# *Modular representations and finite W-algebras*

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# MODULAR REPRESENTATIONS AND FINITE $W$-ALGEBRAS

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NOTES BY

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## 1. Introduction and preliminaries

These notes can be viewed as a short and informal introduction to some aspects of the theory of finite $W$-algebras. In full generality, finite $W$-algebras, also known as *enveloping algebras of Slodowy slices* or simply as *quantized Slodowy slices*, were introduced into the mathematics literature quite recently (in [37]), but they already found some important applications in representation theory and the theory of primitive ideals. Since general interest to these algebras is growing, it seems natural to review some of the main results obtained in the theory so far.

Before we start, let me say that there are very few proofs in these notes, which is a fair reflection of the fact that there were almost no proofs in my Bremen lectures. Some arguments are just indicated and references to complete proofs are given to the interested reader.

One should also mention that there already exist two extensive surveys of the theory of finite $W$-algebras. One survey is a set of notes by Wang, and the other is Losev's contribution to ICM 2010; see [51, 28]. Fortunately, despite some inevitable overlaps the three surveys complement each other rather well. The present notes mostly deal with modular aspects of the theory, Wang's notes discuss combinatorial aspects [4, 5] which we do not touch here, while Losev's survey explains in much detail his approach to the theory via equivariant deformation quantization.

Let $k$ denote an algebraically closed field of characteristic $p \geq 0$. Throughout these notes we shall assume that $k = \mathbb{C}$ if $p = 0$ and $k = \overline{\mathbb{F}}_p$, an algebraic closure of the prime field $\mathbb{F}_p$, if $p > 0$.

We begin with a brief reminder from the theory of linear algebraic groups. The reader is referred to [19] or [1] for a detailed exposition. Recall first that an *affine algebraic group* is an affine algebraic variety $G$ over $k$ together with a structure of a group such that the multiplication map $\mu\colon G \times G \to G$, $(x, y) \mapsto x \cdot y$, and the inverse map $\iota\colon G \to G$, $x \mapsto x^{-1}$, are morphisms of affine varieties. The main examples of interest for us will be linear algebraic groups, that is the groups $\mathrm{GL}(n, k)$ and their Zariski closed subgroups such as $\mathrm{SL}(n, k)$, $\mathrm{SO}(n, k)$, $\mathrm{Sp}(2n, k)$, $\mathrm{T}(n, k)$ (the subgroup of all upper–triangular matrices with nonzero diagonal entries in $\mathrm{GL}(n, k)$), and $\mathrm{D}(n, k)$ (the subgroup of all invertible diagonal matrices in $\mathrm{GL}(n, k)$).

A finite dimensional *rational representation* of $G$ (or a *$G$-module*) is a pair $(V, \rho)$ consisting of a finite dimensional $k$-vector space $V$ together with a morphism of algebraic groups $\rho : G \to \mathrm{GL}(V)$. A *character* of $G$ is a rational representation of $G$ in $k^*$. Let $\mathfrak{g}$ denote the Lie algebra of $G$. Then $G$ acts on $\mathfrak{g}$ via the *adjoint representation* Ad.

A linear algebraic group is called *unipotent* if it consists of unipotent elements. Note that, in prime characteristic $p$, an element $u$ is unipotent if and only if $u^{p^s} = 1$ for some $s \in \mathbb{N}$. A *torus* is a linear algebraic group isomorphic to $\mathrm{D}(n, k)$ for some $n$. A subgroup $B$ of a linear algebraic group is called a *Borel subgroup* of $G$ if $B$ a closed connected solvable subgroup of $G$ maximal for this property. Such a subgroup is isomorphic to a Zariski closed subgroup of $T(n, k)$. A *parabolic subgroup* $P$ of $G$ is a Zariski closed subgroup containing a Borel subgroup of $G$ (there is also a geometric definition of $P$: a parabolic subgroup of $G$ is a Zariski closed subgroup such that the homogeneous space $G/P$ is a complete variety). As $k$ is algebraically closed, we know that all the Borel subgroups of $G$ are $G$-conjugate and so are all maximal tori of $G$. Note also that given a torus $T$ of $G$ there is always a Borel subgroup of $G$ containing $T$.

For a linear algebraic group $G$, we define the *unipotent radical* of $G$, denoted $R_u(G)$, to be the maximal closed connected unipotent subgroup of $G$. A linear algebraic group $G$ is called *reductive* if $R_u(G) = \{1\}$. Recall also that $G$ is called *semisimple* if it has no non-trivial closed connected normal solvable subgroups. In particular if $G$ is semisimple then $G$ is also reductive. We say that a semisimple group $G$ is *simply connected* if $G$ is a direct product of its simple normal subgroups each of which is isomorphic to a universal Chevalley group over $k$; see [44] for more detail. Given a parabolic subgroup $P$ of the reductive group $G$ there is a well defined reductive subgroup $L$ of $P$ such that the multiplication map $L \times R_u(P) \to P$ is an isomorphism. In particular, $P = L \cdot R_u(P)$. This decomposition is often referred to as a *Levi decomposition* of $P$ and $L$ is referred to a *Levi subgroup* of $P$.

We now recall some facts about root data. To any algebraic group $H$, we can attach two sets

$$X^*(H) := \{\text{the rational group homomorphisms from } H \text{ to } k^*\},$$
$$X_*(H) := \{\text{the rational group homomorphisms from } k^* \text{ to } H\}.$$

An element of $X^*(H)$ is called a (rational) *character* of $H$ and an element of $X_*(H)$ is called a *cocharacter* of $H$. Both $X^*(\mathrm{D}(n,k))$ and $X_*(\mathrm{D}(n,k))$ carry natural abelian group structures, moreover, we have that $X^*(\mathrm{D}(n,k)) \cong \mathbb{Z}^n$ and $X_*(\mathrm{D}(n,k)) \cong \mathbb{Z}^n$ (and so both are free abelian groups of the same finite rank). Since any rational homomorphism of the one-dimensional torus $k^*$ to itself has the form $x \mapsto x^m$ for some $m \in \mathbb{Z}$, we can define, for $H = \mathrm{D}(n,k)$, a pairing $\langle \cdot, \cdot \rangle : X^*(H) \times X_*(H) \to \mathbb{Z}$ by setting $\langle \chi, \lambda \rangle := m$ where $(\chi \circ \lambda)(x) = x^m$ for all $x \in k^*$. This pairing is nondegenerate for $H = \mathrm{D}(n,k)$, hence for every algebraic torus $T$.

From now on we denote by $G$ a connected reductive algebraic group over $k$ and write $T$ for a maximal torus of $G$. Let $B$ be a Borel subgroup of $G$ containing $T$, and let $P$ be a parabolic subgroup containing $B$. We set $X^* := X^*(T)$ and $X_* := X_*(T)$. The group $T$ acts on $\mathfrak{g}$ via the adjoint representation Ad. As the action of $T$ on $\mathfrak{g}$ is semisimple, the Lie algebra $\mathfrak{g}$ splits into a finite direct sum of one-dimensional subspaces stable under $\operatorname{Ad} T$. We denote by $\Phi$ the set of all *nonzero* $\alpha \in X^*(T)$ which arise as characters of the adjoint action of $T$ on $\mathfrak{g}$.

The elements of $\Phi$ are called *roots*. The set of roots $\Phi$ is finite and for every $\alpha \in \Phi$ the corresponding root subspace

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \,|\, (\mathrm{Ad}\ t)(x) = \alpha(t)x \ \text{ for all } \ t \in T\}$$

is one-dimensional. The group $W := N_G(T)/T$ is called the *Weyl group* of $G$. This group is finite and acts on both $X^*$ and $X_*$. There exists a scalar product $(\,\cdot\,|\,\cdot\,)$ on the real vector space $\mathbb{E} := X^* \otimes_{\mathbb{Z}} \mathbb{R}$ invariant under the action of the Weyl group $W$ and such that

$$2\frac{(\alpha|\beta)}{(\beta|\beta)} \in \mathbb{Z} \qquad \text{for all } \ \alpha, \beta \in \Phi.$$

For every $\alpha \in \Phi$ there is a cocharacter $\alpha^\vee \in X_*$, called the *coroot* of $\alpha$, which takes values in the derived subgroup of $G$ and has the property that $\langle \nu, \alpha^\vee \rangle = 2(\nu|\alpha)/(\alpha|\alpha)$ for all $\nu \in X^*$. Due to the duality between $X^*$ and $X_*$ described above we may regard $\alpha^\vee$ as an element of $\mathbb{E}^*$, and we often identify $\mathbb{E}^*$ with $\mathbb{E}$ by using the $W$-invariant scalar product $(\,\cdot\,|\,\cdot\,)$.

The set $\Phi$ is a crystallographic root system in the Euclidean space $\mathbb{E}$ (or rather in the $\mathbb{R}$-span of $\Phi$) and $W$ is the Weyl group of $\Phi$. In particular, the group $W$ is generated by the orthogonal reflections $s_\alpha$, $\alpha \in \Phi$, such that $s_\alpha(x) = x - \alpha^\vee(x)\alpha$ for all $x \in \mathbb{E}$. The set $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\} \subseteq \mathbb{E}^*$ is the crystallographic root system dual to $\Phi$.

We conclude this preliminary section with the construction of a standard Levi subgroup and a standard parabolic subgroup. Let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be a basis of simple roots in $\Phi$ and let $\Phi^+ = \Phi^+(\Pi)$ be the positive system of $\Phi$ with respect to $\Pi$. The group $G$ contains a unique Borel subgroup $B$ with $\mathrm{Lie}(B) = \mathfrak{t} \oplus \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$. Given a subset $I$ of $\{1,\dots,l\}$ we denote by $\Phi_I$ the subset of $\Phi$ consisting of all roots of the form $\sum_{i\in I} c_i\alpha_i$. The set $\Phi_I$ is a root subsystem of $\Phi$ and there exists a unique reductive subgroup $L_I$ in $G$ which contains $T$ as a maximal torus and has $\Phi_I$ as its root system with respect to $T$. This group is referred to as the *standard Levi subgroup* of $G$ associated with $I$. Moreover, there is a unique parabolic subgroup $P_I$ in $G$ whose Lie algebra $\mathfrak{p}_I$ is spanned by $\mathfrak{t} := \mathrm{Lie}(T)$ and by all root spaces $\mathfrak{g}_\alpha$ with $\alpha \in \Phi_I \cup \Phi^+$. The Lie algebra of the unipotent radical $R_u(P_I)$ is spanned by all root spaces $\mathfrak{g}_\alpha$ with $\alpha \in \Phi^+ \setminus \Phi_I$. The group $P_I$ is called the *standard parabolic subgroup associated to $I$*. Note that $P_I$ contains the Borel subgroup $B$ described above. Every parabolic subgroup of $G$ is conjugate to precisely one of the standard parabolic subgroups just defined.

Given a rational representation $\rho\colon G \to \mathrm{GL}(V)$ and a vector $v \in V$ we write $Z_G(v)$ for the stabilizer of $v$ in $G$. The group $Z_G(v)$ is Zariski closed in $G$ and $\mathrm{Lie}(Z_G(v)) \subseteq \mathfrak{g}_v$, where $\mathfrak{g}_v = \{X \in \mathfrak{g} \mid (\mathrm{d}\rho)(X)(v) = 0\}$. If $\mathrm{char}(k) = 0$, then the equality $\mathfrak{g}_v = \mathrm{Lie}(Z_G(v))$ holds.

Given a linear algebraic group $H$ we write $(H,H)$ and $H^0$ for the derived subgroup and the connected component of 1 in $H$. Recall that $(H,H)$ is Zariski closed in $H$ and $H^0$ is the largest connected normal subgroup of $H$. If $H$ is connected, then so is $(H,H)$. We write $Z(H)$ for the center of $H$. If $H$ is a semisimple group, then $Z(H)$ is a finite group contained in any maximal torus of $H$.

## 2. Nilpotent orbits

2.1. **Unstable vectors.** Recall that $G$ is a connected reductive algebraic group over $k$. Let $V$ be a finite dimensional rational $G$-module and let $k[V]$ be the algebra of polynomial functions on $V$. It is well known that the invariant algebra $k[V]^G$ is generated as a $k$-algebra with 1 by finitely many homogeneous polynomial functions of positive degree (in positive characteristic this requires the Mumford conjecture proved by Haboush in [17]). Denote by $k[V]^G_+$ the maximal ideal of $k[V]^G$ generated by all $f \in k[V]^G$ with $f(0) = 0$.

Given a subset $X$ of $V$ we write $\overline{X}$ for the closure of $X$ in the Zariski topology of $V$.

**Definition.** A vector $v \in V$ is *$G$-stable* if $\overline{Gv} = Gv$, *$G$-semistable* if $0 \notin \overline{Gv}$ and *$G$-unstable* if $0 \in \overline{Gv}$.

