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Montaldi, James

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Multiplicities of Critical Points of Invariant Functions

James Montaldi

Introduction

The purpose of this expository article is to describe in an elementary and homogeneous manner, the relationship between the geometric and algebraic multiplicities of isolated critical points of holomorphic functions. In particular, I am interested in the setting where the function is invariant under some group action. The emphasis is on functions invariant under actions of finite groups as very little is known if the group is not finite. Most of the results described here are already explicitly in the literature; the only small extension is to functions that are not invariant, but *equivariant* under the action of a group G: a function f satisfying $f(gx) = \vartheta(g)f(x)$ for some homomorphism $\vartheta : G \to \mathbf{C}^*$. The results (in Section 7) on the multiplicity of critical points of homogeneous functions invariant under $\mathbf{C}^*$ are also new. *Caveat*: I will say nothing about the other important invariant of critical points of functions: the Milnor fibre. For this, the interested reader should refer to the original material, namely [8], [20] (for finite group actions), [10] (for $\mathbf{C}^*$-actions) and [9, 13] (for the weighted homogeneous cases).

This article grew out of a series of lectures I gave at the ICMSC in July 1992, preceding the conference. I would particularly like to thank Maria Ruas for inviting me to give the lectures, for organizing a wonderful conference, and finally for encouraging me to write up the lectures for publication in these proceedings. I would also like to thank Mark Roberts and Duco van Straten for the many stimulating discussions I have had with them on the material in these lectures.

TERMINOLOGY AND NOTATION All functions and differential forms will be assumed to be holomorphic, and although we will usually say, "let f be a function on $\mathbf{C}^n$", we will mean that f is defined in a neighbourhood of 0 in $\mathbf{C}^n$. All the actions we consider are linear; consequently the terms representation and action are interchangeable. The motivation for considering only linear actions is that the results we are interested in here are purely local, and locally, near a fixed point, any action can be linearized.

We assume a basic familiarity with the representation theory of finite groups, see for example Serre's book [17]. For a representation V of the group G, we write $[V]$ for its image in the representation ring of G. The representation ring is isomorphic to the ring of virtual characters (the ring generated by the characters of G), and consequently we use the symbol $[V]$ as a character. Thus $[V](g) = \mathrm{tr}(g;V)$, for each $g \in G$. From standard representation theory, $[V \oplus W] = [V] + [W]$ and $[V \otimes W] = [V][W]$.

If S is a finite set acted upon by G, then $[S]$ denotes the associated representation (or rather its image in the representation ring of G); that is, the action induced on $\mathrm{Map}(S, \mathbf{C})$, or again, on the vector space $\{\sum_{s\in S} \lambda_s s \mid \lambda_s \in \mathbf{C}\}$, with $g \cdot (\sum \lambda_s s) = \sum \lambda_s g \cdot s$. A particular case is the action of a finite group G on itself by say left multiplication, giving the *regular representation* $\mathbf{C}.G$ and its character $[G]$. As is well-known, $[G](g) = 0$ for $g \neq e$, and $[G](e) = |G|$.

Note that a 1-dimensional representation can be identified with its character, so we need not distinguish $[\vartheta]$ from ϑ.

If G acts on V, and $g \in G$, then V^g denotes the subspace fixed pointwise by g. The fixed point subspace for the entire group G is denoted V^G.

CONTENTS Section 1 describes the basic method used from commutative/homological algebra to relate algebraic and geometric multiplicities, namely deformations of complexes of modules or sheaves. Section 2 applies this method to isolated critical points of holomorphic functions. Section 3 describes some elementary invariant theory needed to understand critical points of invariant functions, which are the subject of Section 4. Section 5 describes some recent results on critical points of functions that are invariant under actions of $\mathbf{C}^*$. This is the only case where results are known on multiplicities of critical points of functions invariant under groups that are not finite. Finally, Sections 6 and 7 show how one can calculate the algebraic multiplicities of critical points of weighted homogeneous functions, both general functions and invariant functions.

1 Complexes and their deformations

Our proof of the results relating algebraic and geometric multiplicities of critical points of functions in these notes is through complexes of R-modules (or sheaves, if the reader prefers), where R is the ring of (germs at 0 of) analytic functions on $\mathbf{C}^n$. There are two central abstract results that we use: one on deformations of complexes and the other (the acyclicity lemma) on exactness. Before giving these, we first give a brief description of how the two multiplicities are related. Note that similar arguments can be used for counting multipicities of other geometric phenomena.

The algebraic definition of multiplicity is as the dimension (over $\mathbf{C}$) of some R-module $\mathcal{M}_0$ that depends on the geometry in question, here the critical point of an analytic function. This module will be finite dimensional, which is equivalent to it being supported at an isolated point (by the Nullstellensatz). The aim is to show that if the function in question is perturbed, the resulting perturbation $\mathcal{M}_t$ of the module $\mathcal{M}_0$ is such that its dimension remains constant, or rather the sum of the dimensions of the constituent parts remains constant.

This is made precise by including the deformation parameter $t \in C$. Then $\mathcal{M}$ is the family of $\mathcal{M}_t$ as t varies, and is an $S = R\{t\}$-module. If $\mathcal{M}_0$ is finite dimensional, then $\mathcal{M}$ is a finitely generated $\mathbf{C}\{t\}$-module. The crucial point to be established is that it is a *free* $\mathbf{C}\{t\}$-module.

To see this, let $C \subset \mathbf{C}^n \times \mathbf{C}$ be the support[1] of the S-module $\mathcal{M}$, and let $\pi : C \to \mathbf{C}$, $(x,t) \mapsto t$ be the restriction to C of the Cartesian projection which is finite-to-1. If we consider $\mathcal{M}$ as a $\mathbf{C}\{t\}$-module, we can write $\pi_*(\mathcal{M})$. The relationship between $\mathcal{M}$ and $\pi_*(\mathcal{M})$ is given by

$$\pi_*(\mathcal{M})_t = \oplus_{(x,t)\in\pi^{-1}(t)} \mathcal{M}_{(x,t)}, \tag{1.1}$$

for each $t \in \mathbf{C}$. Thus, if $\pi_*(\mathcal{M})$ is a free module, then

$$r(t) = \sum_{(x,t)\in\pi^{-1}(t)} \dim_{\mathbf{C}} \mathcal{M}_{(x,t)}$$

is constant. From here on, we will write $\dim \mathcal{M}$ rather than $\dim_{\mathbf{C}} \mathcal{M}$.

The geometric step is to interpret $\dim \mathcal{M}_{(x,t)}$ when this is as simple as possible; for us at non-degenerate, or generic, critical points.

In the above discussion, and everywhere else, all constructions such as C should really be interpreted as germs to ensure that we only consider critical points of f_t

[1] If M is an R-module, then $x \in \operatorname{supp}(M)$ if the localisation of M at x is non-zero. In terms of sheaves, this means that for any neighbourhood U of x in $\mathbf{C}^n$, the restriction of M to U is non-zero

that approach 0 as $t \to 0$. This will always be tacitly assumed, and I make no further reference to this point.

DEFINITION Let R be a commutative ring with unit — for example the ring $\mathbf{C}\{x_1,\ldots,x_n\}$ of analytic functions or its subring of invariants under a given group action — and let

$$\mathbf{K}_0: \quad 0 \to M_0^0 \xrightarrow{d} M_0^1 \xrightarrow{d} \cdots \xrightarrow{d} M_0^{N-1} \xrightarrow{d} M_0^N \to 0$$

be a complex of finitely generated R-modules. Let $S = R\{t\}$ (so $R \simeq S/tS$). A *deformation of* $\mathbf{K}_0$ *over* $\mathbf{C}$ is a complex $\mathbf{K}$ of finitely generated S-modules

$$\mathbf{K}: \quad 0 \to M^0 \xrightarrow{d} M^1 \xrightarrow{d} \cdots \xrightarrow{d} M^{N-1} \xrightarrow{d} M^N \to 0,$$

where for each $i = 0,\ldots,n$ we have an exact sequence

$$0 \to M^i \xrightarrow{t} M^i \to M_0^i \to 0,$$

where $\xrightarrow{t}$ is multiplication by t, and the differentials d commute with multiplication by t. In particular, this implies that each M^i is torsion-free as a $\mathbf{C}\{t\}$-module. There is thus an exact sequence of complexes:

$$0 \to \mathbf{K} \xrightarrow{t} \mathbf{K} \to \mathbf{K}_0 \to 0.$$

This short exact sequence of complexes gives a long exact sequence in cohomology as follows:

$$\begin{aligned} 0 \to H^0 \xrightarrow{t} H^0 \longrightarrow H_0^0 \longrightarrow H^1 \xrightarrow{t} \cdots \\ \cdots \to H_0^{i-1} \to H^i \xrightarrow{t} H^i \to H_0^i \to H^{i+1} \xrightarrow{t} \cdots \\ \cdots \to H_0^{N-1} \to H^N \xrightarrow{t} H^N \to H_0^N \to 0, \end{aligned}$$

where $H^i = H^i(\mathbf{K})$, and $H_0^i = H^i(\mathbf{K}_0)$. The long exact sequence is obtained by an easy diagram chase (if the reader is unfamiliar with this, he should remind himself of the simplicity of the argument; the map $H_0^{i-1} \to H^i$ is essentially the differential d of the complexes).

Lemma 1.1 *Let* $\mathbf{K}_0$ *be a complex of R-modules, such that all cohomology groups are finite dimensional vector spaces. Let* $\mathbf{K}$ *be a deformation of the complex* $\mathbf{K}_0$*, depending on the parameter* $t \in \mathbf{C}$*. Then,*

1. *the* H^i *are finitely generated* $\mathbf{C}\{t\}$*-modules;*

2. $H_0^i = 0$ *implies* $H^i = 0$ *(though not conversely!);*

3. $H_0^{N-1} = 0$ *implies* H^N *is a free* $\mathbf{C}\{t\}$*-module.*

PROOF: 1. This follows from the preparation theorem (see for example [5]), as H^i is a finitely generated S-module, and

$$\dim(H^i/tH^i) \le \dim(H_0^i) < \infty.$$

2. Suppose $H_0^i = 0$. Then there is an exact sequence $H^i \xrightarrow{t} H^i \to 0$, so that $H^i/tH^i = 0$. It follows from Nakayama's Lemma that $H^i = 0$.

3. This follows immediately from the last row of the long exact sequence above, for then $H^N \xrightarrow{t} H^N$ is injective so H^N is a torsion free $\mathbf{C}\{t\}$-module, and hence free. □

Remark 1.2 One can show more, namely that in the deformation the Euler characteristic of the complex is constant: for each $s \in \mathbf{C}$

$$\chi(\mathbf{K}_s) = \chi(\mathbf{K}_0),$$

where $\mathbf{K}_s$ is the complex induced from $\mathbf{K}$ by putting $M_s^i = M^i/(t-s)M$.

To see this, note that since H^i is finitely generated over $\mathbf{C}\{t\}$, it is the direct sum of a torsion module (which is necessarily a finite dimensional vector space) and a free module. Write accordingly

$$H^i \cong T^i \oplus F^i.$$

Let α_i be the number of generators of T^i, and β_i the number of generators of F^i, so $\beta_i = \mathrm{rk}(F^i) = \mathrm{rk}(H^i)$. Note that multiplication by t respects the decomposition $H^i \cong T^i \oplus F^i$, and $\alpha_i = \dim\ker[T^i \xrightarrow{t} T^i] = \dim\mathrm{coker}[T^i \xrightarrow{t} T^i]$, while $\beta_i = \dim\mathrm{coker}[F^i \xrightarrow{t} F^i]$ and $\ker[F^i \xrightarrow{t} F^i] = 0$. From the long exact sequence in cohomology given above, it follows that

$$\dim(H_0^i) = \alpha_i + \beta_i + \alpha_{i+1}. \tag{1.2}$$

Consequently,

$$\sum_{i=0}^{N} (-1)^i \dim(H_0^i) = \sum_{i=0}^{N} (-1)^i \beta_i,$$

(note that $\alpha_0 = 0$ since by the long exact sequence, H^0 is torsion free). Thus, the Euler characteristic of the complex $\mathbf{K}_0$ depends only on the free part of $H(\mathbf{K})$. This will also be true for any other specialization $\mathbf{K}_s$.

