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Abstract

Nowadays Hopf algebras is one of the well-known tools in combinatorics.

To study the combinatorics of convex polytopes we develop the approach based on the ring $\mathscr{P}$ of combinatorial convex polytopes and flag-vector transformation $\mathscr{F}\colon \mathscr{P} \to \mathrm{QSym}[\alpha]$, where QSym is the ring of quasisymmetric functions. We show that the ring of polytopes has a natural Hopf comodule structure over the Rota-Hopf algebra $\mathscr{R}$ of posets.

As a corollary we build a ring homomorphism $l_\alpha\colon \mathscr{P} \to \mathscr{R}[\alpha]$ such that $F(l_\alpha(P)) = \mathscr{F}(P)^*$, where $F\colon \mathscr{R} \to \mathrm{QSym}$ is the Ehrenborg transformation of posets.

The talk is based on the papers:

[1] V. M. Buchstaber, *Ring of Simple Polytopes and Differential Equations.*, Proceedings of the Steklov Institute of Mathematics, v. 263, 2008, 1–25.

[2] V. M. Buchstaber, N. Yu. Erokhovets, *Ring of polytopes, quasisymmetric functions and Fibonacci numbers.*, see arXiv: 1002.0810 v1 [math CO] 3 Feb 2010.

Contents

Differential ring of combinatorial polytopes

Definition. Two polytopes P_1 and P_2 of the same dimension are said to be *combinatorially equivalent* if there is a bijection between their sets of faces that preserves the inclusion relation.

Definition. A *combinatorial polytope* is a class of combinatorial equivalent polytopes.

Denote by $\mathscr{P}^{2n}$ the free abelian group generated by all n-dimensional combinatorial polytopes.

For $n \geqslant 1$ we have the direct sum

$$\mathscr{P}^{2n} = \sum_{m \geqslant n+1} \mathscr{P}^{2n,2(m-n)},$$

where $P^n \in \mathscr{P}^{2n,2(m-n)}$ if it is a polytope with m facets and $\mathrm{rank}\, \mathscr{P}^{2n,2(m-n)} < \infty$ for any fixed n and m.

Definition. The product of polytopes turns the direct sum

$$\mathscr{P} = \sum_{n\geqslant 0} \mathscr{P}^{2n} = \mathscr{P}^0 + \sum_{m\geqslant 2} \sum_{n=1}^{m-1} \mathscr{P}^{2n,2(m-n)}$$

into a bigraded commutative associative ring, the *ring of polytopes*. The unit is P^0, a point.

The direct product $P_1^n \times P_2^m$ of simple polytopes P_1^n and P_2^m is a simple polytope as well.

Thus the ring $\mathscr{P}_s$ generated by simple polytopes is a subring in $\mathscr{P}$.

A polytope is *indecomposable* if it can not be represented as a product of two other polytopes of positive dimensions.

Theorem. The ring $\mathscr{P}$ is a polynomial ring generated by indecomposable combinatorial polytopes.

Let P^n be a polytope. Denote by dP^n the *disjoint union* of all its facets.

Lemma. There is a linear operator of degree -2

$$d : \mathscr{P} \longrightarrow \mathscr{P},$$

such that

$$d(P_1^{n_1} P_2^{n_2}) = (dP_1^{n_1})P_2^{n_2} + P_1^{n_1}(dP_2^{n_2}).$$

Thus $\mathscr{P}$ is a *differential ring*, and $\mathscr{P}_S$ is a *differential subring* in $\mathscr{P}$.

Examples:

$$dI^n = n(dI)I^{n-1} = 2nI^{n-1},$$

$$d\Delta^n = (n+1)\Delta^{n-1},$$

where Δ^n is the standard n-simplex and $I^n = I \times \cdots \times I$ is the standard n-cube.

Face-polynomial (f-polynomial)

Consider the linear map

$$f\colon \mathscr{P} \longrightarrow \mathbb{Z}[\alpha, t],$$

which sends a polytope P^n to the homogeneous *face-polynomial*

$$f(P^n) = \alpha^n + f_{n-1,1}\alpha^{n-1}t + \cdots + f_{1,n-1}\alpha t^{n-1} + f_{0,n}t^n,$$

where $f_{k,n-k} = f_{k,n-k}(P^n)$ is the number of its k-dim faces.

