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1 Introduction

This material was originally the first part of our preprint [18]. The problem addressed there is that of determining when the category of $\mathcal{O}_X$-modules, where $\mathcal{O}_X$ is a ringed space, is locally finitely presented. The main results of that preprint, with some but not all of the proofs, were reported in [16] and, in 2009, the first author split off the main part of the preprint, retaining the title, for publication. The results remaining here are either background or, at least regarding the main aim of [18], are superceded by the results in the version prepared for publication. Nevertheless, they might be found to be of use so are preserved as this account of the sheafification localisation of presheaves of modules over a ringed space.

The problem addressed in [18] is that of determining when the category of $\mathcal{O}_X$-modules, where $\mathcal{O}_X$ is a ringed space, is locally finitely presented.

In the first section here we give a proof of the, known, result that the category, PreMod-$\mathcal{O}_X$, of presheaves over a presheaf, $\mathcal{O}_X$, of rings is locally finitely presented, with the presheaves of the form $j_0\mathcal{O}_U$ with U open being a generating set of finitely presented objects (2.14). Here j_0 denotes extension by zero in the sense of presheaves. We also prove that if, for each open U, the ring $\mathcal{O}_X(U)$, of sections over U is right coherent then the category PreMod-$\mathcal{O}_X$ is locally coherent (and that, under a flatness hypothesis, the converse is true) (2.18).

In the subsequent section we investigate that torsion theory, localisation at which is the process of sheafification. In particular we identify a finiteness condition (originally introduced in [14]) which guarantees preservation of local finite presentation and use that to derive the following result from the presheaf case. If X is a noetherian topological space and $\mathcal{O}_X$ is any sheaf of rings over X then the category, Mod-$\mathcal{O}_X$, of sheaves of modules over $\mathcal{O}_X$ is locally finitely presented (3.11) (in [18] we show that it is enough that X have a basis of compact open sets).

2 The Category PreMod-$\mathcal{O}_X$

For background on sheaves and presheaves see, for instance, [2], [8], [6], [21]

Let $\mathcal{O}_X$ be a presheaf of rings (all our rings will be associative with an identity $1 \neq 0$) and denote by PreMod-$\mathcal{O}_X$ the category of presheaves over X which are $\mathcal{O}_X$-pre-modules. That is, $M \in$ PreMod-$\mathcal{O}_X$ means that M is a presheaf of abelian groups such that, for each open set $U \subseteq X$, $M(U)$ is a right R_U-module, where $R_U = \mathcal{O}_X(U)$ (we use this notation throughout the paper), and such that, for every inclusion $V \subseteq U \subseteq X$ of open subsets of X, the restriction map, $\mathrm{res}^M_{U,V} : M(U) \longrightarrow M(V)$, is a homomorphism of R_U-modules, where we regard $M(V)$ as an R_U-module *via* $\mathrm{res}^{\mathcal{O}_X}_{U,V} : R_U \longrightarrow R_V$.

Example 2.1. Preshv_k/X, *the category of presheaves of k-vectorspaces over X,, where k is a field has this form. Take $\mathcal{O}_X$ to be the constantly k presheaf: $\mathcal{O}_X(U) = k$, $\mathrm{res}^{\mathcal{O}_X}_{U,V} = \mathrm{id}_k$ for $U \subseteq X$ open.*

Theorem 2.2. *([2, Section I.3])* PreMod-$\mathcal{O}_X$ *is a Grothendieck abelian category.*

An object C of a category $\mathcal{C}$ is **finitely presented** (**fp**) if the representable functor $(C, -) : \mathcal{C} \longrightarrow \mathbf{Ab}$ commutes with direct limits in $\mathcal{C}$. If $\mathcal{C}$ is Grothendieck abelian, then it is sufficient to check that for every directed system $((D_\lambda)_\lambda, (g_{\lambda\mu} : D_\lambda \longrightarrow D_\mu)_{\lambda<\mu})$ in $\mathcal{C}$, with limit $(D, (g_{\lambda\infty} : D_\lambda \longrightarrow D)_\lambda)$, every $f \in (C, D)$ factors through some $g_{\lambda\infty}$. A category $\mathcal{C}$ is **finitely accessible** if the full subcategory, $\mathcal{C}^{\mathrm{fp}}$, of finitely presented objects is skeletally small and if every object of $\mathcal{C}$ is a direct limit of finitely presented objects; if $\mathcal{C}$ also is complete (equivalently, [1, 2.47], cocomplete) then $\mathcal{C}$ is said to be **locally finitely presented** (**lfp**). Abelian categories which are finitely accessible hence, [4, 2.4], Grothendieck and lfp are in many ways as well-behaved as categories of modules over rings. In particular, objects of $\mathcal{C}$ are determined by their "elements" (morphisms from finitely presented objects) and these "elements" have finitary character (as opposed to what one has for merely presentable categories). Such categories have a good model theory and they admit a useful embedding into a related functor category (see e.g., [7], [9], [10], [16]). The category PreMod-$\mathcal{O}_X$ is locally finitely presented (see [3, p. 7] for example, but we give a proof), indeed it is a variety of finitary many-sorted algebras in the sense of [1, Section 3A]; on the other hand Mod-$\mathcal{O}_X$ need not be locally finitely presented (see [18]).

