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Morozov, Andrei and Korovina, Margarita

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Remarks on Σ–definability without the equality test over the Reals [⋆]

Andrei Morozov [1]

Sobolev Institute of Mathematics
Novosibirsk, Russia

Margarita Korovina[2]

IIS SBRAS, Novosibirsk, Russia, and
Universität Siegen, Germany

Abstract

In this paper we study properties of Σ–definability over the reals without the equality test which is one of the main concepts in the logical approach to computability over continuous data [3,4,5]. In [3] it has been shown that a set $B \subset \mathbb{R}^n$ is Σ–definable without the equality test if and only if B is c.e. open. If we allow the equality test, the structure of a Σ–definable subset of $\mathbb{R}^n$ can be rather complicated. The next natural question to consider is the following. Is there an effective procedure producing a set which is a maximal c.e. open subset of a given Σ–definable with the equality subset of $\mathbb{R}^n$? It this paper we give the negative answer to this question.

Keywords: The real numbers, Σ-definability, computably enumerable open sets.

1 Introduction

In some specifications over the reals, in our settings they are Σ-formulas, it is natural to use the equality test. Unfortunately the equality test can not be performed in real exact computations which is reflected in computable analysis. A natural question to ask whether we can reasonably approximate sets which are Σ-definable with the equality test by sets which are Σ-definable without the equality test.

One of the main differences between Σ-definability without equality and Σ-definability with equality is that subsets of $\mathbb{R}^n$ which are Σ-definable without equal-

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[1] Email: morozov@math.nsc.ru

[2] Email: korovina@brics.dk

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ity are always open. In this paper we investigate whether we can reasonably approximate sets of reals which are Σ-definable with equality and not necessarily open by a subsets which are Σ-definable without equality. We show that there is no effective procedure producing a set which is a maximal c.e. open subset of a given Σ–definable with the equality subset of $\mathbb{R}^n$.

2 Basic Definitions and Notions

In this paper we consider the ordered structure of the real numbers

$$\langle \mathbb{R}, 0, 1, +, \cdot, <, = \rangle = \langle \mathbb{R}, \sigma_0 \rangle .$$

We extend the real numbers by the set of hereditarily finite sets $\mathrm{HF}(\mathbb{R})$ which is rich enough for information to be coded and stored. We construct the set of hereditarily finite sets, $\mathrm{HF}(\mathbb{R})$ over the reals, as follows:

(i) $\mathrm{HF}_0(\mathbb{R}) \rightleftharpoons \mathbb{R}$,

(ii) $\mathrm{HF}_{\mathrm{n}+1}(\mathbb{R}) \rightleftharpoons \mathcal{P}_\omega(\mathrm{HF}_{\mathrm{n}}(\mathbb{R})) \cup \mathrm{HF}_{\mathrm{n}}(\mathbb{R})$, where $n \in \omega$ and for every set B, $\mathcal{P}_\omega(B)$ is the set of all finite subsets of B.

(iii) $\mathrm{HF}(\mathbb{R}) = \bigcup_{m \in \omega} \mathrm{HF}_m(\mathbb{R})$.

We define $\mathbf{HF}(\mathbb{R})$ as the following model: $\mathbf{HF}(\mathbb{R}) \rightleftharpoons \langle \mathrm{HF}(\mathbb{R}), R, \sigma_0, \emptyset, \in \rangle \rightleftharpoons \langle \mathrm{HF}(\mathbb{R}), \sigma \rangle$, where the constant $\emptyset$ stands for the empty set and the binary predicate symbol $\in$ has the set-theoretic interpretation. We also add a predicate symbol R for elements of $\mathbb{R}$.

The set of Δ_0*–formulas* is the closure of the set of atomic formulas under $\wedge, \vee, \neg$, bounded quantifiers $(\exists \mathbf{x} \in \mathbf{y})$ and $(\forall \mathbf{x} \in \mathbf{y})$, where $(\exists \mathbf{x} \in \mathbf{y})\ \Psi$ denotes $\exists \mathbf{x}(\mathbf{x} \in \mathbf{y} \wedge\ \Psi)$, $(\forall \mathbf{x} \in \mathbf{y})\ \Psi$ denotes $\forall \mathbf{x}(\mathbf{x} \in \mathbf{y} \to \Psi)$ and $\mathbf{y}$ ranges over sets.

