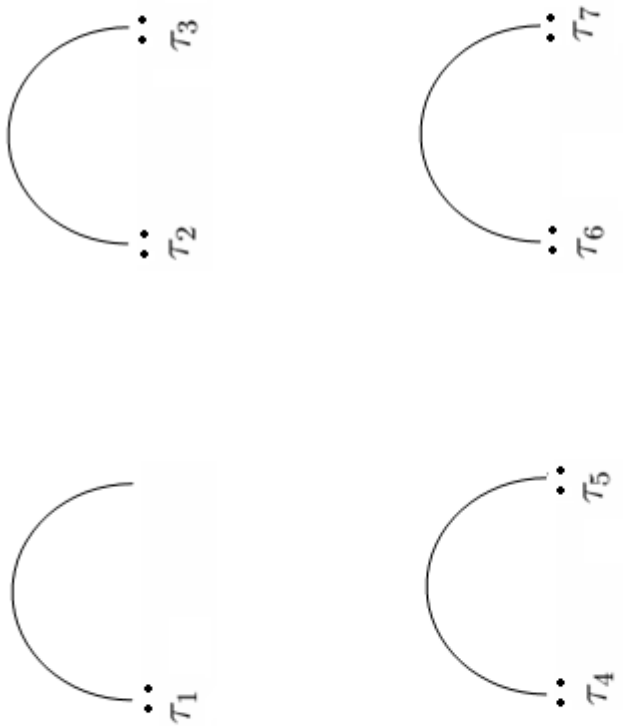

Figure 4.1: For $\mathfrak{s} = [T, \tau]_G$, $\mathrm{SL}_2(\mathbb{Q}_2)$

2. $\mathfrak{s} = [T, 1]_G$

In this case, we take $D = D^{\mathfrak{s}}$ and we know $W = W^{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z}$. So we can easily get $D//W = D/W \sqcup \{-1\} \sqcup \{1\}$. Thus, we have two reducible points in this case, due to $\Psi(T) \cong^L T \cong \mathbb{C}$, we denote these two points by $\{-1\}$ and $\{1\}$

We denote the component D/W, $\{-1\}$ and $\{1\}$ by c_0,c_1,c_2 respectively.

Each unramified quasicharacter χ_s of the maximal torus $T \subset SL(2)$ is given by $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \longmapsto s^{val_F(x)}$ with $s \in \mathbb{C}^\times$.

The unramified unitary principal series is defined as follows:

$$\omega(s) := \mathrm{Ind}(x_s \otimes 1)$$

By [15], we know that $\mathrm{Ind}_P^G(\chi \otimes 1)$ is reducible when the character χ is order of 2. Then we know the representation $\omega(-1)$ is reducible:

$$\omega(-1) = \omega^+(-1) \oplus \omega^-(-1)$$

On the other hand, there is a special representation admitted in $\omega(q^{-1})$. This is Steinberg representation St_2 and actually is a quotient representation of induced representation $\omega(q^{-1})$. Therefore, we exhaust the tempered dual with respect to $\mathfrak{s}$. They are $\mathrm{Ind}_P^G(\chi_s \otimes 1)$ where $s \neq -1$, St_2 and $\omega^+(-1)$, $\omega^-(-1)$.

Now we define a map π_t sending $D//W$ to D/W as the diagram as follow:

$$\begin{array}{ccccccc} D//W & = & \mathbb{C}^\times/(\mathbb{Z}/2\mathbb{Z}) & \sqcup & \{-1\} & \sqcup & \{1\} \\ \downarrow{\scriptstyle \pi_t} & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow \\ D/W & & \mathbb{C}^\times/(\mathbb{Z}/2\mathbb{Z}) & & \{-1\} & & \{t^2\} \end{array}$$

In particular, when we interpolate $t = \sqrt{q}$. we have such map as below:

$$\begin{array}{ccccccc} D//W & = & \mathbb{C}^\times/(\mathbb{Z}/2\mathbb{Z}) & \sqcup & \{-1\} & \sqcup & \{1\} \\ \downarrow{\scriptstyle \pi_{\sqrt{q}}} & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow \\ D/W & & \mathbb{C}^\times/(\mathbb{Z}/2\mathbb{Z}) & & \{-1\} & & \{q\} \end{array}$$

Then we check the diagram is commutative, where the diagram is the relation between the extended quotient and the smooth dual of G:

$$\begin{tikzcd} D//W \arrow[r, "\mu"] \arrow[dr, "\pi_{\sqrt{q}}"'] & \mathrm{Irr}(G)^{\mathfrak{s}} \arrow[d, "inf.ch"] \\ & D/W \end{tikzcd}$$

$$\mu : D/W \sqcup \{-1\} \sqcup \{1\} \longrightarrow \mathrm{Irr}^t(G)^{\mathfrak{s}}$$

The detail of this map is as follow:

$$\mu : \begin{array}{ccccc} \{1\} & \longmapsto & St_2 & \hookrightarrow & \mathrm{Ind}_M^G(\nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}) \\ \{-1\} & \longmapsto & w^+(-1) & \hookrightarrow & \mathrm{Ind}_M^G((-1)^{val \circ det} \otimes 1) \\ z & \longmapsto & \mathrm{Ind}_M^G(z^{val \circ det} \otimes 1)) & & \end{array}$$

Now we investigate the image of $inf.ch$ (infinitesimal character)

$$inf.ch(\mu(1)) = [T, \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}]_G \longmapsto \{q^{-1}, q\} \in D/W$$

$$inf.ch(\mu(-1)) = [T, (-1)^{val \circ det} \otimes 1]_G \longmapsto \{-1\} \in D/W$$

$$inf.ch(\mu(z)) = [T, z^{val \circ det} \otimes 1]_G \longmapsto \{z^{-1}, z\} \in D/W$$

We set $\pi_t(x) = \pi(h_c(t) \cdot x)$. Now, there exists the cocharacter : $h_{c_0}(t) = 1$, $h_{c_1}(t) = 1$ and $h_{c_2}(t) = t^2$. Such the cocharacters make the diagram commutative. For $t \in \mathbb{C}^\times$, the flat family $\mathfrak{X}_t$ is given by $(x+1)(x^{-1}+1)(x+t^2)(x^{-1}+t^2) = 0$.

3. $\mathfrak{s} = [T, \eta]_G$

Here character η is not quadractic. In this case, We take $D = D^{\mathfrak{s}}$ and we know $W^{\mathfrak{s}} = 1$. Hence, we have

$$D//W = D/W$$

On the other hand, we denote

$$\tau(s) := \mathrm{Ind}_M^G(\chi_s \eta \otimes 1)$$

Recall that $\tau(s)$ is reducible if and only if $\chi_s \eta$ is quadratic. By assumption, $\chi_s \eta$ is not of order 2. This means the induced representation $\tau(s)$ is irreducible. Hence the map μ is trivial.

Then we conclude the three cases above and get

Lemma 4.2.1. *The geometric conjecture is true for* $\mathfrak{s} = [T, \sigma]_G$.

4.2.2 $M = G$

For $\mathfrak{s} = [G, \sigma]_G$ where σ is a discrete series of G, it is easy to see that each discrete series contributes a isolated point in tempered dual. In fact, in this case, $D^{\mathfrak{s}}$ is given by $\mathbb{C}^\times/\mathbb{C}^\times \cong 1$. Hence, we can take $D^{\mathfrak{s}}$ as a point. Furthermore, the isotropy group $W^{\mathfrak{s}}$ is trivial since the Weyl group $W(M)$ of M is trivial. It implies that

$$D^{\mathfrak{s}}//W^{\mathfrak{s}} = D^{\mathfrak{s}}/W^{\mathfrak{s}}$$

It means that this case is trivial. Then we conclude that the conjecture is true for $\mathrm{SL}_2(\mathbb{Q}_2)$. Therefore, we conclude that

Theorem 4.2.2. *The geometric conjecture is true for* $\mathrm{SL}_2(\mathbb{Q}_p)$ *for* $p = 2$.

Hence, we can combine the two conclusion above and get

Theorem 4.2.3. *The geometric conjecture is true for* $\mathrm{SL}_2(\mathbb{Q}_p)$.

Chapter 5

Geometric structure: $\mathrm{SL}_3(F)$

In this chapter, we will show that the part (3) of the geometric conjecture for the case $G = \mathrm{SL}_3(F)$ with $F = \mathbb{Q}_p$ is true. For the group $\mathrm{SL}_3(F)$, there are three kinds of standard Levi subgroup M_0, M_1 and M_2 which correspond to the partitions 3, $2+1$ and $1+1+1$, i.e., $M_0 = \mathrm{SL}_3(F)$, $M_1 = (\mathrm{GL}_2(F) \times F^\times) \cap \mathrm{SL}_3(F)$, $M_2 = (F^\times \times F^\times \times F^\times) \cap \mathrm{SL}_3(F)$. Then we will treat these three kinds of standard Levi subgroup respectively. From now on, for convenient, we denote $(\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \cdots \times \mathrm{GL}_{n_r} \cap \mathrm{SL}_n)$ by $n_1 + n_2 + \cdots + n_r$ where $\Sigma n_i = n$. For example, $2+1$ means

$$(\mathrm{GL}_2 \times \mathrm{GL}_1) \cap \mathrm{SL}_3.$$

5.1 Case $M = M_0$

First of all, we consider the case $M = M_0$. As the description above, we know this standard Levi subgroup corresponding to partitions 3. This means, in this case, the Levi factor is the $\mathrm{SL}_3(F)$ itself, i.e. $M = G$. Recall that the Weyl group $W(M)$ of M is given by $N_G(M)/Z_G(M)$. It is easy to get $W(M) = 1$. For convenience, we will use W to denote $W(M)$ for abbreviation. Let $\mathfrak{s} = [M, \sigma]_G$ be the Bernstein component with respect to M where σ is in the discrete series. For computing the extended quotient, we have to determine the isotropy group $W^{\mathfrak{s}}$. In fact, $W^{\mathfrak{s}}$ is a subgroup of W. Since W is trivial, it leads that $W^{\mathfrak{s}}$ is also trivial, i.e. $W^{\mathfrak{s}} = 1$.

Then, we investigate the extended quotient $E^{\mathfrak{s}}//W^{\mathfrak{s}}$. Recall that $E^{\mathfrak{s}} := \{\Psi^{\mathfrak{t}}(M) \otimes \sigma\}$. Since $\Psi^{\mathfrak{t}}(M) = \Psi^{\mathfrak{t}}(\mathrm{SL}_3(F)) = 1$, we know that $E^{\mathfrak{s}}$ is just a point only. Since $W^{\mathfrak{s}}$ is trivial, there is only one conjugacy class. By the property of extended quotient, we get

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}} \cong 1.$$

Now, we consider the tempered representation with respect to $\mathfrak{s}$. By Langlands classification and since σ is discrete series, σ is the unique candidate in the tempered dual $\mathrm{Irr}^{\mathfrak{t}}(G)^{\mathfrak{s}}$. Hence, each discrete series σ contributes an isolated point in tempered dual. Thus, the bijection is explicit:

$$\mu : E^{\mathfrak{s}}//W^{\mathfrak{s}} \longrightarrow \mathrm{Irr}^{\mathfrak{t}}(G)^{\mathfrak{s}}$$

It is given by

$$\mu : 1 \longmapsto \sigma$$

In fact, the image of $inf.ch$ is also easy to figure out:

$$inf.ch \circ (\mu(1)) = [M, \sigma]_G$$

We set the cocharacter $h_c = 1$ and we can see that the following diagram is commutative:

$$\begin{tikzcd} E//W \arrow[r, "\mu"] \arrow[dr, "\pi_{\sqrt{q}}"'] & \mathrm{Irr}(G)^{\mathfrak{s}} \arrow[d, "inf.ch"] \\ & D/W \end{tikzcd}$$

where $\pi_t(x) = \pi(h_c(t) \cdot x)$ for $x \in E//W$. Hence we can see that $\pi_{\sqrt{q}}(x) = \pi(x)$.

We note that in this case there is no any reducible point occurs and we conclude that we have the following lemma:

Lemma 5.1.1. *Part (3) of the geometric conjecture is true for $\mathfrak{s} = [M, \sigma]_G$ where $M = G$.*

5.2 Case $M = M_1$

In this section, we will consider the case $M = M_1$. In the beginning , we know that M_1 corresponds to the partition $2 + 1$. So we have $M_1 \cong (GL_2(F) \times F^\times) \cap \mathrm{SL}_3(F)$.

Then it is easy to know that the Weyl group $W(M)$ of M is trivial. Because of this reason, the isotropy subgroup $W(\sigma)$ have to be trivial. In this case, $E^{\mathfrak{s}}$ is given by $\mathbb{T}^2/\mathbb{T}$. We have the decomposition of extended quotient:

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}}$$

Now we consider the element in tempered dual. We have to mention that the R-group $R(\sigma)$ is a subgroup of $W(M)$. As we know, the Weyl group $W(M)$ is trivial. It implies $R(\sigma) = 1$. So we can conclude that the induced representation $\mathrm{Ind}_M^G(\chi_t \otimes \sigma)$ is irreducible for each unramified unitary character χ_t where $t \in E^{\mathfrak{s}}$. Therefore, we have the bijection:

$$\mu : E^{\mathfrak{s}}//W^{\mathfrak{s}} \longrightarrow \mathrm{Irr}^{\mathfrak{s}}(G)$$

by

$$\mu : t \in E^{\mathfrak{s}}/W^{\mathfrak{s}} \longmapsto \mathrm{Ind}_M^G(\chi_t \otimes \sigma)$$

We have

$$inf.ch \circ (\mu(t)) = [M, \chi_t \otimes \sigma]_G$$

We set the cocharacter $h_c = 1$. It is clear that $\pi_{\sqrt{q}}(x) = \pi(x)$.

Lemma 5.2.1. *Part (3) of the geometric conjecture is true for* $\mathfrak{s} = [M, \sigma]_G$ *where* $M = M_1$.

5.3 Case $M = M_2$

In the last, we consider the case $M = M_2$. By the partition $1 + 1 + 1$, we know that $M_2 \cong T$ where T is the maximal torus of G. Let $\mathfrak{s} = [M, \sigma]_G$ be a Bernstein component of M. As we know, M is maximal torus of $G = \mathrm{SL}_3(F)$. Let $\pi_\sigma|_M \supset \sigma$ where π_σ is a discrete series of $\tilde{M}$. Then we can rewrite π_σ in the form $\pi_\sigma = \pi_1 \otimes \pi_2 \otimes \pi_3$. We note that the representation σ of M is irreducible cuspidal representation and M is the maximal torus. It implies the form of representation π_σ is the same as the form of σ. This means

$$\pi_\sigma = \pi_1 \otimes \pi_2 \otimes \pi_3$$

where π_i is representation(character) of $\mathrm{GL}_1(F) \cong F^\times$.

Then we can list all possibilities of π_σ due to the relations among π_i. This means we have three possibilities here as follows:

- $\pi_\sigma = \pi \otimes \pi \otimes \pi$;
- $\pi_\sigma = \pi_1 \otimes \pi_1 \otimes \pi_2$;
- $\pi_\sigma = \pi_1 \otimes \pi_2 \otimes \pi_3$.

where π_i is not a twist of π_j by a unramified character for $i \neq j$.

In the following, we consider these three cases one by one.

Case 1. Let π_σ be the form of $\pi_\sigma = \pi \otimes \pi \otimes \pi$. Now we suppose $\eta \in \bar{L}(\pi_\sigma)$. Then it follows that

$$\eta\pi \cong \pi$$

It is easy to see that the character η is the trivial character. That means, in this case, the trivial character of F is the unique element in $\bar{L}(\pi_\sigma)$. Furthermore, this character η is also in $X(\pi_\sigma)$. Hence, it leads that $R(\sigma) = 1$. It implies that the representation induced by representation σ is irreducible. At the same time, we point out that the isotropy group $W^{\mathfrak{s}}$ is given by the symmetric group $\mathfrak{S}_3$.

Lemma 5.3.1. *For $\mathfrak{s} = [M, \sigma]_G$ and $\pi_\sigma \cong \pi \otimes \pi \otimes \pi$, the R-group $R(\sigma) = 1$ and $W^{\mathfrak{s}} = \mathfrak{S}_3$.*

Case 2. Secondly, we think about the case: $\pi_\sigma = \pi_1 \otimes \pi_1 \otimes \pi_2$. Here, we use the similar way as we treat on the case 1. Suppose η is an element in $\bar{L}(\pi_\sigma)$. Then we get

$$\begin{cases} \eta\pi_1 \cong \pi_1 \\ \eta\pi_1 \cong \pi_1 \\ \eta\pi_2 \cong \pi_2 \end{cases} \tag{5.1}$$

or

$$\begin{cases} \eta\pi_1 \cong \pi_1 \\ \eta\pi_1 \cong \pi_2 \\ \eta\pi_2 \cong \pi_1 \end{cases} \tag{5.2}$$

In fact, (5.2) contradicts with our assumption since it implies π_1 is equivalent to π_2. Therefore, we know that (5.1) is the only situation. It is immediate to know that $\eta = 1$. Hence $X(\pi_\sigma) = \bar{L}(\pi_\sigma)$. Therefore we know that the R-group $R(\sigma)$ is trivial group. The induced representation of σ is irreducible. In this case, the isotropy group $W^{\mathfrak{s}}$ is given by $\mathfrak{S}_2$. So we can conclude

Lemma 5.3.2. *For $\mathfrak{s} = [M, \sigma]_G$ and $\pi_\sigma \cong \pi_1 \otimes \pi_1 \otimes \pi_2$, the R-group $R(\sigma) = 1$ and $W^{\mathfrak{s}} = \mathfrak{S}_2$.*

Case 3. Finally, let us focus on the case: $\pi_\sigma \cong \pi_1 \otimes \pi_2 \otimes \pi_3$ and let η be a elements in $\bar{L}(\pi_\sigma)$. As above, we will have the two following situations:

$$\begin{cases} \eta\pi_1 \cong \pi_1 \\ \eta\pi_2 \cong \pi_2 \\ \eta\pi_3 \cong \pi_3 \end{cases} \tag{5.3}$$

or

$$\begin{cases} \eta\pi_1 \cong \pi_2 \\ \eta\pi_2 \cong \pi_3 \\ \eta\pi_3 \cong \pi_1 \end{cases} \tag{5.4}$$

For the type (5.3), we will have $\eta = 1$. We denote this type by case 3.1. Otherwise, in the type (5.4), we have the following property

$$\pi_2 \cong \eta\pi_1,\ \pi_3 \cong \eta^2\pi_1,\ \eta^3 = 1$$

Then we denote this type by case 3.2 and rewrite π_σ in the form of $\pi_\sigma = \pi \otimes \eta\pi \otimes \eta^2\pi$. Then it is easy to know that the $\bar{L}(\pi_\sigma)$ consists of 1, η and η^2. The R-group $R(\sigma)$ is given by the cyclic group $\mathbb{Z}/3\mathbb{Z}$ and the isotropy group $W^{\mathfrak{s}}$ is given by $\mathbb{Z}/3\mathbb{Z}$ as well. For case 3.1, $\bar{L}(\pi_\sigma)$ is trivial. We will have $R(\sigma)$ is trivial and the isotropy group $W^{\mathfrak{s}}$ is trivial. We have to mention that character η is ramified. Otherwise, this case would belong to the case 1. Hence, we have

Lemma 5.3.3. *For $\mathfrak{s} = [M, \sigma]_G$ and $\pi_\sigma \cong \pi_1 \otimes \pi_2 \otimes \pi_3$, there are two possibilities:*

1. *Suppose there exist a cubic ramified character η of $\widehat{F^\times}$ such that $\pi_2 \cong \eta\pi_1$ and $\pi_3 \cong \eta^2\pi_1$. Then the R-group $R(\sigma)$ is generated by η and isomorphic to $\mathbb{Z}/3\mathbb{Z}$.*

In particular, we can rewrite $\pi_\sigma = \pi_1 \otimes \eta\pi_1 \otimes \eta^2\pi_1$ *and the isotropy group* $W(\sigma)$ *is given by the cyclic group* $\mathbb{Z}/3\mathbb{Z}$.

2. *Otherwise, if there do not exist such character, then* $R(\sigma)$ *and* $W(\sigma)$ *are trivial.*

Now we conclude all cases we discuss before and organize in the following table. At the same time, we also mention what the R-groups and isotropy groups are in each cases.

Table of R-group for principal series of $\mathrm{SL}_3(F)$				
Case	π_σ	$R(\sigma)$	$W(\sigma) = W^{\mathfrak{s}}(\sigma)$	Condition
1	$\pi \otimes \pi \otimes \pi$	1	$\mathfrak{S}_3$	
2	$\pi_1 \otimes \pi_1 \otimes \pi_2$	1	$\mathfrak{S}_2$	
3.1	$\pi_1 \otimes \pi_2 \otimes \pi_3$	1	1	
3.2	$\pi \otimes \eta\pi \otimes \eta^2\pi$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\eta^3 = 1$

In the following, we consider the geometric structure of each case and try to show the relation between the extended quotient and tempered representations.

In fact, in [21], Plymen mentions a similar result for $\mathrm{SL}_3(F)$.

Case 1: $\pi_\sigma \cong \pi \otimes \pi \otimes \pi$

For this case, we have shown that the isotropy group is the symmetric group $\mathfrak{S}_3$. The group structure is shown by the table below:

$W^{\mathfrak{s}} = \mathfrak{S}_3$	
γ	$\gamma(abc)$
γ_1	abc
γ_2	acb
γ_3	bac
γ_4	bca
γ_5	cab
γ_6	cba

There are three conjugacy classes. They are $\{\gamma_1\}$, $\{\gamma_2.\gamma_3, \gamma_6\}$, $\{\gamma_4, \gamma_5\}$. Now we choose γ_1, γ_2 and γ_4 as the representatives for their own conjugacy classes. The extended quotient is given as follows:

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\gamma_1}/Z(\gamma_1) \sqcup E^{\gamma_2}/Z(\gamma_2) \sqcup E^{\gamma_4}/Z(\gamma_4)$$

where $Z(\gamma_1) = W^{\mathfrak{s}}$, $Z(\gamma_2) = \{\gamma_1, \gamma_2\}$ and $Z(\gamma_4) = \{\gamma_1, \gamma_4, \gamma_5\}$. Now we analysis case by case.