Let $U \subseteq X$ be open and let $G \in$ PreMod-$\mathcal{O}_U$ (where by $\mathcal{O}_U$ we denote the restriction, $\mathcal{O}_X \restriction U$, of the structure presheaf $\mathcal{O}_X$ to U - this is another notation that will be used throughout). Define the functor j_0G on the category of open sets of X by $j_0G(V) = \begin{cases} G(V) & \text{if } V \subseteq U \\ 0 & \text{otherwise} \end{cases}$ on objects V, and in the obvious way on inclusions $W \subseteq V$.

Proposition 2.3. *If $G \in$* PreMod-$\mathcal{O}_U$*, then $j_0G \in$* PreMod-$\mathcal{O}_X$*, indeed j_0 extends to a functor from* PreMod-$\mathcal{O}_U$ *to* PreMod-$\mathcal{O}_X$ *which is exact.*

Proof. Given $f : G \longrightarrow H$ define $j_0f : j_0G \longrightarrow j_0H$ by $j_0f.V = \begin{cases} f_V & \text{if } V \subseteq U, \\ 0 & \text{otherwise} \end{cases}$ (we use f_V instead of the notationally more consistent $fV : GV \longrightarrow HV$). Clearly this does define a functor. For presheaves, exactness is equivalent to exactness at each open set. Clearly j_0 preserves this and hence is an exact functor. $\square$

Remark 2.4. *If G already is a sheaf then one might also consider the extension of G by zero, $j_!G$, which is (see [6, p.68, p.111]) the sheafification of j_0G, but, as the next example shows, this may well be different from j_0G.*

Example 2.5. *Let $X = \{x, y\}$ be a two point set with the discrete topology. Let $U = \{x\}$ and define the presheaf G on U by $GU = \mathbb{Z}$. Then $j_0G.X = 0$ but $j_!G.X \neq 0$ since we have the open cover $X = U \cup (X \setminus U)$ and the compatible sections $1 \in j_!G.U$, $0 \in j_!G(X \setminus U)$ and hence a global section $a \in j_!G.X$ agreeing with these sections (in particular $a \neq 0$).*

Remark 2.6. *(a) $F \in \text{PreMod-}\mathcal{O}_U$ implies $(j_0F)_U \simeq F$, indeed the composite of j_0 then restriction to U is the identity on $\text{PreMod-}\mathcal{O}_U$.*
(b) $G \in \text{PreMod-}\mathcal{O}_X$ implies $j_0(G_U) \leq G$.

These observations follow directly from the definitions.

Proposition 2.7. *If $U \subseteq X$ is open, $F \in \text{PreMod-}\mathcal{O}_U$, $G \in \text{PreMod-}\mathcal{O}_X$ then there is a natural isomorphism $(j_0F, G) \simeq (F, G_U)$. Indeed $j_0 : \text{PreMod-}\mathcal{O}_U \longrightarrow \text{PreMod}\mathcal{O}_X$ is left adjoint to the restriction functor $(-)_U : \text{PreMod-}\mathcal{O}_X \longrightarrow \text{PreMod-}\mathcal{O}_U$. In particular the functor j_0 is full.*

Proof. Given $f : j_0F \longrightarrow G$ define $f' : F \longrightarrow G_U$ by, for $V \subseteq U$, $f'_V = f_V : FV = j_0F.V \longrightarrow GV = G_UV$. Conversely, given $f' : F \longrightarrow G_U$, define $f : j_0F \longrightarrow G$ by $f_V = \begin{cases} f'_V : FV \longrightarrow GV & \text{if } V \subseteq U \\ 0 : FV(=0) \longrightarrow GV & \text{otherwise} \end{cases}$. Clearly these processes are mutually inverse and the isomorphism is natural.

To see that j_0 is full, note that $(j_0F, j_0F') \simeq (F, (j_0F')_U) \simeq (F, F')$, for $F, F' \in \text{PreMod-}\mathcal{O}_X$, by the remark 2.6. $\square$

Corollary 2.8. *For $U \subseteq X$ open, $j_0\text{PreMod-}\mathcal{O}_U$ is a localising subcategory of $\text{PreMod-}\mathcal{O}_X$.*

Proof. By, for instance, [11, 4.6.3] it is enough to check that $j_0\text{PreMod-}\mathcal{O}_U$ is a Serre subcategory and is closed under direct sums in the larger category. These are trivial to check. $\square$

Since we have a localising subcategory, there is a corresponding quotient category and localisation functor, which we now identify.

Corollary 2.9. *The localisation functor corresponding to the localising subcategory $j_0(\text{PreMod-}\mathcal{O}_U)$ is given by $G \mapsto qG$ where $qG.V = \begin{cases} GV & \text{if } V \not\subseteq U \\ 0 & \text{if } V \subseteq U \end{cases}$.*

Proof. First note that $G \in \text{PreMod-}\mathcal{O}_X$ is torsionfree iff for all $F \in \text{PreMod-}\mathcal{O}_U$ we have $(j_0F, G) = 0$, equivalently, $(F, G_U) = 0$. Taking $F = G_U$ we deduce that $G_U = 0$ if G is torsionfree and, conversely, if $G_U = 0$ then G is torsionfree.