We now turn to the acyclicity lemma. Suppose now that

$$\mathbf{K}: \quad 0 \to M^0 \xrightarrow{d} M^1 \xrightarrow{d} \cdots \xrightarrow{d} M^{N-1} \xrightarrow{d} M^N \to 0, \tag{1.3}$$

is a complex of *free* finitely generated R-modules. The cohomology groups of this complex $H^i(\mathbf{K})$ are also R-modules, since d is R-linear. Note that by the Hilbert Nullstellensatz, the hypothesis that the complex $\mathbf{K}_0$ have finite dimensional cohomology groups is equivalent to their support being a finite set.

Lemma 1.3 (Acyclicity Lemma — Basic version) *Suppose the cohomology of the complex (1.3) of free R-modules is supported on an algebraic subset of codimension c, then*

$$H^0 = H^1 = \cdots = H^{c-1} = 0,$$

where $H^i = H^i(\mathbf{K})$.

This famous lemma is due to Peskine and Szpiro. For an elementary self-contained proof see the appendix of [10], and for a more detailed account, see the recent book of J. Strooker [19]. In our use of this lemma, the cohomology of (1.3) will be supported at an isolated point, and $N = n$, so we will have that all cohomology groups except H^n vanish — that is, the complex is *acyclic*.

More general versions of the Acyclicity Lemma replace the freeness hypothesis with one on the depth of the R-modules M^i. That this is the “correct” hypothesis is (hopefully) made clear in [10].

2 Isolated critical points

We are interested principally in two invariants associated to isolated critical points of holomorphic functions. They are the geometric and algebraic multiplicities, denoted μ_{geom} and μ_{alg} respectively. It was shown by Milnor [8] and Palamodov [14] that in fact these are equal. We concentrate on Palamodov’s algebraic/geometric proof; Milnor’s proof is more differentio-topological in nature, relying on the degree of the gradient of the given function.

A *1-parameter deformation* of a function $f(x)$ on V is a function $F(x,t)$, $t \in \mathbf{C}$ such that for each $x \in V$, $F(x,0) = f(x)$. The deformed function $F(\cdot,t)$ is also denoted f_t. All our deformations will be 1-parameter deformations, although it is seldom made explicit. A critical point x of a function f is said to be *non-degenerate* if the second differential of f at x is a non-degenerate quadratic form. We say a function f is non-degenerate if all critical points are non-degenerate. The following result is of central importance.

Proposition 2.1 *Suppose f has an isolated critical point. Then there are deformations F of f with the property that for $t \neq 0$, all the critical points of f_t are non-degenerate.*

This is proved by considering the explicit n-parameter deformation $F : \mathbf{C}^n \times (\mathbf{C}^n)^* \to \mathbf{C}$, $(x,a) \mapsto f_a(x) = f(x) - a(x)$. Then the "catastrophe set" $C(F)$ (those pairs (x,a) corresponding to critical points) is an n-dimensional submanifold of $\mathbf{C}^n \times (\mathbf{C}^n)^*$. Singularities of the projection $C(F) \to (\mathbf{C}^n)^*$ correspond to degenerate singular points. Moreover, the set of singular values of an analytic map is contained in some hypersurface, here $H \subset (\mathbf{C}^n)^*$. To find a non-degenerate deformation, it suffices to take any curve C in $(\mathbf{C}^n)^*$ such that $C \cap H = \{0\}$.

MULTIPLICITIES Let $f : V \to \mathbf{C}$ have an isolated critical point at the origin ($V = \mathbf{C}^n$). Then by the proposition above, there are deformations $F(x,t)$ of f with only non-degenerate critical points in a neighbourhood of 0. The number of such critical points is the *geometric multiplicity* of the critical point of f at 0, denoted $\mu_{\text{geom}} = \mu_{\text{geom}}(f,0)$. The fact that this is independent of the non-degenerate deformation F can be proved directly, but also follows from the results below.

The standard definition of *algebraic multiplicity* is:

$$\mu_{\text{alg}} = \mu_{\text{alg}}(f,0) := \dim_{\mathbf{C}} \left(\frac{R}{Jf} \right),$$

where R is the ring of germs at the origin of analytic functions, and Jf denotes the Jacobian ideal, the ideal generated by the n partial derivatives of f. However, we are going to use an alternative expression for this invariant using differential forms.

Let f have an isolated critical point at x. Define the *multiplicity module* to be the R-module

$$\mathcal{M}(f,0) = \frac{\Omega^n_V}{df \wedge \Omega^{n-1}_V},$$

where Ω^p_V is the R-module of analytic p-forms on V.

Proposition 2.2 *Let $f : V \to \mathbf{C}$ have an isolated critical point at 0, then*

$$\mu_{\text{alg}}(f,0) = \dim_{\mathbf{C}} \mathcal{M}(f,0).$$

PROOF: Indeed more is true: as R-modules, $\mathcal{M}(f)$ and R/Jf are isomorphic. The proof is merely an observation: there is an isomorphism of R-modules

$$\begin{array}{rcl} \psi : R & \to & \Omega^n_V \\ h & \mapsto & h\,dx_1 \wedge \ldots \wedge dx_n, \end{array}$$

(depending of course on a choice of coordinates on V), and under this isomorphism, $\psi(\partial f/\partial x_i) = (-1)^{i-1} df \wedge \widehat{dx_i}$, where $\widehat{dx_i} \in \Omega^{n-1}$ denotes the form $dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n$. Thus ψ induces an isomorphism $R/Jf \to \mathcal{M}(f)$. □

Theorem 2.3 (Milnor [8], Palamodov [14]) *Let $f: V \to \mathbf{C}$ have an isolated critical point at the origin. Then $\mu_{\text{alg}} = \mu_{\text{geom}}$.*

PROOF: Consider the complex of differential forms:

$$0 \to R \to \Omega^1_V \xrightarrow{df\wedge} \Omega^2_V \to \cdots \to \Omega^{n-1}_V \xrightarrow{df\wedge} \Omega^n_V \to 0, \tag{2.1}$$

where the differentials are given by $\alpha \mapsto df \wedge \alpha$, for $\alpha \in \Omega^p$. Given a deformation F of f, this complex has a natural deformation as follows. Let $S = R\{t\}$ and define the S-modules of *relative differential forms*:

$$\Omega^p_{V\times\mathbf{C}/\mathbf{C}} = \frac{\Omega^p_{V\times\mathbf{C}}}{dt \wedge \Omega^{p-1}_{V\times\mathbf{C}}}.$$

This module is isomorphic to the module $\Omega^p_V \otimes_{\mathbf{C}} \mathbf{C}\{t\}$ — forms on V that are parametrized by $t \in \mathbf{C}$. The deformation F thus defines a map

$$dF\wedge : \Omega^p_{V\times\mathbf{C}/\mathbf{C}} \to \Omega^{p+1}_{V\times\mathbf{C}/\mathbf{C}}$$

which corresponds to the differential of F with respect to all but the t variable. Clearly we have $\Omega^p_{V\times\mathbf{C}/\mathbf{C}}/t\Omega^p_{V\times\mathbf{C}/\mathbf{C}} \simeq \Omega^p_V$. Consequently, there is a short exact sequence of complexes:

$$0 \to (\Omega^{\cdot}_{V\times\mathbf{C}/\mathbf{C}}, dF\wedge) \xrightarrow{t} (\Omega^{\cdot}_{V\times\mathbf{C}/\mathbf{C}}, dF\wedge) \to (\Omega^{\cdot}_V, df\wedge) \to 0.$$

Denote $H^n(\Omega^{\cdot}_{V\times\mathbf{C}/\mathbf{C}}, dF\wedge)$ by $\mathcal{M}(F)$.

Since f has an isolated critical point at 0, the cohomology of the complex $(\Omega^{\cdot}, df\wedge)$ is supported at 0. For, if U is a contractible open set away from 0 coordinates can be chosen on U so that $df = dx_1$, and then exactness on U is clear. It follows from the acyclicity lemma that $H^i_0 := H^i(\Omega^{\cdot}, df\wedge) = 0$ for $i < n$, and by definition $H^n_0 := H^n(\Omega^{\cdot}, df\wedge) = \mathcal{M}(f)$.

It follows from Lemma 1.1(3) that $\mathcal{M}(F)$ is a free $\mathbf{C}\{t\}$-module.

It remains to show that if F is a non-degenerate deformation, then for $t \neq 0$, $\dim \mathcal{M}(f_t) = \mu_{\text{geom}}$. Now,

$$\dim \mathcal{M}(f_t) = \sum_{x \in C(f_t)} \dim \mathcal{M}(f_t, x),$$

so we reduce to a local calculation in a neighbourhood of a non-degenerate critical point. By the Morse lemma, coordinates can be chosen locally such that $f_t(u_1, \ldots, u_n) = \sum_i u_i^2$, and so $\mathcal{M}(f_t, u_i = 0) \cong \mathbf{C}$ as required. □

Remark 2.4 (i) If the deformation fails to be non-degenerate, the same proof implies that

$$\mu_{\text{alg}}(f_0,0)=\sum_{x\in C(f_t)}\mu_{\text{alg}}(f_t,x).$$

This can be interpreted as saying that μ_{alg} defines a 'good' notion of multiplicity. We will see below that there are instances where the allowed deformations are never non-degenerate.

(ii) This proof is isomorphic to the proof that Palamodov gave [14]. He considered the Koszul complex on the generators of Jf:

$$\mathbf{K}(Jf):0\to K_n\to K_{n-1}\to\cdots\to K_1\to K_0\to 0,$$

where K_p is the free R-module generated by $\{e_{i_1}\wedge e_{i_2}\wedge\ldots\wedge e_{i_p}\}$, with $1\le i_1\le i_2\le\ldots\le i_p\le n$, which is therefore of rank $\binom{n}{p}$. The differentials in this complex are the R-homomorphisms generated by

$$\begin{aligned} d:K_p &\to K_{p-1},\\ \bigwedge_{i\in I}e_i &\mapsto \sum_{k\in I}(-1)^k\frac{\partial f}{\partial x_k}\bigwedge_{i\in I\setminus\{k\}}e_i, \end{aligned}$$

where I is any index set of length p.

(iii) The theorem was also proved by Milnor, using techniques that are more topological. Briefly, let $\nabla f:\mathbf{C}^n\to\mathbf{C}^n$ be the "holomorphic gradient" of f:

$$\nabla f(x)=\left(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\right).$$

Clearly, critical points of f are zeros of ∇f, and a critical point is non-degenerate if and only if the corresponding zero of ∇f is simple. Clearly $\mu_{\text{geom}}(f,0)$ is the number of solutions of the equation $\nabla f_t=0$, that is the multiplicity of the zero of ∇f at 0. It remains to show that the multiplicity of an isolated zero of a map $g:\mathbf{C}^n\to\mathbf{C}^n$ is given by $\dim(R/\langle g_1,\ldots,g_n\rangle)$. This Milnor does by topological arguments involving the degree of ∇f in [8, Appendix B].