The set of Σ*–formulas* is the closure of the set of Δ_0–formulas under $\wedge$,$\vee$, $(\exists \mathbf{x} \in \mathbf{y})$, $(\forall \mathbf{x} \in \mathbf{y})$ and $\exists$, where $\mathbf{y}$ ranges over sets.

The set of $\Sigma_<$*–formulas* is the subset of Σ–formulas which have positive occurrences of the predicate " $<$ " and don't have occurrences of the predicate " $=$ ".

Definition 2.1 (i) A relation $B \subseteq \mathrm{HF}(\mathbb{R})^n$ is Σ**–definable**, if there exists a Σ–formula Φ such that $\mathbf{x} \in B \leftrightarrow \mathbf{HF}(\mathbb{R}) \models \Phi(\mathbf{x})$.

(ii) A set $B \subseteq \mathrm{HF}(\mathbb{R})$ is Δ**–definable**, if both B and its complement are Σ–definable.

In sequel we tell that a relation is Σ*–definable without equality* if it is definable by a $\Sigma_<$–formula. The following theorem reveals algorithmic properties of Σ–formulas over $\mathbf{HF}(\mathbb{R})$.

Theorem 2.2 *[2,4][Semantic Characterisation of Σ–definability]*

(i) *A set $A \subseteq \mathbb{R}^n$ is Σ–definable if and only if there exists an effective sequence of* *quantifier free formulas in the language* σ_0, $\{\Phi_s(x)\}_{s\in\omega}$, *such that*

$$x \in A \leftrightarrow \mathbf{HF}(\mathbb{R}) \models \bigvee_{s\in\omega} \Phi_s(x).$$

(ii) *A set* $B \subseteq \mathbb{R}^n$ *is* Σ*–definable without equality if and only if there exists an effective sequence of quantifier free formulas in the language* σ_0 *with positive occurrences of "* $<$ *" and without occurrences of "* $=$ *",* $\{\Psi_s(x)\}_{s\in\omega}$, *such that*

$$x \in B \leftrightarrow \mathbf{HF}(\mathbb{R}) \models \bigvee_{s\in\omega} \Psi_s(x).$$

It is worth noting that both of the directions of these characterisations are important. The right directions give us effective procedures which generate formulas approximating Σ–relations. The converse directions provide tools for descriptions of the results of effective infinite approximating processes by finite formulas.
For $\bar{a} \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}$, let $B(\bar{a},\varepsilon) = \{\bar{x} \in \mathbb{R}^n \mid ||\bar{x}-\bar{a}|| < \varepsilon\}$.

Definition 2.3 A set $S \subseteq \mathbb{R}^n$ is called *computably enumerable (c.e.) open* if there exist computable families $(\bar{a}_i)_{i<\omega} \in (\mathbb{Q}^n)^\omega$ and $(\varepsilon_i)_{i<\omega} \in \mathbb{Q}^\omega$ such that $S = \cup_{i<\omega} B(\bar{a}_i,\varepsilon_i)$.

Corollary 2.4 *A set* $S \subseteq \mathbb{R}^n$ *is* Σ*-definable without equality if and only if* S *is c.e. open.*

3 Main Results

We start with consideration of the interior part of Σ-definable set of reals as a reasonable c.e. open approximation to this set.

Hypothesis: if a set $S \subseteq \mathbb{R}$ is Σ-definable then $\mathtt{Int}(S)$, i.e., the interior part of S is Σ-definable without equality.

The following result shows this hypothesis to be false. Moreover, in general case, we cannot even hope for the existence of an internal part of a Σ–definable set which is maximal by inclusion among its Σ–subsets.

