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2010

MIMS EPrint: **2010.25**

ISSN 1749-9097

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February 22, 2010: draft for LMS book v1.0

1 Introduction

Let G be a locally compact group. Here are some important examples:

- $\mathbb{R}$: the additive group of the real line
- $\mathbb{T}$: the group of complex numbers of modulus 1
- $\mathrm{SU}(2)$: the group of 2×2 unitary matrices with determinant 1

- $\mathrm{SL}(2,\mathbb{R})$: the group of 2×2 real matrices with determinant 1
- $\mathrm{SL}(2,\mathbb{C})$: the group of 2×2 complex matrices with determinant 1
- $\mathrm{GL}(2,\mathbb{Q}_p)$: the group of invertible 2×2 matrices with entries in the p-adic field $\mathbb{Q}_p$

Attached to each locally compact group there is a *dual object*: the *unitary dual* $\widehat{G}$. This is the set of equivalence classes of irreducible unitary representations of G. An element of $\widehat{G}$ will be denoted ω.

So as not to cast our net too wide, we will confine our attention to *liminal* groups, see §2. The Plancherel Theorem is an expression of a fundamental duality between G and $\widehat{G}$. This duality is expressed in terms of a chosen Haar measure on G, and the corresponding dual measure on $\widehat{G}$ called Plancherel measure.

Let dg be a chosen left-invariant Haar measure on G, and let ϕ be a test function. Let π be chosen in ω and define

$$\widehat{\phi}(\pi) := \int \phi(g)\pi(g)dg \tag{1}$$

This depends on the choice of π in ω. But the trace is independent of this choice and so we can define

$$\Theta_\omega(\phi) := Tr\ \widehat{\phi}(\pi) \qquad \text{with} \quad \pi \in \omega$$

The Plancherel Theorem is the following statement:

$$\phi(1) = \int \Theta_\omega(\phi)d\nu(\omega) \tag{2}$$

where ν is Plancherel measure on $\widehat{G}$ and ϕ is a test-function on G.

The Haar measure dg is not unique: it is determined up to a scalar $\lambda > 0$. If we replace dg by $\lambda \cdot dg$ then $\Theta_\omega(\phi)$ is replaced by $\lambda \cdot \Theta_\omega(\phi)$ and then ν is replaced by $\lambda^{-1} \cdot \nu$. If we halve the Haar measure, then we double the Plancherel measure. Any Plancherel formula is with reference to a specific choice of Haar measure.

The convolution product of functions f_1 and f_2 will be denoted $f_1 \star f_2$. A simple substitution $\phi = f^* \star f$ then yields the formulation

$$\int |f(g)|^2 dg = \int ||\widehat{f}(\omega)||^2_{HS}\, d\nu(\omega) \tag{3}$$

where $||\cdot||_{HS}$ is the Hilbert-Schmidt norm defined by

$$||A||_{HS}^2 := Tr\, A^* A.$$

If the group G is *commutative*, then each irreducible unitary representation of G is a unitary character, and the unitary dual $\widehat{G}$ is the set of unitary characters of G. In particular, let G be the additive group $\mathbb{R}$. In this case, the unitary dual of $\mathbb{R}$ can be identified with $\mathbb{R}$ itself. This comes about as follows. Let $y \in \mathbb{R}$ and define

$$\chi(x) = e^{-2\pi i x y} \tag{4}$$

This formula defines a unitary character on $\mathbb{R}$, and every character arises in this way.

Let f be a function in $L^1(\mathbb{R})$. We will obtain the classical Fourier transform of f. Eqn.(1) and Eqn. (4) lead to the definition. The Fourier transform of f is the function on $\mathbb{R}$ given by

$$\widehat{f}(y) = \int_{\mathbb{R}} f(x)\chi(x)dx = \int_{\mathbb{R}} e^{-2\pi i x y} f(x)dx$$

If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then $\widehat{f}$ is in $L^2(\mathbb{R})$ and the following formula of Plancherel holds:

$$||\widehat{f}||_2 = ||f||_2. \tag{5}$$

The map $f \mapsto \widehat{f}$ has a unique extension to a continuous, linear map from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ which is an *isometry*, i.e. Plancherel's formula (3) holds for this extension, see [10, Chapter 5]. Once Lebesgue measure has been chosen on $\mathbb{R}$, then Plancherel measure is Lebesgue measure on $\mathbb{R}$.

Let $\mathbb{T}$ be the circle group. The unitary dual of $\mathbb{T}$ can be identified with the additive group $\mathbb{Z}$ via the equation

$$\chi(z) = z^n$$

with $z \in \mathbb{T}, n \in \mathbb{Z}$. In polar coordinates, this equation becomes

$$\chi(e^{i\theta}) = e^{in\theta}$$

If we select normalized Haar measure $d\theta/2\pi$ on $\mathbb{T}$ then Eqn.(1) becomes

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta} f(\theta) d\theta \tag{6}$$

If we now write

$$a_n = \widehat{f}(n)$$

then we recover the classical formula for the nth Fourier coefficient, as in [9, p.41]. Eqn.(3) takes the familiar form of the Parseval identity:

$$||f||^2 = \sum_{n\in\mathbb{Z}} |a_n|^2$$

Plancherel measure on the dual of $\mathbb{T}$ is counting measure on $\mathbb{Z}$.

We can see from these classical examples that the Plancherel formula is a vast generalization, to noncommutative liminal groups, of classical results in Fourier analysis.

We have chosen to use the framework of C^*-algebras in these lectures. This allows a uniform approach to the Plancherel theorem for reductive groups over any local field. With this in view, we begin with a slimline account of C^*-algebras, largely taken from Dixmier's classic book on C^*-algebras [5].

The sections on unitary representations and Plancherel Formula are inspired by Wallach's book on real reductive groups [15].

We should point out that, for reductive p-adic groups, thanks to the influence of Joseph Bernstein, an algebraic approach is possible, see [14].

We describe explicitly the Plancherel formulas for $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{R})$.

The explicit Plancherel formula for p-adic $\mathrm{GL}(n)$ is provided in [1].

It would appear that a uniform Plancherel formula, valid for any local field, has yet to be written. A very special case of such a uniform formula appears in [11].

We would like to thank Anne-Marie Aubert for her careful reading of the manuscript, and for some constructive comments.

2 C^*-algebras

2.1. $*$ *algebras.* Let A be an algebra over the field $\mathbb{C}$ of complex numbers. An *involution* in A is a map $x \longmapsto x^*$ of A into itself such that

- $(x^*)^* = x$
- $(x+y)^* = x^* + y^*$
- $(\lambda x)^* = \overline{\lambda} x^*$
- $(xy)^* = y^* x^*$

for any $x, y \in A$ and $\lambda \in \mathbb{C}$. An algebra over $\mathbb{C}$ endowed with an involution is called a $*$ algebra. The element x^* is often called the *adjoint* of A. A subset of A which is closed under the involution operation is said to be *self-adjoint*.