- $\gamma = \gamma_1$, $E^{\gamma}/Z(\gamma) = E^{\mathfrak{s}}/W^{\mathfrak{s}}$.
- $\gamma = \gamma_2$, $E^{\gamma} = \{(a, b, b) : a, b \in \mathbb{T}\}/\mathbb{T} \cong \{(1, z, z) : z \in \mathbb{T}\} \cong \mathbb{T}$.
- $\gamma = \gamma_4$, $E^{\gamma} = \{(1, 1, 1), (1, \omega, \omega^2), (1, \omega^2, \omega) : \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}\} \cong pt_1 \sqcup pt_2 \sqcup pt_3$. Hence, we have
$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T} \sqcup pt_1 \sqcup pt_2 \sqcup pt_3$$

The following we try to exhaust the tempered dual with respect to $\mathfrak{s}$. We consider the induced representation $\mathrm{Ind}_M^{(2+1)}(\nu^{-\frac{1}{2}} z^{val\circ det}\pi \otimes \nu^{\frac{1}{2}} z^{val\circ det}\pi \otimes \pi)$ for $z \in \mathbb{T}$. As we know, $\mathrm{Ind}_{(1+1)}^{2}(\nu^{-\frac{1}{2}} z^{val\circ det}\pi \otimes \nu^{\frac{1}{2}} z^{val\circ det}\pi)$ are reducible and there exists an irreducible subquotient $z \cdot St_2(\pi)$ which is generalized Steinberg representation. Hence, we have the following inclusion:

$$z \cdot St_2(\pi) \otimes \pi \hookrightarrow \mathrm{Ind}_{(1+1+1)}^{(2+1)}(\nu^{-\frac{1}{2}} z^{val\circ det}\pi \otimes \nu^{\frac{1}{2}} z^{val\circ det}\pi \otimes \pi)$$

Thus, we get the elements in tempered dual. It is

$$\mathrm{Ind}_{(2+1)}^{3}(z \cdot St_2(\pi) \otimes \pi)$$

These representations can be represented by $\mathbb{T}$ and we set the cocharacter by

$$h_c(t) = (t, t^{-1}, 1).$$

As above, similarly, we consider the representation $\mathrm{Ind}_M^G(\nu^{-1}\pi \otimes \pi \otimes \nu\pi)$. This representation is reducible and there is an irreducible subquotient given by $St_3(\pi)$:

$$St_3(\pi) \hookrightarrow \mathrm{Ind}_{(1+1+1)}^{(3)}(\nu^{-1}\pi \otimes \pi \otimes \nu\pi)$$

Hence, we know that

$$St_3(\pi)$$

is in the tempered dual and we identify it by pt_1. For this component, we set the cocharacter:

$$h_c(t) = (t^2, 1, t^{-2}).$$

At last, we turn to focus on the non-special representations. Let $t = (1, \omega, \omega^2)$ where $\omega^3 = 1$ and χ_t be corresponding unramified unitary character. Then the representation $\chi_t \otimes \pi_\sigma$ is given by $\pi \otimes \omega^{val \circ det}\pi \otimes (\omega^2)^{val \circ det}\pi$. Then it is not hard to know that 1, $(\omega)^{val \circ det}$ and $(\omega^2)^{val \circ det}$ are the whole candidates in $\bar{L}(\chi_t \otimes \pi_\sigma)$ and $X(\chi_t \otimes \pi_\sigma) = 1$. Hence, $R(\chi_t \otimes \sigma) \cong \mathbb{Z}/3\mathbb{Z}$ and it means the representation λ induced by $\chi_t \otimes \pi_\sigma$ is reducible and there are three irreducible subrepresentations in this induced representation. We denote these three representations by λ_1, λ_2 and λ_3. We can decompose λ as

$$\lambda = \lambda_1 \oplus \lambda_2 \oplus \lambda_3$$

At the same time, we identify the representations λ_2 and λ_3 by pt_2 and pt_3 and set the cocharacter h_{pt_2} and h_{pt_3} to be trivial.

$$\mu : E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T} \sqcup pt_1 \sqcup pt_2 \sqcup pt_3 \longrightarrow \mathrm{Irr}^t(G)^{\mathfrak{s}}$$

The detail of this map is as follow:

$$z \in \mathbb{T} \longmapsto \mathrm{Ind}^3_{(2+1)}(z \cdot St_2(\pi) \otimes \pi) \hookrightarrow \mathrm{Ind}^{(3)}_{(1+1+1)}(\nu^{-\frac{1}{2}} z^{val \circ det}\pi \otimes \nu^{\frac{1}{2}} z^{val \circ det}\pi \otimes \pi)$$

$$\mu : \quad pt_1 \longmapsto St_3(\pi) \hookrightarrow \mathrm{Ind}^{(3)}_{(1+1+1)}(\nu^{-1}\pi \otimes \pi \otimes \nu\pi)$$

$$pt_2 \longmapsto \lambda_2 \hookrightarrow \mathrm{Ind}^{(3)}_{(1+1+1)}(\pi \otimes \omega^{val \circ det}\pi \otimes (\omega^2)^{val \circ det}\pi)$$

$$pt_3 \longmapsto \lambda_3 \hookrightarrow \mathrm{Ind}^{(3)}_{(1+1+1)}(\pi \otimes \omega^{val \circ det}\pi \otimes (\omega^2)^{val \circ det}\pi)$$

Now we investigate the image of $inf.ch$:

$$\begin{aligned}
inf.ch(\mu(z)) &= [M, \nu^{-\frac{1}{2}} z^{val \circ det} \pi \otimes \nu^{\frac{1}{2}} z^{val \circ det} \pi \otimes \pi]_G \\
&\longmapsto (q^{\frac{1}{2}} z' q^{-\frac{1}{2}} z, 1) \in D^{\mathfrak{s}}/W^{\mathfrak{s}} \\
inf.ch(\mu(pt_1)) &= [M, \nu^{-1} \pi \otimes \pi \otimes \nu \pi]_G \\
&\longmapsto (q, 1, q^{-1}) \in D^{\mathfrak{s}}/W^{\mathfrak{s}} \\
inf.ch(\mu(pt_2)) &= [M, \pi \otimes \omega^{val \circ det} \pi \otimes (\omega^2)^{val \circ det} \pi]_G \\
&\longmapsto (1, \omega, \omega^2) \in E^{\mathfrak{s}}/W^{\mathfrak{s}} \\
inf.ch(\mu(pt_3)) &= [M, \pi \otimes \omega^{val \circ det} \pi \otimes (\omega^2)^{val \circ det} \pi]_G \\
&\longmapsto (1, \omega, \omega^2) \in E^{\mathfrak{s}}/W^{\mathfrak{s}}
\end{aligned}$$

We have the following graph:

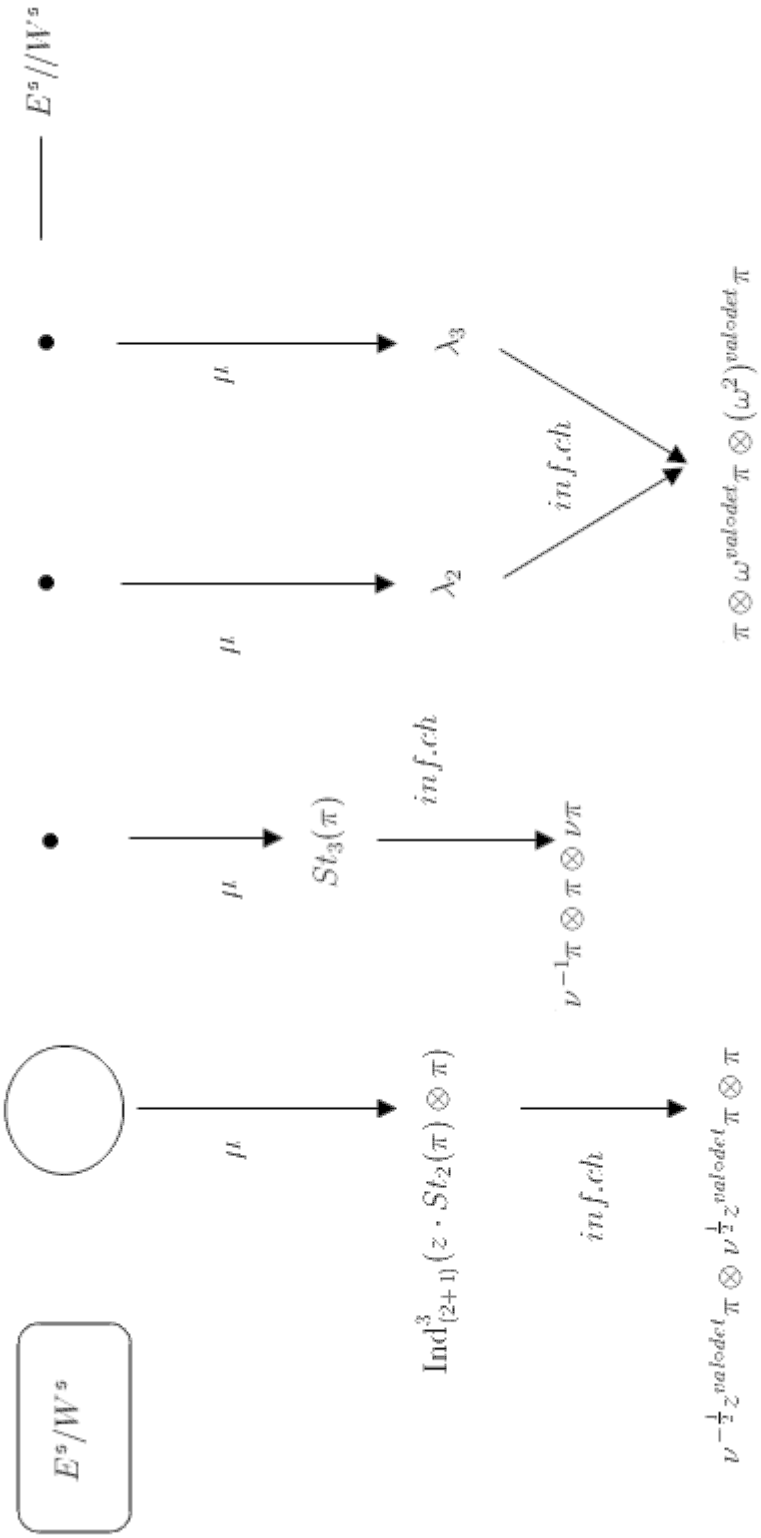

Figure 5.1: $\mathfrak{s} = [M_2, \sigma]_G$ where $\pi_\sigma \cong \pi \otimes \pi \otimes \pi$

Hence, we have

Lemma 5.3.4. *Part (3) of geometric conjecture is true for* $\mathfrak{s} = [M_0, \pi \otimes \pi \otimes \pi]_G$.

Case 2

In this section, we will consider the case $\pi_\sigma \cong \pi_1 \otimes \pi_1 \otimes \pi_2$. Hence , we know the isotropy subgroup $W^{\mathfrak{s}} = \mathfrak{S}_2$. The table below is the structure of $W^{\mathfrak{s}}$.

$W^{\mathfrak{s}} = \mathfrak{S}_2$		
γ	$\gamma(abcd)$	$Z(\gamma)$
γ_1	abc	$W^{\mathfrak{s}}$
γ_2	bac	$W^{\mathfrak{s}}$

This case would be easier to handle because of the simple structure of isotropy subgroup. Then, we focus on the extended quotient with respect to this case. Easily, we know that

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\gamma_1}/Z(\gamma_1) \sqcup E^{\gamma_2}/Z(\gamma_2)$$

Analysis case by case:

- $\gamma = \gamma_1$, $E^{\gamma}/Z(\gamma) = E^{\mathfrak{s}}/W^{\mathfrak{s}}$.
- $\gamma = \gamma_2$, $E^{\gamma} = \{(a, a, c) : a, c \in \mathbb{T}\}/\mathbb{T} \cong \{(z, z, 1) : z \in \mathbb{T}\} \cong \mathbb{T}$and $Z(\gamma_2) = W^{\mathfrak{s}}$

$$E^{\gamma}/Z(\gamma) \cong \mathbb{T}$$

Then, we have

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T}$$

Actually, there is only one type of induced representations which are reducible. Let us consider the induced representation

$$\mathrm{Ind}_{(1+1+1)}^{(2+1)}(\nu^{-\frac{1}{2}} z^{val \circ det} \pi_1 \otimes \nu^{\frac{1}{2}} z^{val \circ det} \pi_1 \otimes \pi_2).$$

It is reducible and there exists an unique irreducible subquotient:

$$z \cdot St_2(\pi_1) \otimes \pi_2 \hookrightarrow \mathrm{Ind}_{(1+1+1)}^{(2+1)}(\nu^{-\frac{1}{2}} z^{val \circ det} \pi_1 \otimes \nu^{\frac{1}{2}} z^{val \circ det} \pi_1 \otimes \pi_2)$$

Therefore, the representation

$$\mathrm{Ind}_{(2+1)}^{(3)}(z \cdot St_2(\pi_1) \otimes \pi_2)$$

is tempered and so it is an element in tempered dual. As z runs through the complex torus $\mathbb{T}$, these tempered representations will be identified by $\mathbb{T}$. At the same time, we set the cocharacter

$$h_c(t) = (t, t^{-1}, 1)$$

Now we build up the bijection μ:

$$\mu : E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T} \longrightarrow \mathrm{Irr}^t(G)^{\mathfrak{s}}$$

The detail of this map is as follow:

$$\mu : \qquad z \in \mathbb{T} \longmapsto \mathrm{Ind}_{(2+1)}^{(3)}(z \cdot St_2(\pi_1) \otimes \pi_2) \hookrightarrow \mathrm{Ind}_M^G(\nu^{-\frac{1}{2}} z^{val \circ det} \pi_1 \otimes \nu^{\frac{1}{2}} z^{val \circ det} \pi_1 \otimes \pi_2)$$

Now we investigate the image of $inf.ch$ (infinitesimal character)

$$inf.ch(\mu(z)) = [M, \nu^{-\frac{1}{2}} z^{val \circ det} \pi_1 \otimes \nu^{\frac{1}{2}} z^{val \circ det} \pi_1 \otimes \pi_2]_G \longmapsto (q^{\frac{1}{2}} z, q^{-\frac{1}{2}} z, 1) \in D^{\mathfrak{s}}/W^{\mathfrak{s}}$$

Therefore, we have $\pi_{\sqrt{q}} = inf.ch \circ \mu$ and the following graph:

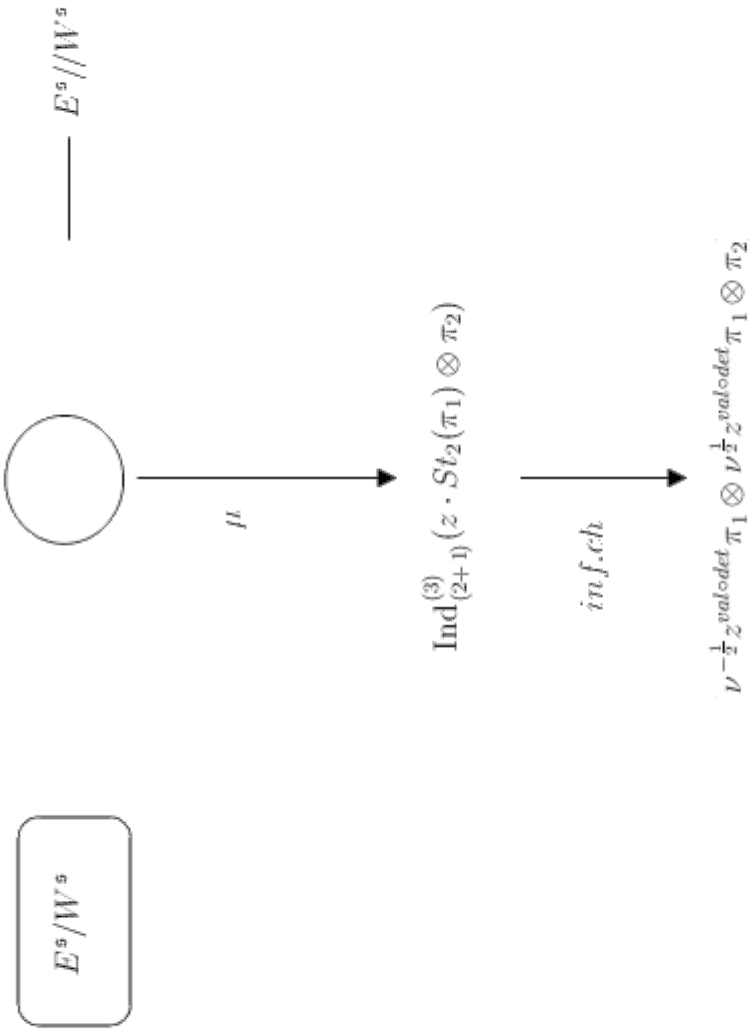

Figure 5.2: $\mathfrak{s} = [M_2, \sigma]_G$ where $\pi_\sigma \cong \pi_1 \otimes \pi_1 \otimes \pi_2$

Lemma 5.3.5. *Part (3) of the geometric conjecture is true for* $\mathfrak{s} = [M_2, \pi_1 \otimes \pi_1 \otimes \pi_2]_G$.

Case 3

At last, we will consider the case $\pi_\sigma \cong \pi_1 \otimes \pi_2 \otimes \pi_3$. In the beginning of this chapter, we have classified this case into two part: case 3.1 and case 3.2. Firstly, we check the case 3.1.

In the case 3.1, we have known that the isotropy group $W^{\mathfrak{s}}$ is trivial. Then, we can get the structure of extended quotient immediately:

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}}$$

Now we focus on the tempered representation with respect to this $\mathfrak{s}$. In fact, all the induced representation $\mathrm{Ind}_M^G(\chi_t \otimes \sigma)$ is irreducible for $t \in E^{\mathfrak{s}}$. Therefore, each induced representation can be represented by each point t. Because of this, we have the bijection trivially:

$$\mu : E^{\mathfrak{s}}//W^{\mathfrak{s}} \longrightarrow \mathrm{Irr}^{\mathfrak{t}}(G)^{\mathfrak{s}}$$

In this case, we set $h_c = 1$. Hence, we have $\pi_{\sqrt{q}}(x) = inf.ch \cdot \mu(x)$. We have

Lemma 5.3.6. *Part (3) of the conjecture is true for* $\mathfrak{s} = [M_2, \pi_1 \otimes \pi_2 \otimes \pi_3]_G$.

Then we will consider the case 3.2. Then we have $\pi_\sigma \cong \pi \otimes \eta\pi \otimes \eta^2\pi$ where η is ramified. We have proved that the R-group with resect to this σ is the cyclic group $\mathbb{Z}/3\mathbb{Z}$. Furthermore, the isotropy group is given by $\mathbb{Z}/3\mathbb{Z}$. In the following,we figure out the extended quotient with respect to $W^{\mathfrak{s}}$. The table below is the structure of $W^{\mathfrak{s}} = \mathbb{Z}/3\mathbb{Z}$.

$W^{\mathfrak{s}} = \mathbb{Z}/3\mathbb{Z}$		
γ	$\gamma(abc)$	$Z(\gamma)$
γ_1	abc	$W^{\mathfrak{s}}$
γ_2	bca	$W^{\mathfrak{s}}$
γ_3	cab	$W^{\mathfrak{s}}$

Since the cyclic group is abelian, each element comprises a single conjugacy classes and the centralizer of each element is the cyclic group itself. Then we can immediately get the extended quotient

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\gamma_1}/W^{\mathfrak{s}} \sqcup E^{\gamma_2}/W^{\mathfrak{s}} \sqcup E^{\gamma_3}/W^{\mathfrak{s}}$$

We analyze case by case.

- $\gamma = \gamma_1$, $E^{\gamma}/W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}}$.
- $\gamma = \gamma_2$, $E^{\gamma} = \{(1,1,1),(1,\omega,\omega^2),(1,\omega^2,\omega)\} \cong pt_1 \sqcup pt_2 \sqcup pt_3$.

$$E^{\gamma}/W^{\gamma} \cong pt_1 \sqcup pt_2 \sqcup pt_3$$

- $\gamma = \gamma_3$, $E^{\gamma} = \{(1,1,1),(1,\omega,\omega^2),(1,\omega^2,\omega)\} \cong pt_1 \sqcup pt_2 \sqcup pt_3$.

$$E^{\gamma}/W^{\gamma} \cong pt_4 \sqcup pt_5 \sqcup pt_6$$

Hence, we have the decomposition of extended quotient $E^{\mathfrak{s}}//W^{\mathfrak{s}}$:

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4 \sqcup pt_5 \sqcup pt_6$$

This coincides the result in [19]. Hence, we apply the Theorem 5.3[19]. Finally, we have

Lemma 5.3.7. *Part (3) of the conjecture is true for* $\mathfrak{s} = [M_2, \pi \otimes \eta\pi \otimes \eta^2\pi]_G$.

Remark: In this case 3.2, the existence of character η of order 3 is very important. In fact, it depends on the local field $\mathbb{Q}_p$ which we consider. Recall that when $p = 3$ or $p \equiv 1 \bmod 3$, there are finitely many characters of order 3. Otherwise, when $p = 2$ or $p \not\equiv 1 \bmod 3$, Levi subgroup M_2 does not admit any character of order 3. For example when $F = \mathbb{Q}_2$, there is no character of order 3. In other word, case 3.2 does not exist when $p = 2$ or $p \not\equiv 1 \bmod 3$.

We conclude 5.3.4, 5.3.5, 5.3.6 and 5.3.7 to get

Theorem 5.3.8. *Part (3) of the geometric conjecture is true for* $\mathrm{SL}_3(\mathbb{Q}_p)$.

