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Artin-Schreier Theory and L-packets in the Principal Series of $SL_2(\mathbb{F}_2((x)))$

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Résumé. French abstract.

Abstract. We survey Artin-Schreier theory, adapted to the local function field $\mathbb{F}_2((x))$. This leads to a neat parametrization of L-packets in the unitary principal series of $SL_2(\mathbb{F}_2((x)))$.

1. Introduction

In this article we consider the local function field $K = \mathbb{F}_2((x))$, the field of Laurent series with coefficients in $\mathbb{F}_2$. This example is particularly interesting because there are countably many quadratic extensions of $\mathbb{F}_2((x))$.

We use Artin-Schreier theory to study the quadratic extensions of $\mathbb{F}_2((x))$. A good account of Artin-Schreier-Witt theory can be found in [8] and [9]. We parametrize such extensions and use the Artin-Schreier symbol to manufacture a quadratic character for each extension.

Let $G = SL_2(K)$ and let T denote the standard maximal torus of G. Let $G^\vee$ denote the Langlands dual, so that $G^\vee = PGL(2, \mathbb{C})$, and let $T^\vee \subset G^\vee$ denote the dual torus. The Weil group attached to K will be denoted W_K.

A quadratic extension $K(\alpha)$ of K determines a quadratic character

$$\chi_a : K^\times \to \mathbb{C}^\times$$

which fits into the sequence

$$W_K \to K^\times \to \mathbb{C}^\times \to T^\vee \to G^\vee$$

and thereby determines an L-parameter ϕ_a. Such L-parameters ϕ_a serve as parameters for the L-packets in the unitary principal series of $SL_2(K)$. The Artin-Schreier symbol creates a particularly neat parametrization of the L-packets.

We anticipate that this result will lead to a description of the unitary principal series of SL_2 as a topological space, along the lines of [5]. In that case, there will be countably many double points, one for each quadratic extension $K(\alpha)$.

2. Artin-Schreier theory

Let K be a field with positive characteristic $ch(K) = p$. Kummer theory describes the Galois groups of cyclic extensions of K whose degree is coprime with p. It is well known that any cyclic extension L/K of degree n, $(n, p) = 1$, is generated by a root α of an irreducible polynomial $x^n - a \in K[x]$. A polynomial $x^n - a$, $a \in K^\times$, is irreducible if and only if the class $[a]$ has order n in the quotient group $K^\times/K^{\times p}$. If $\alpha \in K^s$ is a root of $x^n - a$ then $K(\alpha)/K$ is a cyclic extension of degree n and is called a Kummer extension of K.

Artin-Shreier theory deals with cyclic extensions of degree equal to $ch(K) = p$. It is therefore an analogue of Kummer theory, where the role of the polynomial $x^n - a$ is played by $x^n - x - a$. Essentially, every cyclic extension of K with degree $p = ch(K)$ is generated by a root α of a polynomial $x^p - x - a \in K[x]$. We now state this result more precisely.

From now on we assume that K is a field with positive characteristic p. We fix an algebraic closure $\overline{K}$ of K. We also fix a separable closure K^s of K in $\overline{K}$. Let $\wp$ denote the Artin-schreier morphism

$$\wp : K \to K, x \mapsto x^n - x.$$

Given $a \in K$ denote by $K(\wp^{-1}(a))$ the extension $K(\alpha)$, where $\wp(\alpha) = a, \alpha \in K^s$.

Theorem 2.1. (i) *Let L/K be a finite cyclic extension of degree p. Then, $L = K(\wp^{-1}(a))$, for some $a \in K$.*

(ii) *Given $a \in K$, either $\wp(x) - a \in K[x]$ has one root in K in which case it has all the p roots are in K, or is irreducible.*

(iii) *If $\wp(x) - a \in K[x]$ is irreducible then $K(\wp^{-1}(a))/K$ is a cyclic extension of degree p, with $\wp^{-1}(a) \subset K^s$.*

(See [3] and [4] for more details.)