2.2. A *normed* $*$ *algebra* is a $*$ algebra with norm $\|\cdot\|$ such that

- $\|x\| \geq 0$ with $\|x\| = 0$ if and only if $x = 0$
- $\|x+y\| \leq \|x\| + \|y\|$
- $\|\lambda x\| = |\lambda| \cdot \|x\|$
- $\|xy\| \leq \|x\| \cdot \|y\|$
- $\|x^*\| = \|x\|$

for any $x, y \in A$ and $\lambda \in \mathbb{C}$. If, in addition, A is complete, A is called a Banach $*$ algebra.

2.3 A C^*-*algebra* is a Banach $*$ algebra such that $\|x\|^2 = \|x^* x\|$ for every $x \in A$. The condition $\|x\|^2 = \|x^* x\|$ is called the C^* condition. A C^*-algebra with unit is called a unital C^*-algebra.

The C^* condition hides an absolutely crucial feature by letting one believe that, as in a Banach algebra, there is freedom in the choice of the norm. In fact if a unital $*$ algebra is a C^*-algebra it is so for a *unique* norm, given for any x by the equality:

$$\begin{aligned} \|x\|^2 &= \textit{spectral radius of } x^* x \\ &= \sup\{|\lambda| : x^* x - \lambda \textit{ not invertible}\} \end{aligned}$$

2.4. Examples

2.4.1. Let X be a locally compact Hausdorff space. We say that the complex-valued function f on X *vanishes at infinity* if the following condition holds:

Given $\varepsilon > 0$, *there exists a compact set* $K \subset X$ *such that*

$$|f(x)| < \varepsilon \text{ whenever } x \in X - K.$$

Let A denote the algebra of all complex-valued continuous functions vanishing at infinity on X. Set

$$\begin{aligned} f^*(x) &= \overline{f(x)}, \quad f \in A,\ x \in X \\ \|f\| &= \sup\{|f(x)| : x \in X\} \\ (fg)(x) &= f(x)g(x). \end{aligned}$$

Then A is a commutative C^* algebra denoted $C_0(X)$. The C^* algebra $C_0(X)$ is unital if and only if X is compact.

2.4.2. Let H be a complex Hilbert space and $A = \mathcal{L}(H)$ the algebra of continuous endomorphisms of H. For $S, T \in A$, set

$$\begin{aligned} (T^*\xi|\eta) &= (\xi|T\eta), \quad \xi, \eta \in H, \\ (ST)(\xi) &= S(T\xi), \quad \xi \in H, \\ \|T\| &= \sup\{\|T\xi\| : \|\xi\| \le 1\}. \end{aligned}$$

Then $\mathcal{L}(H)$ is a unital C^* algebra.

2.4.3. Let $\mathfrak{K}(H)$ be the algebra of compact operators on H. This is the closure in $\mathcal{L}(H)$ of the finite-rank operators on H. Then $\mathfrak{K}(H)$ is a C^*-algebra. This C^*-algebra is unital if and only if H is finite-dimensional.

2.5. *Automatic Continuity.* Let A be a Banach $*$ algebra, B a C^* algebra and π a morphism of A into B; this means that π is a morphism of the underlying $*$ algebras, without any condition on the norms. Then $\|\pi(x)\| \le \|x\|$ for every $x \in A$. See [5, 1.3.7]. It follows that an isomorphism of C^*-algebras is automatically isometric.

2.6. Let A and B be C^*-algebras, ϕ a morphism of A into B. Then the image $\phi(A)$ is a sub-C^*-algebra of B.

2.7. Let A be a $*$ algebra and H a Hilbert space. A representation of A in H is a morphism of the $*$ algebra A into the $*$ algebra $\mathcal{L}(H)$. In other words, a representation of A in H is a map π of A into $\mathcal{L}(H)$ such that

$$\begin{aligned}\pi(x+y) &= \pi(x)+\pi(y)\\ \pi(\lambda x) &= \lambda\pi(x)\\ \pi(xy) &= \pi(x)\pi(y)\\ \pi(x^*) &= \pi(x)^*\end{aligned}$$

for $x, y \in A$, $\lambda \in \mathbb{C}$.

2.8. Two representations π and π' of A in H and H' are said to be *equivalent,* and we write $\pi \cong \pi'$, if there is an isomorphism U of the Hilbert space H onto the Hilbert space H' which transforms $\pi(x)$ into $\pi'(x)$ for each $x \in A$. In other words, $U\pi(x) = \pi'(x)U$ for any $x \in A$. Hence the definition of a class of representations. The operator U is an *intertwining operator* for π and π'.

2.9. Let Γ be a finite group acting as automorphisms of the C^*-algebra A. Let A^Γ be the fixed-point set. Then A^Γ is a sub-C^*-algebra of A.

2.10. Irreducible representations. The representation π of the C^*-algebra A in H is *irreducible* if H admits no invariant closed subspaces except 0 and H.

2.11. A C^*-algebra A is *liminal* if, for every irreducible representation π of A and each $x \in A$, $\pi(x)$ is compact.

2.12. The set of equivalence classes of irreducible representations of A is called the *dual* of A, denoted $\hat{A}$.

2.13. *Primitive ideals.* If J is a two-sided ideal in A, then J is *primitive* if it is the kernel of an irreducible representation π of A. The set of primitive ideals in A is denoted $\mathrm{Prim}(A)$. If $\omega \in \hat{A}$, then define $k(\omega) = \mathrm{Ker}\,\pi$ for $\pi \in \omega$. We then have $k : \hat{A} \longrightarrow \mathrm{Prim}(A)$.

If S is a subset of $\mathrm{Prim}(A)$ then we set

$$\begin{aligned}I(S) &= \bigcap_{J\in S} J,\\ \overline{S} &= \{J \in \mathrm{Prim}\,A : I(S) \subset J\}\end{aligned}$$

We then have

- $\overline{\emptyset} = \emptyset$
- $S \subset \overline{S}$

- $\overline{S} = \overline{(\overline{S})}$
- $\overline{S_1 \cup S_2} = \overline{S}_1 \cup \overline{S}_2$

This implies that there is a unique topology on Prim(A) whose closed sets are those S such that $\overline{S} = S$. This topology is the hull kernel topology on Prim(A).

If A is liminal then $k : \hat{A} \to \mathrm{Prim}(A)$ is a bijection. The set $\hat{A}$ is now furnished with the pull-back of the hull kernel topology on Prim(A).

2.14. Let A be a liminal C^*-algebra. Then $\hat{A}$ is a locally quasi-compact space in which the points are closed [5, 3.3.8, 4.4.1].

2.15. Let G be a locally compact, separable, topological group with a fixed choice of Haar measure dg. The Banach space $L^1(G)$ becomes a Banach $*$ algebra when equipped with the convolution product and the involution

$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$$

for $x \in G$ and $f \in L^1(G)$, where Δ denotes the modular homomorphism of G. Any unitary representation π of G on a Hilbert space H extends to a $*$ representation of $L^1(G)$, still denoted by π, via the formula

$$\pi(f) = \int f(g)\pi(g)\, dg$$

with $f \in L^1(G)$. This gives $\|f\|_1 \geq \|\pi(f)\|$, where $\| \ \|_1$ denotes the L^1-norm and $\| \ \|$ the operator norm for H. We therefore define

$$\|f\| = \sup\{\|\pi(f)\| : \pi \in \hat{G}\}, \quad f \in L^1(G).$$

This gives $\|f\|_1 \geq \|f\|$, and we note that $L^1(G)$ need not be complete in this norm. The *full C^*-algebra of G,* denoted $C^*(G)$, is defined to be the completion of $L^1(G)$ with respect to this norm. (Thus $C^*(G)$ is the enveloping C^*-algebra of $L^1(G)$.)