Thus, a vector is $G$–unstable if and only if it is not $G$-semistable. It is also clear that every $G$-stable nonzero vector is $G$-semistable.

**Fact.** A vector $v \in V$ is $G$-unstable if and only if $f(v) = 0$ for every $f \in k[V]^G_+$.

The following result is fundamental for Geometric Invariant Theory. It holds in all characteristics.

**Theorem 1** (The Hilbert–Mumford Criterion)**.** *A vector $v \in V$ is $G$-unstable if and only if there exists a cocharacter $\lambda \in X_*(G)$ with respect to which $v$ is unstable, that is $0 \in \overline{\lambda(k^*)v}$.*

Fix some cocharacter $\lambda \in X_*(G)$. For any $i \in \mathbb{Z}$ define $V(i) := V(i;\lambda) := \{v \in V \mid \lambda(t)v = t^i v \text{ for all } t \in k^*\}$. We then have $V = \bigoplus_{i\in\mathbb{Z}} V(i)$. Indeed, the rational homomorphism $k^* \xrightarrow{\lambda} G \to \mathrm{GL}(V)$ of the torus $k^*$ can be represented by diagonal matrices in some basis of $V$ and all rational characters $k^* \to k^*$ have the form $t \to t^i$ for some $i \in \mathbb{Z}$. For $0 \neq v \in V$ write $v = v_k + v_{k+1} + \cdots + v_s$, where $v_i \in V(i)$ and $v_k \neq 0$. Define $m(v,\lambda) := k = \min\{i \mid v_i \neq 0\}$. From Theorem 1 we get that *a vector $v \in V$ is $G$–unstable if and only if $m(v,\lambda) > 0$ for some cocharacter $\lambda \in X_*(G)$.*

Note that the group $G$ acts on $X_*(G)$. Since all maximal tori in $G$ are conjugate, we can define a $G$-invariant norm map $\|\cdot\| : X_*(G) \to \mathbb{R}_{\geq 0}$ by setting $\|\lambda\| = \|g\lambda g^{-1}\| := \sqrt{(g\lambda g^{-1} \mid g\lambda g^{-1})}$, where $g \in G$ is such that $g\lambda g^{-1} \in X_*$. This map is well defined because the scalar product $(\,\cdot\,|\,\cdot\,)$ is $W$-invariant. We write $0 \in X_*(G)$ for the trivial cocharacter $k^* \to G$.

**Definition.** Let $v$ be a nonzero unstable vector in $V$. We say that $\lambda \in X_*$ is an *optimal* cocharacter for $v$ if

$$\frac{m(v,\lambda)}{\|\lambda\|} \geq \frac{m(v,\mu)}{\|\mu\|} \quad \text{for every } 0 \neq \mu \in X_*(G).$$

The following theorem, proved in [22] and [47], is one of the main results of the Kempf–Rousseau theory.

**Theorem 2.** *Every $G$-unstable vector $v \in V$ admits at least one optimal cocharacter in $X_*(G)$.*

**Remark 1.** If a cocharacter $\lambda \in X_*$ is optimal for a $G$-unstable vector $v \in V$, then necessarily $m(v,\lambda) > 0$.

**Definition.** We say that a nonzero cocharacter $\lambda \in X_*(G)$ is *primitive* if $\lambda = n\mu$ with $n \in \mathbb{N}$ and $\mu \in X_*(G)$ implies $n = \pm 1$.

It is obvious that every nonzero cocharacter has the form $n\mu$ for some $n \in \mathbb{N}$ and some primitive $\mu \in X_*(G)$.

For $0 \neq \lambda \in X_*(G)$ define

$$\mathfrak{p}(\lambda) = \bigoplus_{i\geq 0} \mathfrak{g}(i;\lambda), \quad \mathfrak{l}(\lambda) = \mathfrak{g}(0;\lambda), \quad \mathfrak{u}(\lambda) = \bigoplus_{i>0} \mathfrak{g}(i;\lambda).$$

Then $\mathfrak{p}(\lambda)$ is a parabolic subalgebra of $\mathfrak{g}$, that is, there exists a *unique* parabolic subgroup $P(\lambda)$ with Levi decomposition $P(\lambda) = Z(\lambda)U(\lambda)$

such that $\mathfrak{p}(\lambda) = \mathrm{Lie}(P(\lambda))$, $\mathfrak{l}(\lambda) = \mathrm{Lie}(Z(\lambda))$ and $\mathfrak{u}(\lambda) = \mathrm{Lie}(U(\lambda))$. The group $Z(\lambda)$ coincides with the centralizer of the one-dimensional torus $\lambda(k^*)$ in $G$.

Any maximal torus $T'$ of $Z(\lambda)$ contains $\lambda(k^*)$, so that $\lambda \in X_*(T')$. As $T' = gTg^{-1}$ for some $g \in G$, we can transport the $W$-invariant scalar product $(\,\cdot\,|\,\cdot\,)$ from $\mathbb{E}$ to the real vector space $X_*(T') \otimes_{\mathbb{Z}} \mathbb{R}$. Denote by $T'^{\lambda}$ the subgroup of $T'$ generated by all one-dimensional tori $\mu(k^*)$, where $\mu \in X_+(T')$ is such that $(\mu|\lambda) = 0$ (by construction, $T'^{\lambda}$ is a torus of codimension 1 in $T'$). Define

$$Z^{\perp}(\lambda) := T'^{\lambda}\big(Z(\lambda), Z(\lambda)\big),$$

a connected reductive subgroup of codimension 1 in $Z(\lambda)$. Note that the group $Z^{\perp}(\lambda)$ is independent of the choice of a maximal torus $T'$ in $Z(\lambda)$ since $(gT'g^{-1})^{\lambda} = gT'^{\lambda}g^{-1}$ for all $g \in Z(\lambda)$.

**Remark 2.** It is proved in the Kempf–Rousseau theory that if a cocharacter $\lambda$ is optimal for a $G$-unstable vector $v \in V$, then $Z_G(v) \subseteq P(\lambda)$. Furthermore, if $\lambda$ and $\lambda'$ are two optimal cocharacters for $v$, then $P(\lambda) = P(\lambda')$. This parabolic subgroup is denoted by $P(v)$ and called the *optimal parabolic subgroup* of $v$. In general, the group $P(v)$ depends on the choice of the norm mapping $\|\cdot\|$ on $X_*(G)$, but any two optimal cocharacters for $v$ are always conjugate under the action of $P(v)$.

The following very useful result has a rather long history; see [23], [32], [43], [34]. The most recent proof valid in all characteristics can be found in [49].

**Theorem 3** (The Kirwan–Ness Criterion)**.** *Let $0 \neq v \in V$ and $\lambda \in X_*(G)$ be such that $k := m(v,\lambda) > 0$ and write $v = \sum_{i\geq k} v_i$ with $v_i \in V(i)$ and $v_k \neq 0$. Then the cocharacter $\lambda$ is optimal for $v$ if and only if the vector $v_k$ is semistable with respect to the reductive group $Z^{\perp}(\lambda)$, that is $f(v_k) \neq 0$ for some $f \in k[V(k)]_+^{Z^{\perp}(\lambda)}$.*

2.2. **Nilpotent elements.** Being an algebraic Lie algebra, $\mathfrak{g}$ admits a Jordan–Chevalley decomposition. We denote by $\mathcal{N}(\mathfrak{g})$ the *nilpotent cone* of $\mathfrak{g}$, that is the variety of all nilpotent elements in $\mathfrak{g}$.

If $p > 0$, we impose the following assumptions on $G$ from now on:

(SH1) the group $(G, G)$ is simply connected;
(SH2) the prime $p$ is *good* for the root system $\Phi$, i.e. $p$ does not divide any coefficient of any root of $\Phi^+$ with respect $\Pi$;
(SH3) there exists a finite-dimensional rational representation $\rho\colon G \to \mathrm{GL}(V)$ such that the trace form on $\mathfrak{g}$ given by the formula $(x,y) := \mathrm{tr}\big(\mathrm{d}\rho(x)\cdot \mathrm{d}\rho(y)\big)$ $(x, y \in \mathfrak{g})$ is nondegenerate.

These assumptions are often referred to as *standard hypotheses.* They are satisfied by orthogonal and symplectic Lie algebras when $p \neq 2$, by the exceptional Lie algebras of type other than $\mathrm{E}_8$ when $p \neq 2, 3$, and by Lie algebras of type $\mathrm{E}_8$ when $p \neq 2, 3, 5$. In type A, they are satisfied by

$\mathfrak{sl}_n$ when $p \nmid n$ and by $\mathfrak{gl}_n$ for all $p$ (so we often replace $\mathfrak{sl}_{mp}$ by $\mathfrak{gl}_{mp}$ when dealing with type A). We mention for completeness that when $G$ is a simple algebraic $k$-group with root system $\Phi$ of type other than $\mathrm{A}_{mp-1}$ and $p = \mathrm{char}(k)$ is good for $\Phi$, then the Lie algebra $\mathrm{Lie}(G)$ is simple and isomorphic to $\mathrm{Lie}(\widetilde{G})$, where $\widetilde{G}$ is a simple, simply connected $k$-group with root system $\Phi$. In particular, if $G$ is a simply connected $k$-group of type $\mathrm{B}_n$, $\mathrm{C}_n$ or $\mathrm{D}_n$ and $p \neq 2$, then the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is isomorphic to $\mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n}$, respectively. Moreover, the adjoint action of $G$ on $\mathfrak{g}$ is induced by that of the corresponding classical group.

**Proposition 4.** *An element $x \in \mathfrak{g}$ is $G$-unstable if and only if $x \in \mathcal{N}(\mathfrak{g})$.*

*Proof.* By Theorem 1, if $x$ is $G$-unstable, then $m(v, \lambda) > 0$ for some cocharacter $\lambda \in X_*$. It follows that $x$ lies in the nilradical of a Borel subalgebra of $\mathfrak{g}$, hence is nilpotent.

Conversely, let $x \in \mathcal{N}(\mathfrak{g})$. By a classical result of Richardson [45], if $p$ is good for the root system of $G$, then the number of nilpotent $G$-orbits in $\mathfrak{g}$ is finite. Hence there is a $G$-orbit $\mathcal{O} \subseteq \mathcal{N}(\mathfrak{g})$ for which $\mathcal{O} \cap kx$ is dense in the line $kx$. As $\overline{\mathcal{O}} \cap kx \supseteq \overline{\mathcal{O} \cap kx} = kx$, there is a nonzero $a \in k$ such that the closure of $\mathrm{Ad}(G)(ax)$ contains $kx$. As $Z_G(ax) = Z_G(bx)$ for any nonzero $b \in k$, this entails that all nonzero scalar multiples of $x$ are conjugate in $G$. Since $f(tx) = t^{\deg f} f(x)$ for any homogeneous polynomial function $f$ on $\mathfrak{g}$, this yields that $f(x) = 0$ for all $f \in k[\mathfrak{g}]^G_+$. Therefore, $x$ is $G$-unstable. $\square$

**Remark 3.** Suppose $G$ is a connected reductive $k$-group and $\mathfrak{g} = \mathrm{Lie}(G)$. If $p = \mathrm{char}(k)$ is not good for the root system of $G$, then we say that it is *bad*. It is known that the number of nilpotent $G$-orbits in $\mathfrak{g}$ is finite in all characteristics. For root systems of classical types there are computer-free proofs of this result, but types $\mathrm{F}_4$, $\mathrm{E}_6$, $\mathrm{E}_7$ and $\mathrm{E}_8$ still require extensive computer-aided computations in bad characteristics; see [18]. Due to the main result of [18] one can argue as in the second part of the proof of Proposition 4 to show that all nonzero multiples of any $x \in \mathcal{N}(\mathfrak{g})$ are $G$-conjugate. This result has an analogue for unipotent elements in $G$ suggested by Serre (see [29] for a very recent proof which involves rather deep results on Springer's correspondence).

**Definition.** Suppose $\mathrm{char}(k) = p > 0$ and let $L$ be a Lie algebra over $k$. A mapping $[p]\colon L \to L$, $a \mapsto a^{[p]}$, is called a *p-mapping* if

(1) $\mathrm{ad}\, a^{[p]} = (\mathrm{ad}\, a)^p$ for all $a \in L$,
(2) $(\alpha a)^{[p]} = \alpha^p a^{[p]}$ for all $\alpha \in k$ and $a \in L$,
(3) $(a + b)^{[p]} = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a, b)$,

where

$$(\mathrm{ad}(a \otimes t + b \otimes 1))^{p-1}(a \otimes 1) \;=\; \sum_{i=1}^{p-1} i s_i(a, b) \otimes t^{i-1}$$

in $L \otimes_k k[t]$ for all $a, b \in L$. The pair $(L, [p])$ is referred to as a *restricted Lie algebra*.