We are also interested in infinite extensions of K. Let B be a subgroup of K with finite index such that $\wp(K) \subseteq B \subseteq K$. The composite of two finite abelian Galois extensions of degree p is again a finite abelian Galois extension of exponent p. Therefore, the composite

$$K_B = K(\wp^{-1}(B)) = \prod_{a \in B} K(\wp^{-1}(a))$$

is a finite abelian Galois extension of exponent p. On the other hand, if L/K is a finite abelian Galois extension of exponent p, then $L = K_B$ for some subgroup $\wp(K) \subseteq B \subseteq K$ with finite index.

All such extensions lie in the maximal abelian extension of exponent p, which we denote by

$$K_p = K(\wp^{-1}(K)).$$

K_p/K is an infinite Galois extension and the correspondent Galois group $G_p = Gal(K_p/K)$ is an infinite profinite group.

In the language of [4, p.290] an extension L of K is p-closed if it has no Galois extensions of degree p. Examples of p-closed extensions are the separable closure K^s/K and K_p/K.

The endomorphism $\wp \in End(K)$ of the additive group extends to any extension of K. In particular, it extends to any p-closed extension.

In the sequel we shall use some results about Galois cohomology. We refer to [4] for further details.

Lemma 2.1. *If L/K is p-closed then $\wp : L \to L$ is surjective.*

Proof. For each $a \in L$ the polynomial $\wp(x) - a = x^p - x - a$ is separable. In fact, if α is a root of $\wp(x) - a$ then $\alpha + i$, $1 \leq i \leq p-1$ are the other $p-1$ roots. The result then follows since we assumed L is p-closed, otherwise the split field of $\wp(x) - a$ would be a cyclic extension of L of degree p. $\square$

Lemma 2.2. *Let L/K be a p-closed extension with Galois group $G = Gal(L/K)$. Then $\wp$ is a G-homomorphism with kernel $\mathbb{Z}/p\mathbb{Z}$.*

Proof. Given $\sigma \in G$ and $x \in L$, we have

$$\wp(\sigma(x)) = \sigma(x)^p - \sigma(x) = \sigma(x^p) - \sigma(x) = \sigma(x^p - x) = \sigma(\wp(x)).$$

Moreover,

$$ker\wp = \{x \in L : x^p = x\} = \mathbb{Z}/p\mathbb{Z}.$$

$\square$

Proposition 2.1. *Let L/K be a p-closed extension. Then, there is a canonical isomorphism*

$$K/\wp(K) \xrightarrow{\simeq} Hom(Gal(L/K), \mathbb{Z}/p\mathbb{Z})$$

induced by $a \in K \mapsto (\sigma \in Gal(L/K) \mapsto \sigma(\alpha) - \alpha)$, where $\wp(\alpha) = a$.

Proof. Denote $G = Gal(L/K)$. From lemmas 2.1 and 2.2, we have an exact sequence of G-modules

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow K \xrightarrow{\wp} K \longrightarrow 0.$$

The associated (Galois) cohomology sequence is given by

$$0 \longrightarrow H^0(G, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^0(G, K) \longrightarrow H^0(G, K) \longrightarrow H^1(G, \mathbb{Z}/p\mathbb{Z}) \longrightarrow ...$$

Now,

$$H^0(G, \mathbb{Z}/p\mathbb{Z}) = (\mathbb{Z}/p\mathbb{Z})^G = \mathbb{Z}/p\mathbb{Z}$$

and

$$H^0(G, K) = K^G = K$$

since G acts trivially on both $\mathbb{Z}/p\mathbb{Z}$ and K. By (additive) Hilbert Satz 90 (see [4, p.292]),

$$H^1(G, K) = 0.$$

Hence, the cohomology sequence becomes

$$K \longrightarrow K \longrightarrow H^1(G, \mathbb{Z}/p\mathbb{Z}) \longrightarrow 0$$

and again, since G acts trivially on $\mathbb{Z}/p\mathbb{Z}$, we have (see [7, p.113])

$$K/\wp(K) \cong H^1(G, \mathbb{Z}/p\mathbb{Z}) \cong Hom(G, \mathbb{Z}/p\mathbb{Z}).$$

□

3. Local classfield theory

Let K be a field with a discrete valuation ν. Denote by $\mathfrak{o} = \{x \in K : \nu(x) \geq 0\}$ the ring of integers and by $\mathfrak{p} = \{x \in K : \nu(x) > 0\}$ the maximal ideal of $\mathfrak{o}$. The residue field of K is $\kappa = \mathfrak{o}/\mathfrak{p}$. The valuation is a group homomorphism $\nu : K^\times \to \mathbb{R}^\times$, whose kernel U is called the unit group. The valuation induces a topology on K.