On the other hand, one has the left regular representation λ of $L^1(G)$ on $L^2(G)$:

$$\lambda(f)h = f \star h$$

with $f \in L^1(G)$, $h \in L^2(G)$. The *reduced* C^*-algebra of G, denoted $C^*_r(G)$, is the norm closure of the image $\lambda(L^1(G))$ in the space of all bounded linear operators on $L^2(G)$. The left regular representation induces a canonical surjective $*$ homomorphism [5, 2.15.1]:

$$\lambda : C^*(G) \longrightarrow C^*_r(G)$$

2.16. The group G is *liminal* if $C^*(G)$ is liminal. By (2.15.1), the reduced C^*-algebra $C^*_r(G)$ is liminal if $C^*(G)$ is liminal.

Let $\hat{G}$ be the set of equivalence classes of irreducible unitary representations of G (see section 3.1). Then we have the following bijections:

$$\widehat{G} \cong \widehat{C^*(G)} \cong \mathrm{Prim}(C^*(G)).$$

The topology on the unitary dual $\widehat{G}$ is then, by definition, the pull-back of the hull-kernel topology of $\mathrm{Prim}\, C^*(G)$.

Let σ, τ be irreducible unitary representations of G. For the moment, let $[\sigma]$ denote the equivalence class of σ, so that $[\sigma], [\tau] \in \hat{G}$. A base for the open neighbourhoods of a point $[\tau] \in \hat{G}$ consists of the sets $U(K; \xi_1, \ldots, \xi_n)$, where K is a compact subset of G and $\xi_1, \ldots, \xi_n$ are vectors in the space $H(\tau)$ of the representation τ. By definition, the class $[\sigma]$ belongs to the set $U(K; \xi_1, \ldots, \xi_n)$ if there exist vectors $\eta_1, \ldots, \eta_n$ in the space $H(\sigma)$ such that

$$|\langle \tau(g)\xi_i, \xi_j \rangle - \langle \sigma(g)\eta_i, \eta_j \rangle| < 1$$

for all $g \in K$, $1 \leq i, j \leq n$. It is instructive to see what happens when G is a locally compact *abelian* group. Then each σ is a unitary character. The group G is certainly a liminal group. Then $\hat{G}$ is the Pontryagin dual of G and the topology on $\hat{G}$ is the topology of uniform convergence on compact sets.

2.17. There are two lemmas, which are absolutely crucial.

2.17.1 Harish-Chandra Lemma [15, 3.4.10]. Let G be a real reductive group. Then G is liminal.

2.17.2. Bernstein Lemma [3]. Let G be a reductive p-adic group. Then G is liminal.

2.18. *Topology on the unitary dual $\hat{G}$ of a reductive group G.* From 2.14–2.17 we infer that

• the unitary dual $\hat{G}$ of a reductive group G is a locally quasi-compact space in which points are closed.

2.19. *Noncommutative topology.* The conventional wisdom is that C^*-algebra theory may be viewed as *noncommutative topology.* Each property concerning a locally compact Hausdorff space X can in principle be formulated in terms of the function algebra $C_0(X)$ and will then usually make sense (and hopefully be true) for any noncommutative C^*-algebra. Here is a list of some of the "dualities".

topology $\longleftrightarrow$ algebra
$C_0(X) \longleftrightarrow C^*$-algebra A
proper map $\longleftrightarrow$ morphism
homeomorphism $\longleftrightarrow$ automorphism
measure $\longleftrightarrow$ positive functional
disjoint union $\longleftrightarrow$ direct sum
compact $\longleftrightarrow$ unital
σ-compact $\longleftrightarrow$ σ-unital
open subset $\longleftrightarrow$ ideal
open dense subset $\longleftrightarrow$ essential ideal
closed subset $\longleftrightarrow$ quotient
compactifications $\longleftrightarrow$ unitizations
connected $\longleftrightarrow$ projectionless
2nd countable $\longleftrightarrow$ separable

The idea is that since an algebra isomorphism of $C_0(X)$ onto $C_0(Y)$ induces a homeomorphism of X with Y, *all* topological information about X is stored algebraically in $C_0(X)$. Reference [16, 1.11].

3 Unitary Representations

A homomorphism π of a topological group G into the group of unitary operators on a Hilbert space H ($\neq \{0\}$) is called a *unitary representation* of G if π is *strongly continuous* in the following sense: for any element $x \in H$, the mapping $g \mapsto \pi_g x$ is a continuous mapping from G into H. The Hilbert space H is called the representation space of π and is denoted by $H(\pi)$. Two unitary representations π and π' are said to be *equivalent,* denoted by $\pi \cong \pi'$, if there exists an isometry T from $H(\pi)$ onto $H(\pi')$ that satisfies the equality $T \circ \pi_g = \pi'_g \circ T$ for every g in G. If the representation space $H(\pi)$ contains no closed subspace other than H and $\{0\}$ that is invariant under every π_g, then the unitary representation π is said to be *irreducible.*

Let G be a unimodular locally compact group, let μ be Haar measure on G. Set

$$\begin{aligned} L(y)f(x) &= f(y^{-1}x) \\ R(y)f(x) &= f(xy) \end{aligned}$$

with $f \in L^2(G, \mu)$, $x, y \in G$. Then L is a unitary representation of G called the left regular representation of G, and R is the right regular representation.

These two representations can be combined as follows. Set

$$U(z, y)f(x) = f(z^{-1}xy)$$

with $x, y, z \in G$ and $f \in L^2(G)$. Then U is a unitary representation of the product group $G \times G$. We shall have much more to say about U in §4.

Subrepresentations. Let π be a unitary representation of a topological group G. A closed subspace E of $H(\pi)$ is called π-invariant if E is invariant under every π_g, $g \in G$. Let $E \neq \{0\}$ be a closed invariant subspace of $H(\pi)$ and V_g be the restriction of π_g on E. Then V is a unitary representation of G on the representation space E and is called a *subrepresentation* of π.

Representation of Direct Products. Let G_1, G_2 be topological groups, G the direct product of G_1 and G_2. Let π_1 (resp. π_2) be an irreducible unitary representation of G_1 (resp. G_2) on H_1 (resp. H_2). The Hilbert space tensor product is defined as follows. Form the ordinary algebraic tensor product, denoted here by $H_1 \odot H_2$, and furnish it with the inner product

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle = \langle \xi_1, \eta_1 \rangle \langle \xi_2, \eta_2 \rangle.$$

This is of course a cross-norm in the sense that

$$\|\xi_1 \otimes \xi_2\| = \|\xi_1\| \cdot \|\xi_2\|.$$

The completion of $H_1 \odot H_2$ with respect to this norm is the Hilbert space tensor product $H_1 \otimes H_2$. The tensor product representation $\pi_1 \otimes \pi_2$ of $G_1 \times G_2$ on $H_1 \otimes H_2$ is then specified as follows:

$$(\pi_1 \otimes \pi_2)(g)(\xi_1 \otimes \xi_2) = \pi_1(g_1)\xi_1 \otimes \pi_2(g_2)\xi_2$$

with $g = (g_1, g_2) \in G = G_1 \times G_2$.