Note that $\mathfrak{g} = \mathrm{Lie}(G)$ carries a canonical restricted Lie algebra structure $x \mapsto x^{[p]}$ equivariant under the adjoint action of $G$:

$$\big((\mathrm{Ad}\, g)x\big)^{[p]} = (\mathrm{Ad}\, g)\big(x^{[p]}\big) \quad \text{for all } x \in \mathfrak{g} \text{ and } g \in G.$$

This follows from the fact that $\mathfrak{g}$ is canonically isomorphic to the Lie algebra of all left-invariant derivations of the coordinate algebra $k[G]$.

**Fact.** If $p > 0$, then $x \in \mathcal{N}(\mathfrak{g})$ if and only if $x^{[p]^e} = 0$ for $e \gg 0$.

**Fact.** If $G$ satisfies SH1, SH2, SH3, then the orbit map is *separable* for every $x \in \mathfrak{g}$, that is $\mathrm{Lie}(Z_G(x)) = \mathfrak{g}_x = \big\{y \in \mathfrak{g} \mid [y, x] = 0\big\}$ for all $x \in \mathfrak{g}$.

We shall often identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by using the trace form $(\cdot\,,\,\cdot)$; the corresponding map $\mathfrak{g} \to \mathfrak{g}^*$ is $G$-equivariant.

2.3. **Primitive ideals and associated varieties.** Suppose $k = \mathbb{C}$. As a motivation for the study of nilpotent orbits, we recall some facts about primitive ideals and associated varieties. An ideal $I$ of the universal enveloping algebra $U(\mathfrak{g})$ is called *primitive* if it is the annihilator of some irreducible $\mathfrak{g}$-module $M$. Recall that the *annihilator* of a $\mathfrak{g}$-module $M$ in $U(\mathfrak{g})$ is

$$I_M = \mathrm{Ann}_{U(\mathfrak{g})} M := \{u \in U(\mathfrak{g}) \mid u \cdot m = 0 \text{ for all } m \in M\}.$$

Let $Z(\mathfrak{g})$ denote the center of $U(\mathfrak{g})$, a polynomial algebra in $\mathrm{rk}\,\mathfrak{g} = \dim T$ variables. It is well known that $I$ is a primitive ideal of $U(\mathfrak{g})$ if and only if $I$ is a prime ideal and $Z(\mathfrak{g}) \cap I$ is a maximal ideal of $Z(\mathfrak{g})$; see [11] for example.

By the PBW theorem, the canonical filtration of $U(\mathfrak{g})$ has the property that $\mathrm{gr}(U(\mathfrak{g})) = S(\mathfrak{g})$. If $I$ is any two-sided ideal of $U(\mathfrak{g})$, then $\mathrm{gr}(I)$ is a $G$-stable ideal of $S(\mathfrak{g})$. We define the *associated variety* of $I$ to be the zero locus of $gr(I)$, that is

$$\mathcal{VA}(I) := \{\chi \in \mathfrak{g}^* \mid f(\chi) = 0, \text{ for all } f \in \mathrm{gr}(I)\}.$$

This is a closed $\mathrm{Ad}^*(G)$-stable subset of $\mathfrak{g}^*$. Here we have the following fundamental result which was first proved in full generality by A. Joseph in [21]:

**Theorem 5** (The Irreducibility Theorem)**.** *Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by using the Killing form of $\mathfrak{g}$ and let $I$ be a primitive ideal of $U(\mathfrak{g})$. Then the affine variety $\mathcal{VA}(I)$ is irreducible. Moreover, $\mathcal{VA}(I) = \overline{\mathcal{O}(x)}$ for some nilpotent $G$-orbit $\mathcal{O}(x) \subseteq \mathcal{N}(\mathfrak{g})$.*

If $M$ is an irreducible $\mathfrak{g}$-module, then the associated variety of $I_M$ is an important invariant of the isomorphism class of $M$.

Let $A$ be prime Noetherian ring. An element $a \in A$ is called *regular* if $a$ is not a zero divisor in $A$. By Goldie's theory, the set of all regular elements of $A$ is an Ore set and hence it can be used to construct a ring of fractions $S^{-1}A$. The ring $S^{-1}A$ is prime Artinian, hence isomorphic to $\mathrm{Mat}_n(D)$ for some $n \in \mathbb{N}$ and some skew-field $D$. We write $n = \mathrm{rk}(A)$ and call $n$ the *Goldie rank* of $A$.

One of the important open problems in the theory of primitive ideals is to determine the Goldie ranks of all primitive quotients $U(\mathfrak{g})/I$. An ideal $I$ of $U(\mathfrak{g})$ is called *completely prime* if the ring $U(\mathfrak{g})/I$ is a domain (that is, $xy \in I$ for $x, y \in U(\mathfrak{g})$ implies that either $x \in I$ or $y \in I$). This happens if and only if $\mathrm{rk}\big(U(\mathfrak{g})/I\big) = 1$; see [11] for more detail. Another natural (and old) problem of this theory asks when a primitive ideal of $U(\mathfrak{g})$ is completely prime. This question is open outside type A (for $\mathfrak{g} = \mathfrak{sl}_n$ the problem has been solved in [31]). A partial answer to this question is given by Theorem 22.

2.4. **The Bala–Carter–Pommerening theorem.** We suppose for a moment that $G$ is a semisimple algebraic group. Let $P$ be a parabolic subgroup of $G$ and put $\mathfrak{p} := \mathrm{Lie}(P)$. Write $P = LU_P$, where $U_P = R_u(P)$ and $L$ is the Levi factor of $P$. Set $\mathfrak{l} := \mathrm{Lie}(L)$ and $\mathfrak{u}_P := \mathrm{Lie}(U_P)$. There exists an element $e \in \mathfrak{u}_P$ such that the orbit $\mathrm{Ad}(P)e$ is Zariski dense in $\mathfrak{u}_P$. Such an element is called a *Richardson element* of $\mathfrak{u}_P$. We have that $P = P(\lambda)$, $L = Z(\lambda)$ and $U_P = U(\lambda)$ for some *primitive* $\lambda \in X_*(G)$. Obviously, $\mathfrak{p} = \mathfrak{p}(\lambda) = \bigoplus_{i \geq 0} \mathfrak{g}(i; \lambda)$.

By Richardson, the group $L$ has an open orbit on $U_P/(U_P, U_P)$, which implies that $\dim L \geq \dim U_P/(U_P, U_P)$. We say that a parabolic subgroup $P$ (and a parabolic subalgebra $\mathfrak{p}$) is *distinguished* if the equality $\dim L = \dim U_P/(U_P, U_P)$ holds.

Now suppose $G$ is a connected reductive group. We say that a parabolic subgroup $P$ of $G$ is *distinguished* if the image of $P$ is distinguished in the semisimple algebraic group $G/Z(G)$. If $\mathfrak{p} = \mathrm{Lie}(P)$ and $P$ is a distinguished parabolic subgroup of $G$, then we say that $\mathfrak{p}$ is a *distinguished parabolic subalgebra* of $\mathfrak{g}$.

**Theorem 6** (The Bala–Carter–Pommerening Theorem)**.** *If $G$ satisfies* SH2*, then for every element $e \in \mathcal{N}(\mathfrak{g})$ there exists a Levi subgroup $L$ with Lie algebra $\mathfrak{l}$ and a distinguished parabolic subalgebra $\mathfrak{p}_{\mathfrak{l}} = \bigoplus_{i \geq 0} \mathfrak{l}(i; \lambda)$ of $\mathfrak{l}$ such that $e \in \mathfrak{l}(2; \lambda)$ is a Richardson element in $\mathfrak{u}_{\mathfrak{l}} = \bigoplus_{i > 0} \mathfrak{l}(i; \lambda)$. Furthermore, $\lambda$ is a rational cocharacter of $G$ which takes values in $(L, L)$ and has the property that $\mathfrak{l}(i; \lambda) = \mathfrak{g}(i; \lambda) \cap \mathfrak{l} = 0$ for all odd $i$.*

This important theorem was first proved by P. Bala and R.W. Carter under the assumption that $p = 0$ or $p$ is sufficiently large. It was later proved by K. Pommerening under the assumption that $p$ is good for the root system of $G$. For classical groups and for groups of type $\mathrm{G}_2$ and $\mathrm{F}_4$ the result was known much earlier; we refer to [7], Sect. 5, for a detailed discussion. Pommerening's proof relied heavily on case-by-case considerations. The first conceptual proof valid in good characteristic was found in [38]. It was subsequently simplified by T. Tsujii in [49].

The Bala–Carter–Pommerening theorem allows one to parameterize the nilpotent orbits in $\mathfrak{g}$. Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be a basis of simple roots in $\Phi$. For $I \subseteq \{1, \dots, l\}$ set $\Pi_I = \{\alpha_i \mid i \in I\}$ and let $L_I$ be the standard Levi subgroup associated with $I$. Given a subset $J \subseteq I$ denote by $P_{I,J}$ the standard parabolic subgroup of $L_I$ associated with $J$. Define

$$\mathcal{P}(\Pi) := \big\{(I, J) \mid J \subseteq I \subseteq \{1, \dots, l\},\ P_{I,J} \text{ is distinguished in } L_I\big\}.$$

Two elements $(I, J)$ and $(I', J')$ of $\mathcal{P}(\Pi)$ are said to be *equivalent* if there exists an element $w \in W$ such that $w(\Pi_I) = \Pi_{I'}$ and $w(\Pi_J) = \Pi_{J'}$. We denote by $\big[\mathcal{P}(\Pi)\big]$ the set of all equivalence classes of elements in $\mathcal{P}(\Pi)$.

**Corollary 7.** *There is a natural bijection between the set $\mathcal{N}(\mathfrak{g})/G$ of nilpotent $G$-orbits and $\big[\mathcal{P}(\Pi)\big]$.*

2.5. **Reduction mod $p$.** Now suppose that $p = 0$. Then any nilpotent element $e \in \mathfrak{g}$ can be included into an $\mathfrak{sl}_2$–triple $\{e, h, f\} \subseteq \mathfrak{g}$. The eigenvalues of $\operatorname{ad} h$ are integers, so we get a $\mathbb{Z}$–grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i; h)$ with $e \in \mathfrak{g}(2; h)$. By classical results of Dynkin and Kostant, $e' \in \operatorname{Ad}(G)e$ if and only if $h' \in \operatorname{Ad}(G)h$ (here $\{e', h', f'\} \subseteq \mathfrak{g}$ is another $\mathfrak{sl}_2$-triple). So we may assume that $h \in \mathfrak{t} := \operatorname{Lie}(T)$. Choose a Chevalley basis $\{h_\alpha \mid \alpha \in \Pi\} \cup \{e_\alpha \mid \alpha \in \Phi\}$ in $\mathfrak{g}$ and let $\mathfrak{g}_\mathbb{Z}$ be the $\mathbb{Z}$-span of this basis, a Lie algebra over $\mathbb{Z}$. One knows that $h = \sum_{\alpha \in \Pi} m_\alpha h_\alpha$ for some $m_\alpha \in \mathbb{Z}$ where $h_\alpha \in \mathfrak{t}$ is the differential at 1 of the coroot $\alpha^\vee$. So we may consider the corresponding cocharacter $\sum_{\alpha \in \Pi} m_\alpha \alpha^\vee \in X_*(T)$. Conjugating it to the dominant Weyl chamber by action of $W$ we get a cocharacter $\lambda_\Delta \in X_*(T)$, where $\Delta$ is the so called *Dynkin label* of the $G$-orbit of $e$.

**Remark 4.** The Dynkin label $\Delta$ is also known as the *weighted Dynkin diagram* of $\operatorname{Ad}(G)e$. We may assume, after conjugating $h$ by a suitable $w \in W$, that $\alpha(h) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Pi$. Then $\Delta$ assigns to a simple root $\alpha$ the value $\alpha(h)$. It is well known that $\alpha(h) \in \{0, 1, 2\}$ for all $\alpha \in \Pi$.

All weighted diagrams are classified by Dynkin. Denote the set of all such diagrams by $\mathcal{D}(\Pi)$. There is a bijection between the sets $\mathcal{D}(\Pi)$ and $\big[\mathcal{P}(\Pi)\big]$.

Now return to the $p > 0$ case and assume that $G$ satisfies the hypotheses SH1 and SH2. For each $\Delta \in \mathcal{D}(\Pi)$ we have a cocharacter $\lambda_\Delta \in X_*(T)$. Denote by $\mathfrak{g}(2;\lambda_\Delta)_{\mathrm{reg}}$ the open $Z(\lambda_\Delta)$-orbit in $\mathfrak{g}(2;\lambda_\Delta)$ (such an orbit exists by a result of Richardson; see [46]).

The following result is proved in [38]; it plays an important role in proving Theorem 6.