A local field (K, ν) with discrete valuation ν is a complete field whose residue field κ is finite. K is a non-discrete, totally disconnected locally compact field in its ν-topology, see [2, p.19]. In this article by a local field we always mean a field with discrete valuation. We do not consider the local archimedean local fields $\mathbb{R}$ and $\mathbb{C}$. Examples of local fields are the p-adic fields $\mathbb{Q}_p$ (local fields with characteristic zero) and the field of Laurent series $\mathbb{F}_q((x))$ with coefficients in a finite field $\mathbb{F}_q$, $q = p^s$ (local fields with positive characteristic).

Local classfield theory was developed in the setting of local fields with zero or positive characteristic. We will be mainly interested in local fields with positive characteristic. Denote $G_K = Gal(K^s/K)$ and G_K^{ab} its abelianization. If L/K is a finite Galois extension we denote $Gal(L/K) = G_{L/K}$ and its abelianization by $G_{L/K}^{ab}$.

We briefly review the local reciprocity law for finite extensions of K. Let L/K be a finite extension of order m. Then, there is a natural isomorphism $H^2(G_{L/K}, L^\times) \cong \mathbb{Z}/m\mathbb{Z}$. The canonical generator $u_{L/K}$ of $H^2(G_{L/K}, L^\times)$ is called the fundamental class of L/K. The fundamental class $u_{L/K}$ satisfy the hypotheses of Tate-Nakayama Theorem, see [7, ch IX, §8]. By Tate's Theorem (see [7, p.168]), the cup product with $u_{L/K}$ defines an isomorphism of Tate's cohomology groups

$$\theta_{L/K} : \widehat{H}^{-2}(G_{L/K}, \mathbb{Z}) \longrightarrow \widehat{H}^0(G_{L/K}, L^\times), x \mapsto u_{L/K} \cup x.$$

Since $\widehat{H}^{-2}(G_{L/K},\mathbb{Z}) = G^{ab}_{L/K}$ and $\widehat{H}^{0}(G_{L/K},L^\times) = K^\times/N_{L/K}L^\times$ ([7, p.169], where $N_{L/K}$ is the norm map, there is an isomorphism

$$\theta_{L/K} : G^{ab}_{L/K} \xrightarrow{\simeq} K^\times/N_{L/K}L^\times$$

The local reciprocity law is the inverse map

$$\theta^{-1}_{L/K} = \omega_{L/K} : K^\times/N_{L/K}L^\times \longrightarrow G^{ab}_{L/K}$$

Composing with the canonical morphism, we have

$$K^\times \longrightarrow K^\times/N_{L/K}L^\times \longrightarrow G^{ab}_{L/K}.$$

Given $b \in K^\times$, we denote the composition map by

$$b \mapsto (b, L/K).$$

$(b, L/K)$ is called the Artin symbol.

Example. If L/K is unramified then

$$(x, L/K) = F^{\nu(x)}$$

where F is the Frobenius element, i.e, the canonical generator of $Gal(l/\kappa)$ (we are identifying $Gal(l/\kappa) = Gal(L/K)$) and ν is the valuation of K. See [7, p.197] for more details.

As a consequence of the reciprocity law, if L/K is a finite extension then the index $(K^\times : N_{L/K}L^\times)$ is finite. In general, it divides $[L : K]$, being equal if and only if the extension is abelian. Moreover, the correspondence which associates to each Galois extension L/K the norm subgroup of $K^\times$

$$L \leftrightarrow N_{L/K}L^\times$$

is a bijection.

4. The Artin-Schreier symbol

We briefly recall the case when K has zero characteristic. In that case K is a finite extension of $\mathbb{Q}_p$. If K contains the group μ_p of p-th roots of unity, by Kummer theory there is a pairing

$$K^\times \times G^{ab}_K \longrightarrow \mu_p, (a,\sigma) \mapsto \frac{\sigma(\alpha)}{\alpha},$$

where $\alpha^p = a$ and $\alpha \in K^s$.