Direct Sums. If the representation space H of a unitary representation π is the Hilbert space direct sum $\bigoplus_{\alpha \in I} H_\alpha$ of mutually orthogonal closed invariant subspaces $\{H(\alpha)\}_{\alpha\in I}$ then π is called the *direct sum* of the subrepresentations $\pi(\alpha)$ induced on $H(\alpha)$ by π, and is denoted by

$$\pi = \bigoplus_{\alpha\in I} \pi(\alpha).$$

If π is the direct sum of $\{\pi(\alpha)\}_{\alpha\in I}$ and every $\pi(\alpha)$ is irreducible, then π is said to be *decomposed into the direct sum of irreducible representations.* Decomposition into direct sums of irreducible representations is essentially unique if it exists; that is, if $\pi = \bigoplus_{\alpha\in I} \pi(\alpha) = \bigoplus_{\beta\in J} \theta(\beta)$ are two decompositions of π into direct sums of irreducible representations, then there exists a bijection ϕ from I onto J such that $\pi(\alpha)$ is equivalent to $\theta(\phi(\alpha))$ for every α in I.

Square-Integrable Representations. An irreducible unitary representation π of a unimodular locally compact group G is said to be *square integrable* when for some element $x \neq 0$ in $H(\pi)$, the function

$$\phi(g) = \langle \pi(g)x, x\rangle$$

belongs to $L^2(G, dg)$ where dg is Haar measure on G. If π is square integrable then $\phi_{x,y}(g) = \langle \pi_g x, y\rangle$ belongs to $L^2(G, dg)$ for any x and y in $H(\pi)$. Let π and π' be two square integrable representations of G. Then the following *orthogonality relations* hold:

$$\int (\pi(g)x, y)\overline{(\pi'(g)u, v)}\, dg = \begin{cases} 0 & \text{if } \pi \text{ is not equivalent to } \pi' \\ d_\pi^{-1}(x, u)(y, v) & \text{if } \pi = \pi' \end{cases}$$

When G is compact, every irreducible unitary representation π is square integrable and finite-dimensional. Moreover the scalar d_π above is the degree of π if the total measure of G is normalized to 1. In the general case, the scalar d_π is called the *formal degree* of π and is determined uniquely by the given Haar measure dg.

Let y be an element in $H(\pi)$ with norm 1 and V be the subspace $\{\phi_{x,y} : x \in H(\pi)\}$ of $L^2(G)$. Then the linear mapping

$$T : x \longmapsto \sqrt{d_\pi}\phi_{x,y}$$

is an isometry of $H(\pi)$ onto V. Hence π is equivalent to a subrepresentation of the right regular representation R of G. Conversely, every irreducible subrepresentation of R is square integrable. Thus a square integrable representation is an irreducible subrepresentation of R ($\cong L$). Therefore, in the irreducible decomposition of R, the square integrable representations appear as discrete direct summands.

It will often be necessary to have a more general version of this. If (π, H) is an irreducible unitary representation of G, it is easy to see (Schur's Lemma) that the operators $\pi(z)$, for z in the centre Z_G of G, are scalars of absolute value 1. Hence, if the centre of G is not compact, G can have no square integrable representations in the sense above. We therefore choose a closed subgroup Z of Z_G such that Z_G/Z is compact. (Later, when G is a reductive group over a local field, it is customary to choose Z to be a maximal split torus in Z_G.) For $v \in H$, the function $g \mapsto |\langle \pi(g)v, v\rangle|$ can be viewed as a function on G/Z. We say that π is square integrable (mod centre) if this function satisfies

$$\int_{G/Z} |\langle \pi(g)v, v\rangle|^2 \, d\dot{g} < \infty,$$

where $d\dot{g}$ is a Haar measure on G/Z.

We write $E_2(G)$ for the set of equivalence classes of square integrable (mod centre) representations of G. We sometimes refer to these representations as *discrete series.*

Direct Integrals. Let (S, μ) be a measure space such that

(a) $S = \cup(S_i)$ with S_i measurable and $\mu(S_i) < \infty$;

(b) There is a countable set, M_0, of measurable subsets of S such that if M is the σ-algebra generated by M_0 and if A is a measurable subset of S, then there exists $B \in M$ such that $\mu(A - A \cap B) = 0$ and $\mu(B - A \cap B) = 0$.

A *family of Hilbert spaces over S* is an assignment of a Hilbert space H_s to each $s \in S$. A *section* of the family $\{H_s : s \in S\}$ is a correspondence $s \mapsto v(s)$ of $v(s) \in H_s$ for each $s \in S$. If there exists a set $\mathcal{F}$ of sections of $\{H_s\}$ satisfying the following three conditions, then $\{H_s : s \in S\}$ is called a *measurable family of Hilbert spaces. (i) If $x, y \in \mathcal{F}$ then $s \mapsto \langle x(s), y(s)\rangle$ is measurable*

(ii) If z is a section of $\{H_s\}$ and if $s \mapsto \langle z(s), x(s)\rangle_s$ is measurable for all $x \in \mathcal{F}$, then $z \in \mathcal{F}$.

(iii) There exists a countable subset $\{x_j\}$ of $\mathcal{F}$ such that if $s \in S$, then $\{x_j(s) : j = 1, 2, \ldots\}$ is dense in H_s. The main example is given as follows. Let H be a separable Hilbert space. Take $H_s = H$ for all $s \in S$. Let $\mathcal{F}$ be the space of all functions x from S to H such that $s \mapsto \langle x(s), v\rangle$ is measurable

for all $v \in H$. Let $\{x_j\}$ be a countable dense subset of H looked upon as a subset of $\mathcal{F}$.

Returning to the general case, we have the following observations.

(1) If $v, w \in \mathcal{F}$ then $(s \mapsto v(s) + w(s)) \in \mathcal{F}$

(2) If f is a measurable function on S and if $v \in \mathcal{F}$ then $(s \mapsto f(s)v(s)) \in \mathcal{F}$.

Indeed, if $w \in \mathcal{F}$ then $\langle f(s)v(s), w(s)\rangle_s = f(s)\langle v(s), w(s)\rangle_s$. Now use *(ii)*.

Suppose we have a measurable family of Hilbert spaces $\{H_s\}$ and $\mathcal{F}$ as before. We say that $x \in \mathcal{F}$ is *square-integrable* if

$$\|x\|^2 = \int_S \|x(s)\|_s^2 \, d\mu(s) < \infty.$$

We identify $x, y \in \mathcal{F}$ if

$$\mu(\{s \in S : \|x(s) - y(s)\|_s > 0\}) = 0.$$

Modulo this identification, we use the notation

$$\int_S H_s \, d\mu(s)$$

for the space of all square integrable elements of $\mathcal{F}$. If x, y are square integrable, then we set

$$\langle x, y\rangle = \int_S \langle x(s), y(s)\rangle \, d\mu(s).$$

With this inner product, $\int_S H_s \, d\mu(s)$ is a separable Hilbert space [15, 14.8.2]. If $H_s = H$ for all $s \in S$ and $\mathcal{F}$ is as in the previous section, then we will also write

$$\int_S H_s \, d\mu(s) = L^2(S, H; \mu).$$

If $H = \mathbb{C}$ then $L^2(S, \mathbb{C}; \mu)$ will be written $L^2(S; \mu)$.