**Theorem 8.** *Let $\Delta \in \mathcal{D}(\Pi)$ and $e \in \mathfrak{g}(2;\lambda_\Delta)_{\mathrm{reg}}$. Then*

*(1) the cocharacter $\lambda_\Delta$ is optimal for the $G$-unstable vector $e$;*
*(2) $Z_G(e) \subseteq P(\lambda_\Delta)$ and $R_u\big(Z_G(e)\big) = Z_{U(\lambda_\Delta)}(e)$;*
*(3) the group $C(\Delta, e) := Z_G(e) \cap Z(\lambda_\Delta)$ is reductive;*
*(4) $Z_G(e) = C(\Delta, e) \cdot Z_{U(\lambda_\Delta)}(e)$, a semidirect product;*
*(5) $\mathfrak{g}_e \subseteq \mathfrak{p}(\lambda_\Delta)$ and $\big[\mathfrak{p}(\lambda_\Delta), e\big] = \bigoplus_{i\geq 2} \mathfrak{g}(i;\lambda_\Delta)$.*

**Remark 5.** Parts (2) − (5) generalize classical results of Kostant.

**Remark 6.** The cocharacter $\lambda_\Delta$ is primitive if and only if $\mathfrak{g}(2k+1;\lambda_\Delta) \neq 0$ for some $k$. In that case, $\mathfrak{g}(1;\lambda_\Delta) \neq 0$. When $\mathfrak{g}(2k+1;\lambda_\Delta) = 0$ for all $k$, one says that $e$ is *even*.

**Remark 7.** The weighted diagram $\Delta$ assigns to a root $\gamma = \sum_{\alpha\in\Pi} r_\alpha \alpha$ a value $m := \sum_{\alpha\in\Pi} r_\alpha m_\alpha$. The corresponding root vector $e_\gamma$ belongs to $\mathfrak{g}(m;\lambda_\Delta)$.

It is proved in [38] that for every $e \in \mathcal{N}(\mathfrak{g})$ there exists a unique $\Delta \in \mathcal{D}(\Pi)$ such that $\mathrm{Ad}(G)e \cap \mathfrak{g}(2;\lambda_\Delta) = \mathfrak{g}(2;\lambda_\Delta)_{\mathrm{reg}}$. On the other hand, it is known that if $G$ is a simple, simply connected $k$-group, then $\mathfrak{g}$ identifies with $\mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} k$ as Lie algebras. In conjunction with an earlier result of Elkington [12] this implies, for $G$ simple, that there exists $e' \in \mathfrak{g}_\mathbb{C}(2;\lambda_\Delta)_{\mathrm{reg}} \cap \mathfrak{g}_\mathbb{Z}$ whose image $e' \otimes 1$ in $\mathfrak{g} = \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} k$ lies in $\mathfrak{g}(2;\lambda_\Delta)_{\mathrm{reg}}$ and the equality

$$\dim_k \mathrm{Ad}(G)(e' \otimes 1) = \dim_\mathbb{C} \mathrm{Ad}(G_\mathbb{C})e' \tag{1}$$

holds whenever $p = \mathrm{char(k)}$ is a good prime for the root system $\Phi$.

In what follows, we denote by $\mathcal{O}(\Delta)$ the nilpotent $G$-orbit that intersects with $\mathfrak{g}(2;\lambda_\Delta)_{\mathrm{reg}}$.

2.6. **Sommers' bijection.** In this subsection we assume that $G$ is a simple algebraic group of adjoint type (that is $G \cong \mathrm{Ad}\, G$ as algebraic groups). Working in the characteristic zero setting, E. Sommers has generalized the Bala-Carter bijection discussed in Subsection 2.4; see [48]. In [30, 38], it was shown that his results continue to hold in the case where $p = \mathrm{char}(k)$ is a good prime for the root system $\Phi$.

For $e \in \mathcal{N}(\mathfrak{g})$ we denote by $\Gamma(e)$ the component group $Z_G(e)/Z_G(e)^0$. It follows from Theorem 8 that $\Gamma(e) \cong C(\Delta, e)/C(\Delta, e)^0$.

**Definition.** We say that $H \subseteq G$ is a *pseudo-Levi subgroup* of $G$ if $H = Z_G(s)^0$ for some semisimple element $s \in G$.

If $H = Z_G(s)^0$ for a semisimple element $s \in G$, then it is known that $\operatorname{Lie}(H) = \big\{x \in \mathfrak{g} \mid (\operatorname{Ad} s)x = x\big\}$, and we call $\operatorname{Lie}(H)$ a *pseudo-Levi subalgebra* of $\mathfrak{g}$. Every pseudo-Levi subgroup is $G$-conjugate to *a standard pseudo-Levi subgroup* which is defined as follows:

Let $\widetilde{\Pi} := \{\alpha_0, \alpha_1, \dots, \alpha_l\}$ be the extended set of simple roots, where $\alpha_0$ is the lowest root of $\Phi$ with respect to $\Pi$. For a *proper* subset $J \subseteq \{0, 1, \dots, l\}$ we set $\widetilde{\Pi}_J := \{\alpha_i \mid i \in J\}$ and denote by $\Phi_J$ the set of roots $\gamma$ of the form $\gamma = \sum_{i \in J} m_i\alpha_i$, where $m_i \in \mathbb{Z}$. This is a root system with basis $\widetilde{\Pi}_J$. We write $L_J$ for the subgroup of $G$ generated by $T$ and the unipotent root subgroups $U_\gamma$ with $\gamma \in \Phi_J$. Every pseudo-Levi subgroup of $G$ is conjugate to $L_J$ for some proper subset $J \subseteq \{0, 1, \dots, l\}$.

It is known that if $G$ is of adjoint type and $L = Z_G(s)^0$ for some semisimple element $s \in G$, then the group $Z(L)/Z(L)^0$ is cyclic; see [48, 30, 38]. If $e$ is a nilpotent element of $\mathfrak{g}$ contained in $\operatorname{Lie}(L)$, then $Z(L) \subseteq Z_G(e)$ and $Z(L)^0 \subseteq Z_G(e)^0$. Hence we have a natural group homomorphism $Z(L)/Z(L)^0 \longrightarrow \Gamma(e)$. It turns out that all generators of the cyclic group $Z(L)/Z(L)^0$ map onto conjugate elements of $\Gamma(e)$. Moreover, the following result holds:

**Theorem 9** (Sommers' bijection)**.** *There is a bijection between the $G$-conjugacy classes of pairs $(L, e)$, where $L \subseteq G$ is a pseudo-Levi subgroup of $G$ and $e$ is a distinguished nilpotent element of* $\operatorname{Lie}(L)$*, and the $G$-conjugacy classes of pairs $(e, \mathscr{C})$, where $e$ is a nilpotent element of $\mathfrak{g}$ and $\mathscr{C}$ is a conjugacy class in $\Gamma(e)$, which takes the class of $(L, e)$, where $L = Z_G(s)^0$, to the class of $(e, \mathscr{C}_s)$, where $\mathscr{C}_s$ is the the conjugacy class in $\Gamma(e)$ containing the image of $s$ in $\Gamma(e)$.*

**Remark 8.** Sommers' bijection extends the Bala-Carter one as it maps the pairs $(e, \{1\})$, where $\{1\} \subseteq \Gamma(e)$ is the unit conjugacy class in $\Gamma(e)$, onto the pairs $(L, e)$, where $L$ is a *Levi* subgroup of $G$.

Given two subsets $I, J$ such that $J \subseteq I \subsetneq \{0, 1, \dots, l\}$ we denote by $P_{I,J}$ the standard parabolic subgroup of $L_I$ associated with $J$ and set

$$\mathcal{P}\big(\widetilde{\Pi}\big) := \big\{(I, J) \mid P_{I,J} \text{ is a distinguished parabolic subgroup in } L_I\big\}.$$

Two pairs $(I, J)$ and $(I', J')$ of $\mathcal{P}(\widetilde{\Pi})$ are said to be *equivalent* if $w(\widetilde{\Pi}_I) = \widetilde{\Pi}_{I'}$ and $w(\widetilde{\Pi}_J) = \widetilde{\Pi}_{J'}$ for some $w \in W$. We denote by $\big[\mathcal{P}(\widetilde{\Pi})\big]$ the set of all equivalence classes of elements in $\mathcal{P}(\widetilde{\Pi})$.

**Corollary 10.** *There is a bijection between $\big[\mathcal{P}(\widetilde{\Pi})\big]$ and the set of all $G$-conjugacy classes of pairs $(e, \mathscr{C})$, where $e$ is a nilpotent element of $\mathfrak{g}$ and $\mathscr{C}$ is a conjugate class in $\Gamma(e) = Z_G(e)/Z_G(e)^0$.*

Using Theorem 9 one can prove that the bijection between the nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}$ described in Subsection 2.4 does not alter component groups. This, in turn, enables one to reduce proving Eq. (1) in

Subsection 2.5 to the case where $Z_G(e)$ is a connected unipotent group (the so called *semi-regular* case).

## 3. Representation theory

3.1. **The modular case** ($k = \overline{\mathbb{F}}_p$)**.** Let $\mathfrak{g}$ be a finite dimensional restricted Lie algebra over $k$. Let $U(\mathfrak{g})$ be its universal enveloping algebra and let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. For each $x \in \mathfrak{g}$ the element $x^p - x^{[p]}$ is central and the map $\xi : \mathfrak{g} \to Z(\mathfrak{g})$ given by $x \mapsto x^p - x^{[p]}$ is $p$-linear. That is, for all $\lambda \in k$ and $x, y \in \mathfrak{g}$

$$\xi(x+y) = \xi(x) + \xi(y) \quad \text{and} \quad \xi(\lambda x) = \lambda^p \xi(y).$$

Let $Z_p(\mathfrak{g}) \subseteq Z(\mathfrak{g})$ denote the $k$-subalgebra generated by all $x^p - x^{[p]}$ with $x \in \mathfrak{g}$. If $\{x_1, x_2, \ldots, x_m\}$ is a basis of $\mathfrak{g}$ then the $p$-linearity of $\xi$ implies that $Z_p(\mathcal{L}) = k\big[\xi(x_1), \ldots, \xi(x_m)\big]$. Moreover, it follows from the PBW theorem that $U(\mathfrak{g})$ is a free $Z_p(\mathfrak{g})$-algebra with basis

$$\{x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m} \mid 0 \leq a_i < p \text{ for all } i\}. \tag{2}$$

Let $M$ be a simple $\mathfrak{g}$-module. By Quillen's Lemma, $Z(\mathfrak{g})$ acts on $M$ by scalar operators. This implies that $\dim M \leq p^{(\dim \mathfrak{g})/2}$. Since $k$ is algebraically closed, each $x^p - x^{[p]}$ acts on $M$ as $\chi_M(x)^p \,\mathrm{Id_M}$ for some $\chi_M(x) \in k$. It is immediate from the $p$-linearity of $\xi$ that the function $\chi_M \colon \mathfrak{g} \to \mathfrak{g}$ is $k$-linear, i.e. belongs to $\mathfrak{g}^*$ (this was first observed by Kac and Weisfeiler in [52]). We call $\chi_M$ the *$p$-character* of $M$.

Given $\chi \in \mathfrak{g}^*$ we denote by $I_\chi$ the two-sided ideal of $U(\mathfrak{g})$ generated by all $x^p - x^{[p]} - \chi(x)^p$. The factor-algebra $U_\chi(\mathfrak{g}) = U(\mathfrak{g})/I_\chi$ is called the *reduced enveloping algebra* of $\mathfrak{g}$ corresponding to $\chi$. If $\{x_1, x_2, \ldots, x_m\}$ is a basis of $\mathfrak{g}$ then $U_\chi(\mathfrak{g})$ has basis as in (2). In particular, $\dim U_\chi(\mathfrak{g}) = p^{\dim \mathfrak{g}}$. Furthermore, for each $\chi \in \mathfrak{g}^*$ the Lie algebra $\mathfrak{g}$ admits at least one simple module with $p$-character $\chi$. In fact, since every algebra $U_\xi(\mathfrak{g})$ is finite dimensional, the map $\chi \colon \operatorname{Irr} \mathfrak{g} \to \mathfrak{g}^*$ which assigns to an isomorphism class of a simple $\mathfrak{g}$-module $M$ its $p$-character $\chi_M$ a surjective and has finite fibres.

From now on we assume that $G$ satisfies SH1, SH2, SH3. Then for every $\chi \in \mathfrak{g}^*$ there exists a unique $x \in \mathfrak{g}$ such that $\xi = (x\,,\,\cdot)$. An important result proved by Kac and Weisfeiler [52] and later presented by Friedlander and Parshall in the form of a Morita theorem [13] reduces the general problem of classifying simple $U_\chi(\mathfrak{g})$-modules to the case where the $p$-character $\chi$ is nilpotent, that is has the form $\chi = (e, \cdot)$ for some $e \in \mathcal{N}(\mathfrak{g})$.