Theorem 3.1 *There exists a set* $S \subseteq \mathbb{R}$ *such that*

(i) S *is* Δ*–definable.*

(ii) *Neither the closures nor the inner parts of the sets* S, $\mathbb{R}\setminus S$ *are* Σ*–definable.*

(iii) *If* V *is either* S *or* $\mathbb{R}\setminus S$ *then the class*

$$\{X \subseteq V \mid (X \textit{ is } \Sigma\textit{–definable without equality}\}$$

has no maximal element by inclusion.

(iv) *If* V *is either* S *or* $\mathbb{R}\setminus S$ *then the class*

$$\{X \supseteq V \mid (X \textit{ is closed}) \wedge (X \textit{ is } \Sigma\textit{–definable })\}$$

has no minimal element by inclusion.

Proof. First we need a lemma.

Lemma 3.2 *There exist Σ–definable functions*

$$\alpha_i^+(a,b),\ \alpha_i^-(a,b),\ \beta_i^+(a,b),\ \beta_i^-(a,b)$$

defined on $\omega \times \mathbb{R}^2$ such that

$$a = \alpha_0^-(a,b) < \beta_0^-(a,b) < \alpha_1^-(a,b) < \beta_1^-(a,b) < \ldots \frac{a+b}{2} < \ldots$$

$$\ldots < \beta_1^+(a,b) < \alpha_1^+(a,b) < \beta_0^+(a,b) < \alpha_0^+(a,b) = b$$

and

$$\lim_{i\to\infty} \alpha_i^+(a,b) = \lim_{i\to\infty} \alpha_i^-(a,b) = \lim_{i\to\infty} \beta_i^+(a,b) = \lim_{i\to\infty} \beta_i^-(a,b) = \frac{a+b}{2}.$$

We leave the proof to the reader.

Fix a computable function $f : \omega \to \omega$ whose range is not computable.

First we show how given any two reals a and b, $a < b$, and $n < \omega$, we could separate the interval $[a, b)$ into two Σ–definable sets $A_n(a, b)$ and $B_n(a, b)$ so that $A_n(a,b) \cap B_n(a,b) = \varnothing$, $A_n(a,b) \cup B_n(a,b) = [a,b)$, and $(a+b)/2 \in \mathtt{Int}(A_n(a,b)) \Leftrightarrow n \notin \mathtt{range}\,(f)$.

Denote $c = (a + b)/2$. Let

$$A_n(a,b) = \{c\} \cup \bigcup_{t<\omega,\ n\notin\{f(0),\ldots,f(t)\}} \Big([\alpha_t^-(a,b), \alpha_{t+1}^-(a,b)) \cup [\alpha_{t+1}^+(a,b), \alpha_t^+(a,b))\Big)$$

and let

$$B_n(a,b) = \bigcup_{t<\omega,\ n\in\{f(0),\ldots,f(t)\}} [\alpha_t^-(a,b), \alpha_t^+(a,b)) \setminus \{c\}.$$

Clearly, both sets are Σ–definable, they are disjoint, and their union to $[a, b)$. Moreover, there exist Σ–formulas $\varphi_A(n, a, b, x)$ and $\varphi_B(n, a, b, x)$ such that $A_n(a, b) = \varphi_A(n,a,b,x)^{\mathbf{HF}(\mathbb{R})}[x]$ and $B_n(a,b) = \varphi_B(n,a,b,x)^{\mathbf{HF}(\mathbb{R})}[x]$. Obviously,

$$(a+b)/2 \in \mathtt{Int}(A_n(a,b)) \Leftrightarrow n \notin \mathtt{range}\,(f).$$

Next we show how given any two reals a and b, $a < b$ and $n < \omega$, we could uniformly construct Σ–subsets $C_n(a, b)$ and $C_n(a, b)$ so that $C_n(a, b) \cup D_n(a, b) = [a,b)$, $C_n(a,b) \cap D_n(a,b) = \varnothing$, and $\frac{a+b}{2} \in \mathtt{cl}\,(C_n(a,b)) \Leftrightarrow n \notin \mathtt{range}\,(f)$.