The pairing induces a bijection

$$\left\{ \begin{array}{c} \text{subspaces of } K^\times \\ \text{containing } K^{\times p} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{abelian extensions of } K \\ \text{with exponent } p \end{array} \right\}$$

Now, Kummer theory and the reciprocity law together gives another pairing

$$K^\times \times K^\times \longrightarrow \mu_p, (a,b) \mapsto \frac{(b, L/K)(\alpha)}{\alpha},$$

where $\alpha^p = a$, $\alpha \in K^s$ and $L = K(\alpha)$.

The Hilbert symbol is defined to be

$$(a,b) = \frac{(b, L/K)(\alpha)}{\alpha}.$$

Suppose K is a field with characteristic p. Then K ia a field of Laurent series, $\mathbb{F}_{p^s}((x))$. Recall that Artin-Schreier theory gives an isomorphism of topological groups

$$G_p \overset{\simeq}{\longrightarrow} Hom(K/\wp(K), \mathbb{Z}/p\mathbb{Z}), \sigma \mapsto \varphi_\sigma$$

where $\varphi_\sigma : a + \wp(K) \mapsto \sigma(\alpha) - \alpha$, $\alpha \in K^s$ and $\wp(\alpha) = a$.

Artin-Schreier theory together with the reciprocity law gives a pairing

$$K \times K^\times \longrightarrow \mathbb{Z}/p\mathbb{Z}, (a,b) \mapsto (b, L/K)(\alpha) - \alpha,$$

where $\wp(\alpha) = a$, $\alpha \in K^s$ and $L = K(\alpha)$.

Definition. Given $a \in K$ and $b \in K^\times$, the Artin-Schreier symbol is

$$[a,b) = (b, L/K)(\alpha) - \alpha.$$

It is the analogous of the Hilbert symbol for power series of positive characteristic p.

Proposition 4.1. *The Artin-Schreier symbol is a bilinear map satisfying the following properties:*

(i) $[a_1 + a_2, b) = [a_1, b) + [a_2, b)$;
(ii) $[a, b_1 b_2) = [a, b_1) + [a, b_2)$;
(iii) $[a,b) = 0, \forall a \in K \Leftrightarrow b \in N_{L/K}L^\times, L = K(\alpha)$ *and* $\wp(\alpha) = a$;
(iv) $[a,b) = 0, \forall b \in K^\times \Leftrightarrow a \in \wp(K)$.

(See [3, p.341])

$Hom(-, \mathbb{R}/\mathbb{Z})$ **duality**

Let G be a locally compact abelian group. The Pontrygin dual of G is the group

$$G^\wedge = Hom(G, \mathbb{R}/\mathbb{Z})$$

of all continuous homomorphisms, with the compact-open topology. $G^\wedge$ is an abelian locally compact group.

It is well known that if G is compact (resp., discrete) then $G^\wedge$ is discrete (resp., compact). A simple example is $\mathbb{Z}^\wedge = \mathbb{R}/\mathbb{Z}$ and $(\mathbb{R}/\mathbb{Z})^\wedge = \mathbb{Z}$.

The Pontryagin bidual is defined to be

$$G^{\wedge\wedge} = Hom(G^{\wedge}, \mathbb{R}/\mathbb{Z}).$$

There is a canonical homomorphism

$$\alpha_G : G \to G^{\wedge\wedge}, g \mapsto (\chi \mapsto \chi(g)).$$

The Pontryagin-Van Kampen theorem says that is G is abelian and locally compact, then α_G is an isomorphism of topological groups.

Let G and H be two abelian, locally compact groups and let $\varphi : G \to H$ be a continuous homomorphism. Then φ induces a continuous homomorphism on the Pontryagin duals

$$\varphi^{\wedge} : H^{\wedge} \longrightarrow G^{\wedge}, \chi \mapsto (g \mapsto \chi(\varphi(g)).$$

Suppose H is an open subgroup of G. Then, the canonical injection $\iota : H \hookrightarrow G$ is continuous. It can be shown that the induced map

$$\iota^{\wedge} : G^{\wedge} \longrightarrow H^{\wedge} \tag{4.1}$$

is then surjective.