Thus $f_{n-1,1}$ is the number of facets and $f_{0,n}$ is the number of vertices.

Theorem.

1. The mapping f is a *ring homomorphism*.
2. Let P be a polytope. Then

$$f(dP) = \frac{\partial}{\partial t} f(P)$$

if and only if P is *simple*.

Dehn–Sommerville relations

Theorem. For any *simple polytope* P^n we have

$$f(P^n)(\alpha, t) = f(P^n)(-\alpha, \alpha + t).$$

Theorem. Let $\widehat{f}\colon \mathscr{P}_S \to \mathbb{Z}[\alpha, t]$ be a *linear* map such that

$$\widehat{f}(dP^n) = \frac{\partial}{\partial t}\widehat{f}(P^n) \quad \text{and} \quad \widehat{f}(P^n)|_{t=0} = \alpha^n.$$

Then $\widehat{f}(P^n) = f(P^n)$.

Flag f-vectors

Let P^n be an n-dimensional polytope and
$\omega = \{a_1, \dots, a_k\} \subset \{0, 1, \dots, n-1\}$.
A *flag number* $f_\omega = f_{a_1, \dots, a_k}$ is the number of increasing sequences of faces

$$F^{a_1} \subset F^{a_2} \subset \dots \subset F^{a_k}, \quad \dim F^{a_i} = a_i.$$

For $\omega = \{i\}$ the number $f_{\{i\}} = f_i$ is just the number of i-dimensional faces.

The collection $\{f_\omega\}$ of all the flag numbers is called a *flag f-vector* of the polytope P^n.
By the definition $f_\varnothing = 1$.

For $n \geqslant 1$ let Ψ^n be the set of subsets
$\omega \subset \{0, 1, \dots, n-2\}$ such that ω contains no two consecutive integers.

The cardinality of Ψ^n is equal to the n-th Fibonacci number c_n ($c_n = c_{n-1} + c_{n-2}$, $c_0 = 1$, $c_1 = 1$).

Bayer–Billera group of polytopes

For any polytope P there exists a *cone* CP and a *suspension* BP. These two operations are defined on combinatorial polytopes and can be extended to linear operators on the ring $\mathscr{P}$.
It is natural to set $B\varnothing = 1 = C\varnothing$.

Definition. For $n \geqslant 1$ let Ω^n be the set of n-dimensional polytopes that arises when we apply to the empty set $\varnothing$ words in B and C that end in C^2 and contain no adjacent B's.

Each word of length $n+1$ from the set Ω^n either has the form CQ, $Q \in \Omega^{n-1}$, or BCQ, $Q \in \Omega^{n-2}$, so cardinality of the set Ω^n satisfies the Fibonacci relation $|\Omega^n| = |\Omega^{n-1}| + |\Omega^{n-2}|$. Since $|\Omega^1| = |\{C^2\}| = 1$, and $|\Omega^2| = |\{C^3, BC^2\}| = 2$, we see that $|\Omega^n| = c_n = |\Psi^n|$.

Theorem. (M. Bayer and L. Billera) Let $n \geqslant 1$. Then

1. For all $\omega \subseteq \{0, 1, \dots, n-1\}$ there is a nontrivial linear relation expressing $f_\omega(P)$ in terms of $f_{\omega'}(P)$, $\omega' \in \Psi^n$, which holds for all n-dimensional polytopes.

2. The flag f-vectors of the c_n elements of Ω^n are affinely independent.

Thus the flag f-vectors $\{f_{\omega'}(P^n)\}_{\omega' \in \Psi^n}$ span an $(c_n - 1)$-dimensional affine hyperplane defined by the equation $f_\varnothing = 1$.

Let us identify the words in Ω^n with the sets in Ψ^n in such a way that the word

$$C^{n+1-a_k} B C^{a_k - a_{k-1} - 1} B \dots B C^{a_1 - 1}$$

corresponds to the set $\{a_1 - 3, \dots, a_k - 3\}$.

Let us set $C < B$ and order the words lexicographically.

Consider a $(c_n \times c_n)$-matrix K^n

$$k_{Q,\omega} = f_\omega(Q), \;\; Q \in \Omega^n, \; \omega \in \Psi^n.$$

Theorem. $\det(K^n) = 1$.