Let

$$C_n(a,b) = \bigcup_{t<\omega,\ n\notin\{f(0),\ldots,f(t)\}} \Big([\alpha_t^-(a,b), \beta_t^-(a,b)) \cup [\beta_t^+(a,b), \alpha_t^+(a,b))\Big)$$

and let

$$D_n(a,b) = \bigcup_{t<\omega,\ n\in\{f(0),\dots,f(t)\}} [\beta_t^-(a,b), \beta_t^+(a,b)).$$

Clearly, both the sets are Σ–definable, they are disjoint, and their union equals to $[a,b)$. Moreover, there exist Σ–formulas $\varphi_C(n,a,b,x)$ and $\varphi_D(n,a,b,x)$ such that $C_n(a,b) = \varphi_C(n,a,b,x)^{\mathbf{HF}(\mathbb{R})}[x]$ and $D_n(a,b) = \varphi_D(n,a,b,x)^{\mathbf{HF}(\mathbb{R})}[x]$. Obviously,

$$(a+b)/2 \in \mathtt{cl}\,(C_n(a,b)) \Leftrightarrow n \notin \mathtt{range}\,(f).$$

Now define the set S as follows:

$$S = \bigcup_{i<\omega} A_i(8i, 8i+2) \cup \bigcup_{i<\omega} B_i(8i+2, 8i+4) \cup \bigcup_{i<\omega} C_i(8i+4, 8i+6) \cup \bigcup_{i<\omega} D_i(8i+6, 8i+8).$$

Theorem 2.2 implies

Lemma 3.3 *The set of pairs $\langle\varphi(x), n\rangle$ such that φ is a Σ–formula with at most one free variable x and $\mathbf{HF}(\mathbb{R}) \models \varphi(n)$ is computably enumerable.*

Suppose that the interior of S is Σ–definable. Then we have $8i+1 \in S \Leftrightarrow i \notin \mathtt{range}\,(f)$, which contradicts Lemma 3.3. Similarly, the assumption that the closure of S is Σ–definable leads to the condition $8i+5 \in S \Leftrightarrow i \notin \mathtt{range}\,(f)$, which contradicts Lemma 3.3. The proofs that the sets $\mathtt{Int}(\mathbb{R}\setminus S)$ and $\mathtt{cl}\,(\mathbb{R}\setminus S)$ are not Σ–definable could be done in a similar way. The rest statements of Theorem are easily verified.

□

The next result shows that, in general, one cannot hope even for a reasonable effective transformation of Σ–formulas such that the result of this transformation extracts an open subset of the set defined by the initial formula, and does not change this subset in the case when the initial formula already defines an open subset of $\mathbb{R}$.

Theorem 3.4 *There is no effective transformation $\varphi \mapsto \varphi^\circ$ of Σ–formulas with at most one free variable such that*

(i) *for each such Σ–formula $\varphi(x)$, the set $\varphi^\circ(x)^{\mathbf{HF}(\mathbb{R})}[x]$ is open and holds $\varphi^\circ(x)^{\mathbf{HF}(\mathbb{R})}[x] \subseteq \varphi(x)^{\mathbf{HF}(\mathbb{R})}[x]$;*

(ii) *for each such Σ–formula $\varphi(x)$, if the set $\varphi(x)^{\mathbf{HF}(\mathbb{R})}[x]$ is open then $\varphi^\circ(x)^{\mathbf{HF}(\mathbb{R})}[x] = \varphi(x)^{\mathbf{HF}(\mathbb{R})}[x]$.*

Proof. Let $f : \omega \to \omega$ be a computable function whose range is not computable. Let

$$A_n = \{1\} \cup \bigcup_{t<\omega,\, n\notin\{f(0),\dots,f(t)\}} \Big((\alpha_t^-(0,2), \alpha_{t+1}^-(0,2)] \cup [\alpha_{t+1}^+(0,2), \alpha_t^+(0,2))\Big),$$

where $\alpha_t^{\pm}(0,2)$ are taken from Lemma 3.2. Then clearly, A_n is open if and only if $n \notin \texttt{range}\,(f)$. One can easily ascertain that there exists a computable family $\varphi_n(x)$ of Σ–formulas such that for all $n \in \omega$ holds $\varphi_n(x)^{\mathbf{HF}(\mathbb{R})}[x] = A_n$.