Now suppose that G is torsionfree and $G \leq H$ with H/G torsion. For all open $V \subseteq X$ we have the exact sequence $0 \longrightarrow GV \longrightarrow HV \longrightarrow (H/G)V \longrightarrow 0$. If V is not contained in U then $(H/G)V = 0$ and so $GV = HV$. In the case that $V \subseteq U$ we have $GV = 0$ and hence $HV = (H/G)V$. Therefore the sequence of presheaves splits and we obtain $H = G \oplus H'$ where $H' \simeq (H/G)$, indeed $H = G \oplus j_0(H/G)$.

Therefore every torsionfree sheaf is already injective with respect to the torsion theory and so the quotient category consists of all $G \in \text{PreMod-}\mathcal{O}_X$ with $G_U = 0$ and the localisation functor is as described (since it is just the functor which sends G to G/G_U). $\square$

The corresponding result for the category, $\text{Mod-}\mathcal{O}_X$, of sheaves of modules over a sheaf, $\mathcal{O}_X$, of rings holds true with $j_!\text{Mod-}\mathcal{O}_U$ in place of $j_0\text{PreMod-}\mathcal{O}_U$. In that case the localisation functor is just restriction to $X \setminus U$ and the quotient category is equivalent to $\text{Mod-}\mathcal{O}_{X\setminus U}$ (cf. [8, p. 107, 6.4]).

Lemma 2.10. *Let $G \in \text{PreMod-}\mathcal{O}_X$ and let $f \in (\mathcal{O}_X, G)$. Then f is determined by $f(1) \in GX$, where $1 \in \mathcal{O}_X(X) = R_X$ is the identity element of this ring.*

Proof. Let $U \subseteq X$ be open and let $1_U = \text{res}_{X,U}(1)$ (= the identity element of R_U). We have $f_U \text{res}^{\mathcal{O}_X}_{X,U} = \text{res}^G_{X,U} f_X$ since f is a morphism.

$$\begin{tikzcd} \mathcal{O}_X(X) \arrow[r, "f_X"] \arrow[d, "\text{res}"'] & GX \arrow[d, "\text{res}"] \\ \mathcal{O}_X(U) \arrow[r, "f_U"'] & GU \end{tikzcd}$$

So $f_U 1_U = \text{res}^G_{X,U}(f1)$. If $r \in \mathcal{O}_X(U)$ then, since f_U is R_U-linear, we have $f_U r = (f_U 1_U)r = (\text{res}^G_{X,U}(f1))r$, as claimed. $\square$

Proposition 2.11. *Let $G \in \text{PreMod-}\mathcal{O}_X$. Then there is a natural isomorphism $(\mathcal{O}_X, G) \simeq GX$.*

Proof. Lemma 2.10 gives the map "$\longrightarrow$ ". For the inverse map, given $a \in GX$, define $f \in (\mathcal{O}_X, G)$ as follows. Let $U \subseteq X$ be open. Define $f_U : \mathcal{O}_X(U) \longrightarrow GU$ by: if $r \in \mathcal{O}_X(U)$ set $f_U r = (\text{res}^G_{X,U} a)r$. By the proof of 2.10 this gives a well-defined map and we easily check that the processes are inverse and the isomorphism is natural. $\square$

Proposition 2.12. *$\mathcal{O}_X$ is a finitely presented element of* $\text{PreMod-}\mathcal{O}_X$.

Proof. Let $\{G_\lambda\}_\lambda$ be a directed system in $\text{PreMod-}\mathcal{O}_X$ and let $f \in (\mathcal{O}_X, G)$ where $G = \varinjlim G_\lambda$. Consider $f1 \in GX$. Since $G = \varinjlim G_\lambda$ there is λ and $a \in G_\lambda$ such that $g_{\lambda,\infty} a = f1$, where $g_{\lambda,\infty} : G_\lambda \longrightarrow G$ is the canonical map to the limit.

Define $f_\lambda : \mathcal{O}_X \longrightarrow G_\lambda$ by $f_\lambda 1 = a$ (see 2.11). Then $g_{\lambda,\infty} f_\lambda 1 = g_{\lambda,\infty} a = f1$ so, by 2.11, $f = g_{\lambda,\infty} f_\lambda$ and f factors through some G_λ, as required. $\square$

Essential in the proof above is the fact that in the category of presheaves direct limits are computed at the level of sections: if G is the presheaf direct limit of a directed system $(G_\lambda)_\lambda$ of presheaves then, since this is how we define the direct limit of functors, for each open set U we have that GU is the direct limit of the abelian groups $G_\lambda U$. Even if all the G_λ are sheaves this recipe will yield a presheaf rather than a sheaf in general.