Chapter 6

Geometric structure: $\mathrm{SL}_4(F)$

In this chapter, we fix the local field F to be $\mathbb{Q}_p$, $p \geq 3$. The object we will treat is the group $\mathrm{SL}_4(F)$. In previous chapters, we have discussed the cases $\mathrm{SL}_2(F)$ and $\mathrm{SL}_3(F)$. Due to their sizes (dimensions), it is easier to handle it. On the other hand, the structures of the cases $\mathrm{SL}_2(F)$ and $\mathrm{SL}_3(F)$ are somewhat depended on the prime number 2 and 3. But, in this section, we will consider the first non-prime case. In fact, there is a difference between the prime cases and the non-prime cases.

As before, firstly, we have to classify all the standard Levi subgroups M for $G = \mathrm{SL}_4$. By the partitions of G, we have 5 different Levi subgroups. They are classified by $1+1+1+1$, $2+2$, $1+3$, $1+1+2$, 4. In other words, there are 5 types: $T = (F^\times \times F^\times \times F^\times \times F^\times) \cap \mathrm{SL}_4(F)$, $M_1 = (\mathrm{GL}_2(F) \times \mathrm{GL}_2(F)) \cap \mathrm{SL}_4(F)$, $M_2 = (F^\times \times GL_3(F)) \cap (\mathrm{SL}_4(F))$, $M_3 = (F^\times \times F^\times \times \mathrm{GL}_2(F)) \cap \mathrm{SL}_4(F)$, $M_4 = G = \mathrm{SL}_4(F)$. In this chapter, we just consider the case when the Levi factor is maximal torus. In other word, we focus on the toral part in the tempered dual.

6.0.1 Case: $M = T$

Now we start to consider the Levi subgroup $M = T$ which corresponds to the partition $1+1+1+1$. Actually, it is the maximal torus of $G = \mathrm{SL}_4(F)$. We keep using the method introduced in chapter two . For the Bernstein component $\mathfrak{s} = [M, \sigma]_G$, we let $\pi_\sigma|_M \supset \sigma$ where π_σ is in $\mathcal{E}_2(\tilde{M})$. Then we write π_σ in the form $\pi_\sigma = \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$.

We note that the representation σ of M is discrete series and M is the maximal torus. It implies the form of representation π_σ is the same as the form of σ. This means

$$\pi_\sigma = \sigma = \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$$

Now we will discuss all the cases according the properties of π_i $(i = 1, 2, 3, 4)$. In other words, we have 5 different cases:

- $\pi_\sigma = \pi \otimes \pi \otimes \pi \otimes \pi$;
- $\pi_\sigma = \pi_1 \otimes \pi_1 \otimes \pi_1 \otimes \pi_2$;
- $\pi_\sigma = \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_2$;
- $\pi_\sigma = \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_3$;
- $\pi_\sigma = \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$.

where π_i is not a twist of π_j by a unramified character for $i \neq j$.

In the following, we will discuss each case above one by one.

Case 1: $\pi_\sigma = \pi \otimes \pi \otimes \pi \otimes \pi$

In this case, it would be easy to handle it. Let $\eta \in \bar{L}(\pi_\sigma)$. Then we have

$$\eta\pi_\sigma \cong^w \pi_\sigma$$

We mention that the action of $W(M)$ on π_σ is trivial and therefore the isotropy group $W(\sigma)$ of σ is given by $\mathfrak{S}_4$. It is immediate to get that $\eta \in X(\pi_\sigma)$. Thus, we have $\bar{L}(\pi_\sigma) = X(\pi_\sigma)$. Then, we know that the R-group is trivial.

Lemma 6.0.9. *For $\mathfrak{s} = [M, \sigma]_G$ and $\pi_\sigma \cong \pi \otimes \pi \otimes \pi \otimes \pi$, the R-group $R(\sigma) = 1$ and $W^{\mathfrak{s}} = \mathfrak{S}_4$.*

Case 2: $\pi_\sigma = \pi_1 \otimes \pi_1 \otimes \pi_1 \otimes \pi_2$

Now we carry on the case 2. Let $\eta \in \bar{L}(\pi_\sigma)$. Then we must have

$$\eta\pi_1 \cong \pi_1 \tag{6.1}$$

$$\eta\pi_2 \cong \pi_2 \tag{6.2}$$

Recall that π_1 and π_2 are characters, then we have $\eta = 1$. This leads that η is the trivial character and is the unique element in $\bar{L}(\pi_\sigma)$. Then we have $R(\sigma) = 1$. In this case, it is obvious to see that $W^{\mathfrak{s}} = W(\sigma)$ is given by $\mathfrak{S}_3$.

Lemma 6.0.10. *For $\mathfrak{s} = [M, \sigma]_G$ and $\pi_\sigma \cong \pi_1 \otimes \pi_1 \otimes \pi_1 \otimes \pi_2$, the R-group $R(\sigma) = 1$ and $W^{\mathfrak{s}} = \mathfrak{S}_3$.*

Case 3: $\pi_\sigma = \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_2$

Suppose $\eta \in \bar{L}(\pi_\sigma)$. We have two situations. The first one is

$$\eta\pi_1 \cong \pi_1 \text{ and } \eta\pi_2 \cong \pi_2$$

Then we must have $\eta = 1$.

The second one is that

$$\eta\pi_1 \cong \pi_2 \text{ and } \eta\pi_2 \cong \pi_1$$

Thus we have

$$\eta^2\pi_1 \cong \pi_1 \text{ and } \eta^2\pi_2 \cong \pi_2$$

Finally, η must be the quadratic character. Thus, we can write the $\pi_\sigma = \pi_1 \otimes \pi_1 \otimes \eta\pi_1 \otimes \eta\pi_1$. We have to mention that character η is ramified. Otherwise, this case would belong to case 1. We can conclude that there are two different possibilities in this case.

Lemma 6.0.11. *For $\mathfrak{s} = [M, \sigma]_G$ and $\pi_\sigma \cong \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_2$, there are two possibilities:*

1. *Suppose there exist a ramified quadratic character η of $\widehat{F^\times}$ such that $\pi_1 \cong \eta\pi_2$ and $\pi_2 \cong \eta\pi_1$. The R-group $R(\sigma)$ is generated by η and isomorphic to $\mathbb{Z}/2\mathbb{Z}$. In particular, we can rewrite $\pi_\sigma = \pi \otimes \pi \otimes \eta\pi \otimes \eta\pi$. Furthermore, the isotropy group $W(\sigma)$ is given by group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$.*

2. *Otherwise, if there do not exists such character, then $R(\sigma)$ is trivial and $W(\sigma)$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

Case 4: $\pi_\sigma = \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_3$

Let $\eta \in \bar{L}(\pi_\sigma)$. There are two possibilities. The first one is

$$\eta\pi_1 \cong \pi_1 \ , \ \eta\pi_2 \cong \pi_2 \text{ and } \eta\pi_3 \cong \pi_3$$

This leads η is trivial character.

Another case is

$$\eta\pi_1 \cong \pi_1 \ , \ \eta\pi_2 \cong \pi_3 \text{ and } \eta\pi_3 \cong \pi_2$$

It also follows that $\eta = 1$. Therefore, $\bar{L}(\pi_\sigma) = X(\pi_\sigma)$ We have $R(\sigma) = 1$.

So we can conclude that the R-group is trivial for this case. On the other hand, $W(\sigma)$ is given by $\mathbb{Z}/2\mathbb{Z}$.

Lemma 6.0.12. *For $\mathfrak{s} = [M, \sigma]_G$ and $\pi_\sigma \cong \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_3$, the R-group $R(\sigma) = 1$ and the isotropy group $W^{\mathfrak{s}}$ is given by $\mathbb{Z}/2\mathbb{Z}$.*

Case 5: $\pi_\sigma = \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$

In this case, it will be little intricate due to the relations among $\pi_i (i = 1, 2, 3, 4)$. Similarly, let η be the element in $\bar{L}(\pi_\sigma)$. Then we can construct the following four possibilities:

Type 1: Suppose $\eta \in \bar{L}(\pi_\sigma)$ and satisfies the following conditions:

$$\begin{cases} \eta\pi_1 \cong \pi_2 \\ \eta\pi_2 \cong \pi_3 \\ \eta\pi_3 \cong \pi_4 \\ \eta\pi_4 \cong \pi_1 \end{cases}$$

We will have

$$\eta^4\pi_i \cong \pi_i$$

It leads $\eta^4 = 1$. In fact, 1, η, η^2 ,η^3 are also in $\bar{L}(\pi_\sigma)$. Then we have $R(\sigma) = \mathbb{Z}/4\mathbb{Z}$ and $W(\sigma) = \mathbb{Z}/4\mathbb{Z}$. Similarly as above, we write π_σ in form of $\pi \otimes \eta\pi \otimes \eta^2\pi \otimes \eta^3\pi$.

Type 2: Suppose there are two character η, χ in $\bar{L}(\pi_\sigma)$ and satisfies

$$\begin{cases} \eta\pi_1 \cong \pi_2 \\ \eta\pi_2 \cong \pi_1 \\ \eta\pi_3 \cong \pi_4 \\ \eta\pi_4 \cong \pi_3 \end{cases}$$

and

$$\begin{cases} \chi\pi_1 \cong \pi_3 \\ \chi\pi_2 \cong \pi_4 \\ \chi\pi_3 \cong \pi_1 \\ \chi\pi_4 \cong \pi_2 \end{cases}$$

Then we know that η, χ generate the group $\bar{L}(\pi_\sigma)$. Thus, we have $R(\sigma) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. At the same time, $W(\sigma)$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as well. We will write π_σ in this type by $\pi_\sigma \cong \pi \otimes \eta\pi \otimes \chi\pi \otimes \eta\chi\pi$. Here, η and χ are both ramified.

Type 3: As below, η is the unique nontrivial element in $\bar{L}(\pi_\sigma)$ and

$$\begin{cases} \eta\pi_1 \cong \pi_2 \\ \eta\pi_2 \cong \pi_1 \\ \eta\pi_3 \cong \pi_4 \\ \eta\pi_4 \cong \pi_3 \end{cases}$$

In this type, we have

$$\eta^2\pi_1 \cong \pi_1, \eta^2\pi_2 \cong \pi_2, \eta^2\pi_3 \cong \pi_3, \eta^2\pi_4 \cong \pi_4$$

Then we have $\eta^2 = 1$. So we know that the character η generates the $\bar{L}(\pi_\sigma)$ and $X(\pi_\sigma) = \{1, \eta^2\}$. It leads that $R(\sigma) = \mathbb{Z}/2\mathbb{Z}$ and $W(\sigma) = \mathbb{Z}/2\mathbb{Z}$. Indeed, we can rewrite π_σ by $\pi_1 \otimes \eta\pi_1 \otimes \pi_2 \otimes \eta\pi_2$ where η is ramified.

Type 4: Suppose $\eta \in \bar{L}(\pi_\sigma)$ and satisfies the conditions below:

$$\begin{cases} \eta\pi_1 \cong \pi_1 \\ \eta\pi_2 \cong \pi_2 \\ \eta\pi_3 \cong \pi_3 \\ \eta\pi_4 \cong \pi_4 \end{cases}$$

Then we will have $\eta = 1$. Then this implies $\bar{L}(\pi_\sigma) = X(\pi_\sigma)$. It is immediate to know that $R(\sigma) = 1$. Simply, we write the π_σ in the form of $\pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$.

Remark: all π_i are independent. None of them is equivalent after twisting with some character of $\widehat{F^\times}$.

Lemma 6.0.13. *For $\mathfrak{s} = [M, \sigma]_G$ and $\pi_\sigma \cong \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$, there are four possibilities:*

1. *Suppose there exist a ramified character η of $\widehat{F^\times}$ such that $\pi_2 \cong \eta\pi_1$, $\pi_3 \cong \eta^2\pi_1$, $\pi_4 \cong \eta^3\pi_1$ and $\eta^4 = 1$. The R-group $R(\sigma)$ is generated by η and isomorphic to $\mathbb{Z}/4\mathbb{Z}$. In particular, we can rewrite $\pi_\sigma = \pi \otimes \eta\pi \otimes \eta^2\pi \otimes \eta^3\pi$ and the isotropy group $W(\sigma)$ is given by group $\mathbb{Z}/4\mathbb{Z}$.*

2. *Suppose there are two ramified quadratic characters η and χ and π_σ in the form of $\pi_\sigma \cong \pi \otimes \eta\pi \otimes \chi\pi \otimes \eta\chi\pi$, then $R(\sigma)$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $W(\sigma)$ is also given by $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

3. *Suppose there is a ramified character η such that $\pi_\sigma \cong \pi_1 \otimes \eta\pi_1 \otimes \pi_2 \otimes \eta\pi_2$. The R-group for this type is $\mathbb{Z}/2\mathbb{Z}$ and the isotropy group is $\mathbb{Z}/2\mathbb{Z}$.*

4. *Otherwise, if there do not exist such characters, then $R(\sigma)$ is trivial and $W(\sigma)$ is also trivial.*

Now we conclude all cases we discuss before and organize in the following table. At the same time, we also mention what the R-groups and isotropy groups are in each cases.

Table of R-group for principal series of SL_4				
Case	π_σ	$R(\sigma)$	$W(\sigma) = W^{\mathfrak{s}}(\sigma)$	Condition
1	$\pi \otimes \pi \otimes \pi \otimes \pi$	1	$\mathfrak{S}_4$	
2	$\pi_1 \otimes \pi_1 \otimes \pi_1 \otimes \pi_2$	1	$\mathfrak{S}_3$	
3.1	$\pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_2$	1	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	
3.2	$\pi \otimes \pi \otimes \eta\pi \otimes \eta\pi$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/\mathbb{Z}2 \wr \mathbb{Z}/2\mathbb{Z}$	$\eta^2 = 1$
4	$\pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_3$	1	$\mathbb{Z}/2\mathbb{Z}$	
5.1	$\pi \otimes \eta\pi \otimes \eta^2\pi \otimes \eta^3\pi$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\eta^4 = 1$
5.2	$\pi \otimes \chi\pi \otimes \eta\pi \otimes \eta\chi\pi$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\chi^2 = 1, \eta^2 = 1$
5.3	$\pi_1 \otimes \eta\pi_1 \otimes \pi_2 \otimes \eta\pi_2$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\eta^2 = 1$
5.4	$\pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$	1	1	

From the table above, we have known that there are several kinds of isotropy group for $\mathfrak{s}$, i.e. $W^{\mathfrak{s}} = 1$, $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathfrak{S}_3$ and $\mathfrak{S}_4$. In the following, we will consider the extended quotients. Because of this, we have to realize how is the structure for such isotropy groups with respect to $\mathfrak{s}$.

6.1 Geometric conjecture

In this section, we will discuss the extended quotient $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ with respect to each Bernstein component $\mathfrak{s} = [M, \sigma]_G$ and show the geometric conjecture for the principal series of $\mathrm{SL}_4(F)$. Hence, we will construct the explicit bijection between $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ and $\mathrm{Irr}^{\mathfrak{t}}(G)^{\mathfrak{s}}$. From now on, for convenient, we denote $(\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \cdots \times \mathrm{GL}_{n_r}) \cap \mathrm{SL}_n$ by $n_1 + n_2 + \cdots + n_r$ where $\Sigma n_i = n$. For example, $1 + 1 + 1 + 1$ means

$$(\mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1) \cap \mathrm{SL}_4.$$

Recall that $E^{\mathfrak{s}}$ is given by $\{\Psi^{\mathfrak{t}}(M) \otimes \sigma\}$. Indeed, $\Psi^{\mathfrak{t}}(M) \cong \mathbb{T}^4/\mathbb{T}$. Hence, we identify $E^{\mathfrak{s}}$ by $\mathbb{T}^4/\mathbb{T}$.

6.1.1 Case 1: $\pi_\sigma \cong \pi \otimes \pi \otimes \pi \otimes \pi$

In the pervious section, for the case $\pi_\sigma \cong \pi \otimes \pi \otimes \pi \otimes \pi$, we have shown that $W^{\mathfrak{s}} = \mathfrak{S}_4$. The group $W^{\mathfrak{s}}$ is given as follows:

$W^{\mathfrak{s}} = \mathfrak{S}_4$					
γ	$\gamma(abcd)$	cycle type	γ	$\gamma(abcd)$	cycle type
γ_1	$abcd$	1111	γ_{13}	$adbc$	31
γ_2	$acbd$	211	γ_{14}	$adcb$	211
γ_3	$bacd$	211	γ_{15}	$bdac$	4
γ_4	$bcad$	31	γ_{16}	$bdca$	31
γ_5	$cabd$	31	γ_{17}	$cdab$	22
γ_6	$cbad$	211	γ_{18}	$cdba$	4
γ_7	$abdc$	211	γ_{19}	$dabc$	4
γ_8	$acdb$	31	γ_{20}	$dacb$	31
γ_9	$badc$	22	γ_{21}	$dbac$	31
γ_{10}	$bcda$	4	γ_{22}	$dbca$	211
γ_{11}	$cadb$	4	γ_{23}	$dcab$	4
γ_{12}	$cbda$	31	γ_{24}	$dcba$	22

In fact, for symmetric group $\mathfrak{S}_4$. There are five conjugacy classes due to the type of cycle in $\mathfrak{S}_4$. Indeed, the five conjugacy classes are as follows:

$$\{\gamma_1\}$$

$$\{\gamma_2, \gamma_3, \gamma_6, \gamma_7, \gamma_{14}, \gamma_{22}\}$$

$$\{\gamma_4, \gamma_5, \gamma_8, \gamma_{12}, \gamma_{13}, \gamma_{16}, \gamma_{20}, \gamma_{21}\}$$

$$\{\gamma_9, \gamma_{17}, \gamma_{24}\}$$

$$\{\gamma_{10}, \gamma_{11}, \gamma_{15}, \gamma_{18}, \gamma_{19}, \gamma_{23}\}$$

Then we focus on the extended quotient $E^{\mathfrak{s}}//W^{\mathfrak{s}}$. Now let us recall the property of extended quotient.

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = \bigsqcup_{\gamma} E^{\gamma}/Z(\gamma)$$

and we choose γ_1, γ_2, γ_4, γ_9 and γ_{10} to be the representatives in their conjugacy classes. The centralizers of these representatives are:

$$Z(\gamma_1) = W^{\mathfrak{s}}, Z(\gamma_2) = \{\gamma_1, \gamma_2, \gamma_{22}, \gamma_{24}\}, Z(\gamma_4) = \{\gamma_1, \gamma_4, \gamma_{11}\} \tag{6.3}$$

$$Z(\gamma_9) = \{\gamma_1, \gamma_3, \gamma_7, \gamma_9, \gamma_{17}, \gamma_{18}, \gamma_{23}, \gamma_{24}\}, Z(\gamma_{10}) = \{\gamma_1, \gamma_{10}, \gamma_{17}, \gamma_{19}\} \tag{6.4}$$

Hence, we can decompose the extended quotient as follows:

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\gamma_1}/Z(\gamma_1) \sqcup E^{\gamma_2}/Z(\gamma_2) \sqcup E^{\gamma_4}/Z(\gamma_4) \sqcup E^{\gamma_9}/Z(\gamma_9) \sqcup E^{\gamma_{10}}/Z(\gamma_{10})$$

Now we try to compute each components in extended quotient.

- $\gamma = \gamma_1$, $E^{\gamma}/Z(\gamma_1) = E^{\mathfrak{s}}/W^{\mathfrak{s}}$.
- $\gamma = \gamma_2$, $E^{\gamma} = \{(a, b, b, d) : a, c, d \in \mathbb{T}\}/\mathbb{T} \cong \{(z_1, z_2, z_2, 1) : z_1, z_2 \in \mathbb{T}\} \cong \mathbb{T}^2$.

$$E^{\gamma}/Z(\gamma) \cong \mathbb{T}^2$$

- $\gamma = \gamma_4$, $E^{\gamma} = \{(a, a, a, b) : a, b \in \mathbb{T}\}/\mathbb{T} \cong \{(z, z, z, 1) : z \in \mathbb{T}\} \cong \mathbb{T}$.

$$E^{\gamma}/Z(\gamma) \cong \mathbb{T}$$

- $\gamma = \gamma_9$, $E^{\gamma} = \{(a, a, b, b), (a, -a, b, -b) : a, b \in \mathbb{T}\}/\mathbb{T} \cong \{(1, 1, z, z), (1, -1, z, -z) : z \in \mathbb{T}\} \cong \mathbb{T} \sqcup \mathbb{T}$.

$$E^{\gamma}/Z(\gamma) \cong \mathbb{T} \sqcup \mathbb{T}$$

- $\gamma = \gamma_{10}$, $E^{\gamma} = \{(1, 1, 1, 1), (1, -1, 1, -1), (1, i, -1, -i), (1, -i, -1, i)\} \cong pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4$.

$$E^{\gamma}/Z(\gamma) \cong pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4$$

Hence, we have

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T}^2 \sqcup \mathbb{T} \sqcup \mathbb{T} \sqcup \mathbb{T} \sqcup pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4$$

From now, we start to investigate the elements in the tempered dual $\mathrm{Irr}^{\mathfrak{t}}(G)^{\mathfrak{s}}$.