$Hom(-, \mathbb{Q}/\mathbb{Z})$ **duality**

Given a topological group G, $Hom(G, \mathbb{Q}/\mathbb{Z})$ is the topological group of continuous homomorphisms, where $\mathbb{Q}/\mathbb{Z}$ is endowed with the discrete topology. The Pontryagin duality $Hom(-, \mathbb{R}/\mathbb{Z})$ and $Hom(-, \mathbb{Q}/\mathbb{Z})$ duality do not always coincide. For instance, $Hom(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z}$ whereas $Hom(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$.

However, if G is either a profinite abelian or discrete abelian torsion group, then there is an isomorphism of topological groups

$$Hom(G, \mathbb{R}/\mathbb{Z}) \cong Hom(G, \mathbb{Q}/\mathbb{Z}),$$

for the compact-open topology. Moreover, if G is a profinite abelian group (resp., discrete abelian torsion group) then $Hom(G, \mathbb{Q}/\mathbb{Z})$ is a discrete abelian torsion group (resp., profinite abelain group), see[6, Th. 2.9.6, p.64].

The group $K/\wp(K)$

Proposition 4.2. *Let* $K = \mathbb{F}_q((x))$, $q = p^s$. *Then* $K/\wp(K)$ *is a discrete belian torsion group.*

Proof. The ring of integers decomposes as a (direct) sum

$$\mathfrak{o} = \mathbb{F}_q + \mathfrak{p}$$

and we have

$$\wp(\mathfrak{o}) = \wp(\mathbb{F}_q) + \wp(\mathfrak{p}) \subset \wp(\mathbb{F}_q) + \mathfrak{p}.$$

Now, $\wp(x) = a$ has a solution in $\mathfrak{o}$ by Hensel's lemma whenever $a \in \wp(\mathbb{F}_q) + \mathfrak{p}$. Therefore,

$$\wp(\mathfrak{o}) = \wp(\mathbb{F}_q) + \mathfrak{p}$$

and $\mathfrak{p} \subset \wp(K)$. It follows that $\wp(K)$ is an open subgroup of K and $K/\wp(K)$ is discrete.

Since $\wp(K)$ is annihilated by p, $K/\wp(K)$ is a torsion group. □

The group $K^\times/K^{\times p}$

For every local field K there is a canonical isomorphism

$$K^\times \cong \mathbb{Z} \times \mathcal{U}$$

where $\mathcal{U}$ is the group of units

$$\mathcal{U} = \{x \in K^\times : \nu(x) = 0\}.$$

If K has positive characteristic p, then

$$\mathcal{U} \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots = \prod_{\mathbb{N}} \mathbb{Z}_p,$$

a direct product of countable many copies of the ring of p-adic integers, see [2, p.25].

Proposition 4.3. *$K^\times/K^{\times p}$ is a profinite abelian p-torsion group.*

Proof. Give $\mathbb{Z}$ the discrete topology and $\mathbb{Z}_p$ the p-adic topology. Then, for the product topology, $K^\times = \mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}_p$ is a topological group, locally compact, Hausdorff and totally disconnected.

$K^{\times p}$ decomposes as a product of countable many components

$$K^{\times p} \cong p\mathbb{Z} \times p\mathbb{Z}_p \times p\mathbb{Z}_p \times \ldots = p\mathbb{Z} \times \prod_{\mathbb{N}} p\mathbb{Z}_p.$$

Denote by $y = \prod_n y_n$ and element of $\prod_{\mathbb{N}} \mathbb{Z}_p$, where $y_n = \sum_{i=0}^{\infty} a_{i,n} p^i \in \mathbb{Z}_p$, for every n.

The map

$$\varphi : \mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}_p \to \mathbb{Z}/p\mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z}, (x, y) \mapsto (x(mod p), \prod_n pr_0(y_n))$$

where $pr_0(y_n) = a_{0,n}$ is the projection, is clearly a group homomorphism.

Now, $\mathbb{Z}/p\mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z} = \prod_{n=0}^{\infty} \mathbb{Z}/p\mathbb{Z}$ is a topological group for the product topology, where each component $\mathbb{Z}/p\mathbb{Z}$ has the discrete topology.