3 Invariants

Let G be a finite group acting linearly on $V=\mathbf{C}^n$. If necessary, we will make explicit the representation $\rho:G\to\mathrm{GL}_n(\mathbf{C})$. For any subgroup $H<G$ we denote by V^H or $\mathrm{Fix}(H;V)$ the set of points of V fixed by every element of H:

$$V^H=\mathrm{Fix}(H;V)=\{x\in V\mid h\cdot x=x,\forall h\in H\}.$$

Being the intersection of eigenspaces of the elements of H, V^H is a linear subspace. Recall the *trace formula*:

$$\dim V^H = \frac{1}{|H|}\sum_{h\in H}\mathrm{tr}(h;V). \tag{3.1}$$

Here $\mathrm{tr}(h;V)$ is the trace of the element h as it acts on V.

The action of G on $\mathbf{C}^n$ induces an action on R, by

$$g\cdot f = f\circ g^{-1}.$$

(The inverse power is just to ensure the action is indeed a homomorphism: $(gh)\cdot f = g\cdot(h\cdot f)$.) The G-action on V also induces an action on the modules of differential forms:

$$g\cdot\omega = (g^{-1})^*\omega. \tag{3.2}$$

That is, $(g\cdot\omega)_x(v_1,\dots,v_p) = \omega_{g^{-1}x}(g^{-1}v_1,\dots,g^{-1}v_p)$, where $v_1,\dots,v_p$ are tangent vectors at gx. This action is compatible with exterior differentiation: $d(g\cdot\omega) = g\cdot d\omega$. The action on the module of vector fields is given by $(g\cdot v) = g_*v$.

Since the action of G is linear, then for each $g\in G$ there is a $\lambda(g)\in\mathbf{C}^*$ such that

$$g\cdot dx_1\wedge\ldots\wedge dx_n = \lambda(g)dx_1\wedge\ldots\wedge dx_n.$$

Then $\lambda = \bigwedge^n(V^*)$ is a 1-dimensional representation of G, and the isomorphism $\psi : R\to\Omega^n$ of Proposition 2.2 provides an isomorphism of RG-modules,

$$\lambda\otimes_G R \xrightarrow{\cong} \Omega^n.$$

A function f is said to be *invariant* if $g\cdot f = f$ for all $g\in G$, and more generally *equivariant* if there is a homomorphism $\vartheta : G\to\mathbf{C}^*$ such that

$$f(g\cdot x) = \vartheta(g)f(x). \tag{3.3}$$

In other words, $g\cdot f = \vartheta(g^{-1})f = \vartheta(g)^{-1}f$. The image of ϑ is contained in the cyclic group $C_{|G|}$ of order $|G|$. Such a homomorphism ϑ is called an *abelian character* of the group G; we also refer to it as the *twist* of the equivariant function f.

The set of all invariant functions is a subring of R, denoted R^G, and for a fixed homomorphism ϑ, the set of ϑ-equivariant functions f forms a module over R^G, which we denote R^ϑ. Thus, $R^G = R^1$. Such equivariants are classically called *covariants* or *semi-invariants*, and the modules R^ϑ are called *modules of covariants*.

Let $f : V\to\mathbf{C}$ be an equivariant function, with twist ϑ. Then its differential df is an *equivariant form* since one finds on differentiating (3.3), that $g\cdot df = \vartheta(g)^{-1}df$.

Any finite group has the property that every representation is completely reducible. That is, if V is a representation and $W \subset V$ an invariant subspace, then there is a complementary invariant subspace W' with $V = W \oplus W'$. Moreover, the irreducible representations are all finite dimensional. In particular, R is a G-representation so splits as an infinite sum (or product) of irreducible subspaces. Let χ be the character of a particular irreducible representation of G. Then collecting all the irreducible subspaces of R that have character χ we obtain a subset R^χ of R which, as is easy to see, is an R^G-module called the *isotypic component* of R with character χ. Thus one has the following direct sum decomposition:

$$R = \bigoplus_{\chi} R^\chi,$$

where the sum is over all characters χ of G. The modules R^ϑ for abelian characters are special cases of the R^χ just defined.

Example 3.1 Consider $G = C_N$ acting on $V = \mathbf{C}$ with its natural action, identifying C_N with the Nth roots of unity. The function $f(x) = x^k$ is equivariant, with $\vartheta_k(\omega) = \omega^k$ for each Nth root of unity ω, and we have the decomposition

$$R = \bigoplus_{k=0}^{N-1} R^{\vartheta_k}.$$

More generally, for an abelian group, all characters are abelian (that is, all irreducible representations are 1-dimensional), and one has that R is the direct sum of all the modules of equivariants.

QUOTIENT SPACE The ring of invariants defines the algebraic/analytic structure of the quotient space. However, this space has a more refined structure, namely that of a stratified space. The said stratification is by *orbit type*: the orbit type of a point $x \in V$ is defined to be the conjugacy class (G_x) of the isotropy subgroup $G_x \subset G$ of x. The subset of V consisting of points with isotropy conjugate to a given subgroup H of G is denoted $V^{(H)}$. It is a submanifold of V, and the collection of all such $V^{(H)}$ defines a stratification of V. More precisely, one should take the strata to be the connected components of the $V^{(H)}$. Furthermore, the quotient map $\pi : V \to V/G$ is of constant rank when restricted to each stratum, and the images of the strata define the stratification of V/G by orbit type.

Since G is finite, each stratum is an open subset of V^H for some H. Let $x \in V^{(H)}$, with $G_x = H$. Then there exists an H-invariant neighbourhood S_x of x in V, such that any H-invariant function on S_x can be extended in a unique fashion to a G-invariant function on the image of S_x under the group action. This enables us to localize the study of critical points of invariant functions. Note furthermore that $S_x \cap V^{(H)} = S_x^H$.

There is an important result — known as the *principle of symmetric criticality* — which states that an invariant function f on V has a critical point at x iff the restriction of f to V^{G_x} has a critical point at x. Since π is a submersion on each stratum, this is in turn equivalent to f having a stratified critical point at $[x] \in V/G$. This correspondence is taken advantage of particularly by Bruce and Roberts [3].

The principal of symmetric criticality is proved by noting that if f is invariant, then df_x is fixed by G_x, so is (co)tangent to V^{G_x}. Thus if df_x restricted to V^{G_x} vanishes, then so does df_x.

REFLEXION GROUPS Consider a representation $\rho : G \to \mathrm{GL}(V)$. This is said to be *generated by pseudo-reflexions* if there is a set S of generators of G with each element $s \in S$ having the property that $\mathrm{Fix}(\rho(s);V)$ has codimension 1 in V. If a generator is of order 2 then it is called a reflexion. The action in the previous example is generated by pseudo-reflexions.

It is well-known that the ring of invariants for a pseudo-reflexion group action is a polynomial ring with no relations between the generators. Moreover, each of the modules of equivariants is a free module over this ring, and $R \simeq R^G \otimes_{\mathbf{C}} \mathbf{C}.G$, where $\mathbf{C}.G$ is the regular representation of G. See for example, Chapter V.5 of [2].

REAL ACTIONS Let G be a finite group acting (linearly) on $\mathbf{R}^n$. Then there is a positive definite quadratic form on $\mathbf{R}^n$ invariant under the group action. To see this, let Q be any positive definite quadratic form on $\mathbf{R}^n$, and let $\overline{Q}$ be the average of Q over the group:

$$\overline{Q}(x) = \frac{1}{|G|}\sum_{g\in G} Q(gx).$$

Then $\overline{Q}$ is also positive definite, and so in particular non-degenerate.

Now consider the complexification of the action on $\mathbf{R}^n$. This action also has a non-degenerate (quadratic) invariant function, namely the same quadratic form $\overline{Q}$ considered as a function on $\mathbf{C}^n$. An action on $\mathbf{C}^n$ is said to be a *real action* if it is the complexification of an action on $\mathbf{R}^n$. The existence of a non-degenerate invariant quadratic form in fact characterizes the real actions, as was shown by G. Schwarz [16, Proposition 5.7].

EQUIVARIANT COMPLEXES We describe briefly the effect of an action of a finite group G on the material in Section 1. If R is a ring and G a group, one says that an R-module M is an RG-module if it carries an R-linear action of the group G; that is, if there is a homomorphism of G to the group of automorphisms of M. An

RG-complex is a complex of RG-modules such that the differentials in the complex are equivariant: $d(g\cdot m) = g\cdot(dm)$. It follows then that the cohomology groups are also RG-modules. Furthermore, if there is a short exact sequence of RG-complexes, then the maps in the associated long exact sequence in cohomology commute with the G-actions.

One defines the equivariant Euler characteristic of an RG-complex $\mathbf{K}$ to be

$$\sum_i (-1)^i [H^i(\mathbf{K})].$$

This enjoys the usual properties of Euler characteristics; in particular, if the RG-modules M^i in the complex are finite dimensional, then the equivariant Euler characteristic is equal to $\sum_i (-1)^i [M^i]$.

4 Multiplicities of invariant critical points: Finite Groups

Consider a function f on $\mathbf{C}^n$ with an isolated critical point, and suppose $f \in R^\vartheta$ where ϑ is an abelian character of a finite group G. Let f_t be a 1-parameter deformation of f, with $f_t \in R^\vartheta$ for all t. Note that if $x \in \mathbf{C}^n$ is a critical point of f_t, then so is $g \cdot x$ for all $g \in G$. The results described in this section are mostly due to Mark Roberts [15]; he only deals with the case that f is invariant under a real representation of G, but the extension to the general case we treat here is more or less straightforward.

There are three representations of interest. First, we have G acting on R, and if $f \in R^\vartheta$ is an equivariant, then G preserves the Jacobian ideal $Jf \subset R$. There is therefore an induced action on the R-module R/Jf. Since f has an isolated critical point, this module is a finite dimensional representation of G. Secondly, G acts on the multiplicity module $\mathcal{M}(f)$, and thirdly there is the action on the critical locus $C(f_t)$, and its associated permutation representation. The isomorphism ψ of Proposition 2.2 shows that as RG-modules

$$\mathcal{M}(f) \cong \lambda \otimes (R/Jf). \tag{4.1}$$

The remaining problem is to relate the representations $\mathcal{M}(f)$ or (R/Jf) to the G-action on the critical locus $C(f_t)$.

GENERIC MULTIPLICITY It is important to note that most modules of equivariants contain no non-degenerate functions. As a simple example, consider $G = C_N$ ($N > 1$) acting as in Example 3.1. Any element of R^{ϑ_k} $(0 \le k \le N-1)$ is a function of the form $\sum_{i\geq 0} a_i x^{k+iN}$. For example, for $k=0$, R^G contains non-degenerate functions if and only if $N=2$, while if $k\geq 1$ then $f\in R^{\vartheta_k}$ is never non-degenerate.

Still worse is the possibility that certain modules of equivariants contain no functions with isolated critical points. A simple necessary, though not sufficient, condition for the existence of isolated critical points is that $\dim V^g \leq \frac{1}{2}\dim V$, for all $g\in G$ with $\vartheta(g)\neq 1$ This is because if $\vartheta(g)\neq 1$ then necessarily $f|_{V^g}=0$. Consequently, we can write $f=\sum_{i=1}^{a} x_i f_i$ for some functions f_i, where $x_1=\cdots=x_a=0$ is the equation for V^g. It is easy to see that if this sum has an isolated critical point in V^g then $a\geq n-a$.

The *generic multiplicity* is the local multiplicity of a critical point of an equivariant function that cannot be broken up under equivariant deformation of the function. The generic multiplicity depends purely on the local geometry of the action together with the twist ϑ, and the local geometry is best described by the stratification of V (or equivalently of V/G) by orbit type — see Section 3. Here we give this description for *invariant* functions. For the more general equivariant functions, matters are not so well understood.