Proposition 2.13. *If $U \subseteq X$ is open then $j_0\mathcal{O}_U$ is a finitely presented object of* PreMod-$\mathcal{O}_X$.

Proof. Say $G = \varinjlim G_\lambda$, where the $G_\lambda \in$ PreMod-$\mathcal{O}_X$ form a directed system and let $f : j_0\mathcal{O}_U \longrightarrow G$. By 2.7 $(j_0\mathcal{O}_U, G) \simeq (\mathcal{O}_U, G_U)$ with f corresponding to, say, $f' : \mathcal{O}_U \longrightarrow G_U$.

Now the functor $(-)_U$ commutes with direct limits: if $V \subseteq U$ then $G_U V = GV = \varinjlim(G_\lambda V)$ (as remarked above) $= (\varinjlim G_\lambda \upharpoonright U)V$. So $G \upharpoonright U = \varinjlim(G_\lambda \upharpoonright U)$. Therefore, by 2.12 applied in PreMod-$\mathcal{O}_U$, there is λ and $f'_\lambda : \mathcal{O}_U \longrightarrow (G_\lambda)_U$ with $f' = g'_{\lambda,\infty} f'_\lambda$ (where $g'_{\lambda,\infty} = (g_{\lambda,\infty})_U$ is the relevant colimit map).

Applying j_0 we obtain $j_0 f' = j_0 g'_{\lambda,\infty} j_0 f'_\lambda$. Composing with the canonical maps from $j_0|U$ to G and from $j_0 G_\lambda|U$ to G_λ (and noting that the composition of the first such map with $j_0 f'$ is just f) we obtain a factorisation of f through $g_{\lambda,\infty}$, as required $\square$

Theorem 2.14. *(e.g. [3, p.7 Prop. 6]) The $j_0\mathcal{O}_U$, with $U \subseteq X$ open, form a generating set of finitely presented objects of* PreMod-$\mathcal{O}_X$.

Proof. Let $F \in$ PreMod-$\mathcal{O}_X$ and let $U \subseteq X$ be open. Let $a \in FU$. Then there is $f' : \mathcal{O}_U \longrightarrow F_U$ with $f'_U : \mathcal{O}_U(U) \longrightarrow F_U(U)$ taking 1 to a. Under the adjunction $(\mathcal{O}_U, F_U) \simeq (j_0\mathcal{O}_U, F)$, f' corresponds to $f = f_{U,a}$, say, and $f_U 1_U = a$ (by, e.g., the explicit description of j_0).

Then $\bigoplus_{U \subseteq X} \bigoplus_{a \in FU} f_{U,a} : \bigoplus_U \bigoplus_a j_0\mathcal{O}_U \longrightarrow F$ is, by construction, onto at each open $U \subseteq X$ and hence is an epimorphism, as required. $\square$

Corollary 2.15. *For any space X and presheaf $\mathcal{O}_X$ of rings on X the category* PreMod-$\mathcal{O}_X$ *is a locally finitely presented abelian Grothendieck category.*

Note that it is necessary to use all open sets U: a basis of open sets will not suffice. For consider a presheaf F with $FX \neq 0$ but $FU = 0$ for all proper open subsets U of X. We need $\mathcal{O}_X = j_0\mathcal{O}_X$ to generate this presheaf.

We may also ask when the category of presheaves has stronger finiteness properties, such as being locally noetherian or artinian. Here we consider local coherence, where a locally finitely presented abelian category is **locally coherent** if every finitely generated subobject of each finitely presented object is finitely presented. We say that a ring R is (right) **coherent** if the category, Mod-R, of (right) modules over R is locally coherent.

Let $F \in$ PreMod-$\mathcal{O}_X$, let $U \subseteq X$ be open and let $a \in FU$. Define the presheaf $\langle a \rangle$ by $V \mapsto \begin{cases} \mathrm{res}_{UV}(a).R_V & \text{if } V \subseteq U \\ 0 & \text{otherwise} \end{cases}$. Then $\langle a \rangle \in$ PreMod-$\mathcal{O}_X$ and we have the natural inclusion $\langle a \rangle \leq F$.

We say that an object F in an abelian category is **finitely generated** if $F = \sum_\lambda F_\lambda$ implies $F = \sum_1^n F_{\lambda_i}$ for some $\lambda_1, \dots, \lambda_n$. If the category if locally finitely presented then it is equivalent that F be the image of a finitely presented object.

Lemma 2.16. *The presheaf $F \in$ PreMod-$\mathcal{O}_X$ is finitely generated iff there exist open subsets $U_1, \dots, U_n$ of X and sections $a_i \in FU_i$ such that $F = \sum_1^n \langle a_i \rangle$.*

Proof. ($\Rightarrow$) We have, on comparing at each open set U, $F = \sum_{U \subseteq X} \sum_{a \in FU} \langle a \rangle$ and so F finitely generated implies that F is a sum of finitely many of these.