3.2. **Nilpotent $p$-characters.** Now suppose that $\chi = (e, \cdot)$, where $e \in \mathcal{N}(\mathfrak{g})$, and denote by $\mathcal{O}(e)$ the orbit $\mathrm{Ad}(G)e$. By our discussion in Subsection 2.5, $\mathcal{O}(e) = \mathcal{O}(\Delta)$ for some weighted Dynkin diagram $\Delta \in \mathcal{D}(\Pi)$. Write

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i), \qquad \mathfrak{g}(i) = \mathfrak{g}(i;\lambda_\Delta).$$

Then $e \in \mathfrak{g}(2)_{\mathrm{reg}}$ and Theorem 8 yields

$$\begin{aligned}\dim \mathcal{O}(e) &= \dim \mathfrak{g} - \dim \mathfrak{g}_e\\ &= \sum_{i<0} \dim \mathfrak{g}(i) + \dim \mathfrak{p}(\lambda_\Delta) - \dim \mathfrak{g}_e\\ &= \sum_{i<0} \dim \mathfrak{g}(i) + \dim[\mathfrak{p}(\lambda_\Delta), e].\end{aligned}$$

As the dimension of $[\mathfrak{p}(\lambda_\Delta), e]$ equals $\sum_{i\geq 2} \dim \mathfrak{g}(i)$, it is straightforward to see that

$$\dim \mathfrak{g}_e = \dim \mathfrak{g}(0) + \dim \mathfrak{g}(1).$$

The bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathfrak{g}(-1)$ defined by $\langle x, y\rangle := (e, [x, y])$ is skew-symmetric and nondegenerate. This follows directly from the nondegeneracy of our trace form and the fact that $\mathfrak{g}_e \cap \mathfrak{g}(-1) = 0$. In particular, this implies that the dimension of $\mathfrak{g}(-1)$, and hence of $\mathcal{O}(e)$, is even. We write $\dim \mathcal{O}(e) = 2d(e)$ for $d(e) \in \mathbb{Z}_{\geq 0}$.

3.3. **Rank varieties.** Suppose for a moment that $\mathfrak{g}$ is an arbitrary finite dimensional restricted Lie algebra over $k$ and define

$$\mathcal{N}_p(\mathfrak{g}) := \{x \in \mathfrak{g} \mid x^{[p]} = 0\},$$

a Zariski closed, conical subset of the variety of all $[p]$-nilpotent elements of $\mathfrak{g}$. The *stabilizer* of a linear function $\chi$ on $\mathfrak{g}$ is the restricted subalgebra of $\mathfrak{g}$ consisting of all $x \in \mathfrak{g}$ such that $\chi([x, \mathfrak{g}]) = 0$. If $\mathfrak{g}$ satisfies the standard hypotheses and $\chi = (x,\,\cdot\,)$ for $x \in \mathfrak{g}$, then $\mathfrak{g}_\chi = \mathfrak{g}_x$, of course.

Given a nonzero $x \in \mathcal{N}_p(\mathfrak{g})$ we denote by $U_\chi(x)$ the associated subalgebra of $U_\chi(\mathfrak{g})$ generated by $x$. Since $x^p - x^{[p]} - \chi(x)^p = (x - \chi(x))^p$ for all $x \in \mathcal{N}_p(\mathfrak{g})$ we have a surjective algebra homomorphism $U_\chi(x) \to k[X]/(X^p)$ which sends $x - \chi(x)$ to $X$. From the PBW theorem it follows that this map is an isomorphism, in particular, $\dim U_\chi(x) = p$.

For any finite dimensional $U_\chi(\mathfrak{g})$-module $M$, the *rank variety* $\mathcal{V}_\mathfrak{g}(M)$ of $M$ is a subset of $\mathfrak{g}$ consisting of 0 and all those nonzero $x \in \mathcal{N}_p(\mathfrak{g})$ for which $M$ is not a free $U_\chi(x)$-module. One knows that $\mathcal{V}_\mathfrak{g}(M)$ is a Zariski closed, conical subset of $\mathcal{N}_p(\mathfrak{g})$. Relying on the earlier important work of Jantzen [20], Friedlander and Parshall proved the following result:

**Theorem 11** (see [13], Thm. 6.4)**.** *A finite dimensional $U_\chi(\mathfrak{g})$-module $M$ is projective if and only if $\mathcal{V}_\mathfrak{g}(M) = \{0\}$.*

We stress that this theorem holds for any finite dimensional restricted Lie algebra $\mathfrak{g}$. Let $E_1, E_2, \ldots, E_n$ be representatives of the isomorphism classes of simple $U_\chi(\mathfrak{g})$-modules, and define

$$\mathcal{V}_\chi(\mathfrak{g}) := \bigcup_{i=1}^{n} \mathcal{V}_{\mathfrak{g}}(E_i).$$

It is easy to see that $\mathcal{V}_{\mathfrak{g}}(M) \subseteq \mathcal{V}_\chi(\mathfrak{g})$ for any finite dimensional $U_\chi(\mathfrak{g})$-module $M$. By [36], Prop. 2.2, for any finite dimensional restricted Lie algebra $\mathfrak{g}$ and any $\chi \in \mathfrak{g}^*$ the variety $\mathcal{V}_{\mathfrak{g}}(\chi)$ coincides with the rank variety of $U_\chi(\mathfrak{g})$ regarded as the adjoint $\mathfrak{g}$-module (the action of $x \in \mathfrak{g}$ on $U_\chi(\mathfrak{g})$ is then given by $x.u = xu - ux$ for all $u \in U_\chi(\mathfrak{g})$).

Although it is very difficult to give a more explicit description of the variety $\mathcal{V}_{\mathfrak{g}}(\chi)$ for an arbitrary restricted Lie algebra $\mathfrak{g}$, this problem is completely solved in the case where $\mathfrak{g} = \mathrm{Lie}(G)$ and $G$ satisfies *some* standard hypotheses.

**Theorem 12** (see [36], Thm. 2.4)**.** *If $G$ satisfies* SH1 *and* SH2 *and $\chi$ is any linear function on $\mathfrak{g}$, then*

$$\mathcal{V}_\chi(\mathfrak{g}) \;=\; \mathcal{N}_p(\mathfrak{g}_\chi) \;=\; \{x \in \mathfrak{g}_\chi \mid x^{[p]} = 0\}.$$

As an immediate consequence of Theorem 12 we obtain that the algebra $U_\chi(\mathfrak{g})$ is semisimple if and only if the stabiliser $\mathfrak{g}_\chi$ is a toral subalgebra of $\mathfrak{g}$. One can also use Theorem 12 to describe those $\chi$ for which the algebra $U_\chi(\mathfrak{g})$ has finite representation type, i.e. has finitely many isomorphism classes of indecomposable modules; see [36], Sect. 5.

3.4. **Admissible subalgebras.** Let $\mathfrak{n}$ be a restricted Lie subalgebra of a restricted Lie algebra $\mathfrak{g}$. Then $U_\chi(\mathfrak{n}) := U_{\chi|_{\mathfrak{n}}}(\mathfrak{n})$ is a subalgebra of $U_\chi(\mathfrak{n})$ for every $\chi \in \mathfrak{g}^*$. Moreover, it is immediate from the PBW theorem that $U_\chi(\mathfrak{g})$ is a free $U_\chi(\mathfrak{s})$-module of rank $p^{\dim \mathfrak{g} - \dim \mathfrak{n}}$. It is proved in [13] that a $U_\chi(\mathfrak{g})$-module $M$ is projective over $U_\chi(\mathfrak{n})$ if and only if $\mathfrak{n} \cap \mathcal{V}_{\mathfrak{g}}(M) = \{0\}$.

**Definition.** A restricted Lie subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ is called $\chi$-*admissible* if it satisfies the following conditions:

($\chi$1) $\mathfrak{n}$ is $[p]$-nilpotent, i.e. $x^{[p]^e} = 0$ for all $x \in \mathfrak{n}$, where $e \gg 0$;

($\chi$2) $\chi$ vanishes on the restricted ideal $\sum_{i\ge 1} [\mathfrak{n}, \mathfrak{n}]^{[p]^i}$ of $\mathfrak{n}$;

($\chi$3) $\mathfrak{n} \cap \mathcal{V}_\chi(\mathfrak{g}) = \{0\}$.

If $\mathfrak{n}$ is a $\chi$-admissible subalgebra of $\mathfrak{g}$, then the algebra $U_\chi(\mathfrak{n})$ has a unique maximal ideal which has codimension 1 in $U_\chi(\mathfrak{n})$ and coincides with the Jacobson radical of $U_\chi(\mathfrak{n})$; see [37, 2.3] for detail. It follows that $U_\chi(\mathfrak{n})$ has a unique simple module which we shall denote by $k_\chi$. Moreover, the module $k_\chi = k1_\chi$ is 1-dimensional and the action of $\mathfrak{n}$ on its generator $1_\chi$ is given by

$$x.1_\chi = \chi(x) \cdot 1_\chi \quad \text{for all} \quad x \in \mathfrak{n}.$$

Condition ($\chi 3$) implies that $\mathfrak{n}\cap\mathcal{V}_{\mathfrak{g}}(M)\subseteq\mathfrak{n}\cap\mathcal{V}_{\chi}(\mathfrak{g})=\{0\}$ for any finite dimensional $U_\chi(\mathfrak{g})$-module $M$. It follows that any finite dimensional $U_\chi(\mathfrak{g})$-module $M$ is projective over $U_\chi(\mathfrak{n})$. On the other hand, conditions $\chi 1$) and ($\chi 2$) in conjunction with Engel's theorem show that the Jacobson radical of $U_\chi(\mathfrak{n})$ has codimension 1 in $U_\chi(\mathfrak{n})$. But then it follows from the general theory of associative algebras that the left regular module $U_\chi(\mathfrak{n})$ is indecomposable. This, in turn, yields that that every finite dimensional projective $U_\chi(\mathfrak{n})$-module is free. As consequence, we deduce that any $U_\chi(\mathfrak{g})$-module $M$ is free over $U_\chi(\mathfrak{n})$.

3.5. **Generalized Gelfand-Graev modules.** We recall the notation and conventions of Subsection 3.2 and let $\ell\subseteq\mathfrak{g}(-1)$ be a maximal totally isotropic subspace of $\mathfrak{g}(-1)$ relative to the skew-symmetric bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle$. Define

$$\mathfrak{m}=\ell\oplus\bigoplus_{i\leq-2}\mathfrak{g}(i).$$

This is a restricted Lie subalgebra of $\mathfrak{g}$ contained in $\mathcal{N}(\mathfrak{g})$. As $\dim\ell=\frac{1}{2}\dim\mathfrak{g}(-1)$, $\dim\mathfrak{g}(i)=\dim\mathfrak{g}(-i)$ for all $i$ and $\dim\mathfrak{g}_e=\dim\mathfrak{g}(0)+\dim\mathfrak{g}(1)$ by our discussion in Subsection 3.2 we see that

$$2\dim\mathfrak{m}=\dim\mathfrak{g}-\dim\mathfrak{g}_e=\dim\mathcal{O}(e)=2d(e).$$

As $\mathfrak{g}_e\subseteq\bigoplus_{i\geq0}\mathfrak{g}(i)$ by Theorem 8, the subalgebra $\mathfrak{m}$ intersects trivially with $\mathfrak{g}_e$ as $\mathfrak{g}_e$. Since $\mathcal{V}_\chi(\mathfrak{g})$ is contained in $\mathfrak{g}_e$ by Theorem 12, this implies $\mathfrak{m}\cap\mathcal{V}_\chi(\mathfrak{g})=\{0\}$. Furthermore, one has

$$[\mathfrak{m},\mathfrak{m}]\subseteq[\ell,\ell]\oplus\bigoplus_{i\leq-3}\mathfrak{g}(i).$$

Also, it is well known (and easily seen) that $\mathfrak{g}(i)^{[p]}\subseteq\mathfrak{g}(pi)$ for all $i$. As $\mathfrak{g}(i)$ is orthogonal to $\mathfrak{g}(j)$ with respect to $(\,\cdot\,,\,\cdot\,)$ whenever $i+j\neq0$, the linear function $\chi=(e,\,\cdot\,)$ vanishes on the ideal $\sum_{i\geq1}[\mathfrak{m},\mathfrak{m}]^{[p]^i}$ of $\mathfrak{m}$ (here we use our assumption that $\ell$ is an isotropic subspace of $\mathfrak{g}(-1)$. We thus obtain that $\mathfrak{m}$ is a $d(e)$-dimensional $\chi$-admissible Lie subalgebra of $\mathfrak{g}$.