Suppose $f: S_x\to \mathbf{C}$ is H-invariant with an isloated critical point at x. We perturb f to make it generic in two stages (which of course can be done simultaneously). To begin, we choose an H-invariant splitting $S_x\cong S_x^H\times T$ — note that $T^H=0$. Now, perturb f to f_1 by adding a function independent of T, so that the restriction of f_1 to S_x^H is non-degenerate. In a neighbourhood of each non-degenerate critical point $x_i\in S_x^H$ of f_1, we can apply the (equivariant) splitting lemma to write $f_1=\text{non.deg.}+h_i$, where non.deg. is a non degenerate function on S_x^H, and h_i is an H-invariant function on T. We are now reduced to the local problem of perturbing the $h_i: T\to\mathbf{C}$. *Problem*: for a given representation T of H, how simple can an isolated critical point be? The multiplicity or G-multiplicity, of such a critical point is the *generic multiplicity* in question. Note that since $T^H=0$ there are no linear invariant functions, so that the generic multiplicity is at least 1. For a given representation T, it is not hard to determine the generic multiplicity provided one knows the invariants of low degree. However, it would be nice to have geometrical or representation theoretic estimates for this generic multiplicity.

The results known at present are due to Schwarz [16], and Wall [21]. Schwarz proves that if the representation V of G is real then the representations T of the isotropy subgroups H that arise as above, are also real. For any real action there is an invariant non-degenerate quadratic form, and hence the generic multiplicity for a real representation is 1. Wall considers the case that $\dim V=2$, and there are no fixed point sets of codimension 1 (pseudo-reflexion hyperplanes). He produced a formula for the generic multiplicity in terms of the embedding dimension and the resolution of the quotient space V/G, see Remark 6.8 below. By the reduction procedure described above, Wall's results also apply to points on codimension 2 strata which do not lie on pseudo-reflexion hyperplanes.

It is easy to give the generic multiplicity for generic points on pseudo-reflexion hyperplanes. Here, $\dim T = 1$, and so the isotropy group is $H \cong C_N$ — the cyclic group of order N for some N. The generic invariant is then $f(x) = x^N$, and the generic multiplicity is thus $[\mathcal{M}(f)] = \vartheta_1 + \cdots + \vartheta_{N-1}$, a vector space of dimension $N-1$ with all 1-dimensional representations of C_N present except the trivial one.

It is an important open problem to find further estimates on generic multiplicity.

M. Roberts [15] uses techniques of equivariant jet bundles to prove the equivariant version of Proposition 2.1 for invariants of real actions. The more general version of Proposition 2.1 for invariants can be deduced from a theorem of Lê [7] on Morsifications of isolated critical points on analytic varieties. The analogous statement for equivariants does not follow from Lê's theorem, as they are not functions on the quotient space.

Example 4.1 Consider the cyclic group C_3 of order 3, acting in its natural representation on $\mathbf{C}$ (i.e. by ϑ_1: notation established in Example 3.1), and let $f_t(x) = x^6 - 2tx^3$. The action on $\mathbf{C}.dx$ is ϑ_2. For $t = 0$, we have $\mathcal{M}(f_0) = \mathcal{M}(f_0, 0) =$

$t = 0$ $\qquad$ $t \neq 0$

$\vartheta_2 \mathbf{C}\{x\}/\left\langle x^5 \right\rangle$. This has multiplicity 5; as a representation it is isomorphic to $\vartheta_2(2\vartheta_0 + 2\vartheta_1 + \vartheta_2) = 2\vartheta_0 + \vartheta_1 + 2\vartheta_2$. For $t \neq 0$, we have critical points at 0 and solutions x_1, x_2, x_3 of $x^3 = t$. Then $\mathcal{M}(f_t) = \mathcal{M}(f_t, 0) \oplus \oplus_{i=1}^3 \mathcal{M}(f_t, x_i)$. Let u_1 be a local coordinate about the point x_1, and u_2, u_3 its images under elements of the group, then $\pi_* \mathcal{M}(F)_t$ is given by

$$\mathcal{M}(f_t) = \frac{\mathbf{C}\{x\}}{\langle x^2 \rangle} dx \oplus \frac{\mathbf{C}\{u_1\}}{\langle u_1 \rangle} du_1 \oplus \frac{\mathbf{C}\{u_2\}}{\langle u_2 \rangle} du_2 \oplus \frac{\mathbf{C}\{u_3\}}{\langle u_3 \rangle} du_3.$$

As a representation, $[\mathbf{C}\{x\}/\left\langle x^2 \right\rangle] = \vartheta_0 + \vartheta_1$. Thus,

$$[\mathcal{M}(f_t)] = \vartheta_2(\vartheta_0 + \vartheta_1) + \vartheta_2[\{x_1, x_2, x_3\}] = (\vartheta_0 + \vartheta_2) + (\vartheta_0 + \vartheta_1 + \vartheta_2) = [\mathcal{M}(f_0)].$$

Thus, $[\mathcal{M}(f_t)] = [\mathcal{M}(f_0)]$, and provided we know that any "generic" invariant function has a critical point at 0 of multiplicity 2 (and more precisely of G-multiplicity $\vartheta_0 + \vartheta_2$), we can deduce the representation $[C(f_t)]$. More generally, of course, one has to deal with generic critical points with multiplicity not equal to 1 away from 0 too. We now proceed to prove in general that $[\mathcal{M}(f_t)] = [\mathcal{M}(f_0)]$.

Theorem 4.2 *Let $f = f_0$ be an equivariant function on V with an isolated critical point at 0, and f_t an equivariant deformation of f. Then, as representations of the finite group G,*

$$\mathcal{M}(f_t) \cong \mathcal{M}(f_0).$$

Moreover, if all critical points of f_t are non-degenerate, then

$$[C(f_t)] = \left[\frac{R}{Jf_0}\right] = \lambda^{-1}[\mathcal{M}(f_0)].$$

PROOF: Using the notation of Section 1, recall that $\pi_*\mathcal{M}(F)$ is a free $\mathbf{C}\{t\}$-module, and by the preparation theorem, it is generated as such by any basis for $\mathcal{M}(f_0)$. For any element $g \in G$, let $\mu_t(g)$ be the matrix representing g in the resulting basis of $\mathcal{M}(f_t)$. Then the entries of $\mu_t(g)$ are continuous in t. Since the set of characters of a finite group is finite, it follows that the representation $\mathcal{M}(f_t)$ is constant, up to conjugation.

It follows that $[R/Jf_t] = [R/Jf_0]$, so there remains to relate $[C(f_t)]$ with $[R/Jf_t]$. If all critical points of F_t are non-degenerate then

$$R/Jf_t \cong \oplus_{x\in C(f_t)} R/m_x,$$

where m_x is the maximal ideal of functions vanishing at x. The group G acts on the right hand side as a permutation of the generators $1_x \in (R/m_x)$, which coincides with the action on $C(f_t)$. □

Remark 4.3 (i) Wall proves in [20] that if f is invariant then

$$[\mathcal{M}(f)](g) = (-1)^{n-n(g)} \dim \mathcal{M}_g(f),$$

where $\mathcal{M}_g(f)$ is the multiplicity module of the restriction of f to V^g and $n(g) = \dim V^g$. His proof involves passing to the Milnor fibration so is outside the scope of these lectures. However, we do give a proof in the case of weighted homogeneous functions in Corollary 6.7.
(ii) The representation $[S]$ does not always determine the action of G on a finite set S. Thus one cannot in general read off the action of G on the critical locus of f_t from the theorem. To overcome this problem, Roberts [15] introduces a finer invariant $\rho(f)$ for invariant functions which depends on the multiplicities of the restriction of f to the fixed point subspaces V^H of V. He shows that for real actions, the invariant ρ does indeed determine the action of G on the critical locus C_t. In particular, if every fixed point subspace of V is of the form V^g for some g (as is the case for reflexion groups) then the representation $\mathcal{M}(f)$ determines the G-action on the critical locus of a generic deformation f_t of f. See [15] for details.

5 Multiplicities of invariant critical points: Reductive Groups

The title of this section is rather over-optimistic, for results on multiplicities of critical points of functions invariant under the action of reductive groups are only known for the group $\mathbf{C}^*$, or more generally for finite extensions thereof, see [10]. The first reason that the general reductive case is more difficult than the finite group case is that critical points are no longer isolated, since group orbits no longer consist of isolated points.

In this section, we describe some pertinent geometry of reductive group actions in general, and then proceed to give the known results for actions of $\mathbf{C}^*$.

INVARIANT THEORY Let V be a representation of a reductive group G. (Reductive means that every representation of G is completely reducible; an important class of reductive group is the complexification of compact Lie groups considered as real algebraic groups, [6].) Let R be the ring of invariant functions on V (polynomial or analytic, according to taste — or use).

The quotient space $Y = V/G$ has to be defined with care; it is not so straightforward for reductive groups as for finite groups (or compact groups in the topological setting) since not all orbits are closed. Thus, if the quotient space were defined as the set of orbits, then the natural topology would fail to be Hausdorff (or even T_1). As a set, the quotient space is therefore defined to be the set of *closed* orbits. It can be proved in general, that if $x \in V$, then the closure of the orbit $G \cdot x$ contains exactly one closed orbit. This fact is used to define the quotient map $\pi : V \to Y$, by letting $\pi(x)$ be this unique closed orbit in $\overline{G \cdot x}$. The analytic structure on Y is defined simply by the ring or invariants. This is justified by the fact that the invariant polynomials separate the closed orbits(see [11, Corollary 1.2]).

When dealing with invariants of representations V of reductive groups, an important geometrical construction is the *null cone*. This is defined by

$$\mathcal{Z} = \{x \in V \mid f(x) = f(0) \quad \text{for all invariant functions } f\}.$$

If Y is the quotient space, and $\pi : V \to Y$ the quotient map, then $\mathcal{Z} = \pi^{-1}(0)$. Clearly then, $x \in \mathcal{Z}$ if and only if $0 \in \overline{G \cdot x}$.

Example 5.1 Consider the action of the compact group $\mathbf{SO}(n)$ on the space of symmetric matrices of order n, acting by similarity: $g \cdot A = gAg^T$. It is well known that the invariants are generated by the symmetric functions in the eigenvalues. Thus, $\pi : V \to \mathbf{R}^n$. If we complexify, we have $\mathbf{SO}(n, \mathbf{C})$ acting in the same way on the space of symmetric complex matrices, with quotient map $\pi : V_{\mathbf{C}} \to \mathbf{C}^n$. The null cone is thus the space of symmetric matrices all of whose eigenvalues are zero.

For reductive groups one has two notions similar to that of invariant forms for finite groups. Firstly, the *invariant forms* themselves:

$$\underline{\Omega}^p = \{\omega \in \Omega_V^p \mid g^*\omega = \omega, \quad \forall g \in G\}.$$

These are finitely generated modules over the ring of invariants, and can be interpreted as coherent sheaves on the quotient space Y. However, in contrast to the case for finite groups, the invariant forms are in no sense differential forms on Y; for example, $\underline{\Omega}^p$ has full support on Y for $0 \le p \le \dim V$, and $\dim V > \dim Y$. The other class of forms are the *basic forms* which are more correctly interpreted as forms on Y. These are defined by

$$\Omega_Y^p = \{\omega \in \underline{\Omega}^p \mid i_\theta\omega = 0 \quad \forall \theta \in \mathcal{G}\}.$$

Here $\mathcal{G}$ is the Lie algebra of G, and $i_\theta\omega$ is the contraction of ω with the vector field on V associated to $\theta \in \mathcal{G}$. On the regular part of the G-action where π is a submersion, this means that $\omega \in \Omega_Y^p$ defines a well-defined differential p-form on Y — or rather on its smooth part. In [10], it is shown that for $\mathbf{C}^*$-actions, $\Omega_Y^p = j_*\Omega_U^p$, where $j : U \to Y$ is the inclusion of the smooth part of Y. (The same is probably true for other reductive groups, but I do not know a proof.)