($\Leftarrow$) If $F = \sum_\lambda F_\lambda$ (a directed sum) then choose, for each $i = 1, \dots, n$, some λ_i such that $a_i \in F_{\lambda_i} U_i$ (since $FU_i = \sum_\lambda F_\lambda U_i$ this exists). Then $\langle a_i \rangle \leq F_{\lambda_i}$ and so $F \leq \sum_1^n F_{\lambda_i}$. $\square$

The $j_0 \mathcal{O}_U$ form a generating set of finitely presented presheaves so, if F is finitely presented, then there is an exact sequence $\bigoplus_{j=1}^m j_0 \mathcal{O}_{V_j} \longrightarrow \bigoplus_{i=1}^n j_0 \mathcal{O}_{U_i} \xrightarrow{f} F \longrightarrow 0$ where the first map is given, in terms of components, as $(f_{ij})_{ij}$, where $f_{ij} : j_0 \mathcal{O}_{V_j} \longrightarrow j_0 \mathcal{O}_{U_i}$.

Note that $j_0 \mathcal{O}_U = \langle 1_{R_U} \rangle$ and so $f_{ij} : j_0 \mathcal{O}_{V_j} \longrightarrow j_0 \mathcal{O}_{U_i}$ equals 0 if V_j is not contained in U_i. Let $a_i = f(1_{U_i})$ (writing 1_U for 1_{R_U}) - so $F = \sum_1^n \langle a_i \rangle$.

For each j we have $f(\sum_1^n f_{ij} 1_{V_j}) = 0$. Set $f_{ij} 1_{V_j} = s_{ij} \in R_{V_j}$ so $f(s_{ij}) = \begin{cases} 0 & \text{if } V_j \not\subseteq U_i \\ \mathrm{res}^F_{U_i V_j}(a_i).s_{ij} & \text{if } V_j \subseteq U_i \end{cases}$. Then the relations between the $a_1, \dots, a_n$ are generated by $\sum_{V_j \subseteq U_i} \mathrm{res}^F_{U_i V_j}(a_i).s_{ij} = 0$, for $j = 1, \dots, m$.

From this we get a presentation of each FU (as an R_U-module): the generators of FU are $\{\mathrm{res}^F_{U_i U}(a_i) = a_i^U \text{ say } : U \subseteq U_i\}$ and the relations on these generators are $\{\sum_{i : V_j \subseteq U_i} a_i^U.s_{ij} : U \subseteq V_j\}$.

Lemma 2.17. *If $U' \subseteq U$ are open subsets of X and if $U' \subseteq U_i$ imples $U \subseteq U_i$ for all i and if $U' \subseteq V_j$ implies $U \subseteq V_j$ for all j then $FU' \simeq FU \otimes_{R_U} R_{U'}$.*

Proof. By assumption FU is generated by the a_i^U with $U \subseteq U_i$ and with a relation $\sum_{V_j \subseteq U_i} a_i^U s_{ij} = 0$ for each j with $U \subseteq V_j$. That is, we have a presentation of FU :

$\bigoplus_{j : U \subseteq V_j} R_U \longrightarrow \bigoplus_{i : U \subseteq U_i} R_U \longrightarrow FU \longrightarrow 0$ where the first map is given by the matrix $(t_{ij})_{ij}$ over R_U, where $t_{ij} = \mathrm{res}^{\mathcal{O}_X}_{V_j U}(s_{ij})$ ($= 0$ if $V_j \not\subseteq U_i$).

Similarly we have a presentation of FU' :

$\bigoplus_{j : U' \subseteq V_j} R_U \longrightarrow \bigoplus_{i : U' \subseteq U_i} R_U \longrightarrow FU' \longrightarrow 0$

where the first map is given by the matrix $(t'_{ij})_{ij}$ over $R_{U'}$, where $t'_{ij} = \mathrm{res}^{\mathcal{O}_X}_{V_j U'}(s_{ij}) = \mathrm{res}^{\mathcal{O}_X}_{U U'}(t_{ij})$.

Applying $- \otimes_{R_U} R_{U'}$ to the first sequence we obtain, by right exactness of $\otimes$, and since $R_U \otimes_{R_U} R_{U'} \simeq R_{U'}$, the exact sequence

$\bigoplus_j R_{U'} \longrightarrow \bigoplus_i R_{U'} \longrightarrow FU \otimes_{R_U} R_{U'} \longrightarrow 0$

where the ij-component of the first map is $t_{ij} \otimes_{R_U} 1_{R_{U'}} = \mathrm{res}^{\mathcal{O}_X}_{U U'}(t_{ij}) = t'_{ij}$ and hence $FU' \simeq FU \otimes_{R_U} R_{U'}$. $\square$

Theorem 2.18. *Let $\mathcal{O}_X$ be a presheaf of rings on the space X. If, for each open subset U of X, the ring, $R_U = \mathcal{O}_X(U)$, of sections over U is a right coherent ring then the category,* PreMod-$\mathcal{O}_X$, *of presheaves over $\mathcal{O}_X$ is locally coherent.*

If, for each inclusion $V \subseteq U$ of open sets, the ring R_V is a flat left R_U-module (via the corresponding morphism of rings) then the converse also holds.