Faces-operator

Let P^n be a polytope. Denote by $d_kP^n,\ k \geqslant 0$, the *disjoint union* of all its $(n-k)$-dimensional faces.

Lemma. There is a linear operator of degree $-2k$

$$d_k\colon \mathscr{P} \longrightarrow \mathscr{P}$$

such that

$$d_kP_1^{n_1}P_2^{n_2} = \sum_{i+j=k}(d_iP_1^{n_1})(d_jP_2^{n_2}).$$

Definition. A *faces-operator* for t is the linear map

$$\Phi(t)\colon \mathscr{P} \longrightarrow \mathscr{P}[t] \ :\ \Phi(t)(P^n) = \sum_{k=0}^{\infty} d_kP^nt^k.$$

Theorem.

1. $\Phi(t)$ is a ring homomorphism.
2. $\Phi(t)(P^n) = \exp(td)(P^n)$ if and only if P^n is simple.
3. The composition

$$\Phi(\alpha,t)\colon \mathscr{P} \xrightarrow{\Phi(t)} \mathscr{P}[t] \xrightarrow{\xi(\alpha)} \mathbb{Z}[\alpha,t],$$

where $\xi(\alpha)(P^n) = \alpha^n$ and $\xi(\alpha)t = t$, is the face-polynomial ring homomorphism f.

Flag-vector polynomial

Let $\Phi(t_k)$ be the faces-operator for $t_k,\ k = 1, \ldots, m$. Set $\widehat{\Phi}(t_1) = \Phi(t_1)$ and consider the operators

$$\widehat{\Phi}(t_m)\colon\ \mathscr{P}[t_1, \ldots, t_{m-1}] \longrightarrow \mathscr{P}[t_1, \ldots, t_m],\ m > 1,$$

such that $\widehat{\Phi}(t_m)(P^n) = \Phi(t_m)(P^n)$ and $\widehat{\Phi}(t_m)(t_i) = t_i$, $1 \leqslant i < m$.

Introduce the ring homomorphisms

$$\mathscr{F}(m)\colon\ \mathscr{P} \longrightarrow \mathscr{P}[t_1, \ldots, t_m],\ m \geqslant 1,$$

where $\mathscr{F}(1) = \Phi(t_1)$ and recursively $\mathscr{F}(m),\ m > 1$, is the composition

$$\mathscr{P} \xrightarrow{\mathscr{F}(m-1)} \mathscr{P}[t_1, \ldots, t_{m-1}] \xrightarrow{\widehat{\Phi}(t_m)} \mathscr{P}[t_1, \ldots, t_m].$$

We obtain the operator

$$\mathscr{F}(m) = 1 + \sum_{q \geqslant 1} \sum_{|J|=q} d_J\, \zeta(\boldsymbol{t}^J; m)$$

where $J = (j_1, \ldots, j_k),\ j_i \neq 0,\ i = 1, \ldots, k,\ 1 \leqslant k \leqslant m$, $|J| = j_1 + \cdots + j_k, \quad d_J = d_{j_k} \cdots d_{j_1}, \quad t^J = t_1^{j_1} \cdots t_k^{j_k}$ and

$$\zeta(\boldsymbol{t}^J; m) = \sum_{1 \leqslant l_1 < \cdots < l_k \leqslant m} t_{l_1}^{j_1} \cdots t_{l_k}^{j_k}.$$

Algebra of quasisymmetric functions

Definition. A *composition* J of a number n is an ordered set $J=(j_1,\dots,j_k)$, $j_i \geqslant 1$, such that $n=j_1+j_2+\dots+j_k$. Let us denote $|J|=n$.

The number of compositions of n into exactly k parts is given by the binomial coefficient $\binom{n-1}{k-1}$.

Definition. A *quasisymmetric monomial* in m variables for a composition $J=(j_1,\dots,j_k)$, $k \leqslant m$, is the polynomial

$$\zeta(\boldsymbol{t}^J;m)=\sum_{1\leqslant l_1<\dots<l_k\leqslant m} t_{l_1}^{j_1}\dots t_{l_k}^{j_k}$$

Lemma. The polynomial $f \in \mathbb{Z}[t_1,\dots,t_m]$ is a linear combination of quasisymmetric monomials if and only if $f(t_1,\dots,t_m)$ satisfies the following conditions:

$$f(t_1,t_2,t_3,\dots,t_{m-1},0)=f(0,t_1,t_2,\dots,t_{m-1})=$$
$$=f(t_1,0,t_2,\dots,t_{m-1})=\dots=f(t_1,\dots,t_{m-2},0,t_{m-1}).$$

Example. We have the compositions of 3:

$(1,1,1),\ (1,2),\ (2,1),\ (3)$.

The quasisymmetric monomials of degree 3 in $\mathbb{Z}[t_1,t_2,t_3]$:
$t_1t_2t_3;\ \ t_1t_2^2+t_1t_3^2+t_2t_3^2;\ \ t_1^2t_2+t_1^2t_3+t_2^2t_3;\ \ t_1^3+t_2^3+t_3^3$,
and for $m=3$ we have the conditions:
$f(t_1,t_2,0)=f(0,t_1,t_2)=f(t_1,0,t_2)$.

Let $\mathrm{QSym}^{2n}(m) \subset \mathbb{Z}[t_1, \ldots, t_m]$ be the subgroup generated by the quasisymmetric monomials $\zeta(\boldsymbol{t}^J; m)$ corresponding to all compositions $J = (j_1, \ldots, j_k)$ of n, where $k \leqslant m$. It is easy to see that for $k \leqslant m-1$

$$\zeta(\boldsymbol{t}^J; m)(t_1, \ldots, t_{m-1}, 0) = \zeta(\boldsymbol{t}^J; m-1)(t_1, \ldots, t_{m-1}).$$

Set $\mathrm{QSym}^{2n} = \varprojlim\limits_{m} \mathrm{QSym}^{2n}(m)$.

An algebra of quasisymmetric functions QSym is a graded subring $= \sum\limits_{n\geqslant 0} \mathrm{QSym}^{2n}$ in

$$V = \sum_{n\geqslant 0} V^{2n} = \varprojlim_{m} \mathbb{Z}[t_1, \ldots, t_m],$$

where $\deg t_k = 2$.

Theorem. (M.Hazewinkel, 2001)
The algebra of quasisymmetric functions QSym is a free commutative algebra of polynomials over the integers.

Since $\operatorname{rank} \mathrm{QSym}^{2n} = 2^{n-1}$, $n \geqslant 1$, the numbers β_i of the multiplicative generators of degree $2i$ of QSym can be found by a recursive relation:

$$\frac{1-t}{1-2t} = \prod_{i=1}^{\infty} \frac{1}{(1-t^i)^{\beta_i}}$$

Flag-vector transformation

Set $\mathscr{F}(\boldsymbol{t}) = \varprojlim \mathscr{F}(m)$. Denote by $\mathscr{F}(\alpha;\boldsymbol{t})$ the ring homomorphism

$$\mathscr{P} \xrightarrow{\mathscr{F}(\boldsymbol{t})} \mathscr{P} \otimes \mathrm{QSym} \xrightarrow{\widehat{\varepsilon}(\alpha)} \mathrm{QSym}[\alpha] \subset \mathbb{Z}[\alpha;\boldsymbol{t}],$$

where $\widehat{\varepsilon}(\alpha)$ is the extension of the ring homomorphism

$$\varepsilon(\alpha)\colon \mathscr{P} \longrightarrow \mathbb{Z}[\alpha] \,:\, \varepsilon(\alpha)(P^n) = \alpha^n,\ n \geqslant 0,$$

such that $\widehat{\varepsilon}(\alpha)(t_i) = t_i$.

Lemma. Let P^n be an n-dim polytope. Then

$$\mathscr{F}(P^n)(\alpha;\boldsymbol{t}) = \alpha^n + \sum_{q=1}^{n} \alpha^{n-q} \sum_{|J|=q} f_{\omega(J)}(P^n)\zeta(\boldsymbol{t}^J)$$

is a homogeneous polynomial of degree $2n$.

Here $f_{\omega(J)}(P^n)$ for $J = (j_1, \dots, j_k)$ is the ω-flag number of P^n with $\omega = \omega(J) = (i_1 < \dots < i_k)$, where
$i_1 = n - q,\ \dots,\ i_l = i_{l-1} + j_{k-l+2},\ \dots,\ i_k = i_{k-1} + j_2$
and $q = |J|$.

Definition. The ring homomorphism $\mathscr{F}(\alpha;\boldsymbol{t})$ sends a polytope P^n to the flag-vector polynomial $\mathscr{F}(P^n)(\alpha;\boldsymbol{t})$.

Bayer–Billera relations

Theorem. The image of the homomorphism

$$\mathscr{F}(\alpha, \boldsymbol{t})\colon \mathscr{P}^{2n} \longrightarrow \mathrm{QSym}(m)[\alpha],\ m \geqslant n,$$

consists of all homogeneous polynomials $f(\alpha, t_1, \ldots, t_m)$ of degree n satisfying the equations:

1. $$f(\alpha, t_1, -t_1, t_3, \ldots, t_m) = f(\alpha, 0, 0, t_3, \ldots, t_m);$$
$$f(\alpha, t_1, t_2, -t_2, t_4, \ldots, t_m) = f(\alpha, t_1, 0, 0, t_4 \ldots, t_m);$$
$$\ldots$$
$$f(\alpha, t_1, \ldots, t_{m-2}, t_{m-1}, -t_{m-1}) = f(\alpha, t_1, \ldots, t_{m-2}, 0, 0);$$

2. $$f(-\alpha, t_1, \ldots, t_{m-1}, \alpha) = f(\alpha, t_1, \ldots, t_{m-1}, 0);$$

These equations are a perfected form of the Bayer-Billera (generalized Dehn-Sommerville) relations.

Corollary. The image of the restriction of $\mathscr{F}(\alpha, \boldsymbol{t})$ on $\mathscr{P}_S^{2n}$ consists of all homogeneous polynomials

$$f(\alpha, t_1, \ldots, t_m) = f_1(\alpha, t_1 + \ldots + t_m)$$

where $f_1(\alpha, t)$ is a homogeneous polynomial in two variables of degree n satisfying the equations

$$f_1(-\alpha, \alpha + t) = f_1(\alpha, t).$$

This equation is a perfected form of the classical Dehn-Sommerville relations (see slide 7).

Theorem. The image of the ring homomorphism

$$\mathscr{F}(\alpha, \boldsymbol{t})\colon \mathscr{P} \otimes \mathbb{Q} \longrightarrow \mathrm{QSym}[\alpha] \otimes \mathbb{Q}$$

is a free polynomial algebra with the structure of the graded Hopf algebra dual to the free associative Lie-Hopf algebra $\mathbb{Q}\langle u_1, u_2\rangle$, where $\deg u_i = 2i$ and
$$\Delta u_i = u_i \otimes 1 + 1 \otimes u_i, \quad i = 1, 2.$$

The Rota-Hopf Algebra

Let P be a finite poset with a minimal element $\hat{0}$ and a maximal element $\hat{1}$.

An element y in P *covers* another element x in P, if $x < y$ and there is no z in P such that $x < z < y$.

A poset P is called *graded*, if there exists a rank function $\rho : \mathscr{P} \to \mathbb{Z}$ such that $\rho(\hat{0}) = 0$ and $\rho(y) = \rho(x) + 1$ if y covers x.

Set $\deg P = \rho(P) = \rho(\hat{1})$.

Two finite graded poset are isomorphic if there exists an order preserving bijection between them.

Denote by $\mathscr{R}$ the graded free abelian group with basis the set of all isomorphism classes of finite graded posets.

The structure of the *graded connected Hopf algebra* on $\mathscr{R}$:

The multiplication $P \cdot Q$ is a cartesian product $P \times Q$ of posets P and Q:

Let $x, u \in P$ and $y, v \in Q$. Then $(x, y) \leqslant (u, v)$ if and only if $x \leqslant u$ and $y \leqslant v$.

The unit element in $\mathscr{R}$ is the poset with one element $\hat{0} = \hat{1}$.

The comultiplication is

$$\Delta(P) = \sum_{\hat{0} \leqslant z \leqslant \hat{1}} [\hat{0}, z] \otimes [z, \hat{1}],$$

were $[x, y]$ is the subposet $\{z \in P | x \leqslant z \leqslant y\}$.

The counit ε is

$$\varepsilon(P) = \begin{cases} 1, & \text{if } \hat{0} = \hat{1}; \\ 0, & \text{else} \end{cases}$$

The antipode χ is

$$\chi(P) = \sum_{k \geqslant 0} \sum_{\{c_k\}} (-1)^k [x_0, x_1] \cdot [x_1, x_2] \ldots [x_{k-1}, x_k]$$

where $c_k = (\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1})$

Example. The simplest Boolean algebra $B_1 = \{\hat{0}, \hat{1}\}$ is the face lattice of the point pt

$$\Delta(B_1) = 1 \otimes B_1 + B_1 \otimes 1$$
$$\chi(B_1) = -B_1$$

There is a natural linear mapping $L : \mathscr{P} \longrightarrow \mathscr{R}$
of degree $+1$ that sends a polytope P to its face lattice $L(P)$.

The mapping L is *injective*, but it is *not* a *ring homomorphism*, since it doesn't preserve a unit:
the unit of $\mathscr{P}$ is a point pt and $L(\mathrm{pt}) = B_1$,
while the unit of $\mathscr{R}$ is a one-element set $\{\hat{0} = \hat{1}\}$.

Remark. The face lattice $L(P \times Q)$ of $P \times Q$ contains the empty face $\varnothing$, which can be considered as $\varnothing \times \varnothing$, but evidently has no faces of the form $F \times \varnothing$ or $\varnothing \times G$, where F and G are non-empty faces of P and Q respectively.

Consider the linear operator $*\colon \mathscr{P} \to \mathscr{P}$ such that $*(P)$ is the polytope P^* polar to a polytope P.
There is a natural linear mapping of degree 0

$$l = L*d* : \mathscr{P} \to \mathscr{R} \ : \ l(\mathrm{pt}) = \{\hat{0}\}, \ l(P) = L\left((dP^*)^*\right).$$

- The operation $*d*$ preserves the linear space of all simplicial polytopes.

- For a simple polytope P^n we have
  $*d*(P^n) = f_0(P^n)\Delta^{n-1}$
  and $l(P^n) = f_0 L(\Delta^{n-1}) = f_0\{\hat{0}, \hat{1}\}^n = f_0 B_1^n$,
  where $f_0(P^n)$ is the number of vertices of P^n,
  and $\{\hat{0}, \hat{1}\}^n = B_1^n = B_n$ is a Boolean algebra.

  For example,
  $L(\Delta^n) = B_1^{n+1}$ and $l(\Delta^n) = (n+1)B_1^n$.

Proposition. l is a homomorphism of graded rings.

Proof. We have

$$l(P) = \sum_v [v, P],$$

where $[v, P]$ is the interval between the vertex v and the polytope P in the face lattice $L(P)$.

Then

$$l(P \times Q) = \sum_{v \times w} [v \times w, P \times Q] = \sum_{v \times w} [v, P] \times [w, Q] =$$
$$= \left(\sum_v [v, P] \right) \cdot \left(\sum_w [w, Q] \right) = l(P) \cdot l(Q)$$

Here v, w are vertexes of P and Q respectively.

Definition. Set $\Delta^{-1} = \varnothing$. Consider the linear span of all Boolean algebras $B_n = \{\hat{0}, \hat{1}\}^n = L(\Delta^{n-1}),\ n \geqslant 0$

$$\mathscr{B} = \mathrm{Ls}(1, B_1, B_2, \dots) \subset \mathscr{R}.$$

We have $B_i B_i = B_{i+j}$, so it is a subring in $\mathscr{R}$.
Let us denote $x = B_1$. Since $\Delta x = 1 \otimes x + x \otimes 1$, it is a Hopf subalgebra isomorphic
to the Hopf algebra $\mathbb{Z}[x],\ \ \Delta x = 1 \otimes x + x \otimes 1$.

Proposition. The image of

$$l\colon \mathscr{P}_s \longrightarrow \mathscr{R}$$

is a $\mathbb{Z}$-subalgebra in $\mathscr{B} = \mathbb{Z}[x]$ multiplicatively generated by $2x$ and x^2, that is

$$l(\mathscr{P}_s) = \mathbb{Z}[x_1, x_2]/(x_1^2 - 4x_2)$$

where $\deg x_1 = 1, \deg x_2 = 2$.

Proof. We have $l(\Delta^1) = x_1,\ l(P_m^2) = mx^2 = mx_2$.
Thus $x_1 = l(\Delta^1) = 2x$ and $x_2 = l(P_4^2 - \Delta^2) = x^2$.
On the other hand $l(P^{2n+1}) = f_0(P^{2n+1})B_1^{2n+1}$ where $f_0(P^{2n+1})$ is even. $\square$

Let us denote $\rho(x, y) = \rho(y) - \rho(x)$ for $x \leqslant y$.
Richard Ehrenborg introduced the F-quasi-symmetric function of a graded poset P of rank n

$$F(P) = \sum M_{(\rho(x_0, x_1),\, \rho(x_1, x_2),\, \dots,\, \rho(x_k, x_{k+1}))} = \\ = \sum_{0<a_1<\dots<a_k<n} f_{a_1,\dots,a_k}(P) M_{(a_1,\, a_2-a_1,\, \dots,\, n-a_k)}$$

where the first sum ranges over all chains
$\hat{0} = x_0 < x_1 < \dots < x_{k+1} = \hat{1}$ from $\hat{0}$ to $\hat{1}$, and
$f_{a_1,\dots,a_k}$ are flag numbers.
This mapping induces a Hopf algebra homomorphism

$$F : \mathscr{R} \to \mathrm{QSym}[t_1, t_2, \dots].$$

Definition. For a composition $\omega = (j_1, \dots, j_k)$
let us define the composition $\omega^* = (j_k, \dots, j_1)$.

The correspondence $M_\omega \to (M_\omega)^* = M_{\omega^*}$ defines an involutory ring homomorphism

$$* : \mathrm{QSym}[t_1, t_2, \dots] \to \mathrm{QSym}[t_1, t_2, \dots].$$

Proposition. For a polytope P^n

$$F(l(P^n))^* = \mathscr{F}(0, t_1, t_2, \dots)(P^n) \quad \text{i.e.} \quad * \circ F \circ l = \mathscr{F}|_{\alpha=0}.$$

Proposition. We have

$$L = l \circ (C - \frac{1}{2}[I]),$$

where C is a cone operator and $[I]$ is the operator of multiplication by the interval I. So the images of the maps L and l over $\mathbb{Z}[\frac{1}{2}]$ in dimensions $n \geqslant 1$ coincide.

Remark. The homomorphism l is not injective: we see that on the ring of simple polytopes it remembers only the number of vertices. This Proposition shows that l is invective on the image of the operator $C - \frac{1}{2}[I]$.

Define the operator
$C : \mathrm{QSym}[t_1, \dots, t_m][\alpha] \to \mathrm{QSym}[t_1, \dots, t_{m+1}][\alpha]$
such that $f_{m+1}(CP^n) = (Cf_m)(P^n)$. Then
$(Cg)(\alpha, t_1, \dots, t_{m+1}) =$
$= (\alpha + t_1 + \dots + t_{m+1})(\alpha, t_1, \dots, t_{m+1}) +$
$+ t_{m+1}g(t_{m+1}, t_1, \dots, t_m) + t_m g(t_m, t_1, \dots, t_{m-1}, 0) +$
$+\dots+ t_i g(t_i, t_1, \dots, t_{i-1}, 0, \dots, 0) + \dots + t_1 g(t_1, 0, \dots, 0)$

Corollary.

$$F(L(P))^* = C\mathscr{F}(P)|_{\alpha=0} - \sigma_1 \mathscr{F}(P)|_{\alpha=0},$$

where $\sigma_1 = \sum\limits_{i=1}^{\infty} t_i = M_{(1)}$.

Set $\Lambda = (C - \sigma_1)|_{\alpha=0}$.
The operator Λ has a very simple form on elementary monomials

$$\Lambda(\alpha^{a_1} M_{(n-a_k,\, \dots,\, a_2-a_1)}) = M_{(n-a_k, \dots,\, a_2-a_1,\, a_1+1)}$$

The relation between the Ehrenborg F-quasi-symmetric function and the flag $\mathscr{F}$-polynomial can be illustrated by two commutative diagrams

$$\begin{array}{ccc} \mathscr{P} & \xrightarrow{\mathscr{F}} & \mathrm{QSym}[t_1, t_2, \dots][\alpha] \\ l\big\downarrow & & \big\downarrow \alpha=0 \\ \mathscr{R} & \xrightarrow{*\circ F} & \mathrm{QSym}[t_1, t_2, \dots] \end{array}$$

$$\begin{array}{ccc} \mathscr{P} & \xrightarrow{\mathscr{F}} & \mathrm{QSym}[t_1, t_2, \dots][\alpha] \\ L\big\downarrow & & \big\downarrow \Lambda \\ \mathscr{R} & \xrightarrow{*\circ F} & \mathrm{QSym}[t_1, t_2, \dots] \end{array}$$

Definition. By a *Hopf comodule* (or *Milnor comodule*) over a Hopf algebra X we mean an algebra M with a unit provided M is a comodule over X with a coaction $b\colon M \to X \otimes M$ such that $b(uv) = b(u)b(v)$, i.e. such that b is a homomorphism of rings.

The ring homomorphism l can be extended to a right graded Hopf comodule structure on $\mathscr{P}$

Proposition. The homomorphism $\Delta : \mathscr{P} \to \mathscr{P} \otimes \mathscr{R}$:

$$\Delta(P^n) = \sum_{F \subseteq P^n} F \otimes [F, P^n],$$

defines on $\mathscr{P}$ a right graded Hopf comodule structure over $\mathscr{R}$ such that

$$(\varepsilon \otimes \mathrm{id})\Delta P = 1 \otimes l(P)$$

Corollary.

- Any ring homomorphism $\varphi : \mathscr{P} \to \mathbb{Z}$ defines a ring homomorphism $\mathscr{P} \to \mathscr{R}$

$$\mathscr{P} \xrightarrow{\Delta} \mathscr{P} \otimes \mathscr{R} \xrightarrow{\varphi \otimes \mathrm{id}} \mathbb{Z} \otimes \mathscr{R} \simeq \mathscr{R}$$

- Any homomorphism of abelian groups $\psi : \mathscr{R} \to \mathbb{Z}$ defines a linear operator $\Psi \in \mathscr{L}(\mathscr{P})$

$$\mathscr{P} \xrightarrow{\Delta} \mathscr{P} \otimes \mathscr{R} \xrightarrow{\mathrm{id} \otimes \psi} \mathscr{P} \otimes \mathbb{Z} \simeq \mathscr{P}$$

- In particular, if $\psi : \mathscr{R} \to \mathbb{Z}$ is a multiplicative homomorphism, then Ψ is a ring homomorphism.

Example. Let $\varphi = \xi_\alpha$. Then we obtain the ring homomorphism $l_\alpha : \mathscr{P} \to \mathscr{R}[\alpha]$ defined as

$$l_\alpha(P^n) = (\xi_\alpha \otimes \mathrm{id})\Delta P^n = \sum_{F \subseteq P} \alpha^{\dim F}[F, P]$$

- If we set $\alpha = 0$, then we obtain a usual homomorphism l.

- On the ring of simple polytopes $\mathscr{P}_S$ we have

$$l_\alpha(P^n) = \sum_{F \subseteq P^n} \alpha^{\dim F}\{\hat{0}, \hat{1}\}^{n-\dim F} =$$
$$= \sum_{F \subseteq P^n} \alpha^{\dim F} x^{n-\dim F} = f_1(\alpha, x)$$

  is a homogeneous f-polynomial in two variables.

Set $F(\alpha) = \alpha$. Then we have the ring homomorphism $F\colon \mathscr{R}[\alpha] \to \mathrm{QSym}[t_1, t_1, \dots][\alpha]$.

Proposition. Let P^n be an n-dimensional polytope. Then

$$F(l_\alpha(P^n)) = \mathscr{F}(P^n)^*$$

Proposition. The following diagram commutes:

$$\begin{array}{ccc} \mathscr{P} & \xrightarrow{\mathscr{F}^*} & \mathrm{QSym}[t_1, t_2, \dots][\alpha] \\ \Delta\downarrow & & \Delta\downarrow \\ \mathscr{P} \otimes \mathscr{R} & \xrightarrow{\mathscr{F}^* \otimes F} & \mathrm{QSym}[t_1, t_2, \dots][\alpha] \otimes \mathrm{QSym}[t_1, t_2, \dots] \end{array}$$