ISOLATED CRITICAL POINTS For a reductive group action on V, critical points of invariant functions are almost never isolated (except for invariant non-degenerate quadratic forms in the case of a real action). However, the appropriate notion is that a critical point should be isolated in Y. Note in particular, that if 0 is an isolated critical point in Y of an invariant function f, then f may have critical points throughout the null cone $\mathcal{Z}$. This fact is at the root of the difficulty of the general reductive case. (Note that asking that a critical point in V should be isolated in Y makes sense: if x is a critical point of an invariant function f and the orbit through x is not closed, then any point $y \in \overline{G \cdot x}$ is also a critical point of f.)

Notice that if f is an invariant function, and ω an invariant or basic form, then $df \wedge \omega$ is also invariant or basic, respectively. Thus, associated to an invariant function there are now two complexes of interest:

$$(\underline{\Omega}^{\cdot}, df\wedge): \quad 0 \to R \to \underline{\Omega}^1 \xrightarrow{df\wedge} \underline{\Omega}^2 \xrightarrow{df\wedge} \cdots \xrightarrow{df\wedge} \underline{\Omega}^{n-1} \xrightarrow{df\wedge} \underline{\Omega}^n \to 0,$$

and

$$(\Omega_Y^{\cdot}, df\wedge): \quad 0 \to R \to \Omega_Y^1 \xrightarrow{df\wedge} \Omega_Y^2 \xrightarrow{df\wedge} \cdots \xrightarrow{df\wedge} \Omega_Y^{N-1} \xrightarrow{df\wedge} \Omega_Y^N \to 0,$$

where $N = \dim(Y)$ and R is the ring of invariants. One can show easily that away from the critical locus of f, these complexes are exact (because the complex (2.1) is exact). However, one cannot apply the acyclicity lemma as these R-modules are not free (and do not even satisfy the depth hypothesis for the generalized formulation of the acyclicity lemma). Furthermore, the first complex is too long to have any real chance of being acyclic.

Accordingly as there are two complexes, we can define two multiplicity modules:

$$\underline{\mathcal{M}}(f) = \frac{\underline{\Omega}^n}{df \wedge \underline{\Omega}^{n-1}} \quad \text{and} \quad \mathcal{M}_Y(f) = \frac{\Omega_Y^N}{df \wedge \Omega_Y^{N-1}},$$

and for a deformation F of f there are the corresponding relative versions $\underline{\mathcal{M}}(F)$ and $\mathcal{M}_Y(F)$.

Conjecture Let f_t be a family of G-invariant holomorphic functions on $\mathbf{C}^n$ for some reductive group G, and suppose that f_0 has an isolated critical point on the quotient space. Then the modules $\underline{\mathcal{M}}(F)$ and $\mathcal{M}_Y(F)$ are free.

In particular, this would imply the conjecture of M. Roberts, that for real actions of reductive groups, the dimension of the module $(R/Jf)^G$ is preserved in a deformation.

$\mathbf{C}^*$-ACTIONS Here we give a brief description of the principal results in [10], though to simplify matters we restrict our attention to the case of real $\mathbf{C}^*$-actions. The problem for general reductive groups (even complex tori) is still open. It should be emphasised that the results of [10] apply only to invariants, and not to the more general class of equivariants.

Let $\mathbf{C}^*$ act linearly on $\mathbf{C}^n$. Such an action can be diagonalized, so that $t \in \mathbf{C}^*$ acts on $\mathbf{C}^n$ via the matrix $\mathrm{diag}[t^{w_1}, \ldots, t^{w_n}]$. If all weights are positive or zero then the invariants are just the functions of the variables with weight zero; we therefore assume that there are some positive weights an some negative weights. Let a be the number of strictly positive weights, and b the number of strictly negative weights; so we suppose that $a, b > 0$. Let c be the multiplicity of the weight zero, so that $a+b+c = n$. We denote the positive weights by $\lambda_1, \ldots, \lambda_a$ and the negative weights by $\mu_1, \ldots, \mu_b$. Write $\mathbf{C}^n = \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}^c$, with corresponding coordinates $x_1, \ldots, x_a$, $y_1, \ldots, y_b$, $z_1 \ldots, z_c$.

It is easy to see that the null cone for this action is the union of two linear subspaces $\mathcal{Z} = \mathbf{C}^a \times \{0\} \times \{0\} \cup \{0\} \times \mathbf{C}^b \times \{0\} \subset \mathbf{C}^a \times \mathbf{C}^b \times \mathbf{C}^c$.

Example 5.2 Let $\mathbf{S}^1$ act linearly on $\mathbf{R}^n$. The irreducible real representations of $\mathbf{S}^1$ are either 1-dimensional — the trivial representation — or 2-dimensional:

$$\chi_r: \quad \theta \mapsto \begin{bmatrix} \cos(r\theta) & -\sin(r\theta) \\ \sin(r\theta) & \cos(r\theta) \end{bmatrix}.$$

This representation can be diagonalized over $\mathbf{C}$ to

$$t \mapsto \operatorname{diag}[t^r, t^{-r}].$$

Here t is *a priori* a complex number of modulus 1. However, any holomorphic function which is invariant under this $\mathbf{S}^1$-action on $\mathbf{C}^n$ is also invariant under the corresponding $\mathbf{C}^*$-action defined by allowing t to be any non-zero complex number. Such a $\mathbf{C}^*$-action is said to be *real*, and the real $\mathbf{C}^*$-actions are characterised by the fact that the set of weights is of the form $\{\pm\lambda_1, \dots, \pm\lambda_a\}$.

Suppose now that we have a real action of $\mathbf{C}^*$ on $V = \mathbf{C}^n$, and suppose f is an invariant function with an isolated critical point at 0 in Y. Then there are deformations f_t of f (remaining in the class of invariant functions) such that all group orbits of critical points are non-degenerate. We are therefore in a position to give a simple definition of the notion of *geometric multiplicity*: $\mu_{\text{geom}} = \mu_{\text{geom}}(f, 0)$ is the number of closed group orbits of critical points of f_t near 0 for $t \neq 0$ sufficiently small. (We say a group orbit of critical points is non-degenerate if the restriction of f to a transversal to the orbit has a non-degenerate critical point in the usual sense.)

Theorem 5.3 ([10]) *Suppose we have a real action of $\mathbf{C}^*$ on $\mathbf{C}^n$, and suppose that f is an invariant holomorphic function, with an isolated critical point on Y. Then*

$$\mu_{\text{geom}} = \dim_{\mathbf{C}} \underline{\mathcal{M}}(f).$$

The geometric multiplicity can be expressed in terms of the Jacobian ideal using the isomorphism ψ of Proposition 2.2, since for real actions ψ is $\mathbf{C}^*$-equivariant. Consequently, $\psi(R^G) = \underline{\Omega}^n$, and $\psi((Jf)^G) = df \wedge \underline{\Omega}^{n-1}$. Furthermore, $(R/Jf)^G \cong R^G/Jf^G$ (by elementary linear algebra), and Jf^G can be computed using equivariant vector fields: $Jf^G = \underline{\Theta}(f) = \Theta_Y(f)$. Here $\underline{\Theta}$ is the R^G-module of equivariant vector fields, and Θ_Y is the R^G module of vector fields on the quotient space tangent to the stratification by orbit type, see [16] and [10, Section 5]. Thus we obtain,

Corollary 5.4 *With the above hypotheses,*

$$\mu_{\text{geom}} = \dim_{\mathbf{C}} \left(\frac{R}{Jf}\right)^G = \dim \left(\frac{R^G}{\Theta_Y(f)}\right).$$

PROOF: Here we give an outline of the proof of the theorem as given in [10] (note that there is a change of notation: $\dim(V) = n+1$ in that paper, and V is denoted X). We will write $\underline{H}^i := H^i(\underline{\Omega}^{\cdot}, df\wedge)$ and $H^i_Y := H^i(\Omega^{\cdot}_Y, df\wedge)$. Thus, $\underline{\mathcal{M}}(f) = \underline{H}^n$ and $\mathcal{M}_Y(f) = H^{n-1}_Y$.

The first problem is that both $\underline{H}^n$ and $\underline{H}^{n-1}$ are non-zero; in fact $\dim(\underline{H}^n) - \dim(\underline{H}^{n-1}) = 1$ independently of f. Consequently, one cannot apply Euler characteristic arguments directly to this complex. However, contraction with the vector field ϑ generating the $\mathbf{C}^*$-action defines a homomorphism $\underline{H}^n \to H_Y^{n-1}$, which is an isomorphism (unless the fixed point space in V has codimension 2, in which case there is a 1-dimensional kernel). Thus, if H_Y^{n-1} deforms well in a deformation, then so does $\underline{H}^n$, as required.

Now, the hypotheses of the acyclicity lemma fail for the complex of basic forms $(\Omega^{\cdot}_Y, df\wedge)$, and indeed this complex fails to be acyclic. However, it is "quasi-acyclic" in that for $i < n-1$ the cohomology groups H_Y^i do not depend on f: provided f has an isolated critical point at $0 \in Y$ then

$$H_Y^i = \mathbf{C}, \quad \text{for } i = 3, 5, \ldots, n-2$$

while $H_Y^i = 0$ for all other $i < n-1$ (recall $\dim Y = n-1$.) This result depends on some calculations of the local cohomology of the modules of invariant and basic differential forms.

Let $F(x,t)$ be a deformation of f — as always, assumed to be $\mathbf{C}^*$-invariant. Following the proof of Theorem 2.3, we define the modules of relative basic forms:

$$\Omega^p_{Y\times\mathbf{C}/\mathbf{C}} = \frac{\Omega^p_{Y\times\mathbf{C}}}{dt\wedge\Omega^{p-1}_{Y\times\mathbf{C}}},$$

which is isomorphic to $\Omega^p_Y \otimes_{\mathbf{C}} \mathbf{C}\{t\}$. The function F defines a map $dF : \Omega^p_{Y\times\mathbf{C}/\mathbf{C}} \to \Omega^{p+1}_{Y\times\mathbf{C}/\mathbf{C}}$ as in the ordinary case, which gives rise to a short exact sequence of complexes

$$0 \to (\Omega^{\cdot}_{Y\times\mathbf{C}/\mathbf{C}}, dF\wedge) \xrightarrow{t} (\Omega^{\cdot}_{Y\times\mathbf{C}/\mathbf{C}}, dF\wedge) \to (\Omega^{\cdot}_Y, df\wedge) \to 0.$$

Thus $(\Omega^{\cdot}_{Y\times\mathbf{C}/\mathbf{C}}, dF\wedge)$ is indeed a deformation of $(\Omega^{\cdot}_Y, df\wedge)$.

Write $f_t(\cdot) = F(\cdot, t)$. Since for every t,

$$\dim_{\mathbf{C}} H^{n-2}(\Omega^{\cdot}_Y, df_t\wedge) = 1$$

it follows that H^{n-2} is a free $\mathbf{C}\{t\}$-module of rank 1, and so $H_0^{n-1} \cong H^{n-1}/tH^{n-1}$. Consequently, the map $H_0^{n-2} \to H^{n-1}$ of the long exact sequence of Section 1 (with $N = n-1$) is zero. Thus, by Lemma 1.1, H^{n-1} is torsion free, and

$$\dim \mathcal{M}_Y(f) = \dim \mathcal{M}_Y(f_t).$$

Note that one can argue more simply by conservation of Euler characteristic. For by Remark 1.2 the Euler characteristic of $(\Omega^{\cdot}_Y, df_t\wedge)$ is independent of t. Since the dimensions of their cohomology groups H^i for $i < n-1$ coincide, it follows that so does that of their top cohomology group. □

Remark 5.5 General $\mathbf{C}^*$-actions. If the action is not real, then the main result remains the same, namely that $\dim \underline{\mathcal{M}}(f)$ defines a (fairly) good notion of multiplicity. The complex $(\Omega^\cdot_Y, df\wedge)$ is still quasi-acyclic, so $\underline{\mathcal{M}}(F)$ is free and $\dim \underline{\mathcal{M}}(f)$ is preserved in deformations. The problem is one of "generic multiplicity", as discussed in Section 4. Indeed, the situation here is worse than that in Section 4, as there can be generic orbits of critical points where the multiplicity is 0. This arises for points on pseudo-reflexion hyperplanes, where the quotient space is smooth, and a function with a generic critical point on the hyperplane becomes non-singular on the quotient space. This problem also arises on pseudo-reflexion hyperplanes for finite groups if one only considers invariant functions and invariant forms in the analysis of the critical points.