First of all, we consider the representation induced by $\nu^{-\frac{1}{2}} z_1^{val \circ det} \pi \otimes \nu^{\frac{1}{2}} z_1^{val \circ det} \pi \otimes z_2^{val \circ det} \pi \otimes \pi$ from $(1+1+1+1)$ to $(2+1+1)$ for $z_1, z_2 \in \mathbb{T}$. In fact, as we know, $\mathrm{Ind}^2_{(1+1)}(\nu^{-\frac{1}{2}} z_1^{val \circ det} \pi \otimes \nu^{\frac{1}{2}} z_1^{val \circ det} \pi)$ are reducible and there exists an irreducible subquotient $z_1 \cdot St_2(\pi)$. Then we consider the induced representation $\mathrm{Ind}^4_{(2+1+1)}(z_1 \cdot St_2(\pi) \otimes z_2^{val \circ det} \pi \otimes \pi)$. This representation is irreducible. Hence, we know that the induced representations $\mathrm{Ind}^G_M(\nu^{-\frac{1}{2}} z_1^{val \circ det} \pi \otimes \nu^{\frac{1}{2}} z_1^{val \circ det} \pi \otimes z_2^{val \circ det} \pi \otimes \pi)$ is reducible and there is a unique discrete series. Since z_1 and z_2 are in the unit circle $\mathbb{T}$, it implies that such representations are parameterized by $\mathbb{T}^2$. We denote this component by c_1. Hence, we have the following inclusion

$$\mathrm{Ind}^4_{(2+1+1)}(z_1 \cdot St_2(\pi) \otimes z_2^{val \circ det} \pi \otimes \pi) \hookrightarrow \mathrm{Ind}^G_M(\nu^{-\frac{1}{2}} z_1^{val \circ det} \pi \otimes \nu^{\frac{1}{2}} z_1^{val \circ det} \pi \otimes z_2^{val \circ det} \pi \otimes \pi)$$

We set the cocharacter $h_c(t)$ for this component c_1 by

$$h_{c_1}(t) = (t, t^{-1}, 1, 1)$$

Similarly, we consider the representation induced by $\mathrm{Ind}^{(3)}_{(1+1+1)}(\nu^{-1} z^{val \circ det} \pi \otimes z^{val \circ det} \pi \otimes z^{val \circ det} \nu \pi)$ and it is reducible. Indeed, there is an irreducible subquotient which is tempered:

$$z \cdot St_3(\pi)$$

Now we consider the induced representation $\mathrm{Ind}^G_M(\nu^{-1} z^{val \circ det} \pi \otimes z^{val \circ det} \pi \otimes \nu z^{val \circ det} \pi \otimes \pi)$ and have the following inclusion

$$\mathrm{Ind}^4_{(3+1)}(z \cdot St_3(\pi) \otimes \pi) \hookrightarrow \mathrm{Ind}^G_M(\nu^{-1} z^{val \circ det} \pi \otimes z^{val \circ det} \pi \otimes \nu z^{val \circ det} \pi \otimes \pi)$$

Hence, we know these induced representations are reducible and they are parameterized by $\mathbb{T}$. We denote this component by c_2, we set the cocharacter:

$$h_{c_2}(t) = (t^2, 1, t^{-2}, 1)$$

Now, we carry on the induced representation $\mathrm{Ind}_M^G(\nu^{-\frac{1}{2}}z^{val\circ det}\pi \otimes \nu^{\frac{1}{2}}z^{val\circ det}\pi \otimes \nu^{-\frac{1}{2}}\pi \otimes \nu^{\frac{1}{2}}\pi)$ for $z \in \mathbb{T}$ and $z \neq -1$. This representation is reducible and there is an irreducible subquotient:

$$\mathrm{Ind}_{2+2}^4(z \cdot St_2(\pi) \otimes St_2(\pi)) \hookrightarrow \mathrm{Ind}_M^G(\nu^{-\frac{1}{2}}z^{val\circ det}\pi \otimes \nu^{\frac{1}{2}}z^{val\circ det}\pi \otimes \nu^{-\frac{1}{2}}\pi \otimes \nu^{\frac{1}{2}}\pi)$$

and it is not hard to see that such representations can be parameterized by $\mathbb{T}$ and denote this by c_3. Here, we have to mention that the case when $z = -1$. We will have

$$\mathrm{Ind}_{(2+2)}^4((-1)^{val\circ det} \cdot St_2(\pi) \otimes St_2(\pi))$$

Indeed, this induced representation is reducible since the R-group attached to this representation is generated by $(-1)^{val\circ det}$ and hence there are two irreducible components. We denote it by ρ^+ and ρ^-. Indeed, we count ρ^- in component c_3 and identify ρ^+ by pt_1.

At the same time, we set the cocharacter by

$$h_c(t) = (t, t^{-1}, 1, 1),\ c = c_3,\ pt_1$$

Now we consider the (generalized) Steinberg representation $St_4(\pi)$. In fact, it occurs in the induced representation $\mathrm{Ind}_M^G(\nu^{-\frac{3}{2}}\pi \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}\pi \otimes \nu^{\frac{3}{2}})$ and we have the inclusion:

$$St_4(\pi) \hookrightarrow \mathrm{Ind}(\nu^{-\frac{3}{2}}\pi \otimes \nu^{-\frac{1}{2}}\pi \otimes \nu^{\frac{1}{2}}\pi \otimes \nu^{\frac{3}{2}}\pi)$$

We identify this representation by pt_2 and set the cocharacter for this representation by

$$h_{pt_2}(t) = (t^3, t, t^{-1}, t^{-3})$$

Let $t = (z, -z, 1, -1)$ except $z = i$ and χ_t be corresponding unramified unitary character. Then the representation $\chi_t \otimes \pi_\sigma$ is given by $z^{val\circ det}\pi \otimes (-z)^{val\circ det}\pi \otimes \pi \otimes (-1)^{val\circ det}\pi$. Then it is not hard to know that $(-1)^{val\circ det}\pi$ is an element in $\bar{L}(\chi_t \otimes \pi_\sigma)$ and $X(\chi_t \otimes \pi_\sigma) = 1$. Hence, $R(\chi_t \otimes \sigma) \cong \mathbb{Z}/2\mathbb{Z}$ and it means the representation induced $\lambda(t) = \mathrm{Ind}_M^G(z^{val\circ det}\pi \otimes (-z)^{val\circ det}\pi \otimes \pi \otimes (-1)^{val\circ det}\pi)$ by $\chi_t \otimes \pi_\sigma$ is reducible and there are two irreducible subrepresentations in this induced representation. We

identifies these representation by $\mathbb{T}$ and denote this component by c_4. We denote these two representations by $\lambda(t)^+$ and $\lambda(t)^-$. We can decompose $\lambda(t)$ as

$$\lambda(t) = \lambda(t)^+ \oplus \lambda(t)^-$$

Now, we turn to the point $t = (i, -i, 1-1)$. Then $\bar{L}(\chi_t \otimes \pi_\sigma) =< i^{val \circ det}$ and $X(\pi_\sigma) = 1$. Hence, we know $R(\chi_t \otimes \sigma) = \mathbb{Z}/4\mathbb{Z}$. The order of $R(\chi_t \otimes \sigma)$ is 4. Then the induced representation $\tau = \mathrm{Ind}_M^G(\chi_t \otimes \sigma)$ is reducible and there are 4 irreducible constituents τ_1, τ_2, τ_3 and τ_4. We locate τ_1 to $E^{\mathfrak{s}}/W^{\mathfrak{s}}$ and τ_2 to component c_4 and identify τ_3 and τ_4 by pt_3 and pt_4 respectively. We set the cocharacter

$$h_c(t) = 1,\ c = c_4,\ pt_3,\ pt_4$$

Indeed, for $t = (z_1, z_2, z_3, 1) \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$ except $t = (z, -z, 1, -1)$, the induced representation $\mathrm{Ind}_M^G(\chi_t \otimes \sigma)$ is irreducible. This implies each unramified unitary character can generate a tempered representation of G with respect to $\mathfrak{s}$.

We turn to build up the map μ satisfying the geometric conjecture, i.e.

$$\mu : E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T}^2 \sqcup \mathbb{T} \sqcup \mathbb{T} \sqcup \mathbb{T} \sqcup pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4 \longrightarrow \mathrm{Irr}^{\mathrm{t}}(G)^{\mathfrak{s}}$$

The detail of this map is as follow:

$$
\begin{aligned}
&(z_1, z_2) \in \mathbb{T}^2 \longmapsto \mathrm{Ind}_{(2+1+1)}^{(4)}(z_1 \cdot St_2(\pi) \otimes z_2^{val \circ det}\pi \otimes \pi) \\
&\qquad \hookrightarrow \mathrm{Ind}_M^G(\nu^{-\frac{1}{2}} z_1^{val \circ det}\pi \otimes \nu^{\frac{1}{2}} z_1^{val \circ det}\pi \otimes z_2^{val \circ det}\pi \otimes \pi) \\
&z \in \mathbb{T} \longmapsto \mathrm{Ind}_{(3+1)}^{4}(z \cdot St_3(\pi) \otimes \pi) \\
&\qquad \hookrightarrow \mathrm{Ind}_M^G(\nu^{-1} z^{val \circ det}\pi \otimes z^{val \circ det}\pi \otimes \nu z^{val \circ det}\pi \otimes \pi) \\
\mu : \quad &z \in \mathbb{T} \longmapsto \mathrm{Ind}_{(2+2)}^{4}(z \cdot St_2(\pi) \otimes St_2(\pi)) \\
&\qquad \hookrightarrow \mathrm{Ind}_M^G(\nu^{-\frac{1}{2}} z^{val \circ det}\pi \otimes \nu^{\frac{1}{2}} z^{val \circ det}\pi \otimes \nu^{-\frac{1}{2}}\pi \otimes \nu^{\frac{1}{2}}\pi) \\
&pt_1 \in \mathbb{T} \longmapsto \rho^+ \\
&\qquad \hookrightarrow \mathrm{Ind}_M^G(\nu^{-\frac{1}{2}}(-1)^{val \circ det}\pi \otimes \nu^{\frac{1}{2}}(-1)^{val \circ det}\pi \otimes \nu^{-\frac{1}{2}}\pi \otimes \nu^{\frac{1}{2}}\pi) \\
&pt_2 \in \mathbb{T} \longmapsto St_4(\pi) \\
&\qquad \hookrightarrow \mathrm{Ind}_M^G(\nu^{-\frac{3}{2}}\pi \otimes \nu^{-\frac{1}{2}}\pi \otimes \nu^{\frac{1}{2}}\pi \otimes \nu^{\frac{3}{2}}\pi) \\
&z \in \mathbb{T} \longmapsto \lambda(t)^+ \\
&\qquad \hookrightarrow \mathrm{Ind}_M^G(z^{val \circ det}\pi \otimes (-z)^{val \circ det}\pi \otimes \pi \otimes (-1)^{val \circ det}\pi) \\
&pt_3 \in \mathbb{T} \longmapsto \tau_3 \\
&\qquad \hookrightarrow \mathrm{Ind}_M^G(i^{val \circ det}\pi \otimes (-i)^{val \circ det}\pi \otimes \pi \otimes (-1)^{val \circ det}\pi) \\
&pt_4 \in \mathbb{T} \longmapsto \tau_4 \\
&\qquad \hookrightarrow \mathrm{Ind}_M^G(i^{val \circ det}\pi \otimes (-i)^{val \circ det}\pi \otimes \pi \otimes (-1)^{val \circ det}\pi) \\
&t \in E^{\mathfrak{s}}/W^{\mathfrak{s}} \longmapsto \mathrm{Ind}_M^G(\chi_t \otimes \sigma)
\end{aligned}
$$

Now we investigate the image of $inf.ch$:

$$\begin{aligned}
inf.ch(\mu((z_1,z_2))) &= [M, \nu^{-\frac{1}{2}} z_1^{val\circ det}\pi \otimes \nu^{\frac{1}{2}} z_1^{val\circ det}\pi \otimes z_2^{val\circ det}\pi \otimes \pi]_G \\
&\longmapsto (q^{\frac{1}{2}}z_1, q^{-\frac{1}{2}}z_1, z_2, 1) \in D^{\mathfrak{s}}/W^{\mathfrak{s}} \\
inf.ch(\mu((z))) &= [M, \nu^{-1} z^{val\circ det}\pi \otimes z^{val\circ det}\pi \otimes \nu z^{val\circ det}\pi \otimes \pi]_G \\
&\longmapsto (qz, z, q^{-1}z, 1) \in D^{\mathfrak{s}}/W^{\mathfrak{s}} \\
inf.ch(\mu((z))) &= [M, \nu^{-\frac{1}{2}} z_1^{val\circ det}\pi \otimes \nu^{\frac{1}{2}} z_1^{val\circ det}\pi \otimes z_2^{val\circ det}\pi \otimes \pi]_G \\
&\longmapsto (q^{\frac{1}{2}}z, q^{-\frac{1}{2}}z, q^{\frac{1}{2}}, q^{-\frac{1}{2}}) \in D^{\mathfrak{s}}/W^{\mathfrak{s}} \\
inf.ch(\mu(pt_1)) &= [M, \nu^{-\frac{1}{2}}(-1)^{val\circ det}\pi \otimes \nu^{\frac{1}{2}}(-1)^{val\circ det}\pi \otimes \nu^{-\frac{1}{2}}\pi \otimes \nu^{\frac{1}{2}}\pi]_G \\
&\longmapsto (-q^{\frac{1}{2}}z_1, -q^{-\frac{1}{2}}, q^{\frac{1}{2}}, q^{-\frac{1}{2}}) \in D^{\mathfrak{s}}/W^{\mathfrak{s}} \\
inf.ch(\mu(pt_2)) &= [M, \nu^{-\frac{3}{2}}\pi \otimes \nu^{-\frac{1}{2}}\pi \otimes \nu^{\frac{1}{2}}\pi \otimes \nu^{\frac{3}{2}}\pi]_G \\
&\longmapsto (q^{\frac{3}{2}}, q^{\frac{1}{2}}, q^{-\frac{1}{2}}, q^{-\frac{3}{2}}) \in D^{\mathfrak{s}}/W^{\mathfrak{s}} \\
inf.ch(\mu((z))) &= [M, z^{val\circ det}\pi \otimes (-z)^{val\circ det}\pi \otimes \pi \otimes (-1)^{val\circ det}\pi]_G \\
&\longmapsto (z, -z, 1, -1) \in E^{\mathfrak{s}}/W^{\mathfrak{s}} \\
inf.ch(\mu(pt_3)) &= [M, i^{val\circ det}\pi \otimes (-i)^{val\circ det}\pi \otimes \pi \otimes (-1)^{val\circ det}\pi]_G \\
&\longmapsto (i, -i, 1, -1) \in E^{\mathfrak{s}}/W^{\mathfrak{s}} \\
inf.ch(\mu(pt_4)) &= [M, i^{val\circ det}\pi \otimes (-i)^{val\circ det}\pi \otimes \pi \otimes (-1)^{val\circ det}\pi]_G \\
&\longmapsto (i, -i, 1, -1) \in E^{\mathfrak{s}}/W^{\mathfrak{s}} \\
inf.ch(\mu(t)) &= [M, z_1^{val\circ det}\pi \otimes z_2^{val\circ det}\pi \otimes \nu z_3^{val\circ det}\pi \otimes \pi]_G \\
&\longmapsto (z_1, z_2, z_3, 1) \in E^{\mathfrak{s}}/W^{\mathfrak{s}}
\end{aligned}$$

It is clear that $\pi_{\sqrt{q}} = inf.ch \circ \mu$. The graph below shows the relation between $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ and irreducible tempered representations.

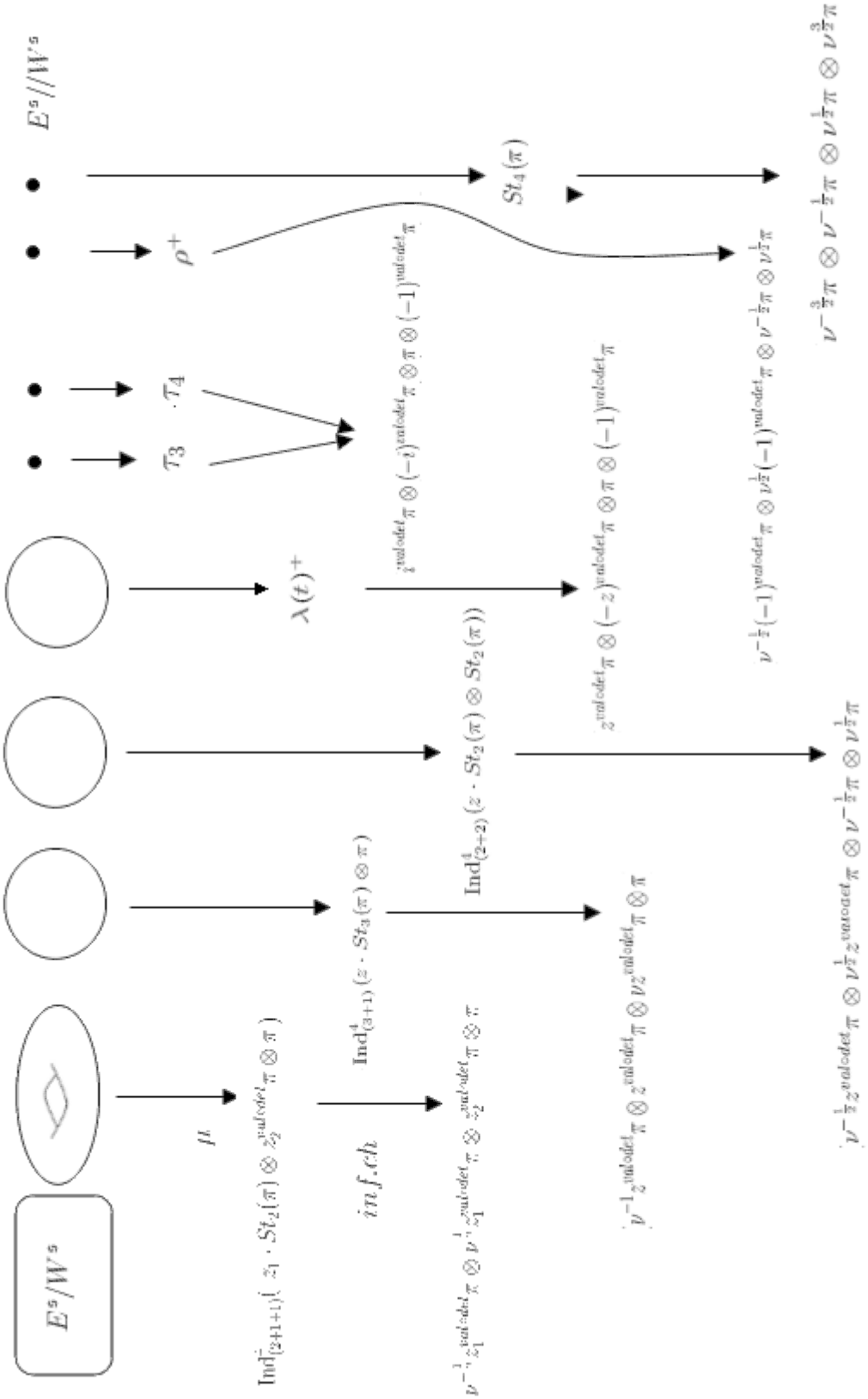

Figure 6.1: $\mathfrak{s} = [M, \pi \otimes \pi \otimes \pi \otimes \pi]_G$

Hence, we have

Lemma 6.1.1. *Part (3) of the geometric conjecture is true for* $\mathfrak{s} = [M, \pi\otimes\pi\otimes\pi\otimes\pi]_G$.

6.1.2 Case 2: $\pi_\sigma \cong \pi_1 \otimes \pi_1 \otimes \pi_1 \otimes \pi_2$

For this case, we have shown that the isotropy group is the symmetric group $\mathfrak{S}_3$. The group structure is shown by the tables below:

$W^{\mathfrak{s}} = \mathfrak{S}_3$	
γ	$\gamma(abcd)$
γ_1	$abcd$
γ_2	$acbd$
γ_3	$bacd$
γ_4	$bcad$
γ_5	$cabd$
γ_6	$cbad$

There are three conjugacy classes. They are $\{\gamma_1\}$, $\{\gamma_2, \gamma_3, \gamma_6\}$, $\{\gamma_4, \gamma_5\}$. Now we choose γ_1, γ_2 and γ_4 as the representatives for their own conjugacy classes. The extended quotient is given as follows:

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\gamma_1}/Z(\gamma_1) \sqcup E^{\gamma_2}/Z(\gamma_2) \sqcup E^{\gamma_4}/Z(\gamma_4)$$

where $Z(\gamma_1) = W^{\mathfrak{s}}$, $Z(\gamma_2) = \{\gamma_1, \gamma_2\}$ and $Z(\gamma_4) = \{\gamma_1, \gamma_4, \gamma_5\}$. Now we analysis case by case.

- $\gamma = \gamma_1$, $E^\gamma/Z(\gamma) = E^{\mathfrak{s}}/W^{\mathfrak{s}}$.
- $\gamma = \gamma_2$, $E^\gamma = \{(a, a, c, d) : a, c, d \in \mathbb{T}\}/\mathbb{T} \cong \{(z_1, z_1, z_2, 1) : z_1, z_2 \in \mathbb{T}\} \cong \mathbb{T}^2$.
- $\gamma = \gamma_4$, $E^\gamma = \{(a, a, a, c) : a, c \in \mathbb{T}\}/\mathbb{T} \cong \{(z, z, z, 1) : z \in \mathbb{T}\} \cong \mathbb{T}$.