The space $\int_S H_s \, d\mu(s)$ will be called the *direct integral* of the family $\{H_s\}$. Notice that the definition involves $\{H_s\}$, μ and $\mathcal{F}$. If S is a countable set and if μ is counting measure then $\int_S H_s \, d\mu(s)$ is just the Hilbert space direct sum of the spaces H_s, denoted $\bigoplus H_s$.

Lemma 3.1. *[15, 14.8.3]. If $S = \bigcup U_i$ with the U_i measurable and pairwise disjoint then*

$$\int_S H_s \, d\mu(s) = \bigoplus_i \int_{U_i} H_s \, d\mu_i(s).$$

Here, $\mu_i = \mu\,|_{U_i}$, $\mathcal{F}_i = \{f|_{U_i} : f \in \mathcal{F}\}$ and we also take the restrictions of the x_i.

An assignment $s \mapsto B_s \in \mathrm{End}(H_s)$ is called an *operator field.* If, for each $v \in \mathcal{F}$, the map $(s \mapsto B_s v(s))$ lies in $\mathcal{F}$, then B is called *measurable.* We say that a bounded operator B on $\mathcal{F} = \int_S H_s\, d\mu(s)$ is *decomposable* if there exists a measurable operator field B_s such that if $v \in \mathcal{F}$,

$$(Bv)(s) = B_s v(s)$$

for μ almost every s.

Let G be a locally compact, separable, topological group. Let (S, μ) be a measure space (as above) and let $\{H_s : s \in S\}$ be a measurable family of Hilbert spaces. Let $\mathcal{F} = \int_S H_s\, d\mu(s)$. Let π_s be a unitary representation of G on H_s for each $s \in S$, such that for each $g \in G$, $s \mapsto \pi_s(g)$ is a measurable operator field. With $g \in G$, let $\pi(g)$ be the decomposable operator corresponding to $\{\pi_s(g)\}$.

If $v, w \in \mathcal{F}$ then

$$\begin{aligned}\langle \pi(g)v, \pi(g)w\rangle &= \int_S \langle \pi_s(g)v(s), \pi_s(g)w(s)\rangle\, d\mu(s)\\ &= \int_S \langle v(s), w(s)\rangle\, d\mu(s)\\ &= \langle v, w\rangle.\end{aligned}$$

Therefore $\pi(g)$ is a unitary operator on $\mathcal{H}$.

Lemma 3.2. *[15, 14.9.2]. $(\pi, \mathcal{H})$ defines a (strongly continuous) unitary representation of G. We call this representation the direct integral of $\{(\pi_s, H_s) : s \in S\}$ and denote it by*

$$\left(\int_S \pi_s\, d\mu(s), \int_S H_s\, d\mu(s)\right).$$

4 Plancherel Formula

We begin by stating the basic decomposition theorem for liminal groups [15, 14.10.5, 14.13.8].

Theorem 4.1. *Decomposition Theorem for Liminal Groups. Let G be a separable, locally compact, liminal, topological group. Let (π, H) be a unitary representation of G. Then there exists a Borel measure σ on $\hat{G}$ and a direct integral of representations $(\int \pi_\omega \, d\sigma(\omega), \int H_\omega \, d\sigma(\omega))$ such that (i) (π, H) is unitarily equivalent to $(\int \pi_\omega \, d\sigma(\omega), \int H_\omega \, d\sigma(\omega))$.*

(ii) There exists a subset N of $\hat{G}$ such that $\sigma(N) = 0$, and if $\omega \in \hat{G} - N$ then (π_ω, H_ω) is equivalent to $(\tilde{\pi}_\omega \otimes I, \tilde{H}_\omega \otimes V_\omega)$, for some Hilbert space V_ω and where $(\tilde{\pi}_\omega, \tilde{H}_\omega) \in \omega$.

(iii) If $\omega \in \hat{G} - N$, set $n(\omega) = \dim V_\omega$. Then n is a σ-measurable function from $\hat{G} - N$ to the extended positive axis $[0, \infty]$. The function n is called the multiplicity function.

(iv) Suppose that μ is another Borel measure on $\hat{G}$ such that (π, H) is unitarily equivalent with $(\int \pi_\omega \, d\mu(\omega), \int H_\omega \, d\mu(\omega))$. Then μ is absolutely continuous with respect to σ and the multiplicity functions are equal σ-almost everywhere.

Two measures ν_1 and ν_2 are *equivalent* if they have the same null sets. The *support* $\text{supp}(\nu)$ of a measure ν is the complement of the largest open set on which ν vanishes. Thus $\text{supp}(\nu)$ is an invariant of the measure class $[\nu]$ of ν. In Theorem 4.1, the pair (π, H) determines a unique measure class $[\sigma]$ on $\hat{G}$. The support $\text{supp}(\sigma)$ is an invariant of this class.

In this section G will denote a separable, locally compact, unimodular, liminal, topological group. Since G is unimodular, $L^2(G)$ is a unitary representation of G under both the left and right regular representations L and R. As in §3, we define a unitary representation U of $G \times G$ on $L^2(G)$ by

$$U(x, z)f(y) = f(x^{-1}yz),$$

for $x, y, z \in G$ and $f \in L^2(G)$. The main result in this section will give a very precise form of the direct integral decomposition of Theorem 4.1 for L and R.

If $\omega \in \hat{G}$, then we fix $(\pi_\omega, H_\omega) \in \omega$. If (π, H) is a unitary representation of G and if H' is the space of continuous linear functionals on H, then we define a representation on H' by $\pi'(g)\lambda = \lambda \circ \pi(g)^{-1}$. If $v \in H$, then we set $\lambda_v(w) = \langle w, v \rangle$. Then the map τ from H to H' given by $\tau(v) = \lambda_v$ is a conjugate linear bijection of H onto H'. We note that

$$\lambda_v(\pi(g)^{-1}w) = \langle \pi(g)^{-1}w, v \rangle = \langle w, \pi(g)v \rangle = \lambda_{\pi(g)v}(w).$$

Thus:

$$\tau \circ \pi(g) = \pi'(g) \circ \tau.$$

We put a Hilbert space structure on H' by $\langle \lambda, \mu_w \rangle = \lambda(w)$. With this Hilbert space structure, $\langle \tau v, \tau w \rangle = \langle w, v \rangle$. If $\omega \in \hat{G}$, then we note that π'_ω is also irreducible. We therefore have an involutive map $\omega \mapsto \omega'$ of $\hat{G}$ onto $\hat{G}$ defined by $\pi'_\omega \in \omega'$. By $H_\omega \otimes H_{\omega'}$ we will mean the tensor product of the Hilbert spaces H_ω and H'_ω, and $\pi_\omega \otimes \pi_{\omega'}$ will stand for the unitary representation $\pi_\omega \otimes \pi'_\omega$ of $G \times G$ on $H_\omega \otimes H_{\omega'}$.