Proof. Suppose that F, with presentation as above, is a finitely presented presheaf and suppose that $G \leq F$ is finitely generated, say $g : \bigoplus_1^p j_0 \mathcal{O}_{W_k} \longrightarrow G$ is epi. Let $b_k = g(1_{W_k})$, so $G = \sum_1^p \langle b_k \rangle$ (and, since $G \leq F$, we have $b_k = \sum_{U_i \supseteq W_k} a_i^{W_k}.t_{ik}$ for some $t_{ik} \in R_{W_k}$). We claim that, under the coherence hypothesis, G is finitely presented.

Let $U \subseteq X$ and consider $GU \leq FU$. Since GU is a finitely generated R_U-submodule of the finitely presented R_U-module FU and since R_U is right coherent, there is a finite generating set of relations on the generators $b_k^U = \mathrm{res}_{W_kU}(b_k)$, where $W_k \supseteq U$, of GU, say $\sum_k b_k^U r_{hk} = 0$ for some $r_{hk} \in R_U$, where $h = 1, \ldots, n_U$.

If $U' \subseteq U$ is such that $U' \subseteq U_i, V_j, W_k$ for any i, j, k implies the same for U then we observe that the module GU' is generated by the $b_k^{U'} = \mathrm{res}_{UU'}(b_k^U)$ and has generating set of relations $\sum_k b_k^{U'} \mathrm{res}_{UU'}^{\mathcal{O}_X} r_{hk} = 0$ for $h = 1, \ldots, n_U$: note that these follow from the above relations which hold in GU.

So, for each U of the form $U_{i_1} \cap \cdots \cap U_{i_N} \cap V_{j_1} \cap \cdots \cap V_{j_M} \cap W_{k_1} \cap \cdots \cap W_{k_P}$ take a finite generating set of relations on the b_k^U $(k = k_1, \ldots, k_p)$.

Then, we claim, the set of all these, as U varies over the finitely many such sets, gives a presentation of G by generators $b_1, \ldots, b_P$ and these finitely many relations.

To prove that, we need show only that these give the correct module, GU, for each U - but that is by construction and the lemma above.

Suppose now that the category of presheaves is locally coherent and that we have the flatness hypothesis. To check that R_U is right coherent it is enough to show that every finitely generated submodule of a finitely generated free module is finitely presented. So let $R_U^{(I)} \longrightarrow R_U^m \longrightarrow M$ where $M \leq R_U^n$ be an exact sequence. For any R_U-module N we have the $\mathcal{O}_U$-module $\bar{N}$ given by $\bar{N}.V = N \otimes_{R_U} R_V$ with the obvious restriction maps. Since tensor product is right exact and since, clearly, $\bar{R_U} = \mathcal{O}_U$ we obtain an exact sequence $\mathcal{O}_U^{(I)} \longrightarrow \mathcal{O}_U^m \longrightarrow \bar{M}$ and then, since j_0 is (right) exact (2.3) an exact sequence $j_0\mathcal{O}_U^{(I)} \longrightarrow j_0\mathcal{O}_U^m \longrightarrow j_0\bar{M} \longrightarrow 0$ $(*)$ in PreMod-$\mathcal{O}_X$. Since each R_V is a flat R_U-module, each morphism $M \otimes_{R_U} R_V \longrightarrow R_U^n \otimes_{R_U} R_V = R_V^n$ is monic and so $j_0\bar{M} \leq j_0\mathcal{O}_U^n$. Then, since $j_0\mathcal{O}_U$ is finitely presented, the index set I can, in the sequence $(*)$, be taken to be finite and, therefore, taking sections over U (an exact functor on presheaves), we see that I can be taken to be finite in the original exact sequence, as required. $\square$

3 Torsion theory on PreMod-$\mathcal{O}_X$ and localising presheaves to sheaves

By a torsion theory we will always mean a hereditary torsion theory - hence one determined by a class of injective objects (see e.g. [11] or [20]).

Proposition 3.1. *Let τ be a hereditary torsion theory on* PreMod-$\mathcal{O}_X$. *Then τ is determined by the collection of Gabriel filters:*
$\mathcal{U}_\tau(j_0\mathcal{O}_U) = \{\mathcal{I} \leq j_0\mathcal{O}_U : j_0\mathcal{O}_U/\mathcal{I} \in \mathcal{T}\}$,
where $\mathcal{T}$ denotes the class of τ-torsion objects.

Proof. The set of objects $j_0\mathcal{O}_U$, as U ranges over all open subsets of X, generates the category and is a set of finitely presented objects. Then it is easy to see (e.g. see [16, 11.1.11]) that the corresponding Gabriel filters determine τ. $\square$

For the purposes of this paper one may take the next proposition as a definition of "finite type" (or see [20]).

Proposition 3.2. *A hereditary torsion theory τ on* PreMod-$\mathcal{O}_X$ *is of finite type iff for all open $U \subseteq X$ the filter $\mathcal{U}_\tau(j_0\mathcal{O}_U)$ has a cofinal set of finitely generated presheaves.*

Proof. This follows from 2.14 and [12, 3.11]. $\square$

For the remainder of this section let τ be the torsion theory corresponding to the sheafification functor and let $\mathcal{T}$, $\mathcal{F}$ denote the corresponding torsion and torsionfree classes respectively (this torsion theory is discussed in [11, Section 4.7] for example). Then $\mathcal{T}$ is the class of presheaves with sheafification equal to 0. We continue for a little longer to assume that $\mathcal{O}_X$ is just a presheaf, rather than a sheaf, of rings.