  Hence, we have

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T}^2 \sqcup \mathbb{T}$$

The following we try to exhaust the tempered dual with respect to $\mathfrak{s}$. We consider the induced representation $\mathrm{Ind}_M^G(\nu^{-\frac{1}{2}} z_1^{val\circ det}\pi_1 \otimes \nu^{\frac{1}{2}} z_1^{val\circ det}\pi_1 \otimes z_2^{val\circ det}\pi_1 \otimes \pi_2)$. In fact, it is reducible and there is an irreducible subquotient which is the tensor product of generalized Steinberg representation and some representations:

$$z_1 \cdot St_2(\pi_1) \otimes z_2^{val\circ det}\pi_1 \otimes \pi_2 \hookrightarrow \mathrm{Ind}_{(1+1+1+1)}^{(2+1+1)}(\nu^{-\frac{1}{2}} z_1^{val\circ det}\pi_1 \otimes \nu^{\frac{1}{2}} z_1^{val\circ det}\pi_1 \otimes z_2^{val\circ det}\pi_1 \otimes \pi_2)$$

Thus, we get the elements in tempered dual. It is

$$\mathrm{Ind}_{(2+1+1)}^4(z_1 \cdot St_2(\pi_1) \otimes z_2^{val\circ det}\pi_1 \otimes \pi_2)$$

These representations can be represented by $\mathbb{T}^2$ and we set the cocharacter by

$$h_c(t) = (t, t^{-1}, 1, 1).$$

As above, similarly, we consider the representation $\mathrm{Ind}_M^G(\nu^{-1} z^{val\circ det}\pi_1 \otimes z^{val\circ det}\pi_1 \otimes \nu z^{val\circ det}\pi_1 \otimes \pi_2)$ for $z \in \mathbb{T}$. It is known that the representation is reducible and there is an irreducible subquotient:

$$z \cdot St_3(\pi_1) \otimes \pi_2 \hookrightarrow \mathrm{Ind}_{(1+1+1+1)}^{(3+1)}(\nu^{-1} z^{val\circ det}\pi_1 \otimes z^{val\circ det}\pi_1 \otimes \nu z^{val\circ det}\pi_1 \otimes \pi_2)$$

Hence, we know that

$$\mathrm{Ind}_{(3+1)}^4(z \cdot St_3(\pi_1) \otimes \pi_2)$$

are in the tempered dual and we identify these by $\mathbb{T}$. For this component, we set the cocharacter:

$$h_c(t) = (t^2, 1, t^{-2}, 1).$$

Now we consider the point $t = (z_1, z_2, z_3, 1) \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$ and χ_t is the unramified unitary character generated by t. In fact, the induced representation $\mathrm{Ind}_M^G(\chi_t \otimes \sigma)$ is irreducible. This implies each unramified unitary character can generate a tempered representation of G with respect to $\mathfrak{s}$.

We turn to build up the map μ satisfying the geometric conjecture, i.e.

$$\mu : E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T}^2 \sqcup \mathbb{T} \longrightarrow \mathrm{Irr}^{\mathfrak{t}}(G)^{\mathfrak{s}}$$

The detail of this map is as follow:

$$
\begin{aligned}
& (z_1, z_2) \in \mathbb{T}^2 \longmapsto \mathrm{Ind}^4_{(2+1+1)}(z_1 \cdot St_2(\pi_1) \otimes z_2^{val \circ det} \pi_1 \otimes \pi_2) \\
& \qquad \hookrightarrow \mathrm{Ind}^G_M(\nu^{-\frac{1}{2}} z_1^{val \circ det} \pi_1 \otimes \nu^{\frac{1}{2}} z_1^{val \circ det} \pi_1 \otimes z_2^{val \circ det} \pi_1 \otimes \pi_2) \\
\mu : \quad & z \in \mathbb{T} \longmapsto \mathrm{Ind}^4_{(3+1)}(z \cdot St_3(\pi_1) \otimes \pi_2) \\
& \qquad \hookrightarrow \mathrm{Ind}^G_M(\nu^{-1} z^{val \circ det} \pi_1 \otimes z^{val \circ det} \pi_1 \otimes \nu z^{val \circ det} \pi_1 \otimes \pi_2) \\
& t \in E^{\mathfrak{s}}/W^{\mathfrak{s}} \longmapsto \mathrm{Ind}^G_M(\chi_t \otimes \sigma)
\end{aligned}
$$

Now we investigate the image of $inf.ch$ (infinitesimal character)

$$
\begin{aligned}
inf.ch(\mu((z_1, z_2))) &= [M, \nu^{-\frac{1}{2}} z_1^{val \circ det} \pi_1 \otimes \nu^{\frac{1}{2}} z_1^{val \circ det} \pi_1 \otimes z_2^{val \circ det} \pi_1 \otimes \pi_2]_G \\
&\longmapsto (q^{\frac{1}{2}} z_1, q^{-\frac{1}{2}} z_1, z_2, 1) \in D^{\mathfrak{s}}/W^{\mathfrak{s}} \\
inf.ch(\mu(z)) &= [M, \nu^{-1} z^{val \circ det} \pi_1 \otimes z^{val \circ det} \pi_1 \otimes \nu z^{val \circ det} \pi_1 \otimes \pi_2]_G \\
&\longmapsto (qz, z, q^{-1} z, 1) \in D^{\mathfrak{s}}/W^{\mathfrak{s}} \\
inf.ch(\mu(t)) &= [M, z_1^{val \circ det} \pi_1 \otimes z_2^{val \circ det} \pi_1 \otimes \nu z_3^{val \circ det} \pi_1 \otimes \pi_2]_G \\
&\longmapsto (z_1, z_2, z_3, 1) \in E^{\mathfrak{s}}/W^{\mathfrak{s}}
\end{aligned}
$$

It is clear that $\pi_{\sqrt{q}} = inf.ch \circ \mu$. The graph below shows the relation between $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ and irreducible tempered representations.

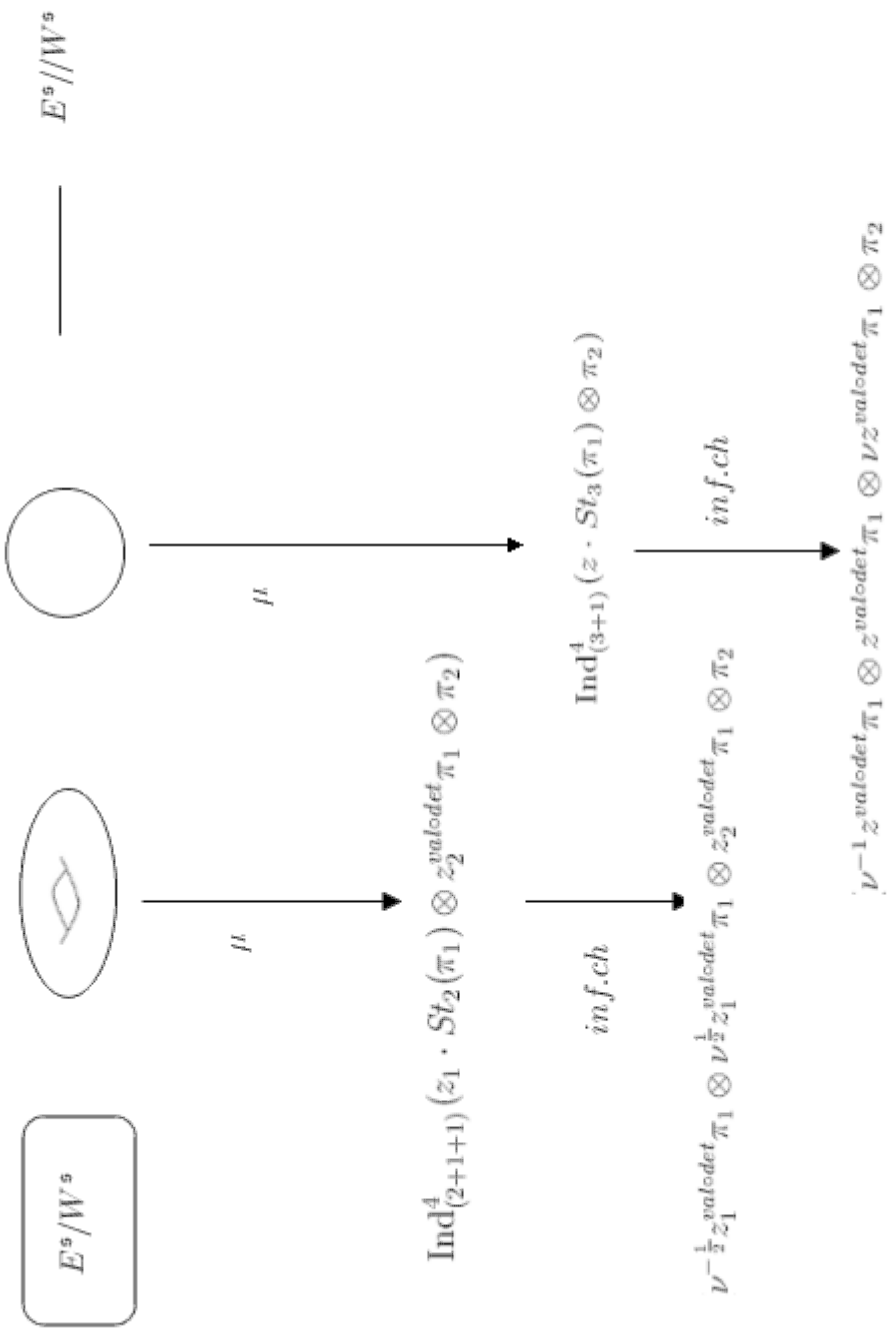

Figure 6.2: $\mathfrak{s} = [M, \pi_1 \otimes \pi_1 \otimes \pi_1 \otimes \pi_2]_G$

Hence, we have

Lemma 6.1.2. *Part (3) of the geometric conjecture is true for* $\mathfrak{s} = [M, \pi_1 \otimes \pi_1 \otimes \pi_1 \otimes \pi_2]_G$.

6.1.3 Case 3: $\pi_\sigma \cong \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_2$

From the table, we have known that there are two distinct cases occurs. Firstly, we discuss the first one case 3.1.

In the case 3.1, the corresponding π_σ is $\pi_\sigma \cong \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_2$. The R-group $R(\sigma)$ is trivial and the isotropy subgroup $W^{\mathfrak{s}}$ is given by $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Now we investigate the extended quotient $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ respect to $\mathfrak{s}$. By the property of extended quotient, we know that

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = \bigsqcup_{\gamma} E^{\gamma}/Z(\gamma)$$

where γ is the representative in its conjugacy class. Since $W^{\mathfrak{s}}$ is abelian in this case, then we know that each element contribute one conjugacy class and the centralizer $Z(\gamma_i)$ of γ_i is W itself. Then, we have

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\gamma_1}/W^{\mathfrak{s}} \sqcup E^{\gamma_2}/W^{\mathfrak{s}} \sqcup E^{\gamma_3}/W^{\mathfrak{s}} \sqcup E^{\gamma_4}/W^{\mathfrak{s}}$$

We will analysis case by case. The group $W^{\mathfrak{s}}$ is given as follows:

$W^{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	
γ	$\gamma(abcd)$
γ_1	$abcd$
γ_2	$bacd$
γ_3	$abdc$
γ_4	$badc$

Now, we compute the each component in extended quotient $E^{\mathfrak{s}}//W^{\mathfrak{s}}$.

- $\gamma = \gamma_1$, $E^{\gamma}/Z(\gamma) = E^{\mathfrak{s}}/W^{\mathfrak{s}}$.

- $\gamma = \gamma_2$, $E^\gamma = \{(a,a,c,d) : a,c,d \in \mathbb{T}\}/\mathbb{T} \cong \{(1,1,z_1,z_2) : z_1,z_2 \in \mathbb{T}\} \cong \mathbb{T}^2$. Then we have
$$E^\gamma/Z(\gamma) \cong \mathbb{T}^2.$$
- $\gamma = \gamma_3$, $E^\gamma = \{(a,b,c,c) : a,b,c \in \mathbb{T}\}/\mathbb{T} \cong \{(z_1,z_2,1,1) : z_1,z_2 \in \mathbb{T}\} \cong \mathbb{T}^2$.
$$E^\gamma/Z(\gamma) \cong \mathbb{T}^2.$$
- $\gamma = \gamma_4$, $E^\gamma = \{(a,a,c,c),(a,-a,c,-c) : a,c \in \mathbb{T}\}/\mathbb{T} \cong \{(1,1,z,z) : z \in \mathbb{T}\} \sqcup \{(1,-1,z,-z) : z \in \mathbb{T}\} \cong \mathbb{T} \sqcup \mathbb{T}$.
$$E^\gamma/Z(\gamma) \cong \mathbb{T} \sqcup \mathbb{T}.$$

Finally, we get the component explicitly in $E^{\mathfrak{s}}//W^{\mathfrak{s}}$:

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T}^2 \sqcup \mathbb{T}^2 \sqcup \mathbb{T} \sqcup \mathbb{T}$$

Then we will try to build the bijection μ between $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ and the set $\mathrm{Irr}^{\mathrm{t}}(G)^{\mathfrak{s}}$ of the irreducible tempered representation of G with respect to $\mathfrak{s}$.

Now we consider the representation induced by $(\nu^{-\frac{1}{2}} z_1^{\mathrm{val}\circ det}\pi_1 \otimes \nu^{\frac{1}{2}} z_1^{\mathrm{val}\circ det}\pi_1 \otimes z_2^{\mathrm{val}\circ det}\pi_2 \otimes \pi_2)$ from $(1+1+1+1)$ to $(2+1+1)$. In fact, this is reducible and there is an irreducible square-integrable representation to be a subquotient of this induced representation. Indeed, the irreducible subquotient is $z_1 \cdot St_2(\pi_1) \otimes z_2^{\mathrm{val}\circ det}\pi_2 \otimes \pi_2$. Therefore, we get a tempered representation of G. It is given by

$$\mathrm{Ind}^4_{(2+1+1)}(z_1 \cdot St_2(\pi_1) \otimes z_2^{\mathrm{val}\circ det}\pi_2 \otimes \pi_2)$$

Indeed, this induced representation is irreducible. We identify these representations by $\mathbb{T}^2$ and set the cocharacter for these representation by

$$h_c(t) = (t, t^{-1}, 1, 1).$$

Then we do the analogy as above and consider the representation induced by $(\pi_1 \otimes z_1^{\mathrm{val}\circ det}\pi_1 \otimes \nu^{-\frac{1}{2}} z_2^{\mathrm{val}\circ det}\pi_2 \otimes \nu^{\frac{1}{2}} z_2^{\mathrm{val}\circ det}\pi_2)$ from $(1+1+1+1)$ to $(2+1+1)$.

Therefore, we get another tempered representation which is given by

$$\mathrm{Ind}^4_{(1+1+2)}(\pi_1 \otimes z_1^{\mathrm{val}\circ det}\pi_1 \otimes z_2 \cdot St_2(\pi_2))$$

We identify these representations by $\mathbb{T}^2$ and set the cocharacter for these by

$$h_c(t) = (1, 1, t, t^{-1}).$$

In the following, we consider the induced representation

$$\mathrm{Ind}^{(2+2)}_{(1+1+1+1)}(\nu^{-\frac{1}{2}} z_1^{\mathrm{val}\circ det}\pi_1 \otimes \nu^{\frac{1}{2}} z_1^{\mathrm{val}\circ det}\pi_1 \otimes \nu^{-\frac{1}{2}}\pi_2 \otimes \nu^{\frac{1}{2}}\pi_2)$$

Actually, this representations is reducible and there is an irreducible subquotient:

$$z \cdot St_2(\pi_1) \otimes St_2(\pi_2)$$

Therefore, the induced representation

$$\mathrm{Ind}^4_{(2+2)}(z \cdot St_2(\pi_1) \otimes St_2(\pi_2))$$

is the element in $\mathrm{Irr}^{\mathfrak{t}}(G)^{\mathfrak{s}}$. Since z varies in $\mathbb{T}$, these elements can be identified by $\mathbb{T}$. For these representations, we set the cocharacter by

$$h_c(t) = (t, t^{-1}, t, t^{-1}).$$

The next is to consider the non-special representations. Let $t = (1, -1, z, -z)$ and χ_t be the corresponding character. We have $\chi_t \otimes \pi_\sigma = \pi_1 \otimes (-1)^{val}\pi_1 \otimes z^{val\circ det}\pi_2 \otimes (-z)^{val\circ det}\pi_2$. It is not hard to get $\bar{L}(\chi_t \otimes \pi_\sigma) =< (-1)^{val\circ det} >$ and $X(\chi_t \otimes \pi_\sigma) = 1$. Thus, $R(\chi_t \otimes \sigma) = \mathbb{Z}/2\mathbb{Z}$. It leads the induced representation $\mathrm{Ind}^G_M(\chi_t \otimes \sigma)$ is reducible and there are two irreducible subrepresentations $\pi(t)^+$, $\pi(t)^-$ of G, i.e.

$$\mathrm{Ind}^G_M(\chi_t \otimes \sigma) = \pi(t)^+ \oplus \pi(t)^-$$

Hence, $\pi(t)^+$ and $\pi(t)^-$ are tempered. They will contribute the elements in $\mathrm{Irr}^{\mathfrak{t}}(G)^{\mathfrak{s}}$. Such induced representations will be identified by $\mathbb{T}$ and we set the character

$$h_c(t) = 1.$$

Indeed, each unitary unramified twist of σ except the type $t = (1, -1, z, -z)$ will be an irreducible tempered representation. In other word, every point t in $E^{\mathfrak{s}}/W^{\mathfrak{s}}$, which does not belong to $(1, -1, z, -z)$, will generate an unitary unramified character of $\mathbb{T}$, i.e. $\chi_t = (z_1^{val}, z_2^{val}, z_3^{val}, z_4^{val})$ and the representation $\mathrm{Ind}_M^G(\chi_t \otimes \sigma)$ induced by $\chi_t \otimes \sigma$ is irreducible. Thus such induced representation contributes an element in $\mathrm{Irr}^{\mathfrak{t}}(G)^{\mathfrak{s}}$.

The detail of this map is as follow:

$$
\begin{aligned}
&(z_1, z_2) \in \mathbb{T}^2 \longmapsto \mathrm{Ind}^4_{(2+1+1)}(z_1 \cdot St_2(\pi_1) \otimes z_2^{val \circ det}\pi_2 \otimes \pi_2) \\
&\qquad \hookrightarrow \mathrm{Ind}^4_{(1+1+1+1)}(\nu^{-\frac{1}{2}} z_1^{val \circ det}\pi_1 \otimes \nu^{\frac{1}{2}} z_1^{val \circ det}\pi_1 \otimes z_2^{val \circ det}\pi_2 \otimes \pi_2) \\
&(z_1, z_2) \in \mathbb{T}^2 \longmapsto \mathrm{Ind}^4_{(1+1+2)}(\pi_1 \otimes z_1^{val \circ det}\pi_1 \otimes z_2 \cdot St_2(\pi_2)) \\
&\qquad \hookrightarrow \mathrm{Ind}^4_{(1+1+1+1)}(\pi_1 \otimes z_1^{val \circ det}\pi_1 \otimes \nu^{-\frac{1}{2}} z_2^{val \circ det}\pi_2 \otimes \nu^{\frac{1}{2}} z_2^{val \circ det}\pi_2) \\
\mu : \quad & z \in \mathbb{T} \longmapsto \mathrm{Ind}^4_{(2+2)}(z \cdot St_2(\pi_1) \otimes St_2(\pi_2)) \\
&\qquad \hookrightarrow \mathrm{Ind}^{(4)}_{(1+1+1+1)}(\nu^{-\frac{1}{2}} z^{val \circ det}\pi_1 \otimes \nu^{\frac{1}{2}} z^{val \circ det}\pi_1 \otimes \nu^{-\frac{1}{2}}\pi_2 \otimes \nu^{\frac{1}{2}}\pi_2) \\
& z \in \mathbb{T} \longmapsto \pi(t)^+ \\
&\qquad \hookrightarrow \mathrm{Ind}^G_M(\pi_1 \otimes (-1)^{val \circ det}\pi_1 \otimes z^{val \circ det}\pi_2 \otimes \nu(-z)^{val \circ det}\pi_2) \\
& t \in E^{\mathfrak{s}}/W^{\mathfrak{s}} \longmapsto \mathrm{Ind}^G_M(\chi_t \otimes \sigma)
\end{aligned}
$$

Now we investigate the image of $inf.ch$ (infinitesimal character)

$$\begin{aligned}
inf.ch(\mu((z_1,z_2))) &= [M, \nu^{-\frac{1}{2}} z_1^{val\circ det}\pi_1 \otimes \nu^{\frac{1}{2}} z_1^{val\circ det}\pi_1 \otimes z_2^{val\circ det}\pi_2 \otimes \pi_2]_G \\
&\longmapsto (q^{\frac{1}{2}} z_1, q^{-\frac{1}{2}} z_1, z_2, 1) \in D^{\mathfrak{s}}/W^{\mathfrak{s}} \\
inf.ch(\mu((z_1,z_2))) &= [M, \pi_1 \otimes z_1^{val\circ det}\pi_1 \otimes \nu^{-\frac{1}{2}} z_2^{val\circ det}\pi_2 \otimes \nu^{\frac{1}{2}} z_2^{val\circ det}\pi_2]_G \\
&\longmapsto (1, z_1, q^{\frac{1}{2}} z_2, q^{-\frac{1}{2}} z_2) \in D^{\mathfrak{s}}/W^{\mathfrak{s}} \\
inf.ch(\mu(z)) &= [M, \nu^{-\frac{1}{2}} z^{val\circ det}\pi_1 \otimes \nu^{\frac{1}{2}} z^{val\circ det}\pi_1 \otimes \nu^{-\frac{1}{2}}\pi_2 \otimes \nu^{\frac{1}{2}}\pi_2]_G \\
&\longmapsto (q^{\frac{1}{2}} z, q^{-\frac{1}{2}} z, q^{\frac{1}{2}}, q^{-\frac{1}{2}}) \in D^{\mathfrak{s}}/W^{\mathfrak{s}} \\
inf.ch(\mu(z)) &= [M, \pi_1 \otimes (-1)^{val\circ det}\pi_1 \otimes z^{val\circ det}\pi_2 \otimes (-z)^{val\circ det}\pi_2]_G \\
&\longmapsto (1, -1, z, -z) \in E^{\mathfrak{s}}/W^{\mathfrak{s}} \\
inf.ch(\mu(t)) &= [M, z_1^{val\circ det}\pi_1 \otimes z_2^{val\circ det}\pi_1 \otimes z_3^{val\circ det}\pi_2 \otimes \pi_2]_G \\
&\longmapsto (z_1, z_2, z_3, 1) \in E^{\mathfrak{s}}/W^{\mathfrak{s}}
\end{aligned}$$

The graph below shows the relation between $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ and irreducible tempered representations.