Theorem 4.2. *Plancherel Theorem for Liminal Groups [15, 14.11.2]. Let G be a separable, locally compact, unimodular, liminal topological group. There exists a Borel measure ζ on $\hat{G}$ and a direct integral of unitary representations $(\int \sigma_\omega \, d\zeta(\omega), \int W_\omega \, d\zeta(\omega))$ satisfying the following three conditions: (1) $(U, L^2(G))$ is unitarily equivalent with $(\int \sigma_\omega \, d\zeta(\omega), \int W_\omega \, d\zeta(\omega))$.*

(2) For ζ-almost every $\omega \in \hat{G}, (\sigma_\omega, H_\omega)$ is unitarily equivalent with

$$(\pi_\omega \otimes \pi_{\omega'}, H_\omega \otimes H_{\omega'}).$$

(3) If $f \in L^1(G) \cap L^2(G)$, then $\pi_\omega(f)$ is of Hilbert-Schmidt class for ζ-almost every $\omega \in \hat{G}$ and

$$\langle f, f \rangle = \int \mathrm{tr}(\pi_\omega(f)^* \pi_\omega(f)) \, d\zeta(\omega).$$

Furthermore (3) uniquely specifies ζ. The measure ζ is called the Plancherel measure *associated with dg.*

Note that $\mathrm{tr}(\pi_\omega(f)^* \pi_\omega(f))$ depends only on ω and f and not on the choice of (π_ω, H_ω). The measure ζ depends on the choice of invariant measure on G.

If we restrict U from $G \times G$ to G (first factor) then we obtain the representation L. The Plancherel theorem provides a unique decomposition of L with multiplicity function

$$n(\omega) = \dim H_\omega.$$

Restricting U from $G \times G$ to G (second factor) we obtain the representation R. The Plancherel theorem provides a unique decomposition of R with multiplicity function

$$n(\omega) = \dim H_\omega$$

If $\phi = f^* \star f$ then

$$\pi_\omega(\phi) = \pi_\omega(f)^* \pi_\omega(f), \quad (7)$$
$$\phi(1) = \langle f, f \rangle, \quad (8)$$

so the Plancherel Formula may be written

$$\phi(1) = \int \operatorname{tr} \pi_\omega(\phi)\, d\zeta(\omega).$$

If ϕ is a test-function then the distribution

$$\phi \longmapsto \operatorname{tr} \pi_\omega(\phi)$$

is the *Harish-Chandra character* Θ_ω. The Plancherel Formula takes the form

$$\phi(1) = \int \Theta_\omega(\phi)\, d\zeta(\omega).$$

The *reduced dual* of G is the support of Plancherel measure ζ. This reduced dual is a closed subset $\hat{G}_r$ of $\hat{G}$.

As in §2, let λ be the left regular representation of G. Regarded as a representation of $C^*(G)$, it has a certain kernel N. Then $\hat{G}_r$ is the set of the $\sigma \in \widehat{C^*(G)}$ whose kernels contain N. In other words, $\hat{G}_r$ is the dual of the C^*-algebra $C^*(G)/N$, itself isomorphic to the C^*-algebra $\lambda(C^*(G))$. This C^*-algebra $\lambda(C^*(G))$ is the norm closure in $\mathcal{L}(L^2(G))$ of the set of left convolution operators by the elements of $L^1(G)$. That is, let $f \in L^1(G)$, $h \in L^2(G)$, set

$$\lambda(f)h = f \star h.$$

Then

$$\lambda(C^*(G)) = \overline{\lambda(L^1(G))} \subset \mathcal{L}(L^2(G)).$$

The C^*-algebra generated by the image of λ is the reduced C^*-algebra $C_r^*(G)$. Its dual is the *reduced dual* of G.

By *Plancherel Formula Level II* we shall mean, in addition, an explicit description of the reduced dual.

By *Plancherel Formula Level III* we shall mean, in addition, an explicit formula for Plancherel measure ζ.

For real reductive groups, Harish-Chandra obtained an explicit formula for Plancherel measure. This formula may be found in [15, 14.12.4].

Wallach makes the following comment [15, 14.12.5]. Harish-Chandra's theorem could be construed as a calculation of the abstract Plancherel measure. However, the theorem is much more than that since it also contains at its heart the full analytic (and a substantial part of the algebraic) theory of tempered representations of real reductive groups.

Let π be a square-integrable representation of G, and let $\pi \in \omega$. Then ω has positive Plancherel measure equal to the formal degree of π:

$$\zeta(\{\omega\}) = d_\pi.$$

A unitary representation π is *integrable* if if admits a matrix coefficient

$$\phi : g \longmapsto (\pi(g)v, v), \qquad v \neq 0,$$

which is integrable, i.e., $\phi \in L^1(G)$.

Let π be an integrable representation and let $\pi \in \omega$. Then ω is an isolated point in the reduced dual $\hat{G}_r$, by [5, 18.4.2].

Example. Let π be a supercuspidal representation of a semisimple p-adic group G, and let $\pi \in \omega$. Then ω is an isolated point in the reduced dual of G.

5 Real reductive groups

We turn to a definitive statement of the *Plancherel formula for real reductive groups,* as in [15, 13.4.1]. Just as in (4.16), we can choose a maximal split torus A_0 in G and define p-pairs as before. Fix a minimal p -pair (P_0, A_0). Let $\mathcal{C}(G)$ be the Harish-Chandra Schwartz space of G: this consists of the functions on G whose derivatives are all rapidly decreasing (defined just as before).

Theorem 5.1. *Let $f \in \mathcal{C}(G)$ be left and right K-finite. Then*

$$f(g) = \sum_{(P,A)} C_A \sum_{\omega} d(\omega) \int_{\mathfrak{a}^*} \Theta_{P,\omega,i\nu}(R(g)f)\mu(\omega, i\nu)\, d\nu.$$

Explanation of terms. The first summation is over all $(P, A) \succ (P_0, A_0)$. Thus $A \subset A_0$ is a split torus containing the maximal central split torus in G. The parabolic subgroup P has Levi component $M_P = Z_G(A)$ and we denote its unipotent radical by N_P. The Levi subgroup M_P has a canonical subgroup ${}^\circ M_P$ defined as the common kernel of the absolute values of all rational characters of M_P. The second summation above is then over $\omega \in E_2({}^\circ M_P)$, the factor C_A is a constant depending only on A, and $d(\omega)$ is the formal degree of ω.

The object $\mathfrak{a}^*$ is the linear dual of the Lie algebra of A. An element $\nu \in \mathfrak{a}^*$ determines a one-dimensional representation of A, which we denote $\exp i\nu$, and the common kernel of these representations is ${}^\circ A = A \cap {}^\circ M_P$. The Langlands decomposition $M_P = {}^\circ M_P A$ thus allows us to define a representation $\omega \otimes \exp i\nu$ of M_P. In these terms, $\mu(\omega, i\nu)$ is Plancherel density on $\mathfrak{a}^*$ and $d\nu$ is Lebesgue measure on $\mathfrak{a}^*$.