On the other hand, the dimension $\dim(R/Jf)^G$ is not in general constant in deformations, but is only upper semicontinuous as is shown by the following example.

Example 5.6 (Mark Roberts) Consider the $\mathbf{C}^*$-action on $\mathbf{C}^n$ with weights $(1,\ldots,1,-1)$, and coordinates $x_1,\ldots,x_{n-1},y$ accordingly. The invariants are thus functions of the $n-1$ variables $u_i = x_i y$. Consider the family of functions

$$f_t(u) = \tfrac{1}{2}\sum_{i=1}^{n-1} u_i^2 - tu_1.$$

Then

$$\frac{\partial f}{x_1} = (u_1 - t)y, \quad \frac{\partial f}{x_i} = u_i y \text{ for } i > 1, \quad \frac{\partial f}{y} = \sum u_i x_i - tx_1.$$

Consequently

$$\frac{R}{Jf_t} = \frac{\mathbf{C}\{x_1,\ldots,x_{n-1},y\}}{\langle (u_1-t)y, u_2 y, \ldots, u_{n-1}y, \sum u_i x_i - tx_1\rangle}.$$

For $t = 0$, this has a single closed critical orbit at 0, and

$$\left(\frac{R}{Jf_0}\right)^G = \frac{\mathbf{C}\{u\}}{\langle u_i u_j\rangle},$$

so $\dim(R/Jf_0) = n$. There are also many critical orbits that are not closed, namely all orbits (other than 0) contained in the null cone

$$\mathcal{Z} = \{y = 0\} \cup \{x_1 = \cdots = x_{n-1} = 0\}.$$

On the other hand, for $t \neq 0$, f_t has 2 critical orbits (both closed) at 0 and at $\{u_1 = t, u_2 = \cdots = u_{n-1} = 0\}$, and

$$\left(\frac{R}{Jf_t}\right)^G \cong \frac{R}{\langle u_1,\ldots u_{n-1}\rangle} \oplus \frac{R}{\langle u_1 - t, u_2, \ldots, u_{n-1}\rangle},$$

which has dimension 2. Thus $\dim(R/Jf_0) = \dim(R/Jf_t)$ if and only if $n = 2$, in which case we have a real action of $\mathbf{C}^*$. See [3] for a geometric interpretation of this loss of multiplicity in terms of the geometry of the quotient space.

If instead we consider the differential forms, we find that

$$d f_t \wedge \underline{\Omega}^{n-1} = \langle u_1 - t, u_2, \dots, u_{n-1} \rangle \, \underline{\Omega}^n,$$

and thus $\underline{\mathcal{M}}(f_t)$ has dimension 1 for all t. Note however, that the critical point at 0 for $t \neq 0$ is not seen by $\underline{\mathcal{M}}(f)$. The same result is found if we consider the basic forms, since if $n > 2$ then $\mathcal{M}_Y(f) \cong \underline{\mathcal{M}}(f)$. Alternatively this can be proved by inspection since on a smooth quotient, the Ω_Y^p coincide with the usual holomorphic forms.

6 Weighted homogeneous functions

In this section we will have continuous recourse to the following simple fact. It can easily be proved by putting T in Jordan canonical form.

Lemma 6.1 *Suppose T is a linear transformation of a vector space W of dimension n. Then T induces transformations T_p of the exterior powers $\bigwedge^p(W)$, and*

$$\sum_{p=0}^{n} (-1)^p \operatorname{tr}(T_p) = \det(I - T),$$

where I is the identity transformation of W.

Now we suppose that the function f is weighted homogeneous of degree d with respect to the weights $w_1, \dots, w_n$. This means that there are coordinates $x_1, \dots, x_n$ and an action of $\mathbf{C}^*$ on $\mathbf{C}^n$:

$$\lambda \cdot (x_1, \dots, x_n) = (\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n),$$

and f satisfies

$$f(\lambda \cdot x) = \lambda^d f(x).$$

The weights w_i are supposed strictly positive, so the action is a so-called good action.

For each integer $i \geq 0$, we have the linear subspace R_i of R consisting of weighted homogeneous polynomials of degree i (with respect to the given weights), and similarly Ω_i^p consists of the p-forms of degree i. For example, if all the weights are equal to 1, then $dx_1 \wedge \dots \wedge dx_p \in \Omega_p^p$. In particular, there are no p-forms of degree less than p.

POINCARÉ SERIES Suppose M is a finitely generated R-module which is *graded*: $M = \oplus_i M_i$ with
$$R_i M_j \subset M_{i+j}.$$
Each graded part M_i of M is finite dimensional, and one defines the *Poincaré series* (or *Hilbert series*) *of* M by
$$P(M,t) = \sum_i \dim_{\mathbf{C}}(M_i) t^i.$$

Remark 6.2 The Poincaré series of a graded module is *a priori* a formal power series, and as such $P(M,t)$ can be interpreted as the character (or trace) of the action of $t \in \mathbf{C}^*$ on M. In a neighbourhood of $t = 0$, this formal power series is convergent and can be expressed as a rational function with denominator $\prod_{i=1}^n (1-t^{w_i})$. If M is finite dimensional over $\mathbf{C}$, then $P(M,t)$ is a polynomial, and $\dim_{\mathbf{C}}(M) = P(M,1)$.

Suppose that M_1 and M_2 are graded modules over R_1 and R_2 respectively. A basic property of Poincaré series is that if $M = M_1 \otimes_{\mathbf{C}} M_2$ is given the induced grading, then $P(M,t) = P(M_1,t)P(M_2,t)$. It follows by induction on n that the Poincaré series of $R = \mathbf{C}[x_1,\dots,x_n]$ with respect to the weights $w_1,\dots,w_n$ is
$$P(R,t) = \prod_{i=1}^n \frac{1}{1-t^{w_i}}.$$
It also follows that
$$P(\Omega^p,t) = P(R,t)P(\textstyle\bigwedge^p(V^*),t).$$
Observe that
$$P(\textstyle\bigwedge^p(V^*),t) = \mathrm{tr}(\tau_p(t)),$$
where $\tau_p(t)$ is the action of $t \in \mathbf{C}^*$ on the p-th exterior power $\bigwedge^p(V^*)$.

If $f \in R_d$, then
$$df \wedge \Omega_i^p \subset \Omega_{d+i}^{p+1}.$$
The differential $df\wedge$ in the complex (2.1) is therefore homogeneous of degree d, and the complex is a *graded complex*. Under the usual assumption that f has an isolated critical point, we can compute the Poincaré series of the multiplicity module $\mathcal{M}(f)$ as follows.

The cohomology of the complex (2.1) is
$$H^i(\Omega^\cdot, f\wedge) = \begin{cases} 0 & \text{if } i \neq n \\ \mathcal{M}(f) & \text{if } i = n. \end{cases}$$

Now, this complex is not of finite dimensional vector spaces. However, for each $r \geq 0$, there is a subcomplex consisting of finite dimensional vector spaces

$$0 \to R_{r-nd} \xrightarrow{df\wedge} \Omega^1_{r-(n-1)d} \xrightarrow{df\wedge} \cdots \xrightarrow{df\wedge} \Omega^{n-1}_{r-d} \xrightarrow{df\wedge} \Omega^n_r \to 0. \tag{6.1}$$

The cohomology of this complex is $H^n = \mathcal{M}(f)_r$, with the other $H^i = 0$. Recall that the Euler characteristic of a complex $\mathbf{K} = (K^\cdot, d)$ of finite dimensional vector spaces satisfies

$$\sum_{i=0}^{n} (-1)^i \dim(H^i(\mathbf{K})) = \sum_{i=0}^{n} (-1)^i \dim(K^i).$$

Thus,

$$\dim \mathcal{M}(f)_r = \sum_{i=0}^{n} (-1)^i \dim(\Omega^{n-i}_{r-id}),$$

and so

$$\begin{aligned} P(\mathcal{M}(f), t) &= \sum_p (-1)^{n-p} t^{d(n-p)} P(\Omega^p, t) \\ &= P(R, t) \prod_i (t^{w_i} - t^d). \end{aligned} \tag{6.2}$$

Here we use Lemma 6.1 with $T = \mathrm{diag}[t^{w_1}, \ldots, t^{w_n}]$.

Using the fact that the isomorphism ψ of Proposition 2.2 is homogeneous of degree $\sum_i w_i$, we obtain the following theorem of Milnor and Orlik [9].

Theorem 6.3 *Suppose f is a weighted homogeneous function on $\mathbf{C}^n$ of degree d, with respect to the weights $w_1, \ldots, w_n$, and has an isolated critical point at 0, then*

$$P(R/Jf, t) = \prod_{i=1}^{n} \left(\frac{1 - t^{d-w_i}}{1 - t^{w_i}} \right).$$

A series of interesting corollaries to this result is given in [1]. They also give an example of weights and degrees in dimension 4 for which there is no function with isolated critical point, and yet the expression $\prod_{i=1}^{n} \left(\frac{1-t^{d-w_i}}{1-t^{w_i}} \right)$ is a polynomial.

Evaluating the expression in the theorem at $t = 1$ gives the following.

Corollary 6.4

$$\mu_{\mathrm{geom}} = \dim_{\mathbf{C}} \mathcal{M}(f) = \prod_i \left(\frac{d - w_i}{w_i} \right).$$

(Note that $d-w_1,\dots,d-w_n$ are the degrees of the partial derivatives of f, i.e. the generators of Jf. There is a more general result, a weighted version of Bezout's Theorem: if I is a weighted homogeneous complete intersection ideal of n generators of degrees $d_1,\dots,d_n$, then $\dim(R/I)=\prod(d_i/w_i)$. See [2] for more details.)

INVARIANT FUNCTIONS We now turn to the case of a weighted homogeneous function which is invariant under the action of a finite group G. We suppose that the action of G preserves the weight spaces, so we can write

$$V=\oplus_k W_k$$

where each W_k is invariant under G, and has weight w_k. We will denote by $\rho_k : G\to \mathrm{GL}(W_k)$ the representaion of G on W_k. Note then that the representation on V^* given by $g\cdot\xi=\xi\circ g^{-1}$, satisfies $V^*\cong\oplus_k W_k^*$, and if ρ_k^* is the representation of G on W_k^* then,

$$\rho_k^*(g)=\rho_k(g)^{-1}.$$

It follows that each of the summands in $R=\oplus_i R_i$ is G-invariant. We say M is a *graded RG-module* if it is an RG-module, which is graded as an R-module, and the grading is such that each M_i is G-invariant. This all amounts to saying that we have an action of $G\times\mathbf{C}^*$ on $\mathbf{C}^n$ and hence on R, and that M is an $R(G\times\mathbf{C}^*)$-module.

The results of this section are due to Orlik and Solomon [13], though they give their formula in the case that all weights are equal, and $\vartheta=1$ (the function is invariant).

The *equivariant Poincaré series* of a graded RG-module M is defined as follows. For each $g\in G$,

$$P_G(M,t)(g):=\sum_r [M_r](g)t^r=\sum_r \mathrm{tr}(g;M_r)t^r,$$

In particular, $P_G(M,t)(e)$ coincides with the ordinary Poincaré series of M. Moreover, if M is finite dimensional (over $\mathbf{C}$) then evaluating at $t=1$ gives the ordinary character of the representation M. Note that, taking Remark 6.2 a step further, $P_G(M,t)(g)$ is the character of $(g,t)\in G\times\mathbf{C}^*$ which is a formal power series in t, and as such it would perhaps be more elegant to write it as $P_{G\times\mathbf{C}^*}(M)\in\mathbf{C}[[t]]\otimes_{\mathbf{C}}\mathrm{Char}(G,\mathbf{C})$, where $\mathrm{Char}(G,\mathbf{C})$ is the ring of complex-valued virtual characters of G.