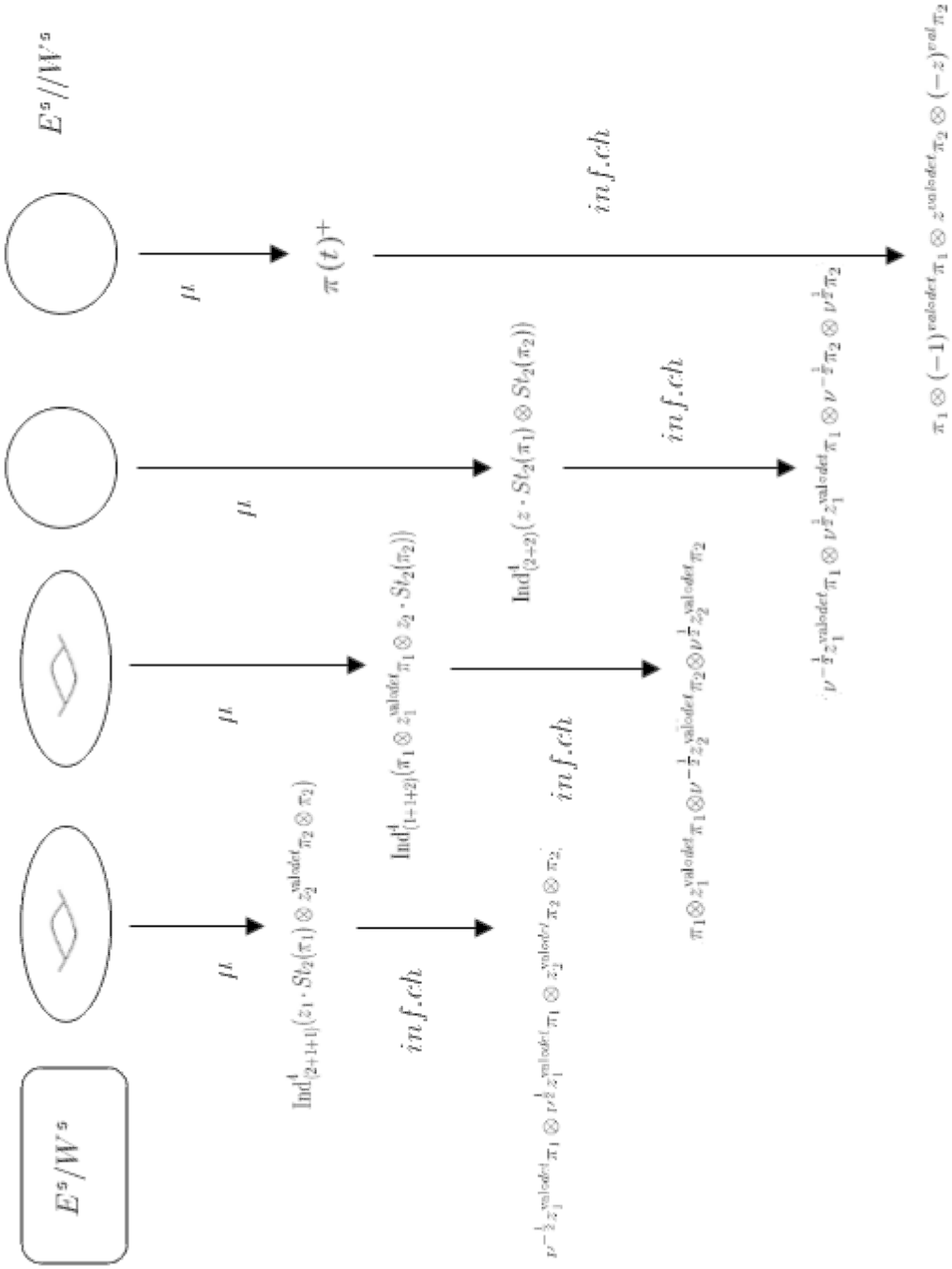

Figure 6.3: $\mathfrak{s} = [M, \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_2]_G$

Therefore, the condition $\pi_{\sqrt{q}} = inf.ch \circ \mu$ is satisfied. We conclude

Lemma 6.1.3. *Part (3) of the geometric conjecture is true for* $\mathfrak{s} = [M, \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_2]_G$.

The next we will discuss the case 3.2. In this case, the representation π_σ is of the form $\pi \otimes \pi \otimes \eta\pi \otimes \eta\pi$. As we know, R-group is $\mathbb{Z}/2\mathbb{Z}$ and the isotropy group is $\mathbb{Z}/\mathbb{Z}2 \wr \mathbb{Z}/2\mathbb{Z}$ generated by $< \gamma_2, \gamma_3, \gamma_4 >$ where $\gamma_2, \gamma_3, \gamma_4$ are explicitly given by the table below:

$W^{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$	
γ	$\gamma(abcd)$
γ_1	$abcd$
γ_2	$bacd$
γ_3	$abdc$
γ_4	$cdab$
γ_5	$badc$
γ_6	$dcab$
γ_7	$cdba$
γ_8	$dcba$

In fact, there are five conjugacy classes:

$$\{\gamma_1\}, \{\gamma_2, \gamma_3\}, \{\gamma_4, \gamma_8\}, \{\gamma_5\}, \{\gamma_6, \gamma_7\}$$

From above, there are five conjugacy classes. The extended quotient $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ is given by

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = \bigsqcup_{\gamma} E^{\gamma}/Z(\gamma)$$

and we choose γ_1, γ_2, γ_4, γ_5 and γ_6 to be the representatives in their own conjugacy classes and their centralizers are as follows:

$$Z(\gamma_1) = W^{\mathfrak{s}}, Z(\gamma_2) = \{\gamma_1, \gamma_2, \gamma_3\}, Z(\gamma_4) = \{\gamma_1, \gamma_4, \gamma_5, \gamma_8\} \tag{6.5}$$

$$Z(\gamma_5) = W^{\mathfrak{s}}, Z(\gamma_6) = \{\gamma_1, \gamma_6, \gamma_7\} \tag{6.6}$$

Then we will have:

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\gamma_1}/Z(\gamma_1) \sqcup E^{\gamma_2}/Z(\gamma_2) \sqcup E^{\gamma_4}/Z(\gamma_4) \sqcup E^{\gamma_5}/Z(\gamma_5) \sqcup E^{\gamma_6}/Z(\gamma_6)$$

Now we analysis case by case.

- $\gamma = \gamma_1$, $E^{\gamma}/W^{\mathfrak{s}} = E/W^{\mathfrak{s}}$.
- $\gamma = \gamma_2$, $E^{\gamma} = \{(a, a, c, d) : a, c, d \in \mathbb{T}\}/\mathbb{T} \cong \{(z_1, z_1, z_2, 1) : z_1, z_2 \in \mathbb{T}\} \cong \mathbb{T}^2$.
- $\gamma = \gamma_4$, $E^{\gamma} = \{(a, b, a, b), (a, b, -a, -b) : a, b \in \mathbb{T}\}/\mathbb{T} \cong \{(z, 1, z, 1), (z, 1, -z, -1) : z \in \mathbb{T}\} \cong \mathbb{T} \sqcup \mathbb{T}$.
- $\gamma = \gamma_5$, $E^{\gamma} = \{(a, a, b, b), (a, -a, b, -b) : a, b \in \mathbb{T}\}/\mathbb{T} \cong \{(1, 1, z, z), (1, -1, z, -z) : z \in \mathbb{T}\} \cong \mathbb{T} \sqcup \mathbb{T}$.
- $\gamma = \gamma_6$, $E^{\gamma} = \{(1, 1, 1, 1), (1, -1, 1, -1), (1, i, -1, -i), (1, -i, -1, i)\} \cong pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4$.

Then, we have the decomposition:

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T}^2 \sqcup \mathbb{T} \sqcup \mathbb{T} \sqcup \mathbb{T} \sqcup \mathbb{T} \sqcup pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4$$

Now, we consider each component in $E^{\mathfrak{s}}/W^{\mathfrak{s}}$. Firstly, we consider the special representations. Now we consider the representation induced by $(\nu^{-\frac{1}{2}} z_1^{\mathrm{val} \circ det} \pi \otimes \nu^{\frac{1}{2}} z_1^{\mathrm{val} \circ det} \pi \otimes z_2^{\mathrm{val} \circ det} \eta\pi \otimes \eta\pi)$ from $(1 + 1 + 1 + 1)$ to $(2 + 1 + 1)$. In fact, this is reducible and there is an irreducible square-integrable representation to be a subquotient of this induced representation. Actually the irreducible subquotient is $z_1 \cdot St_2(\pi) \otimes z_2^{\mathrm{val} \circ det} \eta\pi \otimes \eta\pi$. Therefore, we get a tempered representation of G. It is given by

$$\mathrm{Ind}_{(2+1+1)}^{4}(z_1 \cdot St_2(\pi) \otimes z_2^{\mathrm{val} \circ det} \eta\pi \otimes \eta\pi)$$

Indeed, this induced representation is irreducible. We identify these representations by $\mathbb{T}^2$ and we denote this component by c_1 and set the cocharacter for these representation by

$$h_{c_1}(t) = (t, t^{-1}, 1, 1).$$

In the next, we consider the induced representation

$$\mathrm{Ind}_{(1+1+1+1)}^{(2+2)}(\nu^{-\frac{1}{2}} z^{\mathrm{val}\circ det}\pi_1 \otimes \nu^{\frac{1}{2}} z^{\mathrm{val}\circ det}\pi \otimes \nu^{-\frac{1}{2}}\eta\pi \otimes \nu^{\frac{1}{2}}\eta\pi)$$

Actually, this representations is reducible and there is an irreducible subquotient which is discrete series:

$$z \cdot St_2(\pi) \otimes St_2(\eta\pi)$$

Therefore, the induced representation

$$\mathrm{Ind}_{(2+2)}^{4}(z \cdot St_2(\pi) \otimes St_2(\eta\pi))$$

is the element in $\mathrm{Irr}^{\mathfrak{t}}(G)^{\mathfrak{s}}$. Since z varies in $\mathbb{T}$, these elements can be identified by $\mathbb{T}$ and we denote this component by $c = c_2$. For these representations, we set the cocharacter by

$$h_{c_2}(t) = (t, t^{-1}, t, t^{-1}).$$

In the following, we consider the point $t = (z, 1, -z, -1)$. Thus, $\chi_t \otimes \pi_\sigma = z^{val\circ det}\pi \otimes \pi \otimes (-z)^{val\circ det}\eta\pi \otimes (-1)^{val\circ det}\eta\pi$. We have $\bar{L}(\chi_t \otimes \pi_\sigma) =< (-1)^{val\circ det}\eta >$ and $X(\pi_\sigma) = 1$. Thus $R(\chi_t \otimes \sigma) = \mathbb{Z}/2\mathbb{Z}$. It leads the induced representation $\rho(z)$ is reducible and there are two irreducible constituents $\rho(z)^+$, $\rho(z)^-$, i.e.

$$\rho(z) = \mathrm{Ind}_M^G(z^{val\circ det}\pi \otimes \pi \otimes (-z)^{val\circ det}\eta\pi \otimes (-1)^{val\circ det}\eta\pi = \rho(z)^+ \oplus \rho(z)^-$$

We identify these representations by $\mathbb{T}$ and denote it by c_3.

Then we consider the point $t = (z, 1, z, 1)$. We get $\chi_t \otimes \pi_\sigma = z^{val\circ det}\pi \otimes \pi \otimes z^{val\circ det}\eta\pi \otimes \eta\pi$. We have $\bar{L}(\chi_t \otimes \pi_\sigma) =< \eta >$ and $X(\pi_\sigma) = 1$. Thus $R(\chi_t \otimes \sigma) = \mathbb{Z}/2\mathbb{Z}$. It leads the induced representation $\varrho(z)$ is reducible. There are two irreducible constituents $\varrho(z)^+$, $\varrho(z)^-$, i.e.

$$\varrho(z) = \mathrm{Ind}_M^G(z^{val\circ det}\pi \otimes \pi \otimes z^{val\circ det}\eta\pi \otimes \eta\pi = \varrho(z)^+ \oplus \varrho(z)^-$$

We identify these representations by $\mathbb{T}$ and denote it by c_4.

Finally, we consider the point $t = (1, -1, z, -z)$ except $z = 1, i$. Thus, $\chi_t \otimes \pi_\sigma = \pi \otimes (-1)^{val}\pi \otimes (z)^{val}\eta\pi \otimes (-z)^{val}\eta\pi$. We have $\bar{L}(\chi_t \otimes \pi_\sigma) =< (-1)^{val}\eta >$ and $X(\chi_t \otimes \pi_\sigma) = 1$. Thus $R(\chi_t \otimes \sigma) = \mathbb{Z}/2\mathbb{Z}$. It leads the induced representation $\theta(z)$ is reducible. There are two irreducible constituents $\theta(z)^+$, $\theta(z)^-$, i.e.

$$\theta(z) = \mathrm{Ind}_M^G(\pi \otimes (-1)^{val}\pi \otimes (z)^{val}\eta\pi \otimes (-z)^{val}\eta\pi = \theta(z)^+ \oplus \theta(z)^-$$

We identify these representations by $\mathbb{T}$ and denote it by c_5.

Now we still consider the point $t = (1, -1, z, -z)$ and fix $z = 1$. We have $t = (1, -1, 1, -1)$. Then $\bar{L}(\chi_t \otimes \pi_\sigma) =< (-1)^{val \circ det}, \eta >$ and $X(\chi_t \otimes \pi_\sigma) = 1$. Hence, we know $R(\chi_t \otimes \sigma) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then the induced representation $\xi = \mathrm{Ind}_M^G(\chi_t \otimes \sigma)$ is reducible and there are 4 irreducible constituents ξ_1, ξ_2, ξ_3 and ξ_4. We locate ξ_1 to $E^{\mathfrak{s}}/W^{\mathfrak{s}}$ and ξ_2 to component c_5 and identify ξ_3 and ξ_4 by pt_1 and pt_2 respectively.

In the next, we fix $z = i$. We have $t = (1, -1, i, -i)$. Then $\bar{L}(\chi_t \otimes \pi_\sigma) =< i^{val \circ det}\eta >$ and $X(\chi_t \otimes \pi_\sigma) = 1$. Hence, we know $R(\chi_t \otimes \sigma) = \mathbb{Z}/4\mathbb{Z}$. The order of $R(\chi_t \otimes \sigma)$ is 4. Then the induced representation $\tau = \mathrm{Ind}_M^G(\chi_t \otimes \sigma)$ is reducible and there are 4 irreducible constituents τ_1, τ_2, τ_3 and τ_4. We locate τ_1 to $E^{\mathfrak{s}}/W^{\mathfrak{s}}$ and τ_2 to component c_5 and identify τ_3 and τ_4 by pt_3 and pt_4 respectively.

For components c_3, c_4, c_5, pt_1, pt_2, pt_3 and pt_4, we set the cocharacter $h_c(t) = 1$.

The detail of this map is as follow:

$$(z_1, z_2) \in \mathbb{T}^2 \longmapsto \mathrm{Ind}^4_{(2+1+1)}(z_1 \cdot St_2(\pi) \otimes z_2^{val \circ det} \eta\pi \otimes \eta\pi)$$

$$\hookrightarrow \mathrm{Ind}^4_{(1+1+1+1)}((\nu^{-\frac{1}{2}} z_1^{\mathrm{val} \circ det} \pi \otimes \nu^{\frac{1}{2}} z^{\mathrm{val} \circ det} \pi \otimes z_2^{\mathrm{val} \circ det} \eta\pi \otimes \eta\pi))$$

$$z \in \mathbb{T} \longmapsto \mathrm{Ind}^4_{(4)}(z \cdot St_2(\pi) \otimes St_2(\eta\pi))$$

$$\hookrightarrow \mathrm{Ind}^{(4)}_{(1+1+1+1)}(\nu^{-\frac{1}{2}} z^{\mathrm{val} \circ det} \pi_1 \otimes \nu^{\frac{1}{2}} z^{\mathrm{val} \circ det} \pi \otimes \nu^{-\frac{1}{2}} \eta\pi \otimes \nu^{\frac{1}{2}} \eta\pi)$$

$$\mu : \quad z \in \mathbb{T} \longmapsto \rho(z)^+ \hookrightarrow \mathrm{Ind}^G_M(z^{val \circ det} \pi \otimes \pi \otimes (-z)^{val \circ det} \eta\pi \otimes (-1)^{val \circ det} \eta\pi)$$

$$z \in \mathbb{T} \longmapsto \varrho(z)^+ \hookrightarrow \mathrm{Ind}^G_M(z^{val \circ det} \pi \otimes \pi \otimes z^{val \circ det} \eta\pi \otimes \eta\pi)$$

$$z \in \mathbb{T} \longmapsto \theta(z)^+ \hookrightarrow \mathrm{Ind}^G_M(\pi \otimes (-1)^{val} \pi \otimes (z)^{val} \eta\pi \otimes (-z)^{val} \eta\pi)$$

$$pt_1 \longmapsto \xi_3 \hookrightarrow \theta(1)$$

$$pt_2 \longmapsto \xi_4 \hookrightarrow \theta(1)$$

$$pt_3 \longmapsto \tau_3 \hookrightarrow \theta(i)$$

$$pt_4 \longmapsto \tau_4 \hookrightarrow \theta(i)$$

$$t \in E^{\mathfrak{s}}/W^{\mathfrak{s}} \longmapsto \mathrm{Ind}^G_M(\chi_t \otimes \sigma)$$

Now we investigate the image of $inf.ch$:

$$inf.ch(\mu((z_1,z_2))) = [M, \nu^{-\frac{1}{2}} z_1^{\mathrm{val}\circ det}\pi \otimes \nu^{\frac{1}{2}} z^{\mathrm{val}\circ det}\pi \otimes z_2^{\mathrm{val}\circ det}\eta\pi \otimes \eta\pi]_G$$
$$\longmapsto (q^{\frac{1}{2}} z_1, q^{-\frac{1}{2}} z_1, z_2, 1) \in D^{\mathfrak{s}}/W^{\mathfrak{s}}$$
$$inf.ch(\mu(z)) = [M, \nu^{-\frac{1}{2}} z^{\mathrm{val}\circ det}\pi_1 \otimes \nu^{\frac{1}{2}} z^{\mathrm{val}\circ det}\pi \otimes \nu^{-\frac{1}{2}}\eta\pi \otimes \nu^{\frac{1}{2}}\eta\pi]_G$$
$$\longmapsto (q^{\frac{1}{2}} z, q^{-\frac{1}{2}} z, q^{\frac{1}{2}}, q^{-\frac{1}{2}}) \in D^{\mathfrak{s}}/W^{\mathfrak{s}}$$
$$inf.ch(\mu(z)) = [M, z^{val\circ det}\pi \otimes \pi \otimes (-z)^{val\circ det}\eta\pi \otimes (-1)^{val\circ det}\eta\pi]_G$$
$$\longmapsto (z, 1, -z, -1) \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$$
$$inf.ch(\mu(z)) = [M, z^{val\circ det}\pi \otimes \pi \otimes z^{val\circ det}\eta\pi \otimes \eta\pi]_G$$
$$\longmapsto (z, 1, z, 1) \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$$
$$inf.ch(\mu(z)) = [M, \pi \otimes (-1)^{val\circ det}\pi \otimes (z)^{val\circ det}\eta\pi \otimes (-z)^{val\circ det}\eta\pi]_G$$
$$\longmapsto (1, -1, z, -z) \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$$
$$inf.ch(\mu(pt_1)) = [M, \pi \otimes (-1)^{val\circ det}\pi \otimes \eta\pi \otimes (-1)^{val\circ det}\eta\pi]_G$$
$$\longmapsto (1, -1, 1, -1) \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$$
$$inf.ch(\mu(pt_2)) = [M, \pi \otimes (-1)^{val\circ det}\pi \otimes \eta\pi \otimes (-1)^{val\circ det}\eta\pi]_G$$
$$\longmapsto (1, -1, 1, -1) \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$$
$$inf.ch(\mu(pt_3)) = [M, \pi \otimes i^{val\circ det}\pi \otimes \eta\pi \otimes (-i)^{val\circ det}\eta\pi]_G$$
$$\longmapsto (1, i, 1, -i) \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$$
$$inf.ch(\mu(pt_4)) = [M, \pi \otimes i^{val\circ det}\pi \otimes \eta\pi \otimes (-i)^{val\circ det}\eta\pi]_G$$
$$\longmapsto (1, i, 1, -i) \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$$
$$inf.ch(\mu(t)) = [M, z_1^{val\circ det}\pi \otimes (z_2)^{val\circ det}\pi \otimes (z_3)^{val\circ det}\eta\pi \otimes \eta\pi]_G$$
$$\longmapsto (z_1, z_2, z_3, 1) \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$$

The graph below shows the relation between $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ and irreducible tempered representations. We have the following graph:

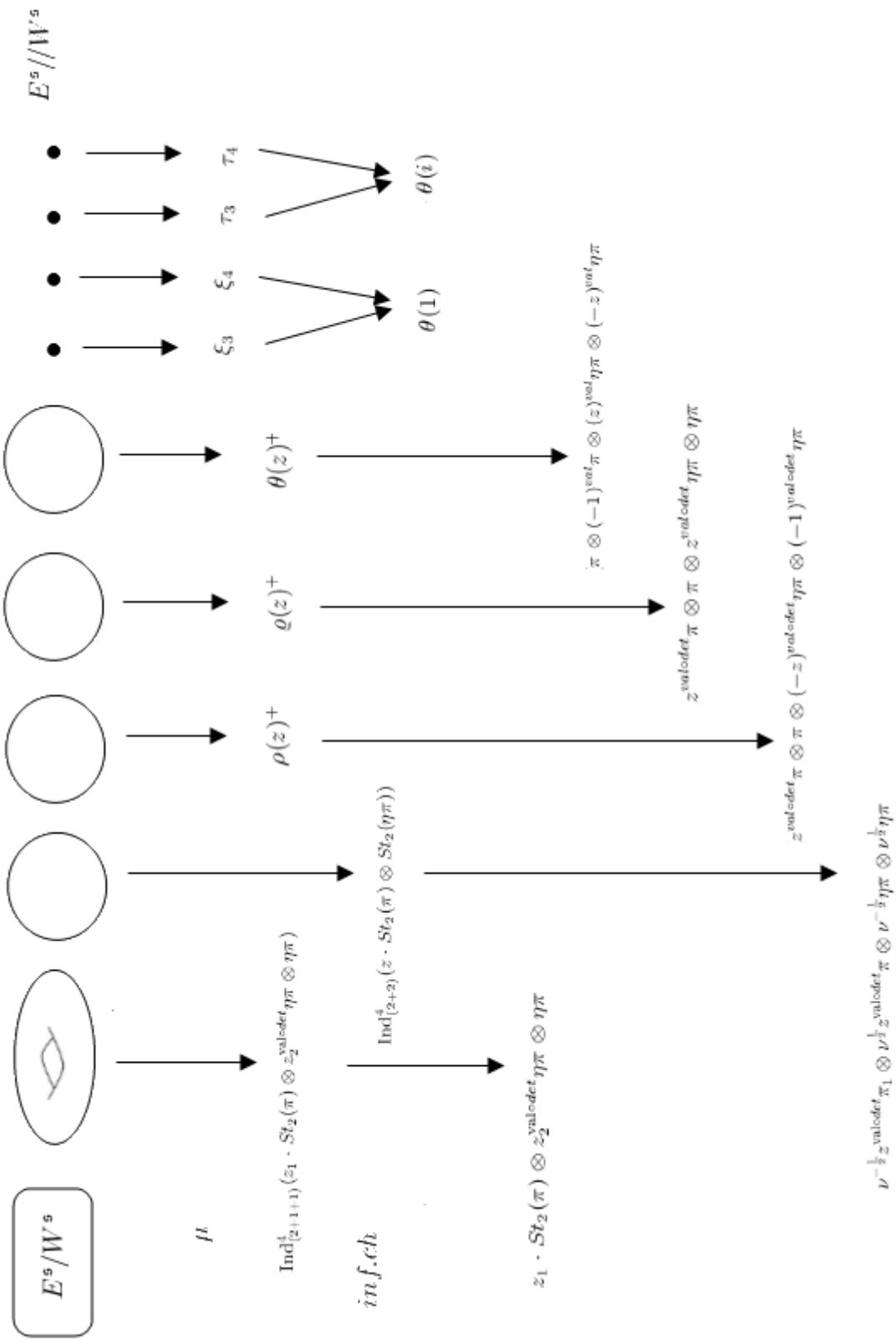

Figure 6.4: $\mathfrak{s} = [M, \pi \otimes \pi \otimes \eta\pi \otimes \eta\pi]_G$

Indeed, the condition

$$\pi_{\sqrt{q}} = inf.ch \circ \mu$$

is satisfied. Hence, we have

Lemma 6.1.4. *Part (3) of the geometric conjecture is true for* $\mathfrak{s} = [M, \pi \otimes \pi \otimes \eta\pi \otimes \eta\pi]_G$.