**Remark 9.** In view of our discussion in Subsection 3.4 every finite dimensional $U_\chi(\mathfrak{g})$-module is free over $U_\chi(\mathfrak{m})$ and hence has dimension divisible by $p^{d(e)}=p^{\frac{1}{2}\dim\mathrm{Ad}^*(G)\chi}$. This was conjectured by Kac and Weisfeiler in [52] and first proved in [35]. A completely different proof, based on a more geometric argument, was later found in [42]. That proof uses neither rank varieties nor the Bala–Carter–Pommerening theorem and works in bad characteristic (for nilpotent $p$-characters).

We now define the (restricted) *generalized Gelfand-Graev module* $Q_\chi^{[p]}$ by setting

$$Q_\chi^{[p]}:=U_\chi(\mathfrak{g})\otimes_{U_\chi(\mathfrak{m})}k_\chi.$$

**Remark 10.** Generalized Gelfand–Graev modules were first introduced by Kawanaka (who coined the name) and Mœglin. Kawanaka defined them for finite Chevalley groups, while Mœglin worked with complex semisimple Lie algebras. When $e$ is regular in $\mathfrak{g}$, i.e when the orbit $\mathrm{Ad}(G)e$ is open in $\mathcal{N}(\mathfrak{g})$, such modules go back a very long way and can be traced to works of Gelfand–Graev, Steinberg, Kostant and others.

The importance of generalized Gelfand–Graev modules in our situation stems from the following theorem:

**Theorem 13** (see [36]). *The module $Q_\chi^{[p]}$ is projective over $U_\chi(\mathfrak{g})$. Moreover, the left regular module $U_\chi(\mathfrak{g})$ is isomorphic to a direct sum of $p^{d(e)}$ copies of $Q_\chi^{[p]}$.*

*Proof.* As the module $Q_\chi^{[p]}$ is induced from $\mathfrak{m}$, it follows from the PBW theorem that $Q_\chi^{[p]}$ is free over $U_\chi(x)$ for all $x \in \mathcal{N}_p(\mathfrak{g}) \setminus \mathfrak{m}$. Therefore, $\mathcal{V}_\mathfrak{g}\big(Q_\chi^{[p]}\big) \subseteq \mathfrak{m}$. Since $\mathcal{V}_\mathfrak{g}\big(Q_\chi^{[p]}\big) \subseteq \mathcal{V}_\chi(\mathfrak{g})$, condition ($\chi$3) in conjunction with Theorem 12 yields $\mathcal{V}_\mathfrak{g}\big(Q_\chi^{[p]}\big) = \{0\}$ showing that $Q_\chi^{[p]}$ is projective over $U_\chi(\mathfrak{g})$. Since every simple $U_\chi(\mathfrak{g})$-module $E$ is free over $U_\chi(\mathfrak{m})$ by our earlier discussion discussion, Frobenius reciprocity yields

$$\dim \mathrm{Hom}_{U_\chi(\mathfrak{g})}\,(Q_\chi^{[p]},\, E) \;=\; \dim \mathrm{Hom}_{U_\chi(\mathfrak{m})}\,(k_\chi,\, E) \;=\; (\dim E)/p^{d(e)}.$$

On the other hand, $\dim \mathrm{Hom}_{U_\chi(\mathfrak{g})}\,(U_\chi(\mathfrak{g}), E) = \dim E$ by the general theory of associative algebras. As this holds for any simple $U_\chi(\mathfrak{g})$-module $E$, the claim follows. □

We set $U^{[p]}(\mathfrak{g}, e) := \mathrm{End}_\mathfrak{g}\big(Q_\chi^{[p]}\big)^{\mathrm{op}}$ and call $U^{[p]}(\mathfrak{g}, e)$ the *restricted finite $W$-algebra* associated with $\mathfrak{g}, e$. As an immediate consequence of Theorem 13 we obtain a nice Morita theorem:

**Corollary 14.** $U_\chi(\mathfrak{g}) \cong \mathrm{Mat}_{p^{d(e)}}\big(U^{[p]}(\mathfrak{g}, e)\big)$ *as $k$-algebras.*

3.6. **The Kazhdan filtration ($k = \overline{\mathbb{F}}_p$ or $k = \mathbb{C}$).** Choose a basis $x_1, x_2, \ldots, x_m$ of $\mathfrak{p}(\lambda_\Delta)$ such that $x_1, x_2, \ldots, x_r$ is a basis of $\mathfrak{g}_e$ and $x_i \in \mathfrak{g}(n_i)$. Let $z_1, \ldots, z_s, z_1', \ldots, z_s'$ be a Witt basis of $\mathfrak{g}(-1)$ with respect to the skew-symmetric form $\langle\,\cdot\,,\,\cdot\,\rangle$ such that our Lagrangian subspace $\ell$ is spanned by the $z_i'$'s. Given a pair $(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}_+^{\mathbf{m}} \times \mathbb{Z}_+^{\mathbf{s}}$ denote by $x^{\mathbf{a}} z^{\mathbf{b}}$ the monomial $x_1^{a_1} \cdots x_m^{a_m} z_1^{b_1} \cdots z_s^{b_s}$.

The *Kazhdan filtration*

$$\cdots \subseteq \mathsf{K}_{i-1}\, U(\mathfrak{g}) \subseteq \mathsf{K}_i\, U(\mathfrak{g}) \subseteq \mathsf{K}_{i+1}\, U(\mathfrak{g}) \subseteq \cdots$$

is an algebra filtration on $U(\mathfrak{g})$ defined by declaring that $x \in \mathfrak{g}(i)$ has filtration degree $i + 2$. For $k = \overline{\mathbb{F}}_p$, this gives rise to a $\mathbb{Z}$-filtration on $U_\chi(\mathfrak{g})$ and turns $Q_\chi^{[p]}$ into a Kazhdan filtered $U_\chi(\mathfrak{g})$-module. By the

PBW theorem, the elements $\{x^{\mathbf{a}}z^{\mathbf{b}} \otimes 1_\chi \mid 0 \le a_i, b_i \le p-1\}$ form a basis of $Q_\chi^{[p]}$. Note that $x^{\mathbf{a}}z^{\mathbf{b}} \otimes 1_\chi$ has Kazhdan degree

$$|(\mathbf{a},\mathbf{b})|_e := 2|\mathbf{a}| + |\mathbf{b}| + \sum_{i=1}^{m} a_i n_i = \sum_{i=1}^{m} a_i(n_i+2) + \sum_{i=1}^{s} b_i$$

(here $|\mathbf{i}|$ stands for the total degree of a $d$-tuple $\mathbf{i} = (i_1, \ldots, i_d) \in \mathbb{Z}_+^d$, i.e. $|\mathbf{i}| = \sum_{j=1}^{d} i_j$). As $n_i \in \mathbb{Z}_+$ for all $i$, the Kazhdan filtration is nonnegative on $Q_\chi^{[p]}$ and its zero part equals $k1_\chi$.

Every nonzero $u \in U^{[p]}(\mathfrak{g}, e)$ is uniquely determined by its value $u(1_\chi) \in Q_\chi^{[p]}$. Write

$$u(1_\chi) = \Big(\sum\nolimits_{|(\mathbf{a},\mathbf{b})|_e \le n} \lambda_{\mathbf{a},\mathbf{b}}\, x^{\mathbf{a}}z^{\mathbf{b}}\Big) \otimes 1_\chi,$$

where $n = n(u) \in \mathbb{Z}_+$ and $\lambda_{\mathbf{a},\mathbf{b}} \in k^\times$ is nonzero for at least one $(\mathbf{a},\mathbf{b})$ with $|(\mathbf{a},\mathbf{b})|_e = n$. For $d \in \mathbb{Z}_+$ define

$$\Lambda_u^d := \big\{(\mathbf{a},\mathbf{b}) \in \mathbb{Z}_+^m \times \mathbb{Z}_+^s \mid |(\mathbf{a},\mathbf{b})|_e = d \text{ and } \lambda_{\mathbf{a},\mathbf{b}} \ne 0\big\},$$

and let $\Lambda_u^{\max}$ be the subset of $\Lambda_u^n$ consisting of all $(\mathbf{a},\mathbf{b})$ for which the total degree $|\mathbf{a}| + |\mathbf{b}|$ assumes its minimal value. Using the fact that $xu(1_\chi) = \chi(x)u(1_\chi)$ for all $x \in \mathfrak{m}$ one obtains the following:

**Lemma 15** (see [37], Lemma 3.2)**.** *If* $(\mathbf{p},\mathbf{q}) \in \Lambda_u^{\max}$ *then* $\mathbf{q} = \mathbf{0}$ *and* $\mathbf{p} = (p_1, \ldots, p_r, 0 \ldots, 0)$ *with* $0 \le p_i \le p-1$ *for all* $i$.

Consequently, the leading component of $u(1_\chi)$ is supported entirely on $\mathfrak{g}_e \subseteq \mathfrak{p}(\lambda_\Delta)$. Now, since $\dim U_\chi(\mathfrak{g}) = p^{\dim \mathfrak{g}}$, Corollary 14 yields

$$\dim U^{[p]}(\mathfrak{g}, e) = p^{\dim \mathfrak{g} - \dim \mathcal{O}(e)} = p^{\dim \mathfrak{g}_e} = p^r.$$

From this it is not hard to deduce that *every* monomial $x_1^{a_1} \cdots x_r^{a_r}$ with $0 \le a_i \le p$ occurs as a leading term of an element of $U^{[p]}(\mathfrak{g}, e)$; see [37, Prop. 3.3] for more details.

**Corollary 16.** *There exist* $\theta_1, \theta_2, \ldots, \theta_r \in U^{[p]}(\mathfrak{g}, e)$ *such that*

$$\theta_i(1_\chi) = \Big(x_i + \sum_{|\mathbf{a},\mathbf{b}|_e = i+2,\ |\mathbf{a}|+|\mathbf{b}| \ge 2} \lambda_{\mathbf{a},\mathbf{b}}^i\, x^{\mathbf{a}}z^{\mathbf{b}} + \sum_{|\mathbf{a},\mathbf{b}|_e < n_i+2} \lambda_{\mathbf{a},\mathbf{b}}^i\, x^{\mathbf{a}}z^{\mathbf{b}}\Big) \otimes 1_\chi.$$

*Furthermore, the algebra* $U^{[p]}(\mathfrak{g}, e)$ *is generated by* $\theta_1, \theta_2, \ldots, \theta_r$.

From now on we assume where $k = \mathbb{C}$. We fix a weighted Dynkin diagram $\Delta$ and consider $\lambda_\Delta \in X_*(T)$. For a nilpotent element $e \in \mathfrak{g}(2; \lambda_\Delta)_{\mathrm{reg}}$ we take the Lie subalgebra $\mathfrak{m}$ introduced before and define

$$Q_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi,$$

where $\chi = (e, \cdot)$. This is now an infinite dimensional (induced) $\mathfrak{g}$-module. Set

$$U(\mathfrak{g}, e) := (\operatorname{End}_{\mathfrak{g}} Q_\chi)^{\mathrm{op}}.$$

This is the *finite $W$-algebra* associated with $\mathfrak{g}, e$. When $e = 0$, we have that $U(\mathfrak{g}, 0) = U(\mathfrak{g})$, and when $e$ is regular nilpotent in $\mathfrak{g}$, a well known result of Kostant [24] implies that that $U(\mathfrak{g}, e) = Z(\mathfrak{g})$, the center of $U(\mathfrak{g})$. As before, we have a Kazhdan filtration on the $\mathfrak{g}$-module $Q_\chi$ and since Corollary 16 holds for all $p \gg 0$ we can "lift" the generating set $\theta_1, \ldots, \theta_r$ to characteristic 0 by solving a non-homogeneous system of linear equations over $\mathbb{Q}$. We then obtain:

**Theorem 17** (see [37], Thm. 4.6)**.**

*(1) For every $1 \le i \le r$ there exists $\Theta_i \in U(\mathfrak{g}, e)$ such that*

$$\Theta_i(1_\chi) = \Big(x_i + \sum_{|\mathbf{a},\mathbf{b}|_e = i+2,\ |\mathbf{a}|+|\mathbf{b}|\ge 2} \lambda^i_{\mathbf{a},\mathbf{b}}\, x^{\mathbf{a}} z^{\mathbf{b}} + \text{lower } \mathsf{K}\text{-terms}\Big) \otimes 1_\chi$$

*(2) The set $\{\Theta_1^{a_1} \cdots \Theta_r^{a_r} \mid a_i \in \mathbb{Z}_+\}$ forms a $\mathbb{C}$-basis of $U(\mathfrak{g}, e)$.*

*(3) If $[x_i, x_j] = \sum_{k=1}^r c_{ij}^k x_k$ in the centralizer $\mathfrak{g}_e$, then $[\Theta_i, \Theta_j] = F_{ij}(\Theta_1, \ldots, \Theta_r) := F_{ij}(\boldsymbol{\Theta})$, where*

$$\begin{aligned} F_{ij}(\boldsymbol{\Theta}) &= \sum_{k=1}^r c_{ij}^k\, \Theta_k + \text{nonlinear terms of } \mathsf{K}\text{-degree } n_i + n_j + 2 \\ &+ \text{lower } \mathsf{K}\text{-degree terms.} \end{aligned}$$

*(4) We have a presentation*

$$U(\mathfrak{g}, e) \cong \mathbb{C}\langle \Theta_1, \ldots, \Theta_r \rangle / \big([\Theta_i, \Theta_j] - F_{ij}(\boldsymbol{\Theta}),\ 1 \le i < j \le r\big),$$

*where $\mathbb{C}\langle \Theta_1, \ldots, \Theta_r \rangle$ is the free associative $\mathbb{C}$-algebra generated by $\Theta_1, \ldots, \Theta_r$.*

Theorem 17 shows that there is an algebra filtration in $U(\mathfrak{g}, e)$ with the property that the corresponding graded algebra $\mathrm{gr}(U(\mathfrak{g}, e))$ is a polynomial algebra in $\mathrm{gr}(\Theta_1), \ldots, \mathrm{gr}(\Theta_r)$ with $\mathrm{gr}(\Theta_i)$ being homogeneous of degree $n_i + 2$.