Moreover, it is compact by Tyconoff Theorem, Hausdorff and totally disconnected [1, TGI.84, Prop. 10]. Therefore, $\prod_{n=0}^{\infty} \mathbb{Z}/p\mathbb{Z}$ is a profinite group.

Since

$$ker\varphi = p\mathbb{Z} \times \prod_{\mathbb{N}} p\mathbb{Z}_p,$$

it follows that there is an isomorphism of topological groups

$$K^\times/K^{\times p} \cong \prod_{\mathbb{N}} p\mathbb{Z}_p,$$

where $K^\times/K^{\times p}$ is given the quotient topology. Therefore, $K^\times/K^{\times p}$ is profinite. □

$Hom(-, \mathbb{Z}/p\mathbb{Z})$ **duality**

Finally, we restrict further to the subcategory of abelian torsion groups. There is a canonical isomorphism

$$Tor_p(\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$$

where $Tor_p(\mathbb{Q}/\mathbb{Z})$denotes the p-torsion subgroup of $\mathbb{Q}/\mathbb{Z}$.

Since $K/\wp(K)$ and $K^\times/K^{\times p}$ are both abelian torsion groups annihilated by $ch(K) = p$, it follows that $Hom(-, \mathbb{Q}/\mathbb{Z})$ duality and $Hom(-, \mathbb{Z}/p\mathbb{Z})$ duality coincide for the groups $K/\wp(K)$ and $K^\times/K^{\times p}$.

Proposition 4.4. *The Artin-Schreier symbol induces isomorphisms of topological groups*

$$K^\times/K^{\times p} \xrightarrow{\simeq} Hom(K/\wp(K), \mathbb{Z}/p\mathbb{Z}), bK^{\times p} \mapsto (a + \wp(K) \mapsto [a, b))$$

and

$$K/\wp(K) \xrightarrow{\simeq} Hom(K^\times/K^{\times p}, \mathbb{Z}/p\mathbb{Z}), a + \wp(K) \mapsto (bK^{\times p} \mapsto [a, b))$$

Proof. From proposition 4.1, the Artin-Schreier symbol is a non-degenerate pairing and so induces monomorphisms

$$\varphi : K^\times/K^{\times p} \hookrightarrow Hom(K/\wp(K), \mathbb{Z}/p\mathbb{Z})$$

and

$$\psi : K/\wp(K) \hookrightarrow Hom(K^\times/K^{\times p}, \mathbb{Z}/p\mathbb{Z}).$$

Since $K/\wp(K)$ is discrete (proposition 4.2), ψ is continuous. By (4.1), we have a surjective map

$$\psi^\wedge : Hom(Hom(K^\times/K^{\times p}, \mathbb{Z}/p\mathbb{Z}), \mathbb{Z}/p\mathbb{Z}) \to Hom(K/\wp(K), \mathbb{Z}/p\mathbb{Z}).$$

defined by $a + \wp(K) \mapsto \chi \circ \psi(a + \wp(K))$.

Now,

$$K^\times/K^{\times p} \cong Hom(Hom(K^\times/K^{\times p}, \mathbb{Z}/p\mathbb{Z})$$

via the map $b \mapsto (\chi \mapsto \chi(b))$.

Composing with $\psi^\wedge$ we have a continuous surjective homomorphism

$$K^\times/K^{\times p} \to Hom(K/\wp(K), \mathbb{Z}/p\mathbb{Z}).$$

This map is precisely φ. It follows that φ is a continuous isomorphism. (Note that this agrees with $K^\times/K^{\times p}$ being profinite.)

Finally, we have

$$K/\wp(K) \cong Hom(Hom(K/\wp(K), \mathbb{Z}/p\mathbb{Z}), \mathbb{Z}/p\mathbb{Z})) \cong Hom(K^\times/K^{\times p}, \mathbb{Z}/p\mathbb{Z}).$$

□

5. The principal series of $SL_2(\mathbb{F}_2((x)))$

In this section we will work with the field $K = \mathbb{F}_2((x))$ of Laurent series with coefficients in $\mathbb{F}_2$. K is a locally compact, totally disconnected, non-discrete field with positive characteristic 2. The ring of integers is $\mathfrak{o} = \mathbb{F}_2[[x]]$, the ring of integral power series in x with coefficients in $\mathbb{F}_2$. The maximal ideal is $\mathfrak{p} = x\mathbb{F}_2[[x]]$ and the residue field is $\kappa = \mathfrak{o}/\mathfrak{p} = \mathbb{F}_2$.