Lemma 3.3. *Let $F \in$* PreMod-$\mathcal{O}_X$. *Then $F \in \mathcal{T}$ iff for every open $U \subseteq X$ and for every $a \in FU$ there is an open cover $\{U_i\}_i$ of U such that, for all i, we have* $\mathrm{res}^F_{U,U_i} a = 0$.

Proof. Clearly such a presheaf is torsion (its sheafification must be 0). Conversely, if $F \in \mathcal{T}$ and if U, a are as given, then a must be identified with $0 \in FU$ by the sheafification process and this can happen only if there is a cover of U as described. $\square$

Corollary 3.4. *Let $G \leq F \in$* PreMod-$\mathcal{O}_X$. *Then G is τ-dense in F (that is, F/G is τ-torsion) iff for every open $U \subseteq X$ and for every $a \in FU$ there is an open cover $\{U_i\}_i$ of U such that, for all i we have* $\mathrm{res}^F_{U,U_i} a \in GU_i$.

Recall that the **stalk** of a (pre)sheaf F at a point x is $F_x = \varinjlim_{x \in U} FU$.

Lemma 3.5. *Let $F \in$* PreMod-$\mathcal{O}_X$. *Then $F \in \mathcal{T}$ iff for every $x \in X$ we have $F_x = 0$.*

Proof. ($\Rightarrow$) $F \in \mathcal{T}$ implies $qF = 0$ where q is the sheafification functor (i.e. the localisation functor associated to τ) and this implies that $(qF)_x = 0$ for all $x \in X$ which, in turn, implies (since stalks are unchanged by sheafification) that $F_x = 0$ for all $x \in X$.

($\Leftarrow$) If every stalk of F is 0 then the same is true for qF and hence $qF = 0$, so $F \in \mathcal{T} = \ker(q)$. $\square$

Lemma 3.6. *Let $F \in$* PreMod-$\mathcal{O}_X$. *Then $F \in \mathcal{F}$ iff F is a mono(=separated) presheaf.*

Proof. We have $F \in \mathcal{F}$ implies F embeds in qF which, clearly, implies that F is a monopresheaf. On the other hand $F \notin \mathcal{F}$ implies $\tau F \neq 0$ but then τF being not a monopresheaf implies the same for F. $\square$

Proposition 3.7. *Let $U \subseteq X$ be open. Then $j_0\mathcal{O}_U$ has a cofinal family of finitely generated τ-dense subpresheaves iff U is compact.*

Proof. ($\Leftarrow$) Suppose that $G \leq F = j_0\mathcal{O}_U$ is τ-dense. Then there is an open cover $\{U_i\}_i$ of U, which, by compactness of U, we may take to be a finite cover $U_1, \dots, U_n$, such that for each i, $\mathrm{res}^F_{U,U_i}(1) \in G(U_i)$, where $1 \in FU$. That is $G \restriction U_i = F \restriction U_i = \mathcal{O}_{U_i}$ for each i.

Let $G_i = j_0\mathcal{O}_{U_i}$. Note that $G_i \le G$ (by the map corresponding to $\mathrm{Id} \in (\mathcal{O}_{U_i}, G \restriction U_i \simeq \mathcal{O}_{U_i}) \simeq (j_0\mathcal{O}_{U_i} = G_i, G)$). So $G_1 + \cdots + G_n \le G$ is a finitely generated subobject of G and, by construction and 3.4, is τ-dense, as required.

($\Rightarrow$) If U is not compact then choose an open cover $\{U_i\}_i$ of U which has no finite subcover and define $G \in \text{PreMod-}\mathcal{O}_X$ by $GV = FV$ if $V \subseteq U_i$ for some i, GV= 0 otherwise. So $G \le F$ and G is τ-dense in F since $\mathrm{res}_{U,U_i}(1) \in GU_i$ for every i.

If there were $G' \le G$ with G' finitely generated and τ-dense in G then there would be, in particular, an open cover $\{V_j\}_j$ with $\mathrm{res}_{U,V_j}(1) \in G'V_j$ for every j. Suppose, using 2.16, that the $a_l \in G'V_l$ with $l = 1, \dots, m$ generate G'. Since $G' \le G$, if $a_l \neq 0$ then $V_l \subseteq U_{i_l}$ for some i_l. But then, if V is an open set not contained in $U_{i_1} \cup \cdots \cup U_{i_m}$, and note that this is a proper subset of U, we must have $G'V = 0$ (since $G' = \sum_1^m \langle a_l \rangle$). So, choosing V_j not contained in this union, we see that $\mathrm{res}_{U,V_j}(1)$ (which is non-zero) is not in $G'V_j$, a contradiction. □

Corollary 3.8. *The localisation "sheafification" is of finite type iff every open subset of X is compact.*

Proof. This is immediate by 3.7 and 3.2. □

A subcategory of an lfp Grothendieck abelian category is **definable** (in the sense of [5]) if it is closed under products, direct limits and pure subobjects, equivalently if it is axiomatisable (in the canonical language of the category) and is closed under products.