6.1.4 Case 4: $\pi_\sigma \cong \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_3$

In this section, we will consider the case $\pi_\sigma \cong \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_3$. Hence, we know the isotropy subgroup $W^{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z}$. The table below is the structure of $W^{\mathfrak{s}}$.

$W^{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z}$		
γ	$\gamma(abcd)$	$Z(\gamma)$
γ_1	$abcd$	$W^{\mathfrak{s}}$
γ_2	$bacd$	$W^{\mathfrak{s}}$

This case would be easier to handle because of the simple structure of isotropy subgroup. Then, we focus on the extended quotient with respect to this case. Easily, we know that

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\gamma_1}/W^{\mathfrak{s}} \sqcup E^{\gamma_2}/W^{\mathfrak{s}}$$

Analysis case by case:

- $\gamma = \gamma_1$, $E^\gamma/Z(\gamma) = E^{\mathfrak{s}}/W^{\mathfrak{s}}$.
- $\gamma = \gamma_2$, $E^\gamma = \{(a, a, c, d) : a, c, d \in \mathbb{T}\}/\mathbb{T} \cong \{(z_1, z_1, z_2, 1) : z_1, z_2 \in \mathbb{T}\} \cong \mathbb{T}^2$.

$$E^\gamma/Z(\gamma) \cong \mathbb{T}^2$$

Then, we have

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T}^2$$

Actually, there are only one type of induced representations which are reducible. Let us consider the induced representation

$$\mathrm{Ind}_{(1+1+1+1)}^{(2+1+1)}(\nu^{-\frac{1}{2}}\pi_1 \otimes \nu^{\frac{1}{2}}\pi_1 \otimes z_2^{val \circ det}\pi_2 \otimes \pi_3).$$

It is reducible and there exists an unique irreducible subquotient:

$$z_1 \cdot St_2(\pi_1) \otimes z_2^{val \circ det} \pi_2 \otimes \pi_3 \hookrightarrow \mathrm{Ind}_{(1+1+1+1)}^{(2+1+1)}(\nu^{-\frac{1}{2}}\pi_1 \otimes \nu^{\frac{1}{2}}\pi_1 \otimes z_2^{val \circ det}\pi_2 \otimes \pi_3)$$

Therefore, the representation

$$\mathrm{Ind}_{(2+1+1)}^{(4)}(z_1 \cdot St_2(\pi_1) \otimes z_2^{val \circ det}\pi_2 \otimes \pi_3)$$

is tempered and so it is an element in tempered dual. As z_1 and z_2 run through $\mathbb{T}^2$, these tempered representations will be identified by $\mathbb{T}^2$. At the same time, we set the cocharacter

$$h_c(t) = (t, t^{-1}, 1, 1)$$

and construct the bijection:

$$\mu : E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T}^2 \longrightarrow \mathrm{Irr}^{\mathfrak{t}}(G)^{\mathfrak{s}}$$

The detail of this map is as follow:

$$\begin{aligned}
\mu : \qquad (z_1, z_2) \in \mathbb{T}^2 &\longmapsto \mathrm{Ind}_{(2+1+1)}^{4}(z_1 \cdot St_2(\pi_1) \otimes z_2^{val \circ det}\pi_2 \otimes \pi_3) \\
&\hookrightarrow \mathrm{Ind}_{(1+1+1+1)}^{4}(\nu^{-\frac{1}{2}}\pi_1 \otimes \nu^{\frac{1}{2}}\pi_1 \otimes z_2^{val \circ det}\pi_2 \otimes \pi_3) \\
t = (z_1, z_2, z_3, 1) &\longmapsto \mathrm{Ind}_M^G(\chi_t \otimes \sigma)
\end{aligned}$$

Now we investigate the image of $inf.ch$ (infinitesimal character)

$$\begin{aligned}
inf.ch(\mu((z_1, z_2))) &= [M, \nu^{-\frac{1}{2}}\pi_1 \otimes \nu^{\frac{1}{2}}\pi_1 \otimes z_2^{val \circ det}\pi_2 \otimes \pi_3]_G \\
&\longmapsto (q^{\frac{1}{2}}z_1, q^{-\frac{1}{2}}z_1, z_2, 1) \in D^{\mathfrak{s}}/W^{\mathfrak{s}} \\
inf.ch(\mu(t)) &= [M, \chi_t \otimes \sigma]_G \\
&\longmapsto (z_1, z_2, z_3, 1) \in E^{\mathfrak{s}}/W^{\mathfrak{s}}
\end{aligned}$$

We have the following graph:

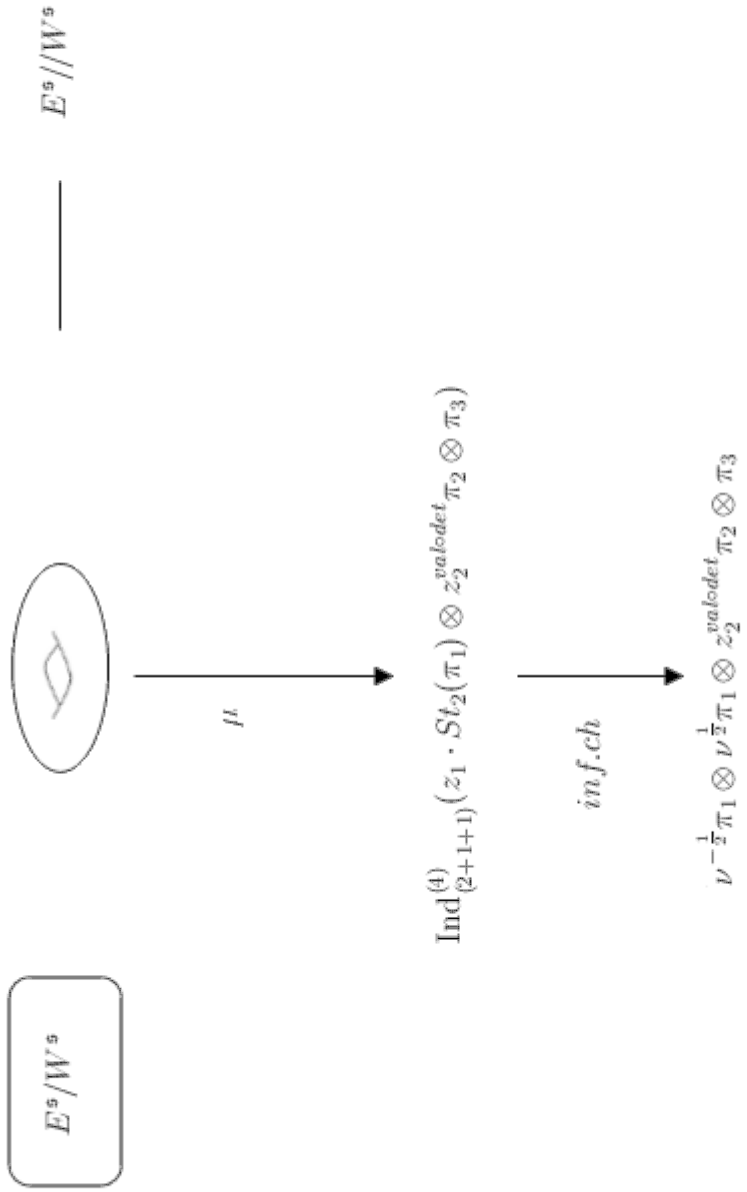

Figure 6.5: $\mathfrak{s} = [M, \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_3]_G$

Indeed, the condition

$$\pi_{\sqrt{q}} = inf.ch \circ \mu$$

is satisfied. Hence, we have

Lemma 6.1.5. *Part (3) of the geometric conjecture is true for* $\mathfrak{s} = [M, \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_3]_G$.

6.1.5 Case 5: $\pi_\sigma \cong \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$

In this section, we will discuss the case when $\pi_\sigma \cong \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$. In fact, from the table of R-group above, we have known that there are four types in this case. First of all, we focus on the case 5.1. Indeed, π_σ is given by the form $\pi \otimes \eta\pi \otimes \eta^2\pi \otimes \eta^3\pi$ where η is ramified. We have proved that the R-group with resect to this is the cyclic group $\mathbb{Z}/4\mathbb{Z}$. Furthermore, the isotropy group $W^{\mathfrak{s}}$ is given by $\mathbb{Z}/4\mathbb{Z}$. In the following,we figure out the extended quotient with respect to $W^{\mathfrak{s}}$. The table below is the structure of $W^{\mathfrak{s}} = \mathbb{Z}/4\mathbb{Z}$.

$W^{\mathfrak{s}} = \mathbb{Z}/4\mathbb{Z}$		
γ	$\gamma(abcd)$	$Z(\gamma)$
γ_1	$abcd$	$W^{\mathfrak{s}}$
γ_2	$bcda$	$W^{\mathfrak{s}}$
γ_3	$cdab$	$W^{\mathfrak{s}}$
γ_4	$dabc$	$W^{\mathfrak{s}}$

Actually, the cyclic group is abelian. So each element comprises a single conjugacy classes and the centralizer of each element is the cyclic group itself. Then we can immediately get the extended quotient

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\gamma_1}/W^{\mathfrak{s}} \sqcup E^{\gamma_2}/W^{\mathfrak{s}} \sqcup E^{\gamma_3}/W^{\mathfrak{s}} \sqcup E^{\gamma_4}/W^{\mathfrak{s}}$$

We analyze case by case.

- $\gamma = \gamma_1$, $E^{\gamma}/W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}}$.

- $\gamma = \gamma_2$, $E^\gamma = \{(1,1,1,1),(1,-1,1,-1),(1,i,-1,-i),(1,i,-1,-i)\} \cong pt \sqcup pt \sqcup pt \sqcup pt$. Hence, we have

$$E^\gamma/W^\gamma \cong pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4$$

- $\gamma = \gamma_3$, $E^\gamma = \{(a,b,-a,b),(a,b,a,b) : a,b \in \mathbb{T}\}/\mathbb{T} \cong \{(1,z,-1,-z),(1,z,1,z) : z \in \mathbb{T}\} \cong \mathbb{T} \sqcup \mathbb{T}$.

$$E^\gamma/W^\gamma \cong \mathbb{T} \sqcup \mathbb{T}$$

- $\gamma = \gamma_4$, $E^\gamma = \{(1,1,1,1),(1,-1,1,-1),(1,i,-1,-i),(1,i,-1,-i)\} \cong pt \sqcup pt \sqcup pt \sqcup pt$.

$$E^\gamma/W^\gamma \cong pt_5 \sqcup pt_6 \sqcup pt_7 \sqcup pt_8$$

Then, we have the decomposition of $E^{\mathfrak{s}}//W^{\mathfrak{s}}$:

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T} \sqcup \mathbb{T} \sqcup pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4 \sqcup pt_5 \sqcup pt_6 \sqcup pt_7 \sqcup pt_8$$

It coincides the result in [19]. By [19, Theorem 5.3], we have

Lemma 6.1.6. *Part (3) of the conjecture is true for* $\mathfrak{s} = [M, \pi \otimes \eta\pi \otimes \eta^2\pi \otimes \eta^3\pi]$.

Then, we will concentrate on the case 5.2. Recall that the representation π_σ is given by $\pi \otimes \chi\pi \otimes \eta\pi \otimes \chi\eta\pi$ where χ and η are ramified quadratic characters. In fact, we consider the field $F = \mathbb{Q}_p$, $p \geq 3$. Actually, there are only two ramified quadratic characters of order 3, the Legendre symbol $(\frac{u}{p})$ and its twist with $(-1)^{val \circ det}$. For convenience, we denote the Legendre symbol by λ and its twist by $(-1)^{val \circ det}\lambda$. Since η and χ have to be distinct, we set

$$\eta = \lambda,\ \chi = (-1)^{val \circ det}\lambda.$$

Indeed, this case would belong to

$$\pi \otimes (-1)^{val \circ det}\lambda\pi \otimes \lambda\pi \otimes (-1)^{val \circ det}\pi$$

It means it should be in case 3.2. Hence, this case does not exist when $F = \mathbb{Q}_3$.

Then we turn to the case 5.3. We know that $\pi_\sigma \cong \pi_1 \otimes \eta\pi_1 \otimes \pi_2 \otimes \eta\pi_2$. The table below is the structure of $W^{\mathfrak{s}}$.

$W^{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z}$		
γ	$\gamma(abcd)$	$Z(\gamma)$
γ_1	$abcd$	$W^{\mathfrak{s}}$
γ_2	$badc$	$W^{\mathfrak{s}}$

Then we focus on the extended quotient with respect to this case. Easily, we know that

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\gamma_1}/W^{\mathfrak{s}} \sqcup E^{\gamma_2}/W^{\mathfrak{s}}$$

Analysis case by case:

- $\gamma = \gamma_1$, $E^{\gamma}/W^{\mathfrak{s}} = E/W^{\mathfrak{s}}$.
- $\gamma = \gamma_2$, $E^{\gamma} = \{(a, a, c, c), (a, -a, c, -c) : a, c \in T\}/T \cong \{(1, 1, z, z), (1, -1, z, -z) : z \in T\} \cong \mathbb{T} \sqcup \mathbb{T}$.

$$E^{\gamma}/W^{\gamma} \cong \mathbb{T} \sqcup \mathbb{T}$$

Hence, we have the decomposition of $E^{\mathfrak{s}}//W^{\mathfrak{s}}$:

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T} \sqcup \mathbb{T}$$

From now, we try to exhaust the tempered dual with respect to $\mathfrak{s}$. First of all, we consider t in the form of $(1, 1, z, z)$ and $(1, -1, z, -z)$. For the point $t = (1, 1, z, z)$, χ_t corresponds to the character $\chi_t = (1, 1, z^{val \circ det}, z^{val \circ det})$. After twisting ,we have the representation $\chi_t \otimes \sigma$. Now we compute the R-group for this representation. This implies we can consider $\chi_t \otimes \pi_\sigma \cong \pi_1 \otimes \eta\pi_1 \otimes z^{val \circ det}\pi_2 \otimes z^{val \circ det}\eta\pi_2$. By computation, we know that the trivial character and η are also contained in $\bar{L}(\chi_t \otimes \pi_\sigma)$. Then we have $R(\chi_t \otimes \sigma) =< 1, \eta >$ and $X(\chi_t \otimes \pi_\sigma) = 1$. We note that the character η is ramified and of order 2. This leads that $R(\sigma) \cong \mathbb{Z}/2\mathbb{Z}$. This implies the representation $\lambda(t)$ induced by $\chi_t \otimes \sigma$ is reducible and can be decomposed as two parts:

$$\lambda(t) = \lambda^+ \oplus \lambda^-$$

Indeed, λ^+ and λ^- are tempered.

Similarly, we consider the point in the form of $t = (1, -1, z, -z)$ and χ_t corresponds to the character $\chi_t = (1, (-1)^{val}, z^{val\circ det}, (-z)^{val\circ det})$. After twisting, we have the representation $\chi_t \otimes \sigma$. Then we can consider $\chi_t \otimes \pi_\sigma \cong \pi \otimes (-1)^{val\circ det}\eta\pi_1 \otimes z^{val\circ det}\pi_2 \otimes (-z)^{val\circ det}\eta\pi_2$. Easily, we can get $R(\chi_t \otimes \sigma) =< (-1)^{\mathrm{val}\circ det}\eta >= \mathbb{Z}/2\mathbb{Z}$. This implies the representation $\rho(t)$ induced by $\chi_t \otimes \sigma$ is reducible and can be decomposed as two parts:

$$\rho(t) = \rho^+ \oplus \rho^-$$

where ρ^+ and ρ^- are tempered representations of G.

Then we check the diagram is commutative, where the diagram is the relation between the extended quotient and the tempered dual of G:

$$\begin{tikzcd} E//W \arrow[r, "\mu"] \arrow[dr, "\pi_{\sqrt{q}}"'] & \mathrm{Irr}(G)^{\mathfrak{s}} \arrow[d, "inf.ch"] \\ & D/W \end{tikzcd}$$

$$\mu : E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{T} \sqcup \mathbb{T} \longrightarrow \mathrm{Irr}^{\mathfrak{t}}(G)^{\mathfrak{s}}$$

The detail of this map is as follow:

$$\mu : \begin{array}{lll} T \longmapsto \lambda^+ \hookrightarrow & \mathrm{Ind}_M^G(\pi_1 \otimes \eta\pi_1 \otimes z^{val\circ det}\pi_2 \otimes z^{val\circ det}\eta\pi_2) \\ T \longmapsto \rho^+ \hookrightarrow & \mathrm{Ind}_M^G(\pi \otimes (-1)^{val\circ det}\eta\pi_1 \otimes z^{val\circ det}\pi_2 \otimes (-z)^{val\circ det}\eta\pi_2) \\ t \longmapsto & \mathrm{Ind}_M^G(\chi_t \otimes \sigma) \end{array}$$

Now we investigate the image of $inf.ch$:

$$\begin{aligned} inf.ch(\mu(T)) &= [M, \pi_1 \otimes \eta\pi_1 \otimes z^{val\circ det}\pi_2 \otimes z^{val\circ det}\eta\pi_2]_G \\ &\longmapsto (1, 1, z, z) \in E^{\mathfrak{s}}/W^{\mathfrak{s}} \\ inf.ch(\mu(T)) &= [M, \pi \otimes (-1)^{val\circ det}\eta\pi_1 \otimes z^{val\circ det}\pi_2 \otimes (-z)^{val\circ det}\eta\pi_2]_G \\ &\longmapsto (1, -1, z, -z) \in E^{\mathfrak{s}}/W^{\mathfrak{s}} \\ inf.ch(\mu(t)) &= [M, \pi \otimes z_1^{val\circ det}\eta\pi_1 \otimes z_2^{val\circ det}\pi_2 \otimes z_3^{val\circ det}\eta\pi_2]_G \\ &\longmapsto (1, z_1, z_2, z_3) \in E^{\mathfrak{s}}/W^{\mathfrak{s}} \end{aligned}$$

The graph below is the relation between extended quotient and tempered dual.