The following condition could be easily verified:

$$1 \in \texttt{Int}(A_n) \Leftrightarrow n \notin \texttt{range}\,(f) \Leftrightarrow A_n \text{ is open.}$$

Suppose now that there exists an effective transformation $^\circ$ satisfying the condition of the theorem. Then we have

$$n \notin \texttt{range}\,(f) \Leftrightarrow \mathbf{HF}(\mathbb{R}) \models \varphi_n^\circ(1),$$

which by Lemma 3.3 implies that the set $\texttt{range}\,(f)$ is computable, which is a contradiction. Theorem is complete. $\square$

Consider an example. Let $\varphi(x)$ be a Σ–formula saying that $(x \in (0,2) \wedge x \neq 1) \vee (x \in (0,2) \wedge x = 1)$.

If we try to satisfy this formula in a direct way with some $x \in (0,1)$, then we should first examine the first part, namely $x \in (0,2) \wedge x \neq 1$. This check will be successful for $x \in (0,1) \cup (1,2)$. Next, we should satisfy the second part, $x \in (0,2) \wedge x = 1$, which either will be unsuccessful for $x \neq 1$ or gets stuck when $x = 1$. Anyway, we could satisfy this formula with elements of the set $(0,1) \cup (1,2)$ only. But it is evident, that $\varphi(x)$ is logically equivalent to the formula $x \in (0,2)$, which also defines an open set.

Thus, we can propose the following uniform way to extract open parts of the formulas, which, we believe, should work more or less reasonably. First we present a Σ–formula $\varphi(x)$ as an infinite disjunction $\bigvee_{i<\omega} \psi_i(x)$ of a computably enumerable family $(\psi_i(x))_{i<\omega}$ of a quantifier–free formulas; the algorithm enumerating members of this disjunction could be found uniformly in $\varphi(x)$. Then we enumerate all pairs $\langle a, b\rangle$ of rationals such that $\forall x \in (a,b) \bigvee_{i<t} \psi_i(x)$, for some t. Show it to be possible. The last condition could be uniformly in a, b, t reduced to an equivalent quantifier–free formula with no free variables, i.e., this formula could be effectively checked uniformly in t, a, b. This yields us an algorithm to enumerate all such pairs $\langle a, b\rangle$. Let $(\langle a_i, b_i\rangle)_{i<\omega}$ be such an enumeration. Now the result of transformation of φ is the infinite c.e. disjunction $\bigvee_{i<\omega}(a_i < x < b_i)$, whose algorithm enumerating its members could be found uniformly in $\varphi(x)$. By the remarks on the uniformity, this infinite disjunction could be presented as an equivalent Σ–formula if needed.

Of course, if we consider the following definition of the set $(0,2)$:

$$x = 1 \vee \bigvee_{n<\omega} \left(\left(0 < x < 1 - \frac{1}{n+1} \right) \vee \left(1 + \frac{1}{n+1} < x < 2 \right) \right),$$

the above algorithm will produce a formula that defines the set $(0,1) \cup (1,2)$, but intuitively, the above definition does not gives us an opportunity to ascertain that $1 \in (0,2)$ as well.

The following result shows that openness of a Σ-definable set is necessary but not sufficient to be Σ-definable without equality.

Theorem 3.5 *There exists an open set $S \subseteq \mathbb{R}$ such that*

(i) *S is Σ–definable;*

(ii) *S is not c.e. open.*

Proof. Fix some computable 1–1 onto mapping $q : \omega \to \mathbb{Q}$ (we denote $q(m) = q_m$). For $n \in \omega$ we let

$$S_n = \bigcup_{i \in W_n} B\left(q_{\ell(i)}, q_{r(i)}\right),$$

where W_n is nth c.e. set. Denote by W_n^t a finite part of W_n enumerated at first t steps. Let

$$S_n^t = \bigcup_{i \in W_n^t} B\left(q_{\ell(i)}, q_{r(i)}\right).$$

Note that each S_n is c.e. open and for each c.e. open set S there exists an n such that $S = S_n$. Moreover, the relation $a \in S_n^t$, $a \in \mathbb{Q}$, $n, t \in \omega$ is computable.