Corollary 3.9. *The class of monopresheaves is a definable subcategory of* PreMod-$\mathcal{O}_X$ *iff every open subset of X is compact.*

Proof. By [12, 3.3] which says that the torsionfree objects for a hereditary torsion theory form a definable class iff the torsion theory is of finite type. The proof is just as for the module case ([13, 2.4]), see [16, 11.1.20]. □

Let $\mathcal{O}_X$ be a sheaf of rings on the space X. We will see that X is sufficiently nice then we can deduce properties of Mod-$\mathcal{O}_X$ from its representation as a localisation of PreMod-$\mathcal{O}_X$.

Let τ be a hereditary torsion theory of finite type on a locally finitely presented Grothendieck abelian category $\mathcal{C}$ which has a generating set $\mathcal{G}$ of finitely presented objects. We say that τ is an **elementary** torsion theory [14] if for every $F \le G \in \mathcal{G}$ with G/F torsion and F finitely generated, we have that F is τ-finitely presented, meaning that if $G'' \in \mathcal{G}$ and $f : G'' \longrightarrow F$ is an epimorphism then $\ker(f)$ has a finitely generated τ-dense subobject. Note that it is enough to show that F contains a finitely presented τ-dense subobject F'. For, let G' be a finitely generated subobject of G'' with $fG' = F'$. Then $G' + \ker(f)$ is τ-dense in G'' and so, since τ is of finite type, it follows that there is a finitely generated τ-dense subobject of G'' lying between $G' + \ker(f)$ and G'. So, without loss of generality, G' is τ-dense in G''. Now, since F' is finitely presented, the kernel of $f \restriction G'$, that is $G' \cap \ker(f)$, is finitely generated and, since it is also τ-dense in $\ker(f)$ (because G' is τ-dense in G''), we have our required conclusion.

The term elementary is used because this is exactly the condition one needs for the localised category $\mathcal{C}_\tau$ to be definable when regarded as a subcategory of $\mathcal{C}$ ([14, 0.1]).

Proposition 3.10. *Let $\mathcal{O}_X$ be a sheaf of rings on the space X and let τ denote the sheafification torsion theory on* PreMod-$\mathcal{O}_X$*. Then τ is an elementary torsion theory iff τ is of finite type iff every open subset of X is compact.*

Proof. It remains to be shown that if every open subset is compact then τ is elementary. Let $F \leq G = j_0\mathcal{O}_U$ with G/F torsion and F finitely generated. As argued above it will be enough to show that F contains a finitely presented τ-dense subobject. Since F is τ-dense in G there is an open cover $(U_\lambda)_\lambda$ of U such that for each λ we have $\mathrm{res}_{U,U_\lambda}1 = 1_\lambda$ is in FU_λ. Since U is compact, we may suppose that the index set is finite and so $F \geq F' = \sum_1^n j_0\langle 1_{U_i}\rangle$, say, where $U_1, \ldots, U_n$ cover U. Then F' is τ-dense in G (and hence in F). But clearly F' is finitely presented, the relations between its generators being generated by the $\mathrm{res}_{U_i,U_i\cap U_j}(1_{U_i}) = \mathrm{res}_{U_j,U_i\cap U_j}(1_{U_j})$ where $1 \leq i < j \leq n$. □

Note that every open set of the space X is compact iff X is **noetherian** (i.e. every descending chain of closed subspaces is finite). Therefore we obtain the following as a corollary.

Theorem 3.11. *Let $\mathcal{O}_X$ be a sheaf of rings on a noetherian space X. Then* Mod-$\mathcal{O}_X$ *is locally finitely presented.*

Proof. If the Grothendieck abelian category $\mathcal{C}$ is locally finitely presented and if τ is an elementary torsion theory on $\mathcal{C}$ then, by the proof of [14, 2.1], see [16, 11.1.26], the localisation of $\mathcal{C}$ at τ is locally finitely presented. □

It is shown in [18] that this condition is stronger than is necessary for Mod-$\mathcal{O}_X$ to be locally finitely presented. Indeed, X being locally noetherian will suffice.

A a consequence of the theorem one has that if $\mathcal{O}_X$ is a sheaf of rings on the noetherian space X then the model theory of sheaves of $\mathcal{O}_X$-modules is just a part of the model theory of presheaves over $\mathcal{O}_X$ and so, being of essentially algebraic character, is well-covered by the existing theory of the model theory of objects of locally finitely presented abelian categories. For modules, this is surveyed in [15], and there are details in [17], [16].

Corollary 3.12. *Let $\mathcal{O}_X$ be a sheaf of rings on the space X. Then* Mod-$\mathcal{O}_X$ *is a definable subcategory of* PreMod-$\mathcal{O}_X$ *iff every open subset of X is compact, that is, iff X is noetherian.*

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