Finally, we view $\omega \otimes \exp i\nu$ as a representation of P via $P \twoheadrightarrow M_P$ and $\Theta_{P,\omega,i\nu}$ is the character of

$$\operatorname{Ind}_P^G \delta_P^{\frac{1}{2}}(\omega \otimes \exp(i\nu) \otimes 1),$$

where δ_P is the modular function of P.

Setting $g = 1$ we obtain

$$f(1) = \sum_P \sum_\omega \int \Theta_{P,\omega,i\nu}(f)\mu(\omega, i\nu)\, d\nu$$

for all test functions f.

This is a Plancherel formula Level III. The support of Plancherel measure is explicitly given, and the Plancherel density is explicitly known [15, 4.12.4].

Now P is *cuspidal* if and only if ${}^\circ M_P$ has a compact Cartan subgroup. This in turn is true if and only if the discrete series of ${}^\circ M_P$ is non-empty.

Let now P be cuspidal, let T_P be a compact Cartan subgroup of ${}^\circ M_P$. The discrete series of ${}^\circ M_P$ is parametrized by Harish-Chandra parameters, i.e., (nonsingular) unitary characters of T_P.

Example. Consider the real reductive group $\mathrm{GL}(n, \mathbb{R})$. Let M be a standard Levi factor, i.e., a block-diagonal subgroup of G. Then ${}^\circ M$ is the block-diagonal subgroup such that each block has determinant ± 1.

If $n \geq 3$ then $\mathrm{SL}(n, \mathbb{R})$ has no compact Cartan subgroup and therefore no discrete series. Therefore the block-diagonal subgroup M can contain only

2×2 blocks or 1×1 blocks (otherwise the discrete series of ${}^\circ M$ is empty). Let M_{max} contain the maximal number of 2×2 blocks. Then the standard Levi factor M contributes to the Plancherel formula (and hence to the reduced C^*-algebra of G) if and only if M has a conjugate contained in M_{max}.

6 Plancherel formula for $\mathrm{SL}(2,\mathbb{C})$

Let $G = \mathrm{SL}(2,\mathbb{C})$. This is a liminal group (see section 2.17). The diagonal subgroup T is a split torus, and is the Levi factor of the standard Borel subgroup B of upper triangular matrices in $\mathrm{SL}(2,\mathbb{C})$. Let $\sigma \in \hat{T}$ and let

$$\pi(\sigma) = \mathrm{Ind}_B^G(\delta_B^{\frac{1}{2}}\sigma).$$

This is normalized L^2-induction. Now $T \cong \mathbb{C}^\times \cong \mathbb{R}\times U(1)$ so that $\hat{T} \cong \mathbb{R}\times\mathbb{Z}$. The unitary principal series $\{\pi(\sigma) : \sigma \in \hat{T}\}$ is therefore parametrized by $\mathbb{R}\times\mathbb{Z}$.

Let $f \in C_c^\infty(G)$. The Plancherel formula (for a suitable normalization of Haar measure) [7, p. 390] says that

$$(2\pi)^3 f(1) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\Theta_{\sigma_n,\,iy}(f)(n^2+y^2)\,dy,$$

where $\sigma = (n,y) \in \mathbb{Z}\times\mathbb{R}$.

This is a Plancherel formula Level III. The support of the Plancherel measure comprises the unitary principal series $\{\pi(\sigma) : \sigma \in \hat{T}\}$. The Plancherel measure is given explicitly. Fix n and consider the set

$$\{(n,y) : y \in \mathbb{R}\} \subset \mathbb{Z}\times\mathbb{R}.$$

On this set the Plancherel measure is

$$(n^2+y^2)\,dy$$

where dy is Lebesgue measure on $\mathbb{R}$. So we have

$$\textit{Plancherel density} = n^2+y^2.$$

7 The Plancherel formula for $\mathrm{SL}(2,\mathbb{R})$

Let $G = \mathrm{SL}(2,\mathbb{R})$. Here is the Plancherel formula [7, p. 401]:

$$2\pi f(1) = \sum_{n\geq 2}(n-1)\Theta_n(f) + \frac{1}{4}\int_{-\infty}^{\infty}\Theta_{+,\,iy}(f)\tanh\frac{\pi y}{2}dy$$

$$+\frac{1}{4}\int_{-\infty}^{\infty}\Theta_{-,\,iy}(f)\coth\frac{\pi y}{2}dy$$

for all $f \in C_c^\infty(\mathrm{SL}(2,\mathbb{R}))$ and Haar measure normalized as in [7, 10.7].

The formula displays the contributions from

- the discrete series
- the even principal series
- the odd principal series

8 Reductive groups over $\mathbb{Q}_p$

Let F be a finite extension of $\mathbb{Q}_p$. Let G be the group of F-points of a connected reductive algebraic group defined over F. We refer to such groups G as *reductive p-adic groups*.

We choose a maximal F-split torus A_0 in G and a minimal F-parabolic subgroup P_0 of G with a Levi component which is the G-centraliser of A_0. By a *p-pair in* G we mean a pair (P, A), where A is an F-split torus in G and P is a parabolic subgroup of G with a Levi component which is the G-centraliser of A. Given two p-pairs (P_i, A_i), we write $(P_1, A_1) \succ (P_2, A_2)$ when $P_1 \supset P_2$ and $A_1 \subset A_2$. In this ordering, the p-pair (P_0, A_0) is minimal; a p-pair (P, A) is called *standard* (relative to our choice of (P_0, A_0)) if $(P, A) \succ (P_0, A_0)$.

The *Schwartz space* $\mathcal{C}(G)$ of G should, at first reading, be thought of as a space of "test-functions". The precise definition is as follows [6, p.93]. Let K be a good maximal compact open subgroup of G, so that $G = P_0.K$. Let δ be the modular function of P_0, extended to G by the equation $\delta(kp) = \delta(p)$, $k \in K$, $p \in P_0$. Let dk be normalized Haar measure on K. We put

$$\Xi(g) = \int_K \delta(gk)^{-1/2}\,dk.$$

Embed G in $GL(n,F)$, for some n. (We recall that the topology on G is inherited from such an embedding.) Let $|\cdot|$ be an absolute value on $\mathbb{M}(n,F)$, for example, $|(x_{ij})| = \max \|x_{ij}\|_F$, where $\| \|_F$ is the absolute value on F. Set $\sigma(g) = \log(\max|g|, |g|^{-1})$.

The Schwartz space $\mathcal{C}(G)$ is then the set of functions $f : G \to \mathbb{C}$, bi-invariant by a compact open subgroup, such that for $r > 0$, there exists $C > 0$ such that

$$|f(g)| \leq C(1+\sigma(g))^{-r}\Xi(g), \quad g \in G.$$

(Functions on G satisfying this condition are said to be *rapidly decreasing.*)

A distribution on G is *tempered* if it extends to $\mathcal{C}(G)$. If a tempered distribution is of the form $f(g)\,dg$, for some function $f : G \to \mathbb{C}$ which is bi-invariant by a compact open subgroup, then there exist $r > 0$, $C > 0$ with $|f(g)| \leq C(1+\sigma(g))^{-r}\Xi(g)$. That is, $f \in \mathcal{C}(G)$.