The equivariant Poincaré series of the ring R is given by

$$P_G(R,t)(g)=\prod_{k=1}^{r}\frac{1}{\det(I_{W_k}-t^{w_k}\rho_k^*(g))}.$$

This can be proved by induction after diagonalizing $\rho(g)$. Note that applying the trace formula (3.1) gives Molien's formula for the Poincaré series $P(R^G,t)$ of the ring of invariants.

Consider again the complex (6.1). This is a complex of finite dimensional representations of G, and the maps are equivariant with a twist ϑ:

$$\Omega^p \xrightarrow{df\wedge} \Omega^{p+1} \otimes \vartheta.$$

For such a complex $\mathbf{K} = (K^{\cdot}, d)$ the equivariant Euler characteristic satisfies

$$\sum_{i=0}^{n} (-1)^p [H^p(\mathbf{K})]\vartheta^p = \sum_{i=0}^{n} (-1)^p [K^p]\vartheta^p.$$

Consequently, in analogy to (6.3), the equivariant Poincaré series of $\mathcal{M}(f)$ is given by,

$$P_G(\mathcal{M}(f),t) = \vartheta^{-n} \sum_{p,r} (-1)^{n-p} \vartheta^p [\Omega^p_{r-(n-p)d}] t^{r-(n-p)d} t^{(n-p)d}.$$

We now repeat the argument prior to Theorem 6.4, but taking the representations into account. Note that, as with ordinary Poincaré series,

$$P_G(\Omega^p,t) = P_G(R,t) P_G(\textstyle\bigwedge^p (V^*),t).$$

Thus

$$P_G(\mathcal{M}(f),t) = \vartheta^{-n} P_G(R,t) \sum_p (-1)^{n-p} \vartheta^p t^{(n-p)d} P_G(\textstyle\bigwedge^p (V^*),t).$$

Applying Lemma 6.1 with

$$T = -\vartheta(g) t^{-d} \operatorname{diag}[t^{w_1}\rho_1^*(g), \dots, t^{w_r}\rho_r^*(g)],$$

we deduce

Theorem 6.5 *Suppose f is a weighted homogeneous function on V with an isolated critical point at 0, and which is an equivariant for the G-action with twist ϑ, then*

$$P_G(\mathcal{M}(f),t)(g) = \prod_k \frac{\det(t^{w_k}\rho_k^*(g) - \vartheta(g)^{-1} t^d I_{W_k})}{\det(I_{W_k} - t^{w_k}\rho_k^*(g))}.$$

The isomorphism $\psi : R/Jf \to \mathcal{M}(f)$ is homogeneous and equivariant, so if we divide by $t^w\lambda(g)$, where $w = \sum_k w_k \dim(W_k)$ we deduce the following result due to Orlik and Solomon [13]. Note that $\lambda(g) = \prod_k \det \rho_k^*(g) = \det \rho^*(g) = \det \rho(g)^{-1}$.

Corollary 6.6 *With the same hypotheses as the theorem,*

$$P_G(R/Jf,t)(g) = \prod_{k=1}^{r} \frac{\det(I_{W_k} - t^{d-w_k}\vartheta(g)^{-1}\rho_k(g))}{\det(I_{W_k} - t^{w_k}\rho_k^*(g))}.$$

For each $g \in G$, let $n(g) = \dim(V^g) =$ the multiplicity of 1 as an eigenvector of $\rho(g)$. And suppose $w_1, \ldots, w_{n(g)}$ are the weights on V^g, then one easily evaluates the expression in the theorem at $t = 1$ to deduce

Corollary 6.7 *Suppose in addition that f is invariant (or more generally that $\vartheta(g) = 1$), then*

$$[\mathcal{M}(f)](g) = (-1)^{n-n(g)} \prod_{i=1}^{n(g)} \left(\frac{d-w_i}{w_i}\right).$$

Remark 6.8 If the action of G is free off the origin (i.e. for all $g \neq e$ one has $V^g = 0$), then $[\mathcal{M}(f)](g) = (-1)^n$ for $g \neq e$. Such a character is easy to interpret. Note that the character $[G]$ of the regular representation $\mathbf{C}.G$ satisfies

$$[G](g) = 0 \quad \text{if } g \neq e$$

and of course $[G](e) = |G|$. Thus for f with an isolated critical point and G acting freely off the origin,

$$[\mathcal{M}(f)] = \nu(f)[G] + (-1)^n\mathbf{C},$$

for some integer $\nu(f)$, where $\mathbf{C}$ is the trivial representation. In [21, Section 5], Wall uses this in the case $\dim(V) = 2$ to produce a formula for the generic multiplicity — see *Generic Multiplicity* in Section 4. Suppose f is a generic critical point and write $\nu(f) = \nu(G,V)$. Wall's formula is in terms of the embedding dimension of V/G and the resolution of V/G, and states that for $\dim V = 2$ and G free off 0,

$$\nu(G,V) = \begin{cases} e-3 & \text{if } G \text{ is cyclic, or } b > 2 \\ e-2 & \text{if } G \text{ is not cyclic and } b = 2, \end{cases}$$

where e is the embedding dimension and $-b$ is the self-intersection number of the central curve of the resolution.

The dimension of the fixed point subspaces $\mathcal{M}(f)^G$ and $(R/Jf)^G$ can be computed by applying the trace formula to the formulae in Theorem 6.5 and Corollary 6.6 respectively.

It is interesting to translate the corollaries of Theorem 6.3 given in [1] into corollaries of the theorem above.

7 Weighted-homogeneous $\mathbf{C}^*$-invariant functions

We continue with the notation used in Section 5 for $\mathbf{C}^*$-invariant functions. In particular, the weights of the $\mathbf{C}^*$-action are $\lambda_i > 0$ and $\mu_j < 0$ (with $i = 1, \ldots, a,\ j = 1, \ldots, b$). In this section we consider $\mathbf{C}^*$-invariant functions on V that are also weighted homogeneous with respect to some set of (strictly positive) weights w_i and v_j. We thus have two $\mathbf{C}^*$-actions, and we suppose that they commute. That is, we have an action of the torus $(\mathbf{C}^*)^2$, and correspondingly every monomial has a *bidegree* (α, d) with $d \geq 0$; in particular, bidegree $(0, d)$ corresponds to invariant functions of degree d with respect to the positive weights w_i, v_j. The monomial x_i has biweight (λ_i, v_i) and the monomial y_j has biweight (μ_j, w_j).

If an invariant function f is weighted homogeneous of degree d, then the good $\mathbf{C}^*$-action defines a grading on $\underline{\mathcal{M}}(f)$, and we wish to compute its Poincaré series. To this end, we reconsider the complex (2.1), each term of which has a bigrading, and for each bidegree (α, e),

$$df \wedge [\Omega^p]_{(\alpha,e)} \subset [\Omega^{p+1}]_{(\alpha,e+d)}.$$

The Poincaré series of a bigraded vector space A with finite dimensional bigraded parts is a formal power series in two variables:

$$P(A; s, t) = \sum_{r,e} \dim(A_{(\alpha,e)}) s^\alpha t^e.$$

(Again, this is the character of the representation of the torus $\mathbf{C}^* \times \mathbf{C}^*$ on A.) Note that the ring $S = \mathbf{C}[x_i, y_j]$ is bigraded with finite dimensional graded parts, since $S_{(\alpha,e)} \subset S_e$ which is finite dimensional (S_e being the subspace of S consisting of functions of degree e with respect to the weights w_i, v_j).

The Poincaré series of the ring $\mathbf{C}[x_i, y_j]$ with biweights $(\lambda_i, w_i), (\mu_j, v_j)$ (with $\lambda_i, w_i, v_j > 0$ and $\mu_j < 0$) converges to

$$P(S; s, t) = \prod_{i=1}^{a} \frac{1}{(1 - s^{\lambda_i} t^{w_i})} \prod_{j=1}^{b} \frac{1}{(1 - s^{\mu_j} t^{v_j})},$$

provided that $|s^{\lambda_i} t^{w_i}| < 1$ and $|s^{\mu_j} t^{v_j}| < 1$ for all i, j. This domain of convergence contains no value of t with $|t| > 1$, and for each t with $|t| < 1$ it is an annulus in s-space: $|t|^{m_2} < |s| < |t^{-1}|^{m_1}$, where $m_1 = \min\{\lambda_i / w_i\}$ and $m_2 = \min\{v_j / (-\mu_j)\}$. In particular, if $|t| < 1$ it contains the circle $|s| = 1$.

Now a polynomial is invariant under $\mathbf{C}^*$ if and only if it is invariant under the maximal compact subgroup $\mathbf{S}^1 \subset \mathbf{C}^*$. To calculate the Poincaré series of the ring R

of invariants, one therefore applies the trace formula (3.1) to the $\mathbf{S}^1$-action:

$$\begin{aligned} P(R)(t) &= \frac{1}{2\pi}\int_0^{2\pi} P(S;e^{i\theta},t)\,d\theta \\ &= \frac{1}{2\pi i}\oint_{|s|=1} P(S;s,t)\,\frac{ds}{s}. \end{aligned}$$

One can use the residue theorem to compute this as a sum of residues at the poles of the rational function $s^{-1}P(S;s,t)$. It seems unlikely that this can be expressed as a simple closed formula for general weights.

Now, by Lemma 6.1 the Poincaré series for the complex $(\Omega^{\cdot}, df\wedge)$ is given by

$$\begin{aligned} P(\Omega^{\cdot}, df\wedge)(s,t) &= P(S)(s,t)\prod_{i=1}^{a}(s^{\lambda_i}t^{w_i}-t^d)\prod_{j=1}^{b}(s^{\mu_j}t^{\nu_j}-t^d) \\ &= \prod_{i=1}^{a}\frac{(s^{\lambda_i}t^{w_i}-t^d)}{(1-s^{\lambda_i}t^{w_i})}\prod_{j=1}^{b}\frac{(t^{\nu_j}-s^{-\mu_j}t^d)}{(s^{-\mu_j}-t^{\nu_j})}. \end{aligned}$$

(Recall $-\mu_j > 0$.) Since f does not in general have an isolated critical point in $\mathbf{C}^n$, this series is not a polynomial.

The Poincaré series $P(\underline{\Omega}^{\cdot}, df\wedge)$ for the complex of invariant forms is deduced from $P(\Omega^{\cdot}, df\wedge)$ by the same method as $P(R)$ is found from $P(S)$: averaging over the $\mathbf{S}^1$-action. Thus for $|t|<1$,

$$P(\underline{\Omega}^{\cdot}, df\wedge)(t) = \frac{1}{2\pi i}\oint_{|s|=1}\rho(s,t)\,ds,$$

where

$$\rho(s,t) = \frac{\prod_{i=1}^{a}(s^{\lambda_i}t^{w_i}-t^d)\prod_{j=1}^{b}(t^{\nu_j}-s^{-\mu_j}t^d)}{s\prod_{i=1}^{a}(1-s^{\lambda_i}t^{w_i})\prod_{j=1}^{b}(s^{-\mu_j}-t^{\nu_j})}. \tag{7.1}$$

For fixed t, the disc $|s|<1$ contains poles of the integrand at $s=0$ and at all solutions of $s^{-\mu_j}=t^{\nu_j}$, for $j=1,\ldots,b$. Thus there are $1+\sum_j|\mu_j|$ poles, counting multiplicity. For $|t|<1$, let

$$S_t = \{s \mid s^{-\mu_j}=t^{\nu_j},\ \text{for some } j=1,\ldots,b\}. \tag{7.2}$$

Then

$$P(\underline{\Omega}^{\cdot}, df\wedge)(t) = \mathrm{res}_{\{s=0\}}\,\rho(s,t) + \sum_{s_0\in S_t}\mathrm{res}_{\{s=s_0\}}\,\rho(s,t).$$

The first summand is easy to compute: $\mathrm{res}_{\{s=0\}}\,\rho(s,t)=t^{ad}$. The other terms are not so easy for general weights. In practice, for given weights they can easily be computed with the aid of a computer package such as Maple. Note that taking the disc $|s| \geq 1$, one obtains an analogous formula, involving the residue of ρ at $s = \infty$ and at solutions of $s^{\lambda_i} = t^{w_i}$ for $i = 1, \ldots, a$. Which formula is chosen in practice would depend on the specific weights and their multiplicities.