**Remark 11.** Let $\{e, h, f\} \subseteq \mathfrak{g}$ be an $\mathfrak{sl}_2$-triple containing $e$. In view of Theorem 17, we may identify the maximal spectrum of $\mathrm{gr}(U(\mathfrak{g}, e))$ with the Slodowy slice $S := e + \mathfrak{g}_f$ at $e$ to the orbit $\mathcal{O}(e)$. In [14], Gan and Ginzburg have determined the Poisson structure on $S$ induced by taking commutators in $U(\mathfrak{g}, e)$ modulo lower terms. It turns out that it is obtained from the Kirillov–Kostant structure on $\mathfrak{g}^*$ by Poisson reduction. This enables one to describe the symplectic leaves of the Poisson variety $S$; see [14] for more detail.

It should be mentioned here that finite $W$-algebras (as well as *classical $W$-algebras*, which we do not discuss here) were first introduced by physicists. More precisely, de Boer and Tjin [9] used the method of Poisson reduction to attach a certain highly nontrivial Poisson algebra to any nilpotent element $e$ in a complex simple Lie algebra $\mathfrak{g}$. They then applied the BRST method to quantize these algebras in the case where $e$ is even (see Remark 6). This was recently generalized to the case of an arbitrary nilpotent element $e \in \mathfrak{g}$ by De Sole–Kac [10] and D'Andrea–DeConcini–De Sole–Heluani–Kac [8] who also proved that the quantum algebra thus obtained can be identified with the algebra $U(\mathfrak{g}, e)$ defined above. This result turned out to be very useful for introducing and studying a category $\mathcal{O}$ for the finite $W$-algebra $U(\mathfrak{g}, e)$. This category, introduced by Brundan–Goodwin–Kleshchev in [3], allows one to reduce some problems of representation theory of $U(\mathfrak{g}, e)$ to the case where the centralizer $\mathfrak{g}_e$ has no nonzero toral subalgebras.

We now give a slightly different description of the finite $W$-algebra $U(\mathfrak{g}, e)$ due to Gan and Ginzburg; see [14]. Set $\mathfrak{m}^\chi := \{x - \chi(x) \mid x \in \mathfrak{m}\}$ a subspace of $U(\mathfrak{g})$. As $Q_\chi$ is a $\mathfrak{g}$-module induced from $\mathfrak{m}$, it is easy to see that

$$U(\mathfrak{g}, e) \cong \big(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}^\chi\big)^{\operatorname{ad}\mathfrak{m}}$$

as algebras. Gan and Ginzburg observed that the algebra $U(\mathfrak{g}, e)$ does not depend upon the choice of a totally isotropic subspace in $\mathfrak{g}(-1)$. Let $\ell$ be any totally isotropic subspace of $\mathfrak{g}(-1)$ and define $\ell^\perp := \{x \in \mathfrak{g}(-1) \mid \langle x, \ell\rangle = 0\}$. Set

$$\mathfrak{m}_\ell := \ell \oplus \bigoplus_{i \le -2} \mathfrak{g}(i) \quad \text{and} \quad \widetilde{\mathfrak{m}}_\ell := \ell^\perp \oplus \bigoplus_{i \le -2} \mathfrak{g}(i).$$

Note that if $\ell$ is Lagrangian, then $\widetilde{\mathfrak{m}}_\ell = \mathfrak{m}_\ell$. It is proved in [14] that

$$U(\mathfrak{g}, e) \cong \big(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\ell^\chi\big)^{\operatorname{ad}\widetilde{\mathfrak{m}}_\ell}$$

as algebras. When $\ell = 0$ we obtain an algebra isomorphism

$$U(\mathfrak{g}, e) \cong \big(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{g}_{\le -2}^\chi\big)^{\operatorname{ad}\mathfrak{g}_{\le -1}},$$

where $\mathfrak{g}_{\le d} := \bigoplus_{i \le d} \mathfrak{g}(i)$. The latter isomorphism shows that the reductive group $C(\Delta, e) = Z_G(e) \cap Z(\lambda_\Delta)$ acts on $U(\mathfrak{g}, e)$ as algebra automorphisms and preserves the Kazhdan filtration.

It is immediate from the definition of $U(\mathfrak{g}, e)$ that there is a natural algebra homomorphism $\psi\colon Z(\mathfrak{g}) \to Z(U(\mathfrak{g}, e))$. According to [37, 6.2], this map is injective. Denote by $\varphi_S$ the restriction to $S$ of the adjoint quotient map $\mathfrak{g} \to \mathfrak{g}/\!\!/G$. By a well known result of Slodowy, $\varphi_S$ is a faithfully flat morphism with normal fibers, while in [37], Sect. 5, it is proved that all scheme-theoretic fibers of $\varphi_S$ are reduced, irreducible complete intersections of dimension $r - l$. Thanks to results of Gan–Ginzburg discussed in Remark 11 this shows that every fiber of $\varphi_S$ has a dense symplectic leaf of the Poisson variety $S$. In conjunction with the reducedness of the fibers of $\varphi_S$ this implies that the Poisson center $\mathcal{Z}_e$ of $\operatorname{gr}(U(\mathfrak{g}, e))$ coincides with $\operatorname{gr}(Z(\mathfrak{g})) \subseteq \operatorname{gr}(U(\mathfrak{g}, e))$.

The argument just outlined is due to Ginzburg; see [39, p. 524]. An alternative proof of the equality $\mathcal{Z}_e = \operatorname{gr}(Z(\mathfrak{g}))$ was later found in [33], Remark 2.1. In view of the flatness of $\varphi_S$ we now obtain the following:

**Theorem 18.** *The map $\psi$ sends $Z(\mathfrak{g})$ isomorphically onto the center of $U(\mathfrak{g}, e)$. The algebra $U(\mathfrak{g}, e)$ is a free module over its center.*

The second part of this theorem extends to the case of an arbitrary $e \in \mathcal{N}(\mathfrak{g})$ a classical result of Kostant which states that $U(\mathfrak{g})$ is a free module over $Z(\mathfrak{g})$ (recall that $U(\mathfrak{g}, 0) = U(\mathfrak{g})$).

3.7. **Skryabin's theorem.** Finite $W$-algebras find rather spectacular applications in the theory of primitive ideals and in some cases help to solve long-standing problems of this theory; see Theorem 26.

Write $\mathfrak{g}$-$\mathsf{mod}$ for the category of all $\mathfrak{g}$-modules and $U(\mathfrak{g}, e)$-$\mathsf{mod}$ for the category of all $U(\mathfrak{g}, e)$-modules. For $\chi = (e, \cdot\,)$ we write $\mathcal{C}_\chi$ for the full subcategory of $\mathfrak{g}$-$\mathsf{mod}$ consisting of all $\mathfrak{g}$-modules $V$ such that $x - \chi(x)$ acts on $V$ as a locally nilpotent operator for all $x \in \mathfrak{m}$. Given $V \in \mathcal{C}_\chi$ we define

$$\mathrm{Wh}_\chi(V) := \{v \in V \mid x.v = \chi(x)v \text{ for all } x \in \mathfrak{m}\}.$$

It is straightforward to see that $\mathrm{Wh}_\chi(V) \neq 0$ for any nonzero $\mathfrak{g}$-module $V$ in $\mathcal{C}_\chi$. As $U(\mathfrak{g}, e) \cong \big(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}^\chi\big)^{\mathrm{ad}\,\mathfrak{m}}$, the vector space $\mathrm{Wh}_\chi(V)$ carries a natural $U(\mathfrak{g}, e)$-module structure.

We thus obtain two functors

$$\mathcal{F}\colon\ U(\mathfrak{g}, e)\text{-}\mathsf{mod} \longrightarrow \mathcal{C}_\chi, \qquad M \longmapsto Q_\chi \otimes_{U(\mathfrak{g},e)} M,$$
$$\mathcal{G}\colon\ \mathcal{C}_\chi \longrightarrow U(\mathfrak{g}, e)\text{-}\mathsf{mod}, \qquad V \longmapsto \mathrm{Wh}_\chi(V).$$

It is known that $Q_\chi$ is free as a left $U(\mathfrak{g}, e)$-module. Therefore, for every $M \in U(\mathfrak{g}, e)$-$\mathsf{mod}$ the $\mathfrak{g}$-module $\mathcal{F}(M) = Q_\chi \otimes_{U(\mathfrak{g},e)} M$ has a nice PBW basis. However, the action of $\mathfrak{g}$ on this basis is extremely difficult to determine, in general.

In the Appendix to [37], Serge Skryabin proved the following important theorem which generalizes one of the main results of Kostant on Whittaker modules; see [24].

**Theorem 19** (Skryabin's equivalences)**.** *The functors $\mathcal{F}$ and $\mathcal{G}$ are quasi-inverse equivalences.*

3.8. **Finite $W$-algebras and primitive ideals.** Theorem 19 implies that a $U(\mathfrak{g}, e)$-module $M$ is irreducible if and only is so is the $\mathfrak{g}$-module $\mathcal{F}(M) = Q_\chi \otimes_{U(\mathfrak{g},e)} M$. This implies that, for any irreducible $U(\mathfrak{g}, e)$-module $M$ the annihilator $I_M := \mathrm{Ann}_{U(\mathfrak{g})}\,\mathcal{F}(M)$ is a primitive ideal of $U(\mathfrak{g})$. In [39], the author determined the associated variety of the primitive ideal $I_M$ in the case where $\dim M < \infty$.

**Proposition 20.** *If $M$ is any finite dimensional irreducible $U(\mathfrak{g}, e)$-module, then $\mathcal{VA}(I_M) = \overline{\mathcal{O}(e)}$.*

Inspired by this result the author predicted that the converse should be true as well:

**Conjecture 1** (see [39], Conjecture 3.2)**.** For every primitive ideal $I$ of $U(\mathfrak{g})$ with $\mathcal{VA}(I) = \overline{\mathcal{O}(e)}$ there exists a finite dimensional irreducible $U(\mathfrak{g}, e)$-module $M$ such that $I = I_M$.

We now have three different proofs of Conjecture 1. It was first proved in [40] under a mild technical assumption on the central character of $I$. The argument used in [40] relied on the characteristic $p$ methods outlined in Subsections 3.4, 3.5 and 3.6. Shortly after [40] was submitted the author realized that a minor modification of his argument solves the general case, and a footnote was added to the final version of [40]. In the meantime a completely different proof of Conjecture 1 was posted on `arXiv` by Ivan Losev. It relied on Losev's new construction of finite $W$-algebras via equivariant Fedosov quantization. Losev's work (now published in [25]) also contains an important Decomposition Theorem which allows one to relate ideals of $U(\mathfrak{g})$ with those of $U(\mathfrak{g}, e)$. Yet another proof of Conjecture 1 was later found by Ginzburg [15] who introduced and studied Harish-Chandra bimodules for quantized Slodowy slices. The author's proof of the general case has finally appeared in [41].

Summarizing, we have the following:

**Theorem 21** (see [25, 41, 15])**.** *For any primitive ideal $I$ of $U(\mathfrak{g})$ with $\mathcal{VA}(I) = \overline{\mathcal{O}(e)}$ there exists a finite dimensional $U(\mathfrak{g}, e)$-module $M$ such that $I = \operatorname{Ann}_{U(\mathfrak{g})}\big(Q_\chi \otimes_{U(\mathfrak{g},e)} M\big)$.*

For any $d \in \mathbb{Z}_+$ the set $\mathfrak{g}_{(d)} := \{x \in \mathfrak{g} \mid \dim \mathfrak{g}_x = d\}$ is Zariski open in its closure. A (locally closed) subset of $\mathfrak{g}$ is called a *sheet* if it coincides with an irreducible component of one of the locally closed subsets $\mathfrak{g}_{(d)}$. All sheets in $\mathfrak{g}$ are classified by Borho; see [2]. Recall that an adjoint orbit $\operatorname{Ad}(G)x$ in $\mathfrak{g}$ is called *rigid* if it coincides with a sheet in $\mathfrak{g}$. It is well known that any rigid orbit in $\mathfrak{g}$ is necessarily nilpotent.