We now discuss the Plancherel formula for reductive p-adic groups. The Bernstein Lemma (2.17.2) guarantees the Plancherel formula for such groups. The statement in Harish-Chandra's Collected Works [6, p. 367] is at first sight rather intimidating, and is as follows.

Let S be a set of representatives for conjugacy classes of F-split tori in G which contain the maximal split torus in the centre of G. We assume that all elements $A \in S$ are "standard", i.e., $A \subset A_0$. For $A \in S$ and $f \in \mathcal{C}(G)$, define

$$f_A(x) = c^{-2}\gamma^{-1}[w]^{-1} \int_{E_2(M)} \mu(\omega)d(\omega)(\Theta_\omega, r(x)f)\,d\omega, \quad x \in G,$$

where M is the centralizer of A in G and

$$c = c(G/A), \quad \gamma = \gamma(G/A), \quad w = w(G/A).$$

Observe that M is a Levi subgroup of G and, in particular, is a reductive p-adic group. There is a unique standard p-pair (P,A) such that M is a Levi component of the parabolic subgroup P. Thus $P = MN$, where N is the unipotent radical of P.

Explanation of terms:

- $w(G/A)$ is the Weyl group $N_G(A)/Z_G(A)$
- $c(G/A)$ and $\gamma(G/A)$ are certain constants

- $[w]$ is the order of $w(G/A)$
- $E_2(M)$ is the discrete series of M, as in §3
- For $\omega \in E_2(M)$, choose a representative $\sigma \in \omega$ and set

$$\pi = \pi(\sigma) = \mathrm{Ind}_P^G \, \delta_P^{\frac{1}{2}} \sigma$$

Here σ has been extended to a representation of P by means of the isomorphism $P/N \cong M$. Then π is unitary. Its G-equivalence class is independent of the choice of σ, and indeed of the parabolic subgroup P with Levi M. Let $C_M^G(\omega)$ denote the class of π and Θ_ω the character of $C_M^G(\omega)$. Then Θ_ω is a tempered and invariant distribution on G, and $(\Theta_\omega, r(x)f)$ is its pairing with $r(x)f$.

- $d(\omega)$ is the formal degree of $\omega \in E_2(M)$
- $r(x)$ is right translation by x
- $\mu(\omega)\, d\omega$ is Plancherel measure on $E_2(M)$

Theorem 8.1. *Harish-Chandra Plancherel Theorem. Let $f \in \mathcal{C}(G)$. Then*

$$f = \sum_{A \in S} f_A$$

is an orthogonal direct sum.

To relate this with our previous description, we set $x = 1$ and strip away the constants c, γ, $[w]$ and $d(\omega)$, i.e., we temporarily absorb these constants in the measure $\mu(\omega)\, d\omega$.

Simplifying notation, we get the pair of statements

$$f_A(1) = \int_{E_2(M)} \mathrm{tr}(\pi(\sigma)f)\mu(\sigma)\, d\sigma \tag{9}$$
$$f(1) = \sum_{A \in S} f_A(1). \tag{10}$$

Combining these statements into a single statement, we get the Harish-Chandra Plancherel Formula

$$\boxed{f(1) = \sum_M \int_{E_2(M)} \mathrm{tr}(\pi(\sigma)f)\mu(\sigma)\, d\sigma.}$$

In this finite sum, one Levi subgroup M is chosen in each conjugacy class in G. The set $E_2(M)$ is given the topology of a locally compact Hausdorff space in which each component is a compact torus: the measure $d\sigma$ restricts to normalized Haar measure on each component [6, p. 355]. Therefore $\mu(\sigma)$ is Plancherel *density.*

Harish-Chandra's article in [6, p. 353 –367], published posthumously, is a sketch-proof.

The Harish-Chandra result is a Plancherel Theorem Level II. The support of Plancherel measure is

$$\bigsqcup_M \{\pi(\sigma) : \sigma \in E_2(M)\}.$$

In this finite disjoint union, one Levi factor is chosen in each conjugacy class.

It is not a Plancherel Theorem Level III, i.e., there is no explicit formula for Plancherel density. However, Harish-Chandra does state a product formula

$$\mu = \prod \mu_\alpha$$

where α runs over all reduced roots of (P, A). The Harish-Chandra Product Formula for Plancherel Measures is discussed in [6, p.92-93]. This product formula is used in a crucial way in the explicit Plancherel formula for $\mathrm{GL}(n)$, see [1].

9 Plancherel formula for $\mathrm{GL}(n, \mathbb{Q}_p)$

The Plancherel formula for $\mathrm{GL}(n)$ is quite complicated, so we will content ourselves with a special case. First, we introduce the p-adic gamma function. Let F be a finite extension of $\mathbb{Q}_p$, and let q_F denote the cardinality of the residue field of F. The p-adic gamma function attached to F is the following meromorphic function of a single complex variable:

$$\Gamma_1(\zeta) = \frac{1 - q_F^\zeta / q_F}{1 - q_F^{-\zeta}}.$$

For more details on the p-adic gamma function, see [13, p.51]. We will change the variable via $s = q_F^\zeta$ and write

$$\Gamma_F(s) = \frac{1 - s/q_F}{1 - s^{-1}},$$

a rational function of s. Let $s \in i\mathbb{R}$ so that s has modulus 1. Then we have

$$1/|\Gamma_F(s)|^2 = \left| \frac{1-s}{1-q_F^{-1}s} \right|^2 .$$

Let T be the standard maximal torus in $\mathrm{GL}(n)$ and let $\widehat{T}$ denote the unitary dual of T. Then $\widehat{T}$ has the structure of a compact torus $\mathbb{T}^n$ (the space of Satake parameters) and the unramified unitary principal series of $\mathrm{GL}(n)$ is parametrized by the quotient $\mathbb{T}^n/S_n$. Let now $t = (z_1, \ldots, z_n) \in \mathbb{T}^n$. Applying the formulas in [1], the Plancherel density $\mu_{G|T}$ is given by

$$\begin{aligned} \mu_{G|T} &= const \cdot \prod_{i<j} \left| \frac{1 - z_j z_i^{-1}}{1 - z_j z_i^{-1}/q} \right|^2 && (11) \\ &= const \cdot \prod_{0<\alpha} \left| \frac{1-\alpha(t)}{1-\alpha(t)/q} \right|^2 && (12) \\ &= const \cdot \prod_{\alpha} 1/\Gamma(\alpha(t)) && (13) \end{aligned}$$

where $q = q_F$ and α is a root of the Langlands dual group $\mathrm{GL}(n, \mathbb{C})$ so that $\alpha_{ij}(t) = z_i/z_j$.

For $\mathrm{GL}(n)$, one connected component in the tempered dual is the compact orbifold $\mathbb{T}^n/S_n$, the symmetric product of n circles. On this component we have the Macdonald formula [11]:

$$d\mu(\omega_\lambda) = const. \cdot d\lambda / \prod_{\alpha} \Gamma(i\lambda(\alpha^\vee))$$

the product over all roots α where $\alpha^\vee$ is the coroot. This formula is a very special case of our formula for $\mathrm{GL}(n)$.

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