Now we simultaneously run ω processes. A process with the number n is assigned to its own interval $(n, n+1)$. At each step, it may generate subintervals of $(n, n+1)$. Namely, at step t, it first generates open intervals $I_{n,t}^- = \left(n, n + \frac{1}{2} - \frac{1}{t+4}\right)$ and $I_{n,t}^+ = \left(n + \frac{1}{2} + \frac{1}{t+4}, n+1\right)$. Next, if $n + \frac{1}{2} \in S_n^t$ then we take the minimal $i \in W_n^t$ such that $n + \frac{1}{2} \in B(q_{\ell(i)}, q_{r(i)}) \subseteq S_n^t$ and generate a new interval $B\left(n + \frac{1}{2}, \varepsilon\right)$ so that there exists a $c_n \in \mathbb{Q}$ such that

$$c_n \in B\left(q_{\ell(i)}, q_{r(i)}\right) \subseteq S_n \setminus \left(B\left(n + \frac{1}{2}, \varepsilon\right) \cup \bigcup_{t' \leqslant t} (I_{n,t'}^- \cup I_{n,t'}^+) \right).$$

We can effectively select such a $\varepsilon = q_k$ and $c_n = q_l$ with minimal possible numbers k and l. If c_n was defined at this step then we stop the nth process forever. Otherwise we pass to the next step.

Now define the set S as the union of all intervals generated by all these processes and single–point sets $\left\{n + \frac{1}{2}\right\}$, $n \in \omega$. Clearly, S is Σ–definable over $\mathbf{HF}(\mathbb{R})$.

We claim that S is open and does not coincide with each S_n, $n \in \omega$ and thus S is not c.e. open. The only points which are not evidiently internal points of S are points of kind $n + \frac{1}{2}$, $n \in \omega$. If $n + \frac{1}{2} \notin S_n$ then the nth process generates infinitely many intervals $I_{n,t}^-$ and $I_{n,t}^+$. One can easily see that in this case

$$\{n + 1/2\} \cup \bigcup_{t \in \omega} \left(I_{n,t}^- \cup I_{n,t}^+\right) = (n, n+1)$$

and thus $n + \frac{1}{2}$ is an internal point of S. If $n + \frac{1}{2} \in S_n$ then at some step an open interval containing this point is generated and thus it is an internal point of S again.

Assume that S coincides with some S_n. If $n + \frac{1}{2} \in S_n$ then by construction $c_n \in S_n \setminus S$. If $n + \frac{1}{2} \notin S_n$ then it remains to note that by definition of S, we have $n + \frac{1}{2} \in S$. It follows that $S \neq S_n$. Theorem is complete. □

4 An equivalent definition of Σ–definability without the equality test

Define the predicate $P^r(p, q) \subseteq \mathbb{R} \times \mathbb{Q}^2$ as follows:

$$P^r(p, q) \overset{df}{\Leftrightarrow} U(p) \wedge U(q) \wedge (p < r < q).$$

Here we use capital letters $X, Y, \ldots$ maybe with indices, as variables for upper indices in the predicates $P^X(x, y)$ and small letters, maybe with indices, for the rest cases. We assume the set of capital variables and the set of small variables to be disjoint.

Define the class of $\Delta_0^{\mathbb{R}}$–formulas as the smallest class of formulas that contains all atomic formulas of the signature $\sigma_{\mathbb{Q}}$, all formulas $P^X(y, z)$, and is closed under conjunctions, disjunctions, and negations.

The class of $\Sigma^{\mathbb{R}}$–formulas is defined as the smallest class of formulas which is closed under conjunctions, disjunctions, and bounded quantifications with small variables $\forall x \in y$ and $\exists x \in y$.

A formula $\Sigma^{\mathbb{R}}$–formula φ is called *positive* $\Sigma^{\mathbb{R}}$*–formula* if all the occurrences of predicates $P^X(x, y)$ in this formula are positive. Such formulas are referred to as $\Sigma_+^{\mathbb{R}}$–formulas.