We are interested in the set of unitary characters of $K^\times$. For $\chi \in \widehat{K^\times}$, we may write $\chi = |.|^{is}\chi_0$, where $s \in \mathbb{T}$ and $\chi_0 \in \widehat{U}$. If $\chi_0 \equiv 1$ we say that χ is unramified.

There is a canonical isomorphism

$$\widehat{K^\times} \cong \widehat{\mathbb{Z} \times U} \cong \mathbb{T} \times \widehat{U}$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle. When $K = \mathbb{F}_2((x))$, we have, according to [2, p.25],

$$U \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times ...$$

with countably infinite many copies of $\mathbb{Z}_2$, the ring of 2-adic integers.

Artin-Schreier theory provides a way to parametrize all the quadratic extensions of $K = \mathbb{F}_2((x))^\times$: there is a bijection between the set of quadratic extensions of $\mathbb{F}_2((x))^\times$ and the group

$$\mathbb{F}_2((x))^\times/\mathbb{F}_2((x))^{\times 2} \cong G_2$$

where G_2 is the Galois group of the maximal abelian extension of exponent 2. Since G_2 is an infinite profinite group, there are countably many quadratic extensions.

To each quadratic extension we attach a unique quadratic character of $\mathbb{F}_2((x))^\times$ using the Artin-Schreier symbol (proposition 4.4). To be more precise, the character

$$\chi = \chi_a = [a, .) \in Hom(K^\times/K^{\times 2}, \mathbb{Z}/2\mathbb{Z})$$

is the quadratic character attached to the quadratic extension $K(\alpha)$ of K, where $\alpha^2 - \alpha = a$, $a \in K$.

The group $SL_2(K)$ is the set of all matrices

$$\left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, a, b, c, d \in K \right\}$$

Let χ denote a unitary character of the multiplicative group of K. Then χ extends to a character, still denoted χ, of the standard Borel subgroup B of $G = SL_2(K)$. We now form the induced representation $Ind_B^G(\chi)$. We use normalized induction.

From now on, χ will be a quadratic character. It is a classical result that the unitary principal series for GL_2 are irreducible. For a representation of GL_2 parabolically induced by $1 \otimes \chi$, Clifford theory tells us that the dimension of the intertwining algebra of its restriction to SL_2 is 2. This is exactly the induced representation of SL_2 by χ. This leads to reducibility of the induced representation $Ind_B^G(\chi)$ into two inequivalent pieces. Thanks to M. Tadic for helpful comments at this point.

Let W_K be the Weil group of our local field K. Let

$$W_K \times SL(2, \mathbb{C}) \to W_K$$

be the projection onto the first factor.

The isomorphism $W_K^{ab} \cong K^\times$ of local classfield theory determines a canonical map

$$W_K \to K^\times$$

Let χ_a be the quadratic character attached to the quadratic extension $K(\alpha)$ of K. The character χ_a is a map

$$K^\times \to \mathbb{C}^\times$$

Let $G = SL_2(K)$ and let T be the standard maximal torus of $SL_2(K)$. Let $G^\vee$ be the Langlands dual of G, and let $T^\vee \subset G^\vee$. Then $G^\vee = PGL(2, \mathbb{C})$. We have an identification

$$\mathbb{C}^\times \cong T^\vee$$

via the map which sends the complex number z to the point in $T^\vee$ with homogeneous coordinates $(z : 1)$. Finally, we have the inclusion

$$T^\vee \to G^\vee$$

Combining all these maps, we have an L-parameter

$$\phi_a : W_K \times SL(2, \mathbb{C}) \to W_K \to K^\times \to \mathbb{C}^\times \to T^\vee \to G^\vee$$

Theorem 5.1. *The L-parameters ϕ_a serve as parameters for the L-packets in the principal series of $SL_2(K)$.*

We anticipate that this theorem will lead to a description of the unitary principal series of SL_2 as a topological space, along the lines of [5]. In that case, there will be countably many double points, one for each quadratic extension $K(\alpha)$.

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