We thus have, in principle, a formula for $P(\underline{\Omega}^{\cdot}, df\wedge)(t)$. It remains to see how this is related to the Poincaré series for the multiplicity module $\underline{\mathcal{M}}(f)$. Let $m = \min\{a,b\}$. The information we require on the cohomology $\underline{H}^{\cdot}$ of the complex $(\underline{\Omega}^{\cdot}, df\wedge)$ is given in [10, Proposition 3.6]. It is shown there that the following are exact sequences:

$$a = b: \ 0 \to \mathbf{C}.[\omega^m] \to \underline{H}^n \to \underline{H}^{n-1} \to 0;$$

$$|a-b| = 1: \ 0 \to \underline{H}^n \to \underline{H}^{n-1} \to \mathbf{C}.[\omega^m] \to 0;$$

$$|a-b| > 1: \ 0 \to \underline{H}^n \to \underline{H}^{n-1} \to 0 \text{ and } \underline{H}^{2m} \cong \mathbf{C}.[\omega^m];$$

and all other $\underline{H}^i = 0$. The map $\underline{H}^n \to \underline{H}^{n-1}$ is given by contraction of differential forms with the vector field ϑ generating the $\mathbf{C}^*$-action, which is of degree 0. The 2-form ω is defined by $\omega(\vartheta, -) = df$, so ω has the same degree as f. Thus, with $M = \max\{a,b\}$,

$$a = b: \ P(\underline{\mathcal{M}}(f)) = (1-t^d)^{-1}(P(\underline{\Omega}^{\cdot}, df\wedge) - t^{(a+1)d});$$

$$a \neq b: \ P(\underline{\mathcal{M}}(f)) = (1-t^d)^{-1}(P(\underline{\Omega}^{\cdot}, df\wedge) - t^{Md});$$

We have therefore proved

Theorem 7.1 *Let f be a weighted homogeneous polynomial of degree d with respect to the weights w_i, v_j that is also invariant under the $\mathbf{C}^*$-action with weights λ_i, μ_j (see above for precise notation). Suppose that f has an isolated critical point in the quotient space. Let ρ be given by (7.1) and S_t by (7.2). Then*

$$P(\underline{\mathcal{M}}(f)) = \mathcal{R} + \mathcal{C}(a,b);$$

where

$$\mathcal{R} = \frac{1}{(1-t^d)} \sum_{s_0 \in S_t} \text{res}_{\{s=s_0\}} \rho(s,t)$$

and

$$\mathcal{C}(a,b) = \begin{cases} t^{ad} & \text{if } a = b \\ 0 & \text{if } a > b \\ (t^{ad} - t^b)/(1-t^d) & \text{if } a < b. \end{cases}$$

Examples 7.2 We give a few examples involving real actions. The case $n = 2$ is trivial, for the quotient space is 1-dimensional and smooth, so our examples are for $n = 4$ and 6.

$(1,1,-1,-1)$: Suppose $\mathbf{C}^*$ acts on $\mathbf{C}^4$ with these weights, and let f be a homogeneous invariant polynomial of degree d with all weights =1. The invariant polynomials are polynomials in the $x_i y_j$, and therefore d must be even. Then

$$\rho(s,t) = \frac{(st - t^d)^2 (t - st^d)^2}{s(1-st)^2 (s-t)^2}.$$

Using Theorem 7.1, one finds

$$P(\underline{\mathcal{M}}(f),t) = t^{2d} + \frac{t^4(1-t^{d-2})}{(1-t^2)^3}\left[1 + t^2 + t^{d-2} - 6t^d + t^{d+2} + t^{2d-2} + t^{2d}\right].$$

This is a polynomial if and only if d is even. Evaluating this at $t = 1$ gives

$$\dim \underline{\mathcal{M}}(f) = 1 + \frac{1}{4}(d-2)(d^2 - 2d + 4),$$

which is an integer if and only if d is even.

$(1,2,-1,-2)$: Consider this C^*-action on $\mathbf{C}^4$. The invariants are functions of $x_1y_1, x_1^2y_2, x_2y_1^2, x_2y_2$. Consider first the case that f is homogeneous of degree d (all $w_i = v_j = 1$). Then

$$\rho(s,t) = \frac{(st - t^d)(s^2t - t^d)(t - st^d)(t - s^2t^d)}{s(1-s)(1-s^2)(s-t)(s^2-t)}.$$

Applying Theorem 7.1 gives

$$P(\underline{\mathcal{M}}(f),t) = t^{2d} + \frac{t^4(1-t^{d-2})}{(1-t)(1-t^2)(1-t^3)} p(t)$$

with

$$p(t) = \left[1 - t + t^2 + t^{d-2} - t^{d-1} - 2t^d - t^{d+1} + t^{d+2} + t^{2d-2} - t^{2d-1} + t^{2d}\right].$$

$P(\underline{\mathcal{M}}(f),t)$ is a polynomial if and only if d is congruent to 0 or 2 modulo 6. Evaluating at $t = 1$ gives

$$\dim \underline{\mathcal{M}}(f) = 1 + \frac{1}{6}(d-2)(d^2 - 2d + 6),$$

Consider now the case that $w_1 = v_1 = 1$, $w_2 = v_2 = 2$ (corresponding to $(\lambda_1 = -\mu_1 = 1,\ \lambda_2 = -\mu_2 = 2)$. Then

$$\rho(s,t) = \frac{(st-t^d)(s^2t^2-t^d)(t-st^d)(t^2-s^2t^d)}{s(1-st)(1-s^2t^2)(s-t)(s^2-t^2)}.$$

Applying Theorem 7.1 and evaluating at $t = 1$ gives

$$\dim \underline{\mathcal{M}}(f) = 1 + \frac{1}{16}(d^3 - 6d^2 + 20d - 32),$$

which is an integer if and only if d is a multiple of 4.

$(1,1,1,-1,-1,-1)$: Now $\mathbf{C}^*$ acts on $\mathbf{C}^6$ with these weights, and f is invariant and homogeneous of degree d with respect to the weights 1. Again, the invariants are generated by the x_iy_j, and d is necessarily even. We have

$$\rho(s,t) = \frac{(st-t^d)^3(t-st^d)^3}{s(1-st)^3(s-t)^3}.$$

From Theorem 7.1, one deduces

$$\dim \underline{\mathcal{M}}(f) = \frac{1}{16}\left(3d^5 - 18d^4 + 48d^3 - 72d^2 + 72d - 32\right).$$

Using generating functions, it is possible to express more explicitly the formula of Theorem 7.1 in the case that all the weights are ± 1.

Consider the real action of $\mathbf{C}^*$ on $\mathbf{C}^{2N}$ with weights ± 1 (so generalizing two of the examples above). Let f be a homogeneous function of degree d invariant under this $\mathbf{C}^*$-action, and with an isolated critical point in the quotient space. The Poincaré series of the multiplicity module of the critical point is given by Theorem 7.1. More explicitly, introducing N into the notation,

$$P(\underline{\mathcal{M}}_N(f),t) = \frac{1}{1-t^d}\left(\frac{1}{2\pi i}\oint_{|s|=1} \rho_N(s,t)\,ds - t^{N+1}\right),$$

with

$$\rho_N(s,t) = \frac{1}{s}\left(\frac{(st-t^d)(t-st^d)}{(1-st)(s-t)}\right)^N.$$

Now form the generating function,

$$G(t,T) = \sum_{N\geq 0} P(\underline{\mathcal{M}}_N(f),t)T^N.$$

For T sufficiently small, this converges to

$$G(t,T)=\frac{1}{(1-t^d)}\left(\frac{1}{2\pi i}\oint_{|s|=1}\frac{ds}{s(1-s\rho_1(s,t)T)}-\frac{t}{1-tT}\right).$$

The integral can be evaluated by the residue theorem. After "clearing the fractions", the denominator of the integrand becomes

$$s[s^2t(t^dT-1)+s(1+t^2(1-T)-t^{2d}T)+t(t^dT-1)].$$

The integrand thus has three poles, at $s=0$, at $s_1\approx t$ and $s_2\approx 1/t$. The first two are within the unit circle, the last is without.

The residue at $s=0$ is simply $1/(1-t^dT)$. The residue at $s=s_1$ is too long to be reproduced here, but when evaluated at $t=1$ gives the following result.

Theorem 7.3 *Let $M(N,d)=\dim(\underline{\mathcal{M}}_N(f))$, where f is a $\mathbf{C}^*$-invariant function, homgeneous of degree d, with an isolated critical point in the quotient. Suppose the $\mathbf{C}^*$-action to be real with weights ±1. Then*

$$\sum_{N\geq 0}M(N,d)T^N=\frac{1}{1-T}+\tfrac{1}{2}(d-2)T(1-T)^{-3/2}(1-(d-1)^2T)^{-1/2}.$$

Remark 7.4 The case of real actions with weights ±1 was considered under a different guise by van Straten [18]. Let H_2 be a homogeneous non-degenerate quadratic form on $\mathbf{C}^{2N}$ invariant under $\mathbf{C}^*$ and H_d a degree d invariant function with an isolated critical point on the quotient space Y (these are not exactly van Straten's hypotheses, but they are equivalent). Consider the map $(H_2^{d/2},H_d):Y\to\mathbf{C}^2$, and let Σ be its singular locus. Suppose further that $\Sigma\cap\{H_2=0\}=\{0\}$ (verified for generic H_d). Let $S(N,d)$ be the number of branches in Y of Σ. Van Straten proves by algebro-geometric methods, that

$$\sum_{N\geq 0}S(N,d)T^N=T(1-T)^{-3/2}[1-(d-1)^2T]^{-1/2}.$$

(Note that van Straten's $n+1$ is our N, and his $2d$ is our d.)

The relation between this result and our Theorem 7.3 is as follows. Consider the deformation $H_\lambda=H_d+\lambda^{d/2-1}H_2$ of H_d (which is homogeneous in $Y\times\mathbf{C}$). Now, if (x,λ) is a critical point of H_λ then x is a singular point of the map $(H_2^{d/2},H_d)$. Let $\Sigma\subset Y\times\mathbf{C}$ be the set of critical points of $H_d+\lambda^{d-2}H_2$, which is 1-dimensional. One branch of this curve is $\{0\}\times\mathbf{C}\subset Y\times\mathbf{C}$. Let Σ' consist of the remaining branches. The projection $\Sigma\to\mathbf{C}$, $([x],\lambda)\mapsto\lambda$ has multiplicity $M(N,d)$

($[x]$ is the $\mathbf{C}^*$-orbit through x). The projection $\Sigma' \to Y$ has multiplicity $\frac{1}{2}(d-2)$, and is onto the critical set of $(H_2^{d/2}, H_d)$. Thus,

$$M(N,d) - 1 = \tfrac{1}{2}(d-2)S(N,d).$$

This provides an alternative proof of Theorem 7.3.

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Institute Non-Linéaire de Nice
1361 route des Lucioles
Sophia Antipolis
06560 Valbonne
France