Inspired by analogy with the modular case, the author has also conjectured the following:

**Conjecture 2** (see [39], Conjecture 3.1)**.** Let $e$ be any nilpotent element of $\mathfrak{g}$.

(1) The algebra $U(\mathfrak{g}, e)$ has an ideal of codimension 1;
(2) The ideals of codimension 1 in $U(\mathfrak{g}, e)$ are finite in number if and only if the orbit $\mathcal{O}(e)$ is rigid in $\mathfrak{g}$;
(3) For any ideal $I_0$ of codimension 1 in $U(\mathfrak{g}, e)$ the primitive ideal $\operatorname{Ann}_{U(\mathfrak{g})}\big(Q_\chi \otimes_{U(\mathfrak{g},e)} U(\mathfrak{g}, e)/I_0\big)$ is completely prime.

This conjecture is now *almost* proved, although Part 1 is still open in two cases.

Firstly, for $\mathfrak{g}$ classical, Part 1 of Conjecture 2 follows from a result of Ranee Brylinski [6] rediscovered by Losev in [25]. Secondly, Part 1 is reduced in [41] to the case where the orbit $\mathcal{O}(e)$ is a rigid (see [27] for a different proof). Thirdly, it is shown it [41] that Part 2 of Conjecture 2 is true provided that Part 1 holds for all non-rigid orbits in $\mathfrak{g}$. The latter was verified by Goodwin–Röhrle–Ubly who relied on (a version of) Theorem 17 and computer-aided computations; see [16].

At the time of writing, Part 1 of Conjecture 2 remains open for two (largest) rigid orbits in Lie algebras of type $E_8$. To be more precise, the algorithm in [16] (implemented in GAP) and subsequent progress achieved by Glenn Ubly in his PhD thesis left open *three* orbits in Lie algebras of type $E_8$. Incidentally, one of them was dealt with by Losev by a different (non-computational) method; see [27]. It seems likely that Losev's approach can be used to settle the remaining two cases as well[1]. In principle, it can also be used to determine the central characters of all 1-dimensional representations of *rigid* finite $W$-algebras.

Finally, Part 3 of Conjecture 2 was proved by Losev [25] as a consequence of the following result which relates the Goldie rank of the primitive quotient $U(\mathfrak{g})/I_M$ with the dimension of $M$.

**Theorem 22** (Losev's inequality). *If $M$ is a finite dimensional irreducible $U(\mathfrak{g}, e)$–module, then*

$$\mathrm{rk}\,\big(U(\mathfrak{g})/I_M\big) \le \dim M.$$

When $\dim M = 1$, it follows from Theorem 22 that $\mathrm{rk}\,\big(U(\mathfrak{g})/I_M\big) = 1$ and hence that the ideal $I_M$ is completely prime (see our discussion in Subsection 2.3).

It would be very important for future applications of $W$-algebras to the theory of primitive ideals to strengthen Theorem 22 even further. There are three nilpotent orbits $\mathcal{O}(e)$ in $\mathfrak{g}$ for which the equality

$$\mathrm{rk}\,\big(U(\mathfrak{g})/I_M\big) = \dim M \tag{3}$$

holds for all finite dimensional irreducible $U(\mathfrak{g}, e)$-modules $M$. The zero orbit has this property because $U(\mathfrak{g}, 0) = U(\mathfrak{g})$ and $I_M$ has finite codimension in $U(\mathfrak{g})$, while Kostant's results on Whittaker modules [24] imply that (3) also holds for the regular nilpotent orbit in $\mathfrak{g}$. Finally, if $e$ lies in the minimal nonzero nilpotent orbit of $\mathfrak{g}$ (i.e. if $e$ is $G$-conjugate to a longest root vector $e_{\alpha_0}$), then (3) follows from [39, Theorem 5.3].

In fact, at present there are no examples of irreducible finite dimensional $U(\mathfrak{g}, e)$-modules $M$ for which (3) does not hold and it seems very likely that (3) does hold for $\mathfrak{g} = \mathfrak{sl}_n$. We mention in passing that proving (3) in this case would enable one to recast combinatorial results on finite dimensional representations of Yangians obtained by Molev,

[1]Very recently the author settled the remaining two cases of Conjecture 2. In conjunction with earlier results this implies that *Conjecture 2 holds for all nilpotent elements in complex simple Lie algebras.*

Nazarov, Tarasov and others as explicit formulae for *some* Goldie rank polynomials in type A.

More generally, one wonders if it is always true that $\mathrm{rk}\,\big(U(\mathfrak{g})/I_M\big)$ divides $\dim M$ and the positive integer

$$\frac{\dim M}{\mathrm{rk}\,\big(U(\mathfrak{g})/I_M\big)}$$

divides the order of the component group $\Gamma(e) = Z_G(e)/Z_G(e)^0$.

**Remark 12.** It is worth mentioning that the author has originally termed the algebras $U(\mathfrak{g}, e)$ the *enveloping algebras of special transverse slices*. It was not clear at the beginning that these algebras were related with finite $W$-algebras of mathematical physics, although a possibility of such a link (first suggested to the author by Rumynin) was discussed in [37, 1.10]. Ginzburg favored the term *quantized Slodowy slices* which is, of course, shorter. The term *finite $W$-algebras* which is short (but cryptic) can be traced back to A.B. Zamolodchikov. It was first consistently used in the representation theory literature by Brundan and Kleshchev who generalized earlier results of E. Ragoucy and P. Sorba on hidden Yangian symmetries of $W$-algebras for $\mathfrak{gl}_n$ and identified the finite $W$-algebras of type A with *truncated shifted Yangians*; see [4]. They also obtained many results on finite dimensional representations of these algebras; see [5]. The notation $U(\mathfrak{g}, e)$ was introduced by Losev and instantly accepted by everyone.

3.9. **Finite dimensional representations of finite $W$-algebras.** Recall from Subsection 3.6 that the reductive group $C(\Delta, e)$ acts on $U(\mathfrak{g}, e)$ algebra automorphisms. This action is rational and preserves the Kazhdan filtration of $U(\mathfrak{g}, e)$. Thus, we can twist the module structure $U(\mathfrak{g}, e) \times M \to M$ of any $U(\mathfrak{g}, e)$-module $M$ by an element $g \in C(\Delta, e)$ to obtain a new $U(\mathfrak{g}, e)$-module $M^g$, with underlying vector space $M$ and the $U(\mathfrak{g}, e)$-action given by $u.m = g(u).m$ for all $u \in U(\mathfrak{g}, e)$ and $m \in M$. It is proved in [41, 4.8], for example, that

$$\mathrm{Ann}_{U(\mathfrak{g})}\big(Q_\chi \otimes_{U(\mathfrak{g},e)} M\big) = \mathrm{Ann}_{U(\mathfrak{g})}\big(Q_\chi \otimes_{U(\mathfrak{g},e)} M^g\big) \quad \big(\forall\, g \in C(\Delta, e)\big).$$

For $d \in \mathbb{Z}_+$ we set $\mathfrak{g}_e(d) := \mathfrak{g}_e \cap \mathfrak{g}(d)$. It is well known (and follows, for instance, from Theorem 8) that $\mathrm{Lie}(C(\Delta, e)) = \mathfrak{g}_e(0)$.

**Proposition 23** (see [39], Lemma 2.4)**.** *There is an algebra embedding $\Theta\colon U(\mathfrak{g}_e(0)) \hookrightarrow U(\mathfrak{g}, e)$ such that the differential of the rational action of $C(\Delta, e)$ on $U(\mathfrak{g}, e)$ coincides with* $(\mathrm{ad} \circ \Theta)_{|\mathfrak{g}_e(0)}$.

As an immediate consequence we obtain:

**Corollary 24.** *Any two-sided ideal of $U(\mathfrak{g}, e)$ is invariant under the action of the connected component $C(\Delta, e)^0$.*

Denote by $\mathcal{X}$ the primitive spectrum of $U(\mathfrak{g})$, put $\mathcal{O} := \mathcal{O}(e)$ and let $\mathcal{X}_{\mathcal{O}}$ stand for the set of all $I \in \mathcal{X}$ with $\mathcal{VA}(I) = \mathcal{O}$. Given an algebra homomorphism $\eta\colon Z(\mathfrak{g}) \to \mathbb{C}$ we write $\mathcal{X}_{\mathcal{O}}^{\eta}$ for the set of all $I \in \mathcal{X}_{\mathcal{O}}$ with $I \cap Z(\mathfrak{g}) = \operatorname{Ker}\eta$. Recall from Theorem 18 that $Z(\mathfrak{g})$ identifies canonically with the center of $U(\mathfrak{g}, e)$). We write $\operatorname{Irr}_\eta U(\mathfrak{g}, e)$ for the set of all isomorphism classes of finite dimensional irreducible $U(\mathfrak{g}, e)$-modules with central character $\eta$. Given $d \in \mathbb{Z}_+$ we let $\operatorname{Irr}_e^d$ denote the set of all isomorphism classes of irreducible $d$-dimensional $U(\mathfrak{g}, e)$-modules. Standard results of geometric invariant theory imply that each set $\operatorname{Irr}_e^d$ carries a natural structure of a quasi-affine algebraic variety; see the proof of Theorem 4.3 in [41] for detail.

In view of Theorem 21, for every algebra homomorphism $\eta\colon Z(\mathfrak{g}) \to \mathbb{C}$ we have a natural surjective map

$$\pi_\eta\colon\ \operatorname{Irr}_\eta U(\mathfrak{g}, e) \twoheadrightarrow \mathcal{X}_{\mathcal{O}}^{\eta}, \qquad [M] \longmapsto I_M.$$

This map is well-defined by [41, Cor. 4.1], for instance, while the above discussion implies that the group $\Gamma(e) = C(\Delta, e)/C(\Delta, e)^0$ acts on the fibers of $\pi_\eta$.

A few years ago the author conjectured that the action of $\Gamma(e)$ on the fibers of $\pi_\eta$ is transitive. This conjecture is now confirmed by Losev who proved the following:

**Theorem 25** (Losev [26]). *For every $e \in \mathcal{N}(\mathfrak{g})$ and every algebra homomorphism $\eta\colon Z(\mathfrak{g}) \to \mathbb{C}$ each fiber of $\pi_\eta$ is a single $\Gamma(e)$-orbit.*

It follows from Theorem 25 that there is a natural bijection

$$\bigsqcup_{d\ge 1} \Big(\operatorname{Irr}_e^d/\Gamma(e)\Big) \stackrel{\sim}{\longrightarrow} \mathcal{X}_{\mathcal{O}}.$$

Since every orbit set $\operatorname{Irr}_e^d/\Gamma(e)$ has a natural structure of a quasi-affine variety and $\mathcal{X} = \bigsqcup_{\mathcal{O}\subseteq\mathcal{N}} \mathcal{X}_{\mathcal{O}}$, we obtain the following interesting consequence of Losev's result:

**Theorem 26** (see [41], Thm. 4.3). *The primitive spectrum of $U(\mathfrak{g})$ is a countable disjoint union of quasi-affine algebraic varieties.*

This solves for complex semisimple Lie algebras an old problem posed by Borho and Dixmier in the early 70s; see [11, Problem 2]. We stress that despite the Duflo–Joseph parametrization of primitive ideals via annihilators of highest weight modules and Springer's representations and various results on the (generalized) Dixmier map, the Borho–Dixmier problem remained open for simple Lie algebras of rank $\geq 3$ (including Lie algebras of type A).

Summarizing, the emerging theory of finite $W$-algebras is already capable of solving some old-standing problems of the theory of primitive ideals. As yet another example, one of the challenging open problems in modular representation theory is a conjecture of J.E. Humphreys which states, for $k = \overline{\mathbb{F}}_p$ and $\chi = (e, \cdot\,)$, that the reduced enveloping algebra $U_\chi(\mathfrak{g})$ always admits a module of dimension $p^{d(e)}$; see Remark 9 for a related discussion. In [41], Humphreys' conjecture was linked to Part 1 of Conjecture 2 and thus reduced for $p \gg 0$ to the case of two rigid nilpotent orbits in Lie algebras of type $\mathrm{E}_8$[2].

Since general interest to finite $W$-algebras is still growing, it looks like more applications to primitive ideals and representations of reductive Lie algebras and classical Lie superalgebras (see [50]) are underway.

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