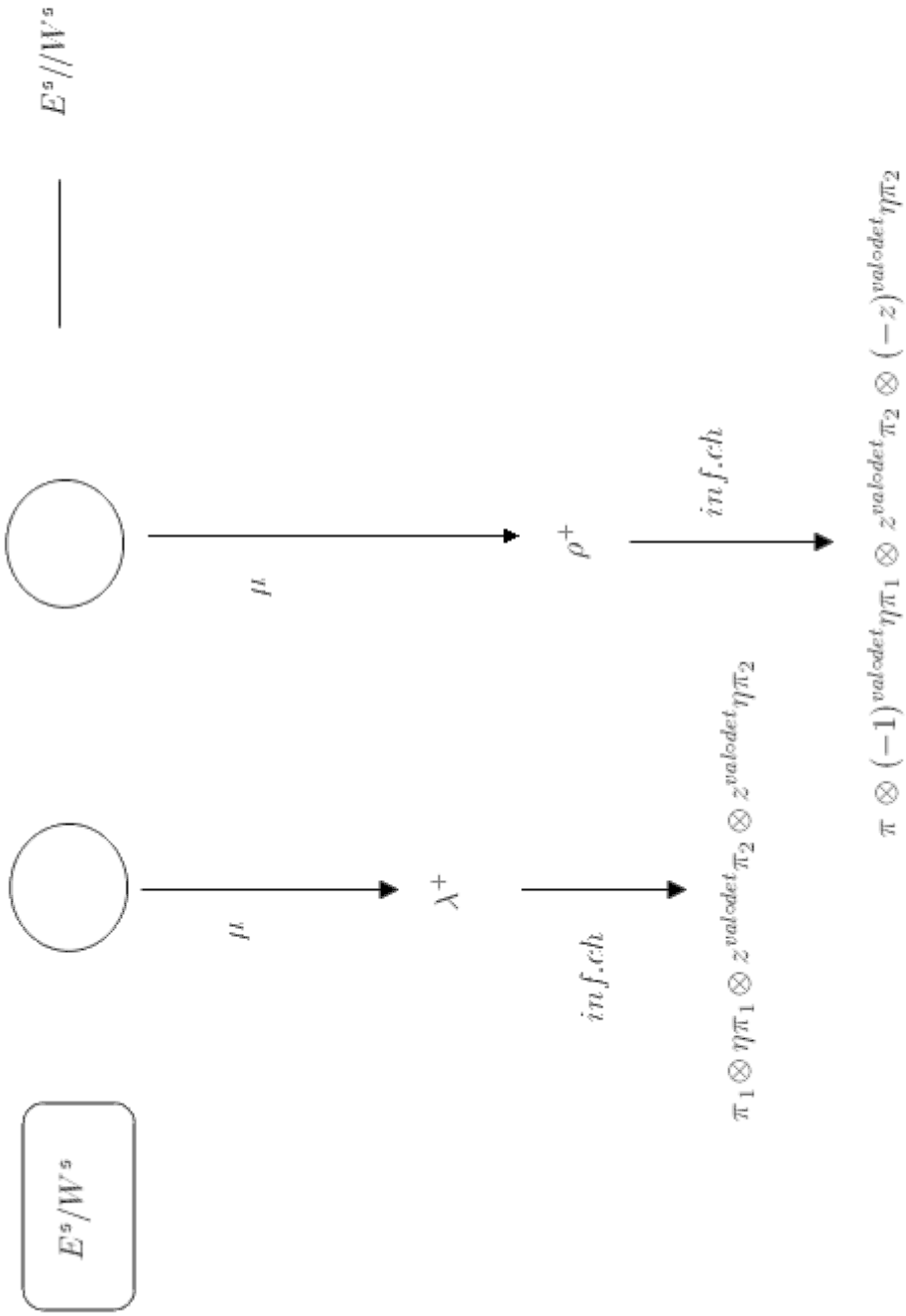

Figure 6.6: $\mathfrak{s} = [M, \pi_1 \otimes \eta\pi_1 \otimes \pi_2 \otimes \eta\pi_2]_G$

Hence, we have

Lemma 6.1.7. *Part (3) of the conjecture is true for* $\mathfrak{s} = [M, \pi \otimes \eta\pi \otimes \eta^2\pi \otimes \eta^3\pi]$.

Now, our next object is the case 5.4. This case would be easily to figure out since the R-group attached to this case is trivial and the isotropy group $W^{\mathfrak{s}}$ is also trivial. Indeed each induced representation by unitary twist with σ is irreducible. This implies every induced representation is irreducible. From the side of extended quotient $E^{\mathfrak{s}}//W^{\mathfrak{s}}$, we know that $E^{\mathfrak{s}}//W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}}$ because $W^{\mathfrak{s}} = 1$. The bijection μ between $E^{\mathfrak{s}}//W^{\mathfrak{s}}$ and $\mathrm{Irr}^{\mathfrak{t}}(G)^{\mathfrak{s}}$

$$\mu : E//W \mapsto \mathrm{Irr}^{\mathfrak{t}}(G)^{\mathfrak{s}}$$

is given by

$$t \in E^{\mathfrak{s}}/W^{\mathfrak{s}} \mapsto \mathrm{Ind}(\chi_t \otimes \sigma)$$

Lemma 6.1.8. *Part 3 of the conjecture is true for* $\mathfrak{s} = [M, \sigma]$ *where* $\pi_\sigma \cong \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$

Theorem 6.1.9. *The part (3) of geometric conjecture is true for* $\mathfrak{s} = [M, \sigma]$*, i.e. the toral case.*

Proof. Combine 6.1.1, 6.1.2, 6.1.3, 6.1.4, 6.1.5, 6.1.6, 6.1.7 and 6.1.8 above. □

6.2 A note on the extended quotient

Note: In order to prove the conjecture for the special linear group SL_n, we have to find out the parametrization for the special representation (Unluckily, it seems not canonical!). If such parametrization is determined, it would be easier to crack the conjecture except some multiple points (It is trouble one.)

From now on, we will consider the case when the Levi $M = T$, as usual, take $\mathfrak{s} = [T, \sigma]_G$ be the Bernstein component.

Lemma 6.2.1. *Suppose* $\pi_\sigma = \pi \otimes \pi \otimes \cdots \otimes \pi$*. Each special representation in the set* $\mathrm{Irr}^{\mathfrak{t}}(G)^{\mathfrak{s}}$ *is parameterized by a partition* λ *of* n*.*

Proof. First of all, we define the partition for n. λ is called a partition of n if

$$\lambda = [\lambda_1, \lambda_2, \cdots, \lambda_r],\ \Sigma\lambda_i = n \text{ and } \lambda_i \geq 1 \in \mathbb{Z}$$

At the same time, we introduce a concept *dominant* if $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_r$ for a partition λ. We denote Λ the set of the partition of n and Λ^+ the set of dominant partition. In fact, it makes sense because, in this case, the isotropy group is given by $\mathfrak{S}_n$ which acts on the Levi as permutation groups. Thus, we can rearrange a non-dominant partition to a dominant one by suitable group actions.

For example $n = 14$, $[8, 1, 2, 3]$ is a partition, $[8, 3, 2, 1]$ is dominant partition.

For every dominant partition $\lambda = [\lambda_1, \lambda_2, \cdots, \lambda_k, \cdots \lambda_r]$. We can construct a corresponding representation canonically. In other word, for such partition, the representation is in the form of

$$St_{\lambda_1}(\pi) \otimes St_{\lambda_2}(\pi) \otimes \cdots \otimes St_{\lambda_{k-1}}(\pi) \otimes \underbrace{\pi \otimes \cdots \otimes \pi}_{n-\Sigma_{i=1}^{k-1}\lambda_i}$$

Then we can twist an unramified unitary character of $\mathrm{GL}_{\lambda_1} \times \mathrm{GL}_{\lambda_2} \times \cdots \times \mathrm{GL}_{\lambda_{k-1}} \times F^\times \times \cdots \times F^\times$ to this representation by partition.
Then we have

$$St_{\lambda_1}(z_1\pi) \otimes St_{\lambda_2}(z_2\pi) \otimes \cdots \otimes St_{\lambda_{k-1}}(z_{k-1}\pi) \otimes \underbrace{z_k\pi \otimes \cdots \otimes z_r\pi}_{n-\Sigma_{i=1}^{k-1}\lambda_i}$$

It is clear that these representation are parameterized by r-torus $\mathbb{T}^r$. Now we restrict theses representations to SL_n. Hence, the parameters space is given by the quotient $\mathbb{T}^r/\mathbb{T}$ since SL_n is determinant 1. □

In fact, it is well adapted with the extended quotient for this case. As we know, the extended quotient can be decomposed to the disjoint union of some ordinary quotients (we call such parts "components") via the conjugacy classes of the action group. We know the action on the quotient is given by a finite group, which is isotropy subgroup of Weyl group. In this case, such isotropy subgroup is given by the symmetric group $\mathfrak{S}_n$. It is well known that the conjugacy classes are depended on the length of cycles in $\mathfrak{S}_n$.

6.3 Multiple character segments

In this section, we will give a theorem related to the reducibility of induced representation. In general, R-group is not easy to compute. Hence, maybe we will ask the question: What kind of the structures R-group can be? In fact, we have shown that they can be trivial group, cyclic group or product of cyclic group. Now we try to give a result related to R-group which will be the product of cyclic group. First of all, we introduce a new concept given by Goldberg in his paper [14].

Definition 6.3.1. *(**Multiple character segments**). Let $\pi \in \mathcal{E}_2(\widetilde{G}_m)$. Let $\eta_1, \eta_2, \cdots, \eta_l, l \geq 2$, be a collection of characters of $F^\times$. Let $o(\eta_i)$ be the order of η_i modulo $X(\pi)$, i.e., $\eta_i^{o(\eta_i)} \otimes \pi \cong \pi$. Suppose that*

1. *$\pi \otimes \eta_1^{i_1}\eta_2^{i_2}\cdots\eta_l^{i_l} \notin X(\pi)$ unless $\eta_j^{i_j} \in X(\pi)$ for each j;*

2. *$gcd(o(\eta_1), o(\eta_2), \cdots, o(\eta_l)) > 1$.*

Let $\Omega(\pi, \eta_1, \eta_2, ..., \eta_l) = \{\pi \otimes \eta_1^{i_1}\eta_2^{i_2}\cdots\eta_l^{i_l} | 0 \leq i_j < o(\eta_j), j = 1, 2, \cdots, l\}$. We call the collection $\Omega(\pi, \eta_1, \eta_2, ..., \eta_l)$ a multiple character segment for π.

Definition 6.3.2. *Let $\widetilde{G} = \mathrm{GL}_m$. Suppose $\widetilde{P} = \widetilde{M}N$ is a standard parabolic of $\widetilde{G}$. A discrete series representation ρ of $\widetilde{M}$ is said to contain a multiple character segment Ω for π if, up to permutation of the blocks of M,*

$$\rho \cong (\bigotimes_{\tau \in \Omega} \tau) \otimes \rho'$$

for some ρ'.

The following theorem shows when the R-group of representation can be a product of cyclic groups. It depends on the order of the characters in the multiple character segment.

Theorem 6.3.3. *Suppose π_σ contains one multiple character segment and $\pi_\sigma|_M \supset \sigma$. Then*

$$R(\sigma) = \prod_{i=1}^{l} \mathbb{Z}/O(\eta_i)$$

Proof. Since π_σ contain one multiple character segment, this means each term is of the form

$$\pi \otimes \eta_1^{i_1}\eta_2^{i_2}\cdots\eta_l^{i_l}$$

$$\pi_\sigma = (\bigotimes_{\tau\in\omega}\tau)\otimes\rho'$$

where ρ' is free of multiple character segments

Without loss of generality, we choose j such that $1 \le j \le l$. The following we will prove $\eta_j \in \overline{L}(\pi_\sigma)$. After twisting the representation π_σ with the character η_j, then we have

$$\pi \otimes \eta_1^{i_1}\eta_2^{i_2}\cdots\eta_j^{i_j}\cdots\eta_l^{i_l} \longmapsto \pi \otimes \eta_1^{i_1}\eta_2^{i_2}\cdots\eta_j^{i_j+1}\cdots\eta_l^{i_l}$$

Actually, such a procedure will generate a cycle. We can explain the cycle by the following diagram:

Here, we denote the product of character $\eta_1^{i_1}\eta_2^{i_2}\cdots\eta_{j-1}^{i_{j-1}}\eta_{j+1}^{i_{j+1}}\cdots\eta_l^{i_l}$ by Δ(The missing term is a power of η_j), where $0 \le i_r < o(\eta_r), r = 1, 2, \cdots, j-1, j+1, \cdots, l$.

$$\begin{tikzcd}
\Delta\eta_j^0 = \Delta\eta_j^{o(\eta_j)} \arrow[r, "\eta_j"] & \Delta\eta_j^1 \arrow[r, "\eta_j"] & \Delta\eta_j^2 \arrow[r, "\eta_j"] & \cdots \arrow[r, "\eta_j"] & \Delta\eta_j^k \arrow[d] \\
\Delta\eta_j^{o(\eta_j)-1} \arrow[u] & \Delta\eta_j^{o(\eta_j)-2} \arrow[l, "\eta_j"] & \cdots \arrow[l, "\eta_j"] & \Delta\eta_j^{k+2} \arrow[l, "\eta_j"] & \Delta\eta_j^{k+1} \arrow[l, "\eta_j"]
\end{tikzcd}$$

We can see that the length of the cycle is the order $o(\eta_j)$ of η_j and the diagram has shown that the permutation of the component of representation π_σ.

Form this, we know $\eta_j^h \in \overline{L}(\pi_\sigma)$ for $0 \le h < o(\eta_j)$. This means $< \eta_j >$ belongs to $\overline{L}(\pi_\sigma)$. In fact, here j is independence. This implies the $\overline{L}(\pi_\sigma)$ is generated by $o(\eta_j)$ for every character η_j. So $R(\sigma) = \overline{L}(\pi_\sigma)/X(\pi_\sigma) = \prod_{i=1}^{l}\mathbb{Z}/O(\eta_i)\mathbb{Z}$. $\square$

Example: Let $m \ge 1$, and let $G = \mathrm{SL}_{4m}$. Let $\widetilde{M} \backsimeq (\widetilde{G}_m)^8$. Let $\pi \in \mathbb{E}_2(M)$. Suppose π_0 contain a multiple character segment. This implies that there are two characters η_1 and η_2 as the element in Ω. We set $o(\eta_1) = 2$ and $o(\eta_2) = 4$. We have $gcd(o(\eta_1), o(\eta_2)) = 2$. For convenience, let

$$\pi_0 = \pi\otimes(\pi\otimes\eta_1)\otimes(\pi\otimes\eta_2)\otimes(\pi\otimes\eta_2^2)\otimes(\pi\otimes\eta_2^3)\otimes(\pi\otimes\eta_1\eta_2)\otimes(\pi\otimes\eta_1\eta_2^2)\otimes(\pi\otimes\eta_1\eta_2^3)$$

We denote the above representation by bracket $(1, \eta_1, \eta_2, \eta_2^2, \eta_2^3, \eta_1\eta_2, \eta_1\eta_2^2, \eta_1\eta_2^3)$ and label each component by $1, 2, \cdots, 8$. Firstly, we investigate the case when twisting with η_1.

$(1,2,3,4,5,6,7,8) \to (2,1,6,7,8,3,4,5) \to (1,2,3,4,5,6,7,8)$ We can realize that then length of this cycle is 2. This is depended on the order of η_1.

Then we think about the case when multiplying η_2,
$(1,2,3,4,5,6,7,8) \to (3,6,5,4,1,7,8,2) \to (4,7,5,1,3,8,2,6) \to (5,8,1,3,4,2,6,7) \to (1,2,3,4,5,6,7,8)$. Similarly, the length of this cycle is $4 = o(\eta_2)$. Then by the theorem, we can conclude that the R-group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

6.4 Note on $\mathrm{SL}_4(\mathbb{Q}_2)$

In this section, we intend to discuss a special case when $F = \mathbb{Q}_2$. Recall that the case 5.2 for $\mathrm{SL}_4(\mathbb{Q}_p)$, $p \geq 3$ does not exists since it cannot admit two ramified quadratic characters which it is not the twist with an unramified character each other. Recall that the representation π_σ is given by $\pi \otimes \chi\pi \otimes \eta\pi \otimes \chi\eta\pi$. In this case, the R-group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and we also know that the isotropy group is $W^{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as the following table.

$W^{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$		
γ	$\gamma(abcd)$	$Z(\gamma)$
γ_1	$abcd$	$W^{\mathfrak{s}}$
γ_2	$badc$	$W^{\mathfrak{s}}$
γ_3	$cdab$	$W^{\mathfrak{s}}$
γ_4	$dcba$	$W^{\mathfrak{s}}$

- $\gamma = \gamma_1$, $E^\gamma/W^{\mathfrak{s}} = E/W^{\mathfrak{s}}$.
- $\gamma = \gamma_2$, $E^\gamma = \{(1,1,z,z), (1,-1,z,-z) : z \in \mathbb{T}\}$.

$$E^\gamma/W^\gamma \cong \mathbb{I} \sqcup \mathbb{I}$$

- $\gamma = \gamma_3$, $E^\gamma = \{(1, z, 1, z), (1, z, -1, -z) : z \in \mathbb{T}\}$.

$$E^\gamma/W^\gamma \cong \mathbb{I} \sqcup \mathbb{I}$$

- $\gamma = \gamma_4$, $E^\gamma = \{(1, z, z, 1), (1, z, -z, -1) : z \in \mathbb{T}\}$.

$$E^\gamma/W^\gamma \cong \mathbb{I} \sqcup \mathbb{I}$$

Therefore, we can decompose the extended quotient as follows:

$$E^{\mathfrak{s}}//W^{\mathfrak{s}} \cong E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup \mathbb{I}$$

where $\mathbb{I}$ is the unit interval in complex plane.

We start from the two disjoint varieties $(1, 1, z, z)$ and $(1, -1, z, -z)$. We assume $z \in \{1, -1\}$. Firstly, we check the character χ_t generated by $t = (1, 1, z, z)$. Then we have $\chi_t = (1, 1, z^{\mathrm{val}\circ det}, z^{\mathrm{val}\circ det})$ and $\chi_t \otimes \pi_\sigma \cong \pi \otimes \chi\pi \otimes \eta z^{\mathrm{val}\circ det}\pi \otimes \chi\eta z^{\mathrm{val}\circ det}\pi$. In fact, χ is the element of $\bar{L}(\chi_t \otimes \pi_\sigma)$ except the trivial character. Roughly speaking, the reason is, in this component, the variety fixes the terms which twists with character $z^{\mathrm{val}\circ det}$. Easily, we have $R(\chi_t \otimes \pi_\sigma) =< \chi >= \mathbb{Z}/2\mathbb{Z}$. This implies for each $t = (1, 1, z, z)$, the representation induced $\delta_1(z)$ by $\chi_t \otimes \pi_\sigma$ is reducible.

Similarly, the character χ_t generated by $t = (1, -1, z, -z)$ is given by $\chi_t = (1, (-1)^{\mathrm{val}\circ det}, z^{\mathrm{val}\circ det}, (-z)^{\mathrm{val}\circ det})$. We have $\chi_t \otimes \pi_\sigma \cong \pi \otimes (-1)^{\mathrm{val}\circ det}\chi\pi \otimes \eta z^{\mathrm{val}\circ det}\pi \otimes \chi\eta(-z)^{\mathrm{val}\circ det}\pi$. Then we know $\bar{L}(\chi_t \otimes \pi_\sigma) = \{1, (-1)^{\mathrm{val}\circ det}\chi\}$. Indeed, $R(\chi_t \otimes \sigma) = \mathbb{Z}/2\mathbb{Z}$. This means for each $t = (1, -1, z, -z)$ the representation induced by $\chi_t \otimes \pi_\sigma$ is reducible.

$(1, z, 1, z)$ and $(1, z, -1, -z)$ are two disjoint varieties. Following the similar method, we know

$$t = (1, z, 1, z) \to \chi_t = (1, z^{\mathrm{val}\circ det}, 1, z^{\mathrm{val}\circ det})$$

and we have $R(\chi_t \otimes \sigma) =< \eta >= \mathbb{Z}/2\mathbb{Z}$.

For $t = (1, z, -1, -z)$, we get the corresponding unitary character χ_t as follows

$$t = (1, z, -1, -z) \to \chi_t = (1, z^{\mathrm{val}\circ det}, (-1)^{\mathrm{val}\circ det}, (-z)^{\mathrm{val}\circ det})$$

and we will have $R(\chi_t \otimes \sigma) =< (-1)^{\mathrm{val}\circ det}\eta >= \mathbb{Z}/2\mathbb{Z}$.

In this component, $(1, z, z, 1)$ and $(1, z, -z, -1)$ are two disjoint varieties. Following the similar method, we know

$$t = (1, z, z, 1) \to \chi_t = (1, z^{\mathrm{val}\circ det}, z^{\mathrm{val}\circ det}, 1)$$

and we have $R(\chi_t \otimes \sigma) =< \chi\eta >= \mathbb{Z}/2\mathbb{Z}$.

For $t = (1, z, -z, -1)$, we get the corresponding unitary character χ_t as follows

$$t = (1, z, -z, -1) \to \chi_t = (1, z^{\mathrm{val}\circ det}, (-z)^{\mathrm{val}\circ det}, (-1)^{\mathrm{val}\circ det})$$

and we will have $R(\chi_t \otimes \sigma) =< (-1)^{\mathrm{val}\circ det}\eta\chi >= \mathbb{Z}/2\mathbb{Z}$.

For convenience, we denote

$$(1, 1, z, z) - (a)$$
$$(1, -1, z, -z) - (b)$$
$$(1, z, 1, z) - (c)$$
$$(1, z, -1, -z) - (d)$$
$$(1, z, z, 1) - (e)$$
$$(1, z, -z, -1) - (f)$$

Now, we investigate the points

$$(1, 1, 1, 1)$$
$$(1, -1, 1, -1)$$
$$(1, 1, -1, -1)$$
$$(1, -1, -1, 1)$$

It is easy to check that

$$(1, 1, 1, 1) \in (a), (c), (e)$$
$$(1, -1, 1, -1) \in (b), (c), (f)$$
$$(1, 1, -1, -1) \in (a), (d), (f)$$
$$(1, -1, -1, 1) \in (b), (d), (e)$$

In fact, for such points, the R-group $R(\chi_t \otimes \sigma)$ is given by $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. This implies, for each $\mathrm{Ind}_M^G(\chi_t \otimes \sigma)$, there are 4 irreducible constituents. The extended quotient is the disjoint union of the ordinary quotient and six unit intervals. The six intervals are sent to the edges of a tetrahedron by the canonical projection

$$\pi : E^{\mathfrak{s}}//W^{\mathfrak{s}} \to E^{\mathfrak{s}}/W^{\mathfrak{s}}$$

The preimage of the interior of one edge is the union of two open intervals (the one corresponding to the given edge and one in the ordinary quotient), replicating the fact that the R-group has order 2, while the preimage of a vertex is the union of three endpoints of intervals and one point in the ordinary quotient, replicating the fact that the R-group has order 4 here. The 1-skeleton of the tetrahedron is perfect model of reducibility and confirms the ABP-conjecture in this case.

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