For each $\Sigma_+^{\mathbb{R}}$–formula $\varphi(X_1, \ldots, X_m, y_1, \ldots, y_n)$ and for each $r_1, \ldots, r_m \in \mathbb{R}$, $q_1, \ldots, q_n \in \mathbf{HF}(\mathbb{Q})$, the relation $\mathbf{HF}(\mathbb{Q}) \models \varphi(r_1, \ldots, r_m,\, q_1, \ldots, q_n)$ is defined in a natural way by induction.

We say that *a set* $S \subseteq \mathbb{R}^n$ *is* $\Sigma_+^{\mathbb{R}}$*–definable* if there exists a $\Sigma_+^{\mathbb{R}}$–formula $\varphi(\bar{X}, \bar{y})$ and a tuple of parameters $\bar{q} \in \mathbf{HF}(\mathbb{Q})$ such that $S = \{\bar{r} \in \mathbb{R} \mid \mathbf{HF}(\mathbb{Q}) \models \varphi(\bar{r}, \bar{q})\}$. Taking into account that all elements of $\mathbf{HF}(\mathbb{Q})$ are Σ–definable over $\mathbf{HF}(\mathbb{Q})$, we may assume that the tuple $\bar{q}$ is empty.

Let $\bar{q} = \langle q_0', q_0'', q_1', q_1'', \ldots, q_m', q_m'' \rangle \in \mathbb{Q}^{2m}$. We define the $B(\bar{q})$ to be the set

$$B(\bar{q}) = \left\{ \bar{r} = \langle r_1, \ldots, r_m \rangle \in \mathbb{R}^m \mid \bigwedge_{i=1}^{m} (q_i' < r_i < q_i'') \right\}.$$

Theorem 4.1 *A set* $S \subseteq \mathbb{R}^m$ *is* $\Sigma_+^{\mathbb{R}}$*–definable if and only if there exists a computable function* $f : \omega \to \mathbb{Q}^{2m}$ *such that* $S = \bigcup_{i<\omega} B(f(i))$*. Moreover, given a* $\Sigma_+^{\mathbb{R}}$*–formula, one can effectively construct an algorithm to compute this function* f*.*

It follows that the concepts of Σ–definability without equality test and that of $\Sigma_+^{\mathbb{R}}$–definability are the same. Indeed, Theorem 4.1 and Theorem 2.2 show that any $\Sigma_+^{\mathbb{R}}$–set is computably enumerable without equality test. On the other hand, if a set is computably enumerable without equality test then it could be defined by a computable infinite disjunction $\bigvee_{i<\omega} \psi_i(\bar{x})$ of finite conjunctions of formulas of the kind $f(\bar{x}) < g(\bar{x})$. Using decidability of the elementary theory of $\mathbb{R}$, we can for each such formula $\psi_i(\bar{x})$, effectively enumerate the set S_i of all $\bar{q} \in \mathbb{Q}^{2m}$ such that $\forall \bar{x} \in B(\bar{q})\psi_i(\bar{x})$, moreover, it could be easily verified that $\psi_i(\bar{x})$ is equivalent to $\bigvee_{\bar{q}\in S_i}(\bar{x} \in B(\bar{q}))$, which proves our statement.

References

[1] Barwise, J., "Admissible sets and Structures," Springer Verlag, Berlin, 1975.

[2] Ershov, Yu. L., "Definability and computability," Plenum, New-York, 1996.

[3] Korovina, Margarita V., and Oleg V. Kudinov, *The Uniformity Principle for Σ-definability with Applications to Computable Analysis*, In Proceedings of CiE'07, Lecture Notes in Computer Science **4497** (2007), 416–425.

[4] Korovina, Margarita V., *Computational aspects of Σ-definability over the real numbers without the equality test*, In Proceedings of CSL'03, Lecture Notes in Computer Science **2803** (2003), 330–344.

[5] Korovina, Margarita V., and Oleg V. Kudinov, *Semantic characterisations of second-order computability over the real numbers*, In Proceedings of CSL'01, Lecture Notes in Computer Science **2142** (2